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Spatial estimates for transient and steady-state solutions in transversely isotropic plates of Mindlin-type

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Abstract

This paper concerns with a state of bending for a linear transversely isotropic plate model based on the Mindlin assumption on the displacement. By using appropriate families of line-integral measures, we are able to establish results about the spatial behaviour of transient and steady-state solutions. All the results are obtained under relaxed hypotheses on the positive definiteness of the elasticity tensor.

Keywords: Mindlin plates; Bending; Transverse isotropy; Strong elliptic elasticity tensor; Spatial behaviour.

1 Introduction

The theory concerning elastic plates is useful in a wide range of practical applications, from building materials to electronic production; for this reason, properties of such particular mechanical structure are investigated in several articles, see e.g. Naghdi (1972), Green and Naghdi (1967), Lagnese and Lions (1988). The classical linear theory based on Kirchhoff's elastic strain-displacement relations completely neglects the effects of transverse shear forces, cf. Lagnese and Lions (1988) and Nowinski (1978). For this reason, mathematical discrepancies due to such assumption have led studies toward increasingly refined models that take into account not only the deflection of plate's middle section, but also transverse shear deformations.

The following assumptions on the displacement field characterize the Mindlin-type thin elastic plate theory, cf. Mindlin (1951) and Reissner (1944)

$$\begin{aligned} u_\alpha(x_1, x_2, x_3, t) &= u_\alpha^0(x_1, x_2, t) + x_3 v_\alpha(x_1, x_2, t), \\ u_3(x_1, x_2, x_3, t) &= w(x_1, x_2, t), \end{aligned}$$

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with respect to rectangular coordinates x_1, x_2, x_3 such that the x_1x_2 -plane represents the middle plane of the plate. Due to the linearity of the problem in concern, we can consider separately the extensional motion, characterized by u_1^0 and u_2^0 , and the state of bending, represented by v_1, v_2 and w . In this context, we can remark that the theory based on the Mindlin-model is developed by Constanda (1990) for the elastostatic bending of a thin plate, including the effects of transverse shear deformation. Furthermore, a dynamic model for small deformations of a thin thermoelastic plate is described by Schiavone and Tait (1993).

The Reissner–Mindlin and Kirchhoff–Love models are the two most common models of a thin linearly elastic plate. It is often remarked in the engineering literature that the Reissner–Mindlin model is more accurate, particularly for thin plates and when transverse shear plays a significant role, see Hughes (1987). In fact, both Mindlin (1951) and Reissner (1945, 1947) have independently proposed a plate theory that incorporates the effect of transverse shear deformation for analyzing thick plates.

Moreover, Arnold et al. (2002) show that the Reissner–Mindlin plate bending model has a wider range of applicability than the Kirchhoff–Love model for the approximation of clamped linearly elastic plates. In fact, under the assumption that the body force density is constant in the transverse direction, they prove that the Reissner–Mindlin model solution converges to the three-dimensional linear elasticity solution in the relative energy norm for the full range of surface loads.

On the other hand, the theory for transversely isotropic materials shows a good applicability for plate mathematical models that take into account transverse shear deformations (see, for details, the paper by Paroni et al. (2006)). Transversely isotropic materials, characterized by 5 elastic constants, have interesting applications in civil, mechanical and aerospace engineering. The main feature of such structures is represented by the existence of an axis of rotational symmetry, that is an axis for which any rotation about it does not change its properties; the plane perpendicular to this axis of rotational symmetry is called plane of isotropy. An example for this kind of materials is represented by laminates made of randomly oriented chopped fibers placed in a certain plane; mechanical properties of a bundled structure have no preferential direction in that plane. Each plane containing the axis of rotation is a plane of symmetry and therefore transversely isotropic materials admit an infinite number of elastic symmetries. Properties of such materials are widely investigated by Gurtin (1972), Padovani (2002), Merodio and Ogden (2003), Chiriță (2006), Chiriță et al. (2007) and Zhang et al. (2006). In addition, Simmonds (2004) obtains an exact 3-D Lévy-type solution for bending of an elastic slab taking into account transversal isotropy. In particular, Merodio and Ogden (2003), Chiriță (2006) and Chiriță et al. (2007) establish the necessary and sufficient conditions for strong ellipticity characterizing a transversely isotropic elastic solid.

For a transversely isotropic plate described through the Mindlin-model and without positive definiteness assumptions on the elasticity tensor, Passarella and Zampoli (2007) have established a uniqueness result and a Galerkin representation of solution of the field equations. Furthermore, under the hypotheses of positive definiteness of the elasticity tensor, they prove a variational theorem of Gurtin

type and a minimum principle.

In the present work, we take into account the linear theory characterizing a state of bending for (bounded and unbounded) plates of Mindlin-type, under the strong ellipticity condition on the elasticity tensor. In particular, using the time-weighted surface power function method (cf. Chiriță and Ciarletta (1999, 2003) and Ciarletta et al. (2005)), we derive the first-order differential inequalities satisfied by a family of appropriate integral measures depending on a specific parameter. Then, we obtain a spatial decay estimate of Saint-Venant's type with time-independent decay rate and we prove the existence of a domain of influence.

Moreover, we consider a (bounded and semi-infinite) strip subjected to prescribed harmonic data on its end. Starting from an idea shown by Ciarletta et al. (2005) and Chiriță (1995), we introduce a set of appropriate line-integral measures associated with the amplitude of time-harmonic vibrations. We derive a differential inequality describing the behaviour of steady-state solutions under the hypothesis that frequency of harmonic vibrations is lower than a certain critical value.

2 Formulation of the problem

Throughout this paper, we study the behaviour of a homogeneous and transversely isotropic elastic solid that occupies at time $t = 0$ the right cylinder \bar{B} of length $2h$ with the (bounded and unbounded) cross-section $\bar{\Sigma}$ and the smooth lateral boundary Π . We call B and Σ be the interiors of \bar{B} and $\bar{\Sigma}$, we choose the rectangular Cartesian coordinate frame in such a way that Ox_1x_2 is the middle plane and thus the faces of the plate are situated at $x_3 = \pm h$. Moreover, we suppose that Σ is a simply connected region, L is the boundary of Σ and that $h \ll \text{diam } \Sigma$.

We employ the usual summation and differentiation conventions. Latin subscripts (unless otherwise specified) are understood to range over the integers $\{1, 2, 3\}$, whereas Greek subscripts are confined to the range $\{1, 2\}$; summation over repeated subscripts is implied. Superposed dots or subscripts preceded by a comma mean partial derivative with respect to the time or the corresponding Cartesian variables. In this connection, we will disregard regularity questions, simply understanding a degree of smoothness sufficient to ensure analysis to be valid.

According to the linear theory of elastodynamics, the field equations are

$$\begin{array}{ll}
 \text{equations of motion} & t_{ji,j} + f_i = \rho \ddot{u}_i, \quad \text{on } B \times (0, \infty) \\
 \text{stress-strain relation} & t_{ij} = C_{ijkl} e_{kl} \\
 \text{strain-displacement relation} & e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})
 \end{array} \tag{1}$$

where u_i are the components of the displacement vector, t_{ij} are the components of the stress tensor, f_i are the components of the body force vector, ρ is the reference mass density and C_{ijkl} are the components of the symmetric elasticity tensor. We assume (see Gurtin (1972)) that the direction of

transversal isotropy is the one indicated by the x_3 coordinate axis and that

$$\begin{aligned} c_{ij} = C_{ijj} = c_{ji}, \quad i, j \text{ (not summed)} \in \{1, 2, 3\}, \quad c_{11} = c_{22}, \quad c_{13} = c_{23}, \\ c_{44} = c_{55} = C_{2323} = C_{1313}, \quad c_{66} = C_{1212} = \frac{c_{11} - c_{12}}{2}. \end{aligned} \quad (2)$$

Apart from the terms obtained using symmetries (2), these are the only non-zero components of C_{ijkl} . We consider the class of transversely isotropic materials having a strongly elliptic elasticity tensor, that is

$$C_{ijkl}m_i m_k n_j n_l > 0 \quad \text{for every non-zero vector } m_i, n_i.$$

The strong ellipticity condition becomes

$$\begin{aligned} c_{66} (n_1 m_2 - n_2 m_1)^2 + c_{11} (n_1 m_1 + n_2 m_2)^2 + 2(c_{13} + c_{44}) (n_1 m_1 + n_2 m_2) n_3 m_3 + \\ + c_{33} n_3^2 m_3^2 + c_{44} (n_3^2 m_1^2 + n_1^2 m_3^2 + n_3^2 m_2^2 + n_2^2 m_3^2) > 0 \end{aligned}$$

or equivalently (see e.g. Merodio and Ogden (2003) and Chiriță (2006))

$$c_{11} > 0, \quad c_{11} > c_{12}, \quad c_{33} > 0, \quad c_{44} > 0, \quad |c_{13} + c_{44}| < c_{44} + \sqrt{c_{11} c_{33}}. \quad (3)$$

Gurtin (1972) proves that \mathbf{C} is positive definite if

$$C_{ijkl}\phi_{ij}\phi_{kl} > 0 \quad \text{for every non-zero symmetric tensor } \phi_{ij}$$

and then it is strongly elliptic too.

