The Cauchy Data in Spacetimes With Singularities

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THE CAUCHY DATA IN SPACETIMES WITH SINGULARITIES

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Abstract. This article shows a method to formulate well-posed Cauchy problems for spacetimes with singularities. The formulation is identical to the prescriptions of General Relativity at the points where the metric is non-singular. The standard formulation of General Relativity doesn't prescribe the topology at the points where the metric is singular, regarding them as not belonging to the spacetime. This allows us to extend the non-singular part of the spacetime to a manifold foliated as a product between a three-dimensional manifold corresponding to the space, and a time coordinate. On the spacelike hypersurfaces from such a foliation, Einstein's field equation can be replaced by an equation with which it is equivalent at non-singular points, but which extends smoothly, by continuity, to the singular points. We provide examples for the neutral and charged, rotating and non-rotating, primordial and non-primordial, evaporating and eternal black holes. In these examples we show that the conformal structure thus find is globally hyperbolic, hence the Cauchy data is well-defined and preserved during the time evolution, and the singularities are not harmful to the time evolution.

Contents

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“I was borne violently into the channel of the Ström, and in a few minutes, was hurried down the coast into the ‘grounds’ of the fishermen.”

Edgar Allan Poe, A Descent into the Maelström, 1841

Introduction

0.1. **The singularity theorems.** Despite the successes of General Relativity, one of its consequences seems to question its full correctness: the occurrence of singularities in the black holes. It is often said that General Relativity predicts, because of these singularities, it's own breakdown [1, 2, 3, 4, 5, 6]. Such singularities, following from the *singularity theorems* of Penrose and Hawking [7, 8, 1, 9], refer to the spacetime *geodesic incompleteness*.

Initially, there was some confusion regarding the singularities. In 1916, when Schwarzschild proposed [10, 11] his solution to Einstein's equation, representing a black hole, it was believed that the event horizon is singular. Only in 1924, when Eddington proposed another coordinate system which removed the singularity at the event horizon, it was understood that it was only apparent, being due to the choice of the coordinate system. But the singularity at the center of the black hole remained independent of the coordinates, and the singularity theorems showed that any black hole would have such a singularity.

0.2. **The black hole information paradox.** Shortly (in its proper time) after an object passes through the event horizon, it reaches the singularity of the black hole. All the information contained in it seems to vanish in the singularity.

On the other hand, the equations governing the physical laws are in general reversible, guaranteeing that no information can be lost. But according to Hawking [12, 2] the black hole may emit radiation and evaporate. If the black hole evaporates completely, it seems to leave behind no trace of the information it swallowed. Moreover, it seems to be possible for an originally pure state to end up being mixed, because the density matrix of the particles in the black hole's exterior is obtained by tracing over the particles lost in the black hole with which they were entangled. This means that the unitarity appears to be violated, and the contradiction becomes even more acute.

0.3. **The meaning of singularities.** The singularities in General Relativity are places where the evolution equations cannot work, because the fields involved become infinite. If the Cauchy surface on which the fields are defined is broken, then the equations cannot be developed.

From geometric viewpoint, these singularities are points where the metric becomes singular, and the geodesics become incomplete. We don't know to extend the fields to such points, consequently we remove them from the spacetime.

Actually, we can rewrite the fields involved, and the equations defining them, so that the fields remain finite at any point [13]. The equations will remain equivalent to Einstein's equation at the points where the metric is non-singular.

With this modification, the singular points can then be kept in the spacetime. We need to do this with care, because it may happen that a coordinate singularity coincides with an actual singularity, and in this case there will be ambiguities when recovering the topology of the spacetime

Once we will have the fields and the topology repaired, we will have to check that we can choose a foliation the spacetime so that the evolution equations are well-defined.

Consequently, a natural interpretation of the singularities emerges, which makes them harmless for the physical law, in particular for the information conservation.

We will illustrate this solution on the black hole solutions known from the literature.

1. Canceling the singularities of the Einstein tensor

Some of the tensor fields involved in Einstein's field equation become infinite at the singularity points. These infinities can be removed by multiplying them with quantities which tend to 0. For example, Einstein and Rosen used the following method (for which they credited Mayer). The method consists in multiplying the Riemann and the Ricci tensors by a power of $\det g$. This way, all instances of g^{ab} in the expression of $\det g^2 \mathrm{Ric}$ are replaced by $\det g g^{ab}$, the adjugate matrix of g_{ab} ([13], p. 74). To cancel the singularities of the Einstein tensor due to the degeneracy of the metric, we also need to cancel the infinities of the curvature scalar, and for this it is enough to multiply the Einstein tensor by $\det g^3$.