A state of bending for an elastic plate is characterized by the following relations

$$u_\alpha(x_1, x_2, x_3, t) = -u_\alpha(x_1, x_2, -x_3, t), \quad u_3(x_1, x_2, x_3, t) = u_3(x_1, x_2, -x_3, t)$$

for every $(x_1, x_2, x_3) \in B$, $t \in [0, \infty)$; moreover, we assume that the body loads obey to the relations

$$f_\alpha(x_1, x_2, x_3, t) = -f_\alpha(x_1, x_2, -x_3, t), \quad f_3(x_1, x_2, x_3, t) = f_3(x_1, x_2, -x_3, t).$$

In the context of the theory of thin plates of uniform thickness such that u_i vary smoothly with respect to x_3 , we consider as independent variables:

$$v_\alpha(x_1, x_2, t) = \frac{3}{2h^3} \int_{-h}^h x_3 u_\alpha(x_1, x_2, x_3, t) dx_3, \quad w(x_1, x_2, t) = \frac{1}{2h} \int_{-h}^h u_3(x_1, x_2, x_3, t) dx_3.$$

If we use the following notations

$$\begin{aligned} M_{ij}(x_1, x_2, t) &= \frac{3}{2h^3} \int_{-h}^h x_3 t_{ij}(x_1, x_2, x_3, t) dx_3, \\ \tau_{ij}(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^h t_{ij}(x_1, x_2, x_3, t) dx_3, \\ F(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^h f_3(x_1, x_2, x_3, t) dx_3 + \frac{1}{h} t_{33}(x_1, x_2, h, t), \\ H_\alpha(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^h x_3 f_\alpha(x_1, x_2, x_3, t) dx_3 + t_{3\alpha}(x_1, x_2, h, t); \end{aligned}$$

we obtain $M_{\alpha 3} = 0$, $\tau_{\alpha\beta} = 0$, $\tau_{33} = 0$ since $t_{\alpha\beta}$, t_{33} are odd functions and $t_{\alpha 3}$ is even with respect to x_3 and we get the equations of motion

$$\frac{h^2}{3}M_{\beta\alpha,\beta} - \tau_{\alpha 3} + H_\alpha = \rho\frac{h^2}{3}\ddot{v}_\alpha, \quad \tau_{\beta 3,\beta} + F = \rho\ddot{w} \quad \text{on } \Sigma \times (0, \infty). \quad (4)$$

In the present paper, we restrict our attention to the state of bending characterized by

$$u_\alpha(x_1, x_2, x_3, t) = x_3 v_\alpha(x_1, x_2, t), \quad u_3(x_1, x_2, x_3, t) = w(x_1, x_2, t), \quad (5)$$

and we assume that

$$F(x_1, x_2, t) = 0, \quad H_\alpha(x_1, x_2, t) = 0.$$

Now, we can write the following equations of motion

$$\frac{h^2}{3}M_{\beta\alpha,\beta} - \tau_{\alpha 3} = \rho\frac{h^2}{3}\ddot{v}_\alpha, \quad \tau_{\beta 3,\beta} = \rho\ddot{w} \quad \text{on } \Sigma \times (0, \infty), \quad (6)$$

the constitutive equations

$$\begin{aligned} M_{11} &= c_{11}\epsilon_{11} + c_{12}\epsilon_{22}, & M_{12} &= M_{21} = (c_{11} - c_{12})\epsilon_{12}, \\ M_{22} &= c_{12}\epsilon_{11} + c_{11}\epsilon_{22}, & \tau_{\alpha 3} &= \tau_{3\alpha} = 2c_{44}\xi_\alpha, \end{aligned} \quad (7)$$

and the geometrical equations

$$\epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \xi_\alpha = \frac{1}{2}(w_{,\alpha} + v_\alpha). \quad (8)$$

The system (6) - (8) can be rewritten in terms of $\{v_\alpha, w\}$

$$\begin{aligned} \frac{h^2}{3} \frac{c_{11} - c_{12}}{2} v_{\alpha,\beta\beta} + \frac{h^2}{3} \frac{c_{11} + c_{12}}{2} v_{\beta,\beta\alpha} - c_{44}(w_{,\alpha} + v_\alpha) &= \rho\frac{h^2}{3}\ddot{v}_\alpha, \\ c_{44}w_{,\beta\beta} + c_{44}v_{\beta,\beta} &= \rho\ddot{w}. \end{aligned} \quad (9)$$

We consider the problem \mathcal{P} defined by (9), the homogeneous initial conditions

$$v_\alpha(x_1, x_2, 0) = 0, \quad w(x_1, x_2, 0) = 0, \quad \dot{v}_\alpha(x_1, x_2, 0) = 0, \quad \dot{w}(x_1, x_2, 0) = 0, \quad \text{on } \bar{\Sigma} \quad (10)$$

and the boundary conditions (Dirichlet problem)

$$v_\alpha = \check{v}_\alpha, \quad w = \check{w}, \quad \text{on } L_1 \times [0, +\infty), \quad v_\alpha = 0, \quad w = 0, \quad \text{on } L_2 \times [0, +\infty), \quad (11)$$

where L_1 and L_2 are disjoint and complementary nonempty subsets of L and \check{v}_α and \check{w} are assigned fields representing external data \mathcal{D} of the problem in concern. Throughout this paper we assume that a smooth solution of the problem \mathcal{P} exists.

The internal energy density of a transversely isotropic elastic plate (considering a state of bending) is a positive definite quadratic form if and only if $c_{44} > 0$, $c_{11} > 0$, $|c_{12}| < c_{11}$. From strong ellipticity conditions (3), we observe that it is $c_{12} < c_{11}$, thus there is a relaxation upon constitutive coefficients with respect to positiveness conditions.

3 Spatial behaviour of transient solutions and estimates related to κ -measure

In this section, we establish results describing the spatial behaviour of solutions of given data on the interval $[0, T]$ under the strong ellipticity condition on the elasticity tensor.

By using the equality $v_{\alpha,12} = v_{\alpha,21}$, we can rewrite the system (9) in the following form

$$\frac{h^2}{3}\widehat{M}_{\beta\alpha,\beta} - \tau_{\alpha 3} = \rho\frac{h^2}{3}\ddot{v}_\alpha, \quad \tau_{\beta 3,\beta} = \rho\ddot{w} \quad \text{on } \overline{\Sigma} \times (0, \infty), \quad (12)$$

where $\widehat{M}_{\beta\alpha}$ are given by

$$\begin{aligned} \widehat{M}_{11} &= c_{11}v_{1,1} + \frac{c_{11} - \kappa}{2}v_{2,2}, & \widehat{M}_{12} &= \frac{c_{12} + \kappa}{2}v_{1,2} + \frac{c_{11} - c_{12}}{2}v_{2,1}, \\ \widehat{M}_{21} &= \frac{c_{11} - c_{12}}{2}v_{1,2} + \frac{c_{12} + \kappa}{2}v_{2,1}, & \widehat{M}_{22} &= \frac{c_{11} - \kappa}{2}v_{1,1} + c_{11}v_{2,2}. \end{aligned} \quad (13)$$

Here the parameter κ is introduced for mathematical convenience, more precisely, in order to get an appropriate family of measures which can cover the whole class of strongly elliptic elastic materials with transverse isotropy. For example, if $\kappa = -c_{12}$ then we obtain $\widehat{M}_{\beta\alpha} = \frac{c_{11} - c_{12}}{2}v_{\alpha,\beta} + \frac{c_{11} + c_{12}}{2}\delta_{\alpha\beta}v_{\tau,\tau}$, and if $\kappa = c_{11}$ then we get $\widehat{M}_{\beta\alpha} = \frac{c_{11} - c_{12}}{2}v_{\alpha,\beta} + \frac{c_{11} + c_{12}}{2}v_{\beta,\alpha}$. Moreover, if we choose $\kappa = c_{11} - 2c_{12}$, then $\widehat{M}_{\alpha\beta} = M_{\alpha\beta}$.

We define the support D_T^* of external data \mathcal{D} on the time interval $[0, T]$, that is

$$D_T^* = \{\mathbf{x} \in L : \exists s \in [0, T] \quad \text{so that } \check{v}_1(\mathbf{x}, s) \neq 0 \text{ or } \check{v}_2(\mathbf{x}, s) \neq 0 \text{ or } \check{w}(\mathbf{x}, s) \neq 0\}$$

and, for convenience, we assume that D_T^* is a bounded regular curve of L . We consider the following sets

$$D_r = \left\{ \mathbf{x} \in \overline{\Sigma} : D_T^* \cap \overline{S(\mathbf{x}, r)} \neq \emptyset \right\}, \quad \Sigma_r = \overline{\Sigma} \setminus D_r, \quad \Sigma(r_1, r_2) = \Sigma_{r_2} \setminus \Sigma_{r_1},$$

where $r \geq 0, r_2 \leq r_1$ and $\overline{S(\mathbf{x}, r)}$ is the closed disk with radius r and center at \mathbf{x} . Further, let L_r be the subcurve of $\partial\Sigma_r$ contained inside $\overline{\Sigma}$ and whose unit normal vector \mathbf{n} is forwarded to the exterior of D_r . For any positive parameter λ , we define the following function

$$\mathcal{I}_\kappa(r, t) = - \int_0^t \int_{L_r} e^{-\lambda s} \left[\tau_{\beta 3}(s)\dot{w}(s) + \frac{h^2}{3}\widehat{M}_{\beta\alpha}(s)\dot{v}_\alpha(s) \right] n_\beta dl ds, \quad r \geq 0, \quad t \in [0, T]. \quad (14)$$

Now, with the help of Eqs. (12), (13) we obtain

$$\left[\tau_{\beta 3}\dot{w} + \frac{h^2}{3}\widehat{M}_{\beta\alpha}\dot{v}_\alpha \right]_{,\beta} = \frac{\partial}{\partial t} [\mathcal{T} + \mathcal{W}], \quad (15)$$

where

$$\begin{aligned} \mathcal{T} &= \frac{\rho}{2} \left(\dot{w}^2 + \frac{h^2}{3}\dot{v}_\alpha\dot{v}_\alpha \right), & \mathcal{W} &= \frac{1}{2} \left(\mathcal{W}_0 + \frac{h^2}{3}\mathcal{W}_1 + \frac{h^2}{3}\mathcal{W}_2 \right), \\ \mathcal{W}_0 &= 4c_{44}(\xi_1^2 + \xi_2^2), & \mathcal{W}_1 &= c_{11}(v_{1,1}^2 + v_{2,2}^2) + (c_{11} - \kappa)v_{1,1}v_{2,2}, \\ \mathcal{W}_2 &= \frac{c_{11} - c_{12}}{2}(v_{2,1}^2 + v_{1,2}^2) + (c_{12} + \kappa)v_{2,1}v_{1,2}. \end{aligned} \quad (16)$$

We use the definitions of Σ_r and $\mathcal{I}_\kappa(r, t)$, the divergence theorem and Eq. (15) in order to write, for $r_2 < r_1$,

$$\begin{aligned} \mathcal{I}_\kappa(r_1, t) - \mathcal{I}_\kappa(r_2, t) &= - \int_0^t \int_{\Sigma(r_1, r_2)} e^{-\lambda s} \frac{\partial}{\partial s} [\mathcal{T}(s) + \mathcal{W}(s)] d\sigma ds = \\ &= - \int_{\Sigma(r_1, r_2)} e^{-\lambda t} [\mathcal{T}(t) + \mathcal{W}(t)] d\sigma - \lambda \int_0^t \int_{\Sigma(r_1, r_2)} e^{-\lambda s} [\mathcal{T}(s) + \mathcal{W}(s)] d\sigma ds. \end{aligned} \quad (17)$$

We can then prove that function $\mathcal{I}_\kappa(r, t)$ satisfies some properties.

Theorem 1. Let ρ be greater than zero, the strong ellipticity condition (3) be valid and κ satisfy the relation

$$-c_{11} < \kappa < \min \{3c_{11}, c_{11} - 2c_{12}\}. \quad (18)$$

For a solution $\{v_\alpha, w\}$ of initial-boundary value problem \mathcal{P} and called \widehat{D}_T the bounded support of external data \mathcal{D} on the time interval $[0, T]$, it is possible to prove that

1) $\mathcal{I}_\kappa(r, t)$ is a continuous differentiable function on $r \geq 0$ and

$$\frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t) = - \int_{L_r} e^{-\lambda t} \left[\tau_{\beta 3}(t) \dot{w}(t) + \frac{h^2}{3} \widehat{M}_{\beta \alpha}(t) \dot{v}_\alpha(t) \right] n_\beta dl, \quad (19)$$

$$\frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t) = - \int_{L_r} e^{-\lambda t} [\mathcal{T}(t) + \mathcal{W}(t)] dl - \lambda \int_0^t \int_{L_r} e^{-\lambda s} [\mathcal{T}(s) + \mathcal{W}(s)] dl ds. \quad (20)$$

2) $\mathcal{I}_\kappa(r, t)$ is a non-increasing function with respect to r , i.e.

$$\mathcal{I}_\kappa(r_1, t) \leq \mathcal{I}_\kappa(r_2, t) \quad r_2 < r_1. \quad (21)$$

3) $\mathcal{I}_\kappa(r, t)$ satisfies the following first-order differential inequalities

$$|\mathcal{I}_\kappa(r, t)| + \frac{c_\kappa}{\lambda} \frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t) \leq 0 \quad \text{and} \quad \left| \frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t) \right| + c_\kappa \frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t) \leq 0, \quad (22)$$

where

$$c_\kappa = \sqrt{\frac{\eta_\kappa}{\rho}} \quad \text{with} \quad \eta_\kappa = \max \left\{ c_{44}, \frac{c_{11} + \kappa}{2}, \frac{3c_{11} - \kappa}{2}, \frac{c_{11} - 2c_{12} - \kappa}{2} \right\}. \quad (23)$$

4) $\mathcal{I}_\kappa(r, t)$ is a non-negative function and

$$0 \leq \mathcal{I}_\kappa(r, t) = \int_{\Sigma_r} e^{-\lambda t} [\mathcal{T}(t) + \mathcal{W}(t)] d\sigma + \lambda \int_0^t \int_{\Sigma_r} e^{-\lambda s} [\mathcal{T}(s) + \mathcal{W}(s)] d\sigma ds. \quad (24)$$

Proof. From a direct differentiation of Eq. (14) with respect to the variable t we get Eq. (19), while from Eq. (17) and through a differentiation with respect to r , we have Eq. (20).

The kinetic energy \mathcal{T} and the quadratic form \mathcal{W} are positive definite if $\rho > 0$ and hypotheses (3), (18) hold, thus Eq. (17) implies that $\mathcal{I}_\kappa(r, t)$ is a non-increasing function with respect to r .