Unfortunately, for important black hole solutions of Einstein's equation, one or more components of the metric tensor become infinite. In this case the metric already causes enough problems, in addition to those caused by the lack of the inverse. Luckily, in the cases of physical relevance, even though one or more components of the metric become infinite, the determinant of the metric becomes 0 in such a way that there is a power of $\det g$ which smoothens that component, and ultimately the Einstein tensor.

To see this, let's take the Kerr-Newman solution of Einstein's equations, in Boyer-Lindquist coordinates [14],[15], p. 313:

$$\begin{aligned}\mathrm{d}s^2 &= \frac{\Delta - a^2\sin^2\vartheta}{\Sigma}\mathrm{d}t^2 - \frac{2a\sin^2\vartheta(r^2+a^2-\Delta)}{\Sigma}\mathrm{d}t\mathrm{d}\varphi \\ &+\frac{(r^2+a^2)^2-\Delta a^2\sin^2\vartheta}{\Sigma}\sin^2\vartheta\mathrm{d}\varphi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\vartheta^2,\end{aligned} \tag{1}$$

where

$$\Sigma := r^2 + a^2\cos^2\vartheta, \tag{2}$$

$$\Delta := r^2 + a^2 + e^2 - 2Mr, \tag{3}$$

and e is the electric charge, a the angular momentum, M the mass. By making $e = 0$ we obtain the Kerr black holes (rotating or not and neutral). By taking $a = 0$, the black hole is non-rotating, being described by the Reissner-Nordström solution. The Schwarzschild solution is obtained when $a = 0$ and $e = 0$.

We can see that we can multiply equation (1) with $\Sigma\Delta$ and obtain a non-singular expression. This expression is dependent on the particular coordinate system, but we are only interested into making the evolution equations valid. Alternatively, we can take a scalar field which is equal to $\Sigma\Delta$ in this coordinate system. Passing to other coordinate systems will change the values of Δ and Σ, but the expression (1) will still remain non-singular.

Then, we can rewrite Einstein's equation multiplied with the appropriate factor which cancels the singularities in the metric tensor, and by using the adjugate instead of the inverse metric, in a way similar to that prescribed in [13]. We obtain a system of equations which reduces to Einstein's equation at the points where the metric is non-singular and non-degenerate.

Since the other black hole exact solutions, namely Schwarzschild, Reissner-Nordström and Kerr, are particular cases of the Kerr-Newman solution, we conclude that this approach works for them too.

The resulting equations still make sense as partial differential equations in the components of the metric tensor. Of course, the invariants of Semi-Riemannian Geometry have geometric meanings so long as the metric tensor is non-degenerate. But once we allow it to become degenerate, or worse, singular, the invariants like covariant derivative and curvature become undefined. The quantities which replace them if we multiply the equations with $\det g$ or other factors cannot give them this meaning. Maybe a new kind of geometry is required to restore the ideas of covariant derivative and curvature for singular metrics. We did a modest step in this direction, by developing a Singular Semi-Riemannian Geometry for a class of metrics which can become degenerate [16, 17], but the more general case of singular metrics remains to be researched.

For the purpose of the present work, it is enough to make the partial differential equations smooth and well-posed. The smoothness can be obtained with this method of multiplying by a suitable power of $\det g$, or other quantity playing the same role.

In the following, we need to explore the arena on which are defined the fields – the spacetime – to make sure that the singularities don't alter the well-posedness of the the field equations.

2. The topology of singularities

The spacetime is not a fixed background for the fields. There is a dynamical interplay between the matter and the geometry and topology of spacetime. Although the equations are made finite and even smooth, we need to make sure that the topology of spacetime is not dramatically altered by the occurrence of the singularities. In the following, we will see that the existence of singularities can lead to ambiguities from topological viewpoint. The reason is that, when developing locally the field equations, the singularities of the coordinate system may coincide with the genuine singularities. In this case, we cannot resolve uniquely the topology of the singularities. Each type of singularity which appears from the field equations needs to be interpreted with care, in order to choose the correct topology.

In the following we shall see that the main black hole solutions of Einstein's equations can be interpreted so that the time evolution is not jeopardized. Two main factors help us to do this:

(1) Remove the singularities from the field equations.
(2) Make sure that the time evolution takes place on a space whose topology doesn't change in time.