We can estimate $\mathcal{I}_\kappa(r, t)$ and $\frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t)$ in terms of $\frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t)$. At the beginning, we denote by \mathcal{A}_α the matrix associated to the quadratic form \mathcal{W}_α and we consider the functional $\mathcal{F}[\mathcal{A}_\alpha; \varphi, \gamma] \equiv \varphi \cdot \mathcal{A}_\alpha \gamma$ for the variables $\varphi = \{\varphi_1, \varphi_2\}$ and $\gamma = \{\gamma_1, \gamma_2\}$. The hypotheses (3), (18) imply

$$k_m^{(\alpha)} (\gamma_1^2 + \gamma_2^2) \leq \mathcal{F}[\mathcal{A}_\alpha; \gamma, \gamma] \leq k_M^{(\alpha)} (\gamma_1^2 + \gamma_2^2), \quad (25)$$

where

$$\begin{aligned} k_m^{(1)} &= \min \left\{ \frac{c_{11} + \kappa}{2}, \frac{3c_{11} - \kappa}{2} \right\}, & k_m^{(2)} &= \min \left\{ \frac{c_{11} + \kappa}{2}, \frac{c_{11} - 2c_{12} - \kappa}{2} \right\}, \\ k_M^{(1)} &= \max \left\{ \frac{c_{11} + \kappa}{2}, \frac{3c_{11} - \kappa}{2} \right\}, & k_M^{(2)} &= \max \left\{ \frac{c_{11} + \kappa}{2}, \frac{c_{11} - 2c_{12} - \kappa}{2} \right\}. \end{aligned} \quad (26)$$

Through Schwarz inequality and Eqs. (13), (16), we can see that

$$\mathcal{F}[\mathcal{A}_\alpha; \varphi, \gamma] \leq [\mathcal{F}[\mathcal{A}_\alpha; \varphi, \varphi]]^{1/2} [\mathcal{F}[\mathcal{A}_\alpha; \gamma, \gamma]]^{1/2} \quad (27)$$

and

$$\begin{aligned} \mathcal{F}[\mathcal{A}_1; \widehat{\mathbf{M}}^{(1)}, \widehat{\gamma}^{(1)}] &= \widehat{M}_{11}^2 + \widehat{M}_{22}^2, & \mathcal{F}[\mathcal{A}_1; \widehat{\gamma}^{(1)}, \widehat{\gamma}^{(1)}] &= \mathcal{W}_1, \\ \mathcal{F}[\mathcal{A}_2; \widehat{\mathbf{M}}^{(2)}, \widehat{\gamma}^{(2)}] &= \widehat{M}_{21}^2 + \widehat{M}_{12}^2, & \mathcal{F}[\mathcal{A}_2; \widehat{\gamma}^{(2)}, \widehat{\gamma}^{(2)}] &= \mathcal{W}_2, \end{aligned} \quad (28)$$

where

$$\widehat{\mathbf{M}}^{(1)} = \{\widehat{M}_{11}, \widehat{M}_{22}\}, \quad \widehat{\gamma}^{(1)} = \{v_{1,1}, v_{2,2}\}, \quad \widehat{\mathbf{M}}^{(2)} = \{\widehat{M}_{21}, \widehat{M}_{12}\}, \quad \widehat{\gamma}^{(2)} = \{v_{1,2}, v_{2,1}\}.$$

Collecting Eqs. (16), (25) - (28), we deduce that

$$\begin{aligned} \tau_{\beta 3} \tau_{\beta 3} &= c_{44} \mathcal{W}_0, \\ \widehat{M}_{11}^2 + \widehat{M}_{22}^2 &\leq \mathcal{F}^{1/2}[\mathcal{A}_1; \widehat{\mathbf{M}}^{(1)}, \widehat{\mathbf{M}}^{(1)}] \mathcal{W}_1^{1/2} \leq [k_M^{(1)} (\widehat{M}_{11}^2 + \widehat{M}_{22}^2)]^{1/2} \mathcal{W}_1^{1/2}, \\ \widehat{M}_{21}^2 + \widehat{M}_{12}^2 &\leq \mathcal{F}^{1/2}[\mathcal{A}_2; \widehat{\mathbf{M}}^{(2)}, \widehat{\mathbf{M}}^{(2)}] \mathcal{W}_2^{1/2} \leq [k_M^{(2)} (\widehat{M}_{21}^2 + \widehat{M}_{12}^2)]^{1/2} \mathcal{W}_2^{1/2} \end{aligned} \quad (29)$$

and, consequently, we can conclude

$$\tau_{\beta 3} \tau_{\beta 3} + \frac{h^2}{3} \widehat{M}_{\beta\alpha} \widehat{M}_{\beta\alpha} \leq c_{44} \mathcal{W}_0 + \frac{h^2 k_M^{(1)}}{3} \mathcal{W}_1 + \frac{h^2 k_M^{(2)}}{3} \mathcal{W}_2 \leq 2\eta_\kappa \mathcal{W}, \quad (30)$$

where $\eta_\kappa = \max \{c_{44}, k_M^{(1)}, k_M^{(2)}\}$ is given by Eq. (23). Using Cauchy-Schwarz and arithmetic-geometric mean inequalities, it is

$$\left| \left[\tau_{\beta 3} \dot{w} + \frac{h^2}{3} \widehat{M}_{\beta\alpha} \dot{v}_\alpha \right] n_\beta \right| \leq \frac{\epsilon \rho}{2} \left[\dot{w}^2 + \frac{h^2}{3} \dot{v}_\alpha \dot{v}_\alpha \right] + \frac{1}{2\epsilon \rho} \left[\tau_{\beta 3} \tau_{\beta 3} + \frac{h^2}{3} \widehat{M}_{\beta\alpha} \widehat{M}_{\beta\alpha} \right]. \quad (31)$$

Furthermore, we use estimates (30), (31) into Eqs. (14) and (19) in order to obtain

$$|\mathcal{I}_\kappa(r, t)| \leq \epsilon \int_0^t \int_{L_r} e^{-\lambda s} \left[\mathcal{T}(s) + \frac{\eta_\kappa}{\epsilon^2 \rho} \mathcal{W}(s) \right] dl ds, \quad (32)$$

$$\left| \frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t) \right| \leq \epsilon \int_{L_r} e^{-\lambda t} \left[\mathcal{T}(t) + \frac{\eta_\kappa}{\epsilon^2 \rho} \mathcal{W}(t) \right] dl. \quad (33)$$

Remembering Eq. (20) and setting $\epsilon = c_\kappa = \sqrt{\frac{\eta_\kappa}{\rho}}$ into (32), (33), we verify differential inequalities (22).

If the plate is bounded, then r ranges in $[0, \ell]$, with $\ell = \max_{\mathbf{x} \in \bar{\Sigma}} \left\{ \min_{\mathbf{y} \in D_T^*} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \right\}$. In view of the homogeneous initial conditions and by taking into consideration the definition for L_r from (14) we deduce that

$$\mathcal{I}_\kappa(r, 0) = 0, \quad \mathcal{I}_\kappa(\ell, t) = 0, \quad r \in [0, \ell], \quad t \in [0, T].$$

Thus, by setting $r_1 = \ell$, $r_2 = r \in [0, \ell]$ into relations (17), (21), we arrive to Eq. (24).

On the other hand, if the plate is unbounded, then the variable r ranges in $[0, \infty)$. Eq. (22)₂ can be rewritten as follows

$$\frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t) + c_\kappa \frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t) \leq 0, \quad -\frac{\partial \mathcal{I}_\kappa}{\partial t}(r, t) + c_\kappa \frac{\partial \mathcal{I}_\kappa}{\partial r}(r, t) \leq 0. \quad (34)$$

If we choose a pair (r_0, t_0) such that $t_0 \in [0, T]$ and $r_0 \geq c_\kappa t_0$ and we put $t = t_0 + \frac{r - r_0}{c_\kappa}$ into inequality (34)₁, we obtain

$$\frac{d}{dr} \left[\mathcal{I}_\kappa \left(r, t_0 + \frac{r - r_0}{c_\kappa} \right) \right] \leq 0 \quad \Rightarrow \quad \mathcal{I}_\kappa \left(r_1, t_0 + \frac{r_1 - r_0}{c_\kappa} \right) \leq \mathcal{I}_\kappa \left(r_2, t_0 + \frac{r_2 - r_0}{c_\kappa} \right), \quad r_1 \geq r_2,$$

so that, considering $r_1 = r_0$ and $r_2 = r_0 - c_\kappa t_0$, we deduce

$$\mathcal{I}_\kappa(r_0, t_0) \leq \mathcal{I}_\kappa(r_0 - c_\kappa t_0, 0) = 0. \quad (35)$$

Similarly, if we set $t = t_0 - \frac{r - r_0}{c_\kappa}$ in inequality (34)₂, then

$$\frac{d}{dr} \left[\mathcal{I}_\kappa \left(r, t_0 - \frac{r - r_0}{c_\kappa} \right) \right] \leq 0 \quad \Rightarrow \quad \mathcal{I}_\kappa \left(r_1, t_0 - \frac{r_1 - r_0}{c_\kappa} \right) \leq \mathcal{I}_\kappa \left(r_2, t_0 - \frac{r_2 - r_0}{c_\kappa} \right), \quad r_1 \geq r_2$$

and thus, for $r_1 = r_0 + c_\kappa t_0$ and $r_2 = r_0$,

$$\mathcal{I}_\kappa(r_0, t_0) \geq \mathcal{I}_\kappa(r_0 + c_\kappa t_0, 0) = 0. \quad (36)$$

We can notice from relations (35) and (36) that

$$\mathcal{I}_\kappa(\infty, t) \equiv \lim_{r_0 \rightarrow \infty} \mathcal{I}_\kappa(r_0, t_0) = 0. \quad (37)$$

Finally, through Eqs. (17), (21), (37) we can arrive to inequality (24) and the proof is complete. ■

The results obtained up to now lead us to formulate the following theorem.

Theorem 2. Let hypotheses of previous theorem be still valid. Then, for each fixed $t \in [0, T]$, we have the following results.

1. *Spatial decay:*

$$\mathcal{I}_\kappa(r, t) \leq \exp \left(-\frac{\lambda}{c_\kappa} r \right) \mathcal{I}_\kappa(0, t), \quad 0 \leq r < c_\kappa t. \quad (38)$$

2. *Domain of influence:*

$$v_\alpha = w = 0, \quad \text{on } \Sigma_r \times [0, T], \quad c_\kappa t \leq r. \quad (39)$$

Proof. Since $\mathcal{I}_\kappa(r, t)$ is a non-negative function, then we can write from Eq. (22)₁

$$\frac{\partial}{\partial r} \left[\exp\left(\frac{\lambda}{c_\kappa} r\right) \mathcal{I}_\kappa(r, t) \right] \leq 0 \quad \Rightarrow \quad \mathcal{I}_\kappa(r, t) \leq \exp\left(-\frac{\lambda}{c_\kappa} r\right) \mathcal{I}_\kappa(0, t).$$

Now, putting $r = c_\kappa t$ into (22)₂, we have

$$\frac{d}{dt} [\mathcal{I}_\kappa(c_\kappa t, t)] \leq 0 \quad \Rightarrow \quad \mathcal{I}_\kappa(c_\kappa t, t) \leq \mathcal{I}_\kappa(0, 0) = 0, \quad 0 \leq t. \quad (40)$$

On the other hand, we deduce from Eqs. (21), (24) that

$$0 \leq \mathcal{I}_\kappa(r, t) \leq \mathcal{I}_\kappa(c_\kappa t, t), \quad r \geq c_\kappa t. \quad (41)$$

Thus, Eqs. (40), (41) imply

$$\mathcal{I}_\kappa(r, t) = 0, \quad \text{for } r \geq c_\kappa t. \quad (42)$$

Obviously, Eq. (39) follows from Eqs. (24), (42) and from the positive definiteness of \mathcal{T} and \mathcal{W} . ■

Now, according to Gurtin (1972), we depict the domain of influence of external given data at time T as the set of the points of $\bar{\Sigma}$ that can be reached by signals propagating from the support \hat{D}_T on the time interval $[0, T]$, with speeds equal to or less than the maximum speed of propagation

$$c = \sqrt{\frac{\eta}{\rho}} \quad \text{with } \eta = \inf_{\kappa} \eta_\kappa$$

where $\kappa \in]-c_{11}, \min\{3c_{11}, c_{11} - 2c_{12}\}[$; in particular, $|c_{12}| < c_{11} \Rightarrow \kappa \in]-c_{11}, c_{11} - 2c_{12}[$, while $c_{12} \leq -c_{11} \Rightarrow \kappa \in]-c_{11}, 3c_{11}[$.

4 Spatial behaviour of steady-state solutions and estimates related to κ -measure

Throughout this section we study the problem of spatial behaviour of steady vibrations for transversely isotropic elastic plates having a strongly elliptic elasticity tensor. We assume that Σ is a rectangular strip and choose a Cartesian frame reference such that the middle surface of the plate $\bar{\Sigma}$ is defined by

$$\bar{\Sigma} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, \ell_1], x_2 \in [0, \ell_2]\}, \quad (43)$$

where ℓ_1, ℓ_2 are some positive constants and $\ell_2 < \infty$.

We discuss the problem of steady-state vibrations assuming that $\{v_\alpha, w\}$ are separable with respect to space and time variables and that the time dependence is periodic, that is

$$v_\alpha = \text{Re}[\zeta_\alpha(x_1, x_2; \omega)e^{i\omega t}], \quad w = \text{Re}[\psi(x_1, x_2; \omega)e^{i\omega t}],$$

where $\text{Re}[f]$ represents the real part of f , $\omega \in \mathbb{R}^+$ is the prescribed frequency of oscillation and ζ_α, ψ are complex vector functions. The equations of motion (9) imply that the amplitude $\{\zeta_\alpha, \psi\}$ satisfies

the following system of differential equations

$$\begin{aligned} \frac{h^2}{3} \left[\frac{c_{11} - c_{12}}{2} \zeta_{\alpha,\beta\beta} + \frac{c_{11} + c_{12}}{2} \zeta_{\beta,\beta\alpha} \right] - c_{44}(\psi_{,\alpha} + \zeta_\alpha) + \rho \frac{h^2 \omega^2}{3} \zeta_\alpha &= 0, \\ c_{44}(\psi_{,\beta\beta} + \zeta_{\beta,\beta}) + \rho \omega^2 \psi &= 0. \end{aligned} \quad (44)$$

In what follows, we assume prescribed harmonic vibrations insisting on the end located at $x_1 = 0$, so that we consider the problem \mathcal{P}_0 defined by Eqs. (44), homogeneous initial conditions and the following boundary conditions

$$\begin{aligned} \zeta_\alpha(x_1, 0) &= 0, \quad \psi(x_1, 0) = 0, \quad x_1 \in [0, \ell_1], \\ \zeta_\alpha(x_1, \ell_2) &= 0, \quad \psi(x_1, \ell_2) = 0, \quad x_1 \in [0, \ell_1], \\ \zeta_\alpha(\ell_1, x_2) &= 0, \quad \psi(\ell_1, x_2) = 0, \quad x_2 \in [0, \ell_2], \\ \zeta_\alpha(0, x_2) &= \check{\zeta}_\alpha(x_2), \quad \psi(0, x_2) = \check{\psi}(x_2), \quad x_2 \in [0, \ell_2] \end{aligned} \quad (45)$$

where $\check{\zeta}_\alpha$ and $\check{\psi}$ are prescribed continuous functions such that

$$\check{\zeta}_\alpha(0) = 0, \quad \check{\psi}(0) = 0, \quad \check{\zeta}_\alpha(\ell_2) = 0, \quad \check{\psi}(\ell_2) = 0.$$

The system (44) can be rewritten in the following form

$$\frac{h^2}{3} \Gamma_{\beta\alpha,\beta} - \chi_\alpha + \rho \frac{h^2 \omega^2}{3} \zeta_\alpha = 0, \quad \chi_{\beta,\beta} + \rho \omega^2 \psi = 0 \quad \text{on } \bar{\Sigma} \times (0, \infty) \quad (46)$$

where

$$\begin{aligned} \Gamma_{11} &= c_{11} \zeta_{1,1} + \frac{c_{11} - \kappa}{2} \zeta_{2,2}, & \Gamma_{12} &= \frac{c_{12} + \kappa}{2} \zeta_{1,2} + \frac{c_{11} - c_{12}}{2} \zeta_{2,1}, \\ \Gamma_{21} &= \frac{c_{11} - c_{12}}{2} \zeta_{1,2} + \frac{c_{12} + \kappa}{2} \zeta_{2,1}, & \Gamma_{22} &= \frac{c_{11} - \kappa}{2} \zeta_{1,1} + c_{11} \zeta_{2,2} \end{aligned} \quad (47)$$