The former condition is ensured by the section §1. In the current section, we shall see that the latter condition can be ensured for the typical black hole solutions of Einstein's equation.

2.1. **The final space-like singularity from Schwarzschild's solution.** Let's take the Penrose diagram of a Schwarzschild black hole (Figure 1).

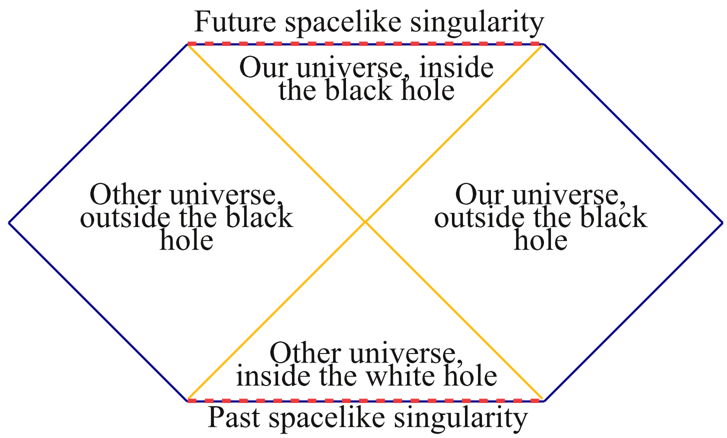

FIGURE 1. The maximally extended Schwarzschild solution, in Penrose coordiantes.

This diagram actually represents the maximally extended Schwarzschild solution, in Penrose coordinates. This extended solution is interpreted as containing, together with the universe containing the black hole, another universe, in which there is a white hole.

We can foliate this spacetime, so that the evolution equations are well-posed, as we can see in Figure 2.

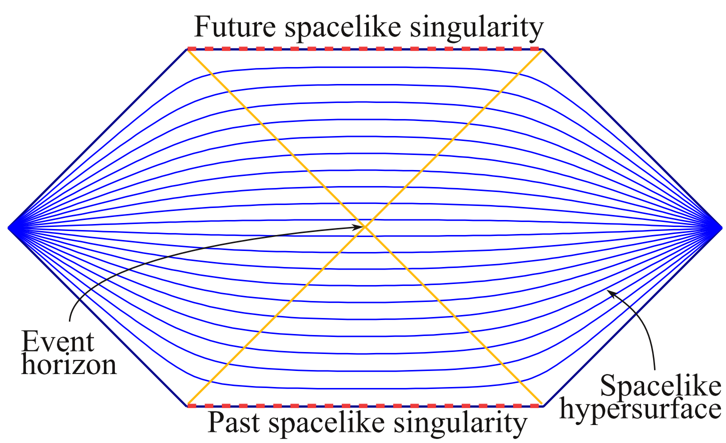

FIGURE 2. Space-like foliation of the maximally extended Schwarzschild solution.

This foliation is obtained with the help of the Schwarz-Christoffel mapping. There is a version of the Schwarz-Christoffel mapping which maps the strip

$$\mathcal{S} := \{z \in \mathbb{C} | \mathrm{Im}(z) \in [0, 1]\} \tag{4}$$

to a polygonal region from $\mathbb{C}$, with the help of the formula

$$f(z) = A+C\int^{\mathcal{S}} \exp\left[\frac{\pi}{2}(\alpha_- - \alpha_+)\zeta\right]\prod_{k=1}^{n}\left[\sinh\frac{\pi}{2}(\zeta - z_k)\right]^{\alpha_k-1} \mathrm{d}\zeta, \tag{5}$$

where $z_k \in \partial\mathcal{S} := \mathbb{R}\times\{0,i\}$ are the prevertices of the polygon, and $\alpha_-, \alpha_+, \alpha_k$ are the measures of the angles of the polygon, divided by π (*cf. e.g.* [18]). The vertices having the angles α_- and α_+ have as prevertices the ends of the strip, which are at infinite. The foliation is given by the level curves $\{\mathrm{Im}(z) = \mathrm{const.}\}$. In this article, all polygons to which we will apply the Schwarz-Christoffel mapping have a common property. They all have $\alpha_- = \alpha_+$ and the edges inclined at most at $\frac{\pi}{4}$, alternating in such a way that the level curves with $\mathrm{Im}(z) \in (0,1)$ have at each point tangents making an angle strictly between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. This ensures that our foliations are indeed spacelike.