and

$$\chi_\alpha = c_{44}(\psi_{,\alpha} + \zeta_\alpha). \quad (48)$$

We define the following function

$$\mathcal{J}_\kappa(x_1) = \int_{L_{x_1}} \left[\bar{\chi}_1 \psi + \chi_1 \bar{\psi} + \frac{h^2}{3} (\bar{\Gamma}_{1\alpha} \zeta_\alpha + \Gamma_{1\alpha} \bar{\zeta}_\alpha) \right] dx_2, \quad x_1 \in [0, \ell_1], \quad (49)$$

where superposed bars denote complex conjugates and $L_{x_1} = \{(x_1, x_2): x_2 \in [0, \ell_2]\}$. Considering Eqs. (47) - (49), we have

$$\begin{aligned} \mathcal{J}_\kappa(x_1) &= \int_{L_{x_1}} \left\{ c_{44} [(\bar{\psi}_{,1} + \bar{\zeta}_1) \psi + (\psi_{,1} + \zeta_1) \bar{\psi}] + \frac{h^2}{3} [c_{11} (\bar{\zeta}_{1,1} \zeta_1 + \zeta_{1,1} \bar{\zeta}_1) + \right. \\ &\quad \left. + \frac{c_{11} - \kappa}{2} (\bar{\zeta}_{2,2} \zeta_1 + \zeta_{2,2} \bar{\zeta}_1) + \frac{c_{12} + \kappa}{2} (\bar{\zeta}_{1,2} \zeta_2 + \zeta_{1,2} \bar{\zeta}_2) + \frac{c_{11} - c_{12}}{2} (\bar{\zeta}_{2,1} \zeta_2 + \zeta_{2,1} \bar{\zeta}_2)] \right\} dx_2. \end{aligned} \quad (50)$$

Arithmetic-geometric and Schwarz inequalities lead to

$$\begin{aligned}
 \int_{L_{x_1}} (\bar{\zeta}_{1,1}\zeta_1 + \zeta_{1,1}\bar{\zeta}_1) dx_2 &\leq \left(\int_{L_{x_1}} \zeta_{1,1}\bar{\zeta}_{1,1} dx_2 \right)^{1/2} \left(\frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \zeta_{1,2}\bar{\zeta}_{1,2} dx_2 \right)^{1/2} \leq \frac{\ell_2}{2\pi} \int_{L_{x_1}} \zeta_{1,\alpha}\bar{\zeta}_{1,\alpha} dx_2, \\
 \int_{L_{x_1}} (\bar{\zeta}_{2,2}\zeta_1 + \zeta_{2,2}\bar{\zeta}_1) dx_2 &\leq \left(\int_{L_{x_1}} \zeta_{2,2}\bar{\zeta}_{2,2} dx_2 \right)^{1/2} \left(\frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \zeta_{1,2}\bar{\zeta}_{1,2} dx_2 \right)^{1/2} \leq \frac{\ell_2}{2\pi} \int_{L_{x_1}} \zeta_{\alpha,2}\bar{\zeta}_{\alpha,2} dx_2, \\
 \int_{L_{x_1}} [(\bar{\zeta}_1 + \bar{\psi}_{,1})\psi + (\zeta_1 + \psi_{,1})\bar{\psi}] dx_2 &\leq \left(\int_{L_{x_1}} (\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) dx_2 \right)^{1/2} \left(\frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \psi_{,2}\bar{\psi}_{,2} dx_2 \right)^{1/2} \\
 &\leq \frac{\ell_2}{2\pi} \int_{L_{x_1}} (\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) dx_2 + \frac{\ell_2}{2\pi} \int_{L_{x_1}} \psi_{,2}\bar{\psi}_{,2} dx_2, \\
 \int_{L_{x_1}} (\bar{\zeta}_{1,2}\zeta_2 + \zeta_{1,2}\bar{\zeta}_2) dx_2 &\leq \left(\int_{L_{x_1}} \zeta_{1,2}\bar{\zeta}_{1,2} dx_2 \right)^{1/2} \left(\frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \zeta_{2,2}\bar{\zeta}_{2,2} dx_2 \right)^{1/2} \leq \frac{\ell_2}{2\pi} \int_{L_{x_1}} \zeta_{\alpha,2}\bar{\zeta}_{\alpha,2} dx_2, \\
 \int_{L_{x_1}} (\bar{\zeta}_{2,1}\zeta_2 + \zeta_{2,1}\bar{\zeta}_2) dx_2 &\leq \left(\int_{L_{x_1}} \zeta_{2,1}\bar{\zeta}_{2,1} dx_2 \right)^{1/2} \left(\frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \zeta_{2,2}\bar{\zeta}_{2,2} dx_2 \right)^{1/2} \leq \frac{\ell_2}{2\pi} \int_{L_{x_1}} \zeta_{2,\alpha}\bar{\zeta}_{2,\alpha} dx_2,
 \end{aligned} \tag{51}$$

where we have used the well-known membrane problem and boundary conditions (45)_{1,2}, i.e.

$$\frac{\pi^2}{\ell_2^2} \int_{L_{x_1}} \zeta_{\alpha}\bar{\zeta}_{\alpha} dx_2 \leq \int_{L_{x_1}} \zeta_{\alpha,2}\bar{\zeta}_{\alpha,2} dx_2, \quad \frac{\pi^2}{\ell_2^2} \int_{L_{x_1}} \psi\bar{\psi} dx_2 \leq \int_{L_{x_1}} \psi_{,2}\bar{\psi}_{,2} dx_2. \tag{52}$$

By means of Eqs. (50), (51) we obtain

$$\begin{aligned}
 |\mathcal{J}_{\kappa}(x_1)| &\leq \int_{L_{x_1}} \left[\frac{\ell_2 c_{44}}{2\pi} (\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + \frac{\ell_2 c_{44}}{2\pi} \psi_{,2}\bar{\psi}_{,2} + \frac{h^2 \ell_2 c_{11}}{6\pi} \zeta_{1,\alpha}\bar{\zeta}_{1,\alpha} + \right. \\
 &\quad \left. + \frac{h^2 \ell_2 (c_{11} - \kappa)}{12\pi} \zeta_{\alpha,2}\bar{\zeta}_{\alpha,2} + \frac{h^2 \ell_2 (c_{12} + \kappa)}{12\pi} \zeta_{\alpha,2}\bar{\zeta}_{\alpha,2} + \frac{h^2 \ell_2 (c_{11} - c_{12})}{12\pi} \zeta_{2,\alpha}\bar{\zeta}_{2,\alpha} \right] dx_2.
 \end{aligned} \tag{53}$$

This relation allows us to prove the following theorem about properties of function \mathcal{J}_{κ} .

Theorem 3. Let hypotheses of Theorem 1 be still valid and let frequency ω obey to

$$\omega < \omega_m \tag{54}$$

where

$$\omega_m = \frac{\pi}{\ell_2} \left[\frac{2\rho}{\mu_{\kappa}} \left(1 + \frac{6\ell_2^2}{\pi^2 h^2} \right) \right]^{-1/2}, \quad 0 < \mu_{\kappa} = \min \{ c_{44}, k_m^{(1)}, k_m^{(2)} \}. \tag{55}$$

The function \mathcal{J}_{κ} is non-decreasing and it satisfies the following inequality

$$\nu_{\kappa} |\mathcal{J}_{\kappa}(x_1)| \leq \mathcal{J}'_{\kappa}(x_1), \quad \text{with } \nu_{\kappa} = \min_{i=1,\dots,6} \frac{P_i}{P_i}, \tag{56}$$

where

$$\begin{aligned}
 P_1 &= \frac{\ell_2}{2\pi} c_{44}, \quad P_2 = \frac{\ell_2}{2\pi} c_{44}, \quad P_3 = \frac{\ell_2}{2\pi} c_{11}, \quad P_4 = \frac{\ell_2}{4\pi} (c_{11} - c_{12}), \\
 P_5 &= \frac{\ell_2}{4\pi} (3c_{11} + c_{12}), \quad P_6 = \frac{\ell_2}{2\pi} c_{11}
 \end{aligned} \tag{57}$$

and

$$\begin{aligned} p_1 &= 2\mu_\kappa, & p_2 &= 2\mu_\kappa \frac{h^2\pi^2}{2(6\ell_2^2 + \pi^2 h^2)} \left(1 - \frac{\omega^2}{\omega_m^2}\right), & p_3 &= 2\mu_\kappa, & p_4 &= 2\mu_\kappa, \\ p_5 &= 2\mu_\kappa \left(1 - \frac{\omega^2}{\omega_m^2}\right), & p_6 &= \mu_\kappa \left(1 - \frac{\omega^2}{\omega_m^2}\right). \end{aligned} \quad (58)$$

Proof. Differentiating \mathcal{J}_κ we obtain

$$\begin{aligned} \mathcal{J}'_\kappa(x_1) &= \int_{L_{x_1}} \left[\bar{\chi}_1 \psi_{,1} + \chi_{1,\bar{1}} \bar{\psi}_{,1} + \bar{\chi}_{1,1} \psi + \chi_{1,1} \bar{\psi} + \right. \\ &\quad \left. + \frac{h^2}{3} (\bar{\Gamma}_{1\alpha} \zeta_{\alpha,1} + \Gamma_{1\alpha} \bar{\zeta}_{\alpha,1} + \bar{\Gamma}_{1\alpha,1} \zeta_\alpha + \Gamma_{1\alpha,1} \bar{\zeta}_\alpha) \right] dx_2. \end{aligned} \quad (59)$$

Eqs. (46) allow us to show that

$$\frac{h^2}{3} \Gamma_{1\alpha,1} = -\frac{h^2}{3} \Gamma_{2\alpha,2} + \chi_\alpha - \rho \frac{h^2 \omega^2}{3} \zeta_\alpha, \quad \chi_{1,1} = -\chi_{2,2} - \rho \omega^2 \psi; \quad (60)$$

thus, with an integration by parts, boundary conditions (45) and Eq. (48), we arrive to

$$\mathcal{J}'_\kappa(x_1) = 2 \int_{L_{x_1}} \left[\widetilde{\mathcal{W}} - \rho \omega^2 \left(\frac{h^2}{3} \bar{\zeta}_\alpha \zeta_\alpha + \bar{\psi} \psi \right) \right] dx_2, \quad (61)$$

where

$$\begin{aligned} \widetilde{\mathcal{W}} &= \widetilde{\mathcal{W}}_0 + \frac{h^2}{3} \widetilde{\mathcal{W}}_1 + \frac{h^2}{3} \widetilde{\mathcal{W}}_2, & \widetilde{\mathcal{W}}_0 &= c_{44} (\bar{\zeta}_\alpha + \bar{\psi}_{,\alpha}) (\zeta_\alpha + \psi_{,\alpha}), \\ \widetilde{\mathcal{W}}_1 &= c_{11} (\bar{\zeta}_{1,1} \zeta_{1,1} + \bar{\zeta}_{2,2} \zeta_{2,2}) + \frac{c_{11} - \kappa}{2} (\bar{\zeta}_{1,1} \zeta_{2,2} + \zeta_{1,1} \bar{\zeta}_{2,2}), \\ \widetilde{\mathcal{W}}_2 &= \frac{c_{11} - c_{12}}{2} (\zeta_{2,1} \bar{\zeta}_{2,1} + \zeta_{1,2} \bar{\zeta}_{1,2}) + \frac{c_{12} + \kappa}{2} (\bar{\zeta}_{2,1} \zeta_{1,2} + \zeta_{2,1} \bar{\zeta}_{1,2}). \end{aligned} \quad (62)$$

The quadratic form $\widetilde{\mathcal{W}}$ is positive definite if relations (3) and (18) are verified. It is then easy to see that

$$\mu_\kappa \left[(\bar{\zeta}_\alpha + \bar{\psi}_{,\alpha}) (\zeta_\alpha + \psi_{,\alpha}) + \frac{h^2}{3} \bar{\zeta}_{\alpha,\beta} \zeta_{\alpha,\beta} \right] \leq \widetilde{\mathcal{W}} \quad (63)$$

and $k_m^{(\alpha)}$ are defined into Eqs. (26). Considering Eqs. (52), (61), (63), we get

$$\begin{aligned} \mathcal{J}'_\kappa(x_1) &\geq 2\mu_\kappa \int_{L_{x_1}} \left[(\bar{\zeta}_\alpha + \bar{\psi}_{,\alpha}) (\zeta_\alpha + \psi_{,\alpha}) + \frac{h^2}{3} \bar{\zeta}_{\alpha,\beta} \zeta_{\alpha,\beta} - \right. \\ &\quad \left. - \frac{\rho \omega^2 \ell_2^2}{\pi^2 \mu_\kappa} \left(\frac{h^2}{3} \zeta_{\alpha,2} \bar{\zeta}_{\alpha,2} + \psi_{,2} \bar{\psi}_{,2} \right) \right] dx_2. \end{aligned} \quad (64)$$

Using the relations (52), we can observe that

$$\begin{aligned} \frac{6\ell_2^2}{\pi^2 h^2} \int_{L_{x_1}} \left[(\zeta_2 + \psi_{,2}) (\bar{\zeta}_2 + \bar{\psi}_{,2}) + \frac{h^2}{6} \bar{\zeta}_{2,2} \zeta_{2,2} \right] dx_2 &\geq \frac{\ell_2^2}{\pi^2} \int_{L_{x_1}} \zeta_{2,2} \bar{\zeta}_{2,2} dx_2 \geq \\ &\geq \int_{L_{x_1}} \zeta_2 \bar{\zeta}_2 dx_2. \end{aligned} \quad (65)$$

Further, using Schwarz and arithmetic-geometric mean inequalities, it follows

$$(\bar{\zeta}_2 + \bar{\psi}_2) (\zeta_2 + \psi_2) \geq (1 - \epsilon) \zeta_2 \bar{\zeta}_2 + \left(1 - \frac{1}{\epsilon}\right) \psi_2 \bar{\psi}_2, \quad \text{for every } \epsilon > 0$$

and, for $\epsilon = 2$, it is

$$(\bar{\zeta}_2 + \bar{\psi}_2) (\zeta_2 + \psi_2) \geq -\zeta_2 \bar{\zeta}_2 + \frac{1}{2} \psi_2 \bar{\psi}_2; \quad (66)$$

thus

$$\begin{aligned} \int_{L_{x_1}} \left[(\zeta_2 + \psi_2)(\bar{\zeta}_2 + \bar{\psi}_2) + \frac{h^2}{6} \zeta_{2,2} \bar{\zeta}_{2,2} \right] dx_2 &\geq \int_{L_{x_1}} (\zeta_2 + \psi_2)(\bar{\zeta}_2 + \bar{\psi}_2) dx_2 \geq \\ &\geq \int_{L_{x_1}} \left(-\zeta_2 \bar{\zeta}_2 + \frac{1}{2} \psi_2 \bar{\psi}_2 \right) dx_2. \end{aligned} \quad (67)$$

Combining Eqs. (65), (67) we obtain

$$\int_{L_{x_1}} \left[(\zeta_2 + \psi_2)(\bar{\zeta}_2 + \bar{\psi}_2) + \frac{h^2}{6} \zeta_{2,2} \bar{\zeta}_{2,2} \right] dx_2 \geq \frac{h^2 \pi^2}{2(6\ell_2^2 + \pi^2 h^2)} \int_{L_{x_1}} \psi_2 \bar{\psi}_2 dx_2. \quad (68)$$

Moreover, Eqs. (64), (68) imply

$$\begin{aligned} \mathcal{J}'_{\kappa}(x_1) &\geq \mu_{\kappa} \int_{L_{x_1}} \left\{ 2(\bar{\zeta}_1 + \bar{\psi}_1) (\zeta_1 + \psi_1) + \frac{h^2 \pi^2}{6\ell_2^2 + \pi^2 h^2} \left[1 - \omega^2 \frac{2\rho\ell_2^2}{\pi^2 \mu_{\kappa}} \left(1 + \frac{6\ell_2^2}{\pi^2 h^2} \right) \right] \bar{\psi}_2 \psi_2 + \right. \\ &+ \frac{2h^2}{3} \bar{\zeta}_{1,1} \zeta_{1,1} + \frac{2h^2}{3} \bar{\zeta}_{2,1} \zeta_{2,1} + \frac{2h^2}{3} \left(1 - \omega^2 \frac{\rho\ell_2^2}{\pi^2 \mu_{\kappa}} \right) \bar{\zeta}_{1,2} \zeta_{1,2} + \\ &\left. + \frac{h^2}{3} \left(1 - \omega^2 \frac{2\rho\ell_2^2}{\pi^2 \mu_{\kappa}} \right) \bar{\zeta}_{2,2} \zeta_{2,2} \right\} dx_2. \end{aligned} \quad (69)$$

If we note that the critical frequency ω_m defined by Eq. (55) is such that

$$\min \left\{ \frac{\ell_2}{\pi} \sqrt{\frac{2\rho}{\mu_{\kappa}} \left(1 + \frac{6\ell_2^2}{\pi^2 h^2} \right)}, \frac{\ell_2}{\pi} \sqrt{\frac{\rho}{\mu_{\kappa}}}, \frac{\ell_2}{\pi} \sqrt{\frac{2\rho}{\mu_{\kappa}}} \right\} = \frac{1}{\omega_m}$$

then we have

$$\begin{aligned} \mathcal{J}'_{\kappa}(x_1) &\geq \int_{L_{x_1}} \left[p_1 (\bar{\zeta}_1 + \bar{\psi}_1) (\zeta_1 + \psi_1) + p_2 \bar{\psi}_2 \psi_2 + \frac{h^2}{3} p_3 \bar{\zeta}_{1,1} \zeta_{1,1} + \right. \\ &\left. + \frac{h^2}{3} p_4 \bar{\zeta}_{2,1} \zeta_{2,1} + \frac{h^2}{3} p_5 \bar{\zeta}_{1,2} \zeta_{1,2} + \frac{h^2}{3} p_6 \bar{\zeta}_{2,2} \zeta_{2,2} \right] dx_2 \geq 0. \end{aligned} \quad (70)$$

This relation implies that \mathcal{J}_{κ} is a non-decreasing function. On the other hand, Eq. (53) implies

$$\begin{aligned} |\mathcal{J}_{\kappa}(x_1)| &\leq \int_{L_{x_1}} \left[P_1 (\bar{\zeta}_1 + \bar{\psi}_1) (\zeta_1 + \psi_1) + P_2 \psi_2 \bar{\psi}_2 + \frac{h^2}{3} P_3 \zeta_{1,1} \bar{\zeta}_{1,1} + \right. \\ &\left. + \frac{h^2}{3} P_4 \zeta_{2,1} \bar{\zeta}_{2,1} + \frac{h^2}{3} P_5 \zeta_{1,2} \bar{\zeta}_{1,2} + \frac{h^2}{3} P_6 \zeta_{2,2} \bar{\zeta}_{2,2} \right] dx_2 \end{aligned} \quad (71)$$

and consequently

$$\begin{aligned} |\mathcal{J}_{\kappa}(x_1)| &\leq \frac{1}{\nu_{\kappa}} \int_{L_{x_1}} \left[p_1 (\bar{\zeta}_1 + \bar{\psi}_1) (\zeta_1 + \psi_1) + p_2 \psi_2 \bar{\psi}_2 + \frac{h^2}{3} p_3 \zeta_{1,1} \bar{\zeta}_{1,1} + \right. \\ &\left. + \frac{h^2}{3} p_4 \zeta_{2,1} \bar{\zeta}_{2,1} + \frac{h^2}{3} p_5 \zeta_{1,2} \bar{\zeta}_{1,2} + \frac{h^2}{3} p_6 \zeta_{2,2} \bar{\zeta}_{2,2} \right] dx_2. \end{aligned} \quad (72)$$

Finally, Eqs. (70), (72) lead to Eq. (56). ■

We consider the class of steady-state vibrations $\{\zeta_\alpha, \psi\}$ for which

$$\begin{aligned} \mathcal{E}(x_1) = \int_{S_{x_1}} & \left[P_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + P_2\psi_{,2}\bar{\psi}_{,2} + \frac{h^2}{3}P_3\zeta_{1,1}\bar{\zeta}_{1,1} + \right. \\ & \left. + \frac{h^2}{3}P_4\zeta_{2,1}\bar{\zeta}_{2,1} + \frac{h^2}{3}P_5\zeta_{1,2}\bar{\zeta}_{1,2} + \frac{h^2}{3}P_6\zeta_{2,2}\bar{\zeta}_{2,2} \right] da < \infty, \end{aligned} \quad (73)$$

where $S_{x_1} = \bar{\Sigma} \cap [x_1, \infty[\times]0, \ell_2]$. We can prove from Eqs. (50), (71), (73) that $\mathcal{E}(x_1)$ is a non-increasing function and

$$\begin{aligned} -\mathcal{E}'(x_1) = \int_{L_{x_1}} & \left[P_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + P_2\psi_{,2}\bar{\psi}_{,2} + \frac{h^2}{3}P_3\zeta_{1,1}\bar{\zeta}_{1,1} + \right. \\ & \left. + \frac{h^2}{3}P_4\zeta_{2,1}\bar{\zeta}_{2,1} + \frac{h^2}{3}P_5\zeta_{1,2}\bar{\zeta}_{1,2} + \frac{h^2}{3}P_6\zeta_{2,2}\bar{\zeta}_{2,2} \right] da \geq |\mathcal{J}_\kappa(x_1)|. \end{aligned} \quad (74)$$

When the strip $\bar{\Sigma}$ is bounded, then the hypothesis (73) is trivially verified and Eqs (45)₃, (74) imply

$$\mathcal{E}'(\ell_1) = 0 \quad \Rightarrow \quad \mathcal{J}_\kappa(\ell_1) = 0. \quad (75)$$

Moreover, if the strip $\bar{\Sigma}$ is semi-infinite and the hypothesis (73) is satisfied, then Eq. (74) implies

$$\lim_{x_1 \rightarrow \infty} \mathcal{E}'(x_1) = 0 \quad \Rightarrow \quad \mathcal{J}_\kappa(\infty) \equiv \lim_{x_1 \rightarrow \infty} \mathcal{J}_\kappa(x_1) = 0. \quad (76)$$

Now, we can establish a theorem furnishing information about the spatial behaviour of the amplitude of vibrations, provided that the frequency of the harmonic vibrations is lower than the critical value ω_m .

Theorem 4. Let hypotheses of Theorem 3 be still true and let $\{\zeta_\alpha, \psi\}$ be the steady-state vibrations for which Eq. (73) holds. Then, the function $-\mathcal{J}_\kappa$ is a measure of the amplitude $\{\zeta_\alpha, \psi\}$ of the harmonic vibration, that is $-\mathcal{J}_\kappa \geq 0$ and $-\mathcal{J}_\kappa = 0$ implies $\{\zeta_\alpha, \psi\} = \{0, 0\}$. Moreover, the following decay estimate of Saint-Venant type, with exponential decay factor $\frac{1}{\nu_\kappa}$, is valid

$$0 \leq -\mathcal{J}_\kappa(x_1) \leq -\mathcal{J}_\kappa(0) \exp\left(-\frac{x_1}{\nu_\kappa}\right). \quad (77)$$

Proof. In both bounded and semi-infinite cases, from hypothesis (54) about frequency ω and from the fact that \mathcal{J}_κ is non-decreasing, Eqs. (75), (76) imply

$$\mathcal{J}_\kappa(x_1) \leq 0, \quad \text{for all } x_1$$

and inequality (56) becomes

$$\mathcal{J}'_\kappa(x_1) + \frac{1}{\nu_\kappa} \mathcal{J}_\kappa(x_1) \geq 0. \quad (78)$$

Through an integration of Eq. (78), we get the decay estimate (77). ■

To conclude, we can remark that, in the context of steady-state vibrations for which $\mathcal{E}(x_1)$ is bounded and the excitation frequency is lower than the critical value ω_m , Eq. (77) describes an exponential decay with the greatest decay factor $\frac{1}{\nu}$, where $\nu = \inf_\kappa \nu_\kappa$.

5 Remark

The investigation performed is based on the assumption concerning the strong ellipticity of elasticity tensor. The results obtained under such hypothesis are thus valid also for classes of particular materials characterized by reversed properties, like negative Poisson's ratio and negative stiffness (auxetic or anti-rubber materials). These particular structures (see, for example, Park and Lakes (2007)) expand laterally when stretched, in contrast to ordinary materials. While optimal bounds for the spatial estimates are not so easy to obtain, in the present paper we offer the crude bounds in the spatial estimates of Theorem 2 and Theorem 4. Our results can be compared with other results of the literature (see e.g. Sanchez Palencia (1988), (1989)).

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