To obtain the foliation from Figure 2, we take the prevertices to be

$$(-\infty, -a, a, +\infty, a+i, -a+i)\,, \tag{6}$$

where $a > 0$ is a real number. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right). \tag{7}$$

This foliation contains the past white hole, which one may consider unphysical. We can make a similar foliation, this time without the white hole, if we use the prevertices

$$(-\infty, -a, 0, a, +\infty, b+i, -b+i)\,, \tag{8}$$

where $0 < b < a$ are positive real numbers. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right). \tag{9}$$

The resulted foliation is represented in Figure 3.

The Schwarzschild solution has the particularity that its singularity is spacelike. Although in some coordinate systems (like the original Schwarzschild coordinates) the singularity is apparently timelike and one dimensional, it is in fact spacelike. If we choose the appropriate coordinates (*e.g.* Kruskal-Szekeres coordinates or Penrose coordinates), we can see that it is actually a spacelike hypersurface.

2.2. Timelike singularities of charged and rotating black holes.

The stationary solutions of Einstein's equations representing charged black holes were discovered by Reissner and Nordström. The Penrose

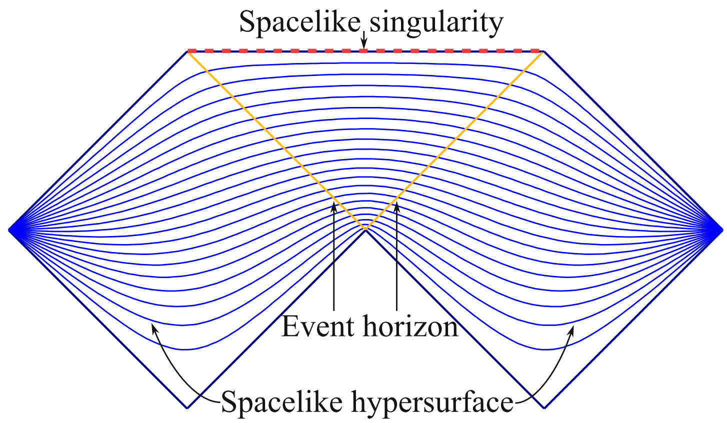

FIGURE 3. Space-like foliation of the Schwarzschild solution.

coordinates of the maximally extended solutions are represented in Figure 4 A for non-extremal black holes, and in Figure 4 B for extremal ones.

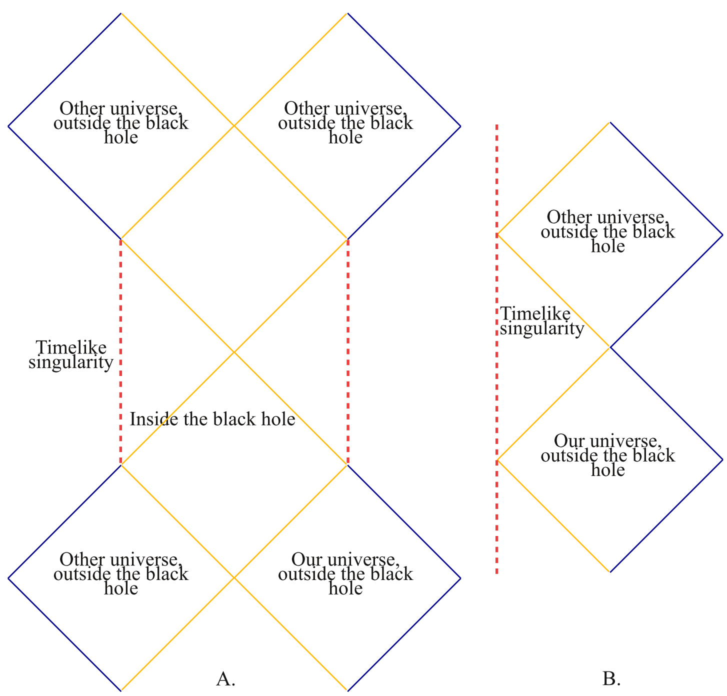

FIGURE 4. A. Reissner-Nordström stationary black holes. B. Extremal Reissner-Nordström stationary black holes.

If we take as prevertices of the Schwarz-Christoffel mapping (5) the set

$$(-\infty, -a, 0, a, +\infty, i)\,, \tag{10}$$

where $0 < a$ is a positive real number. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \tag{11}$$

By an appropriate choices of a we get the foliation represented in diagram 5, respectively 6, corresponding to the Figures 4 A, respectively 4 B.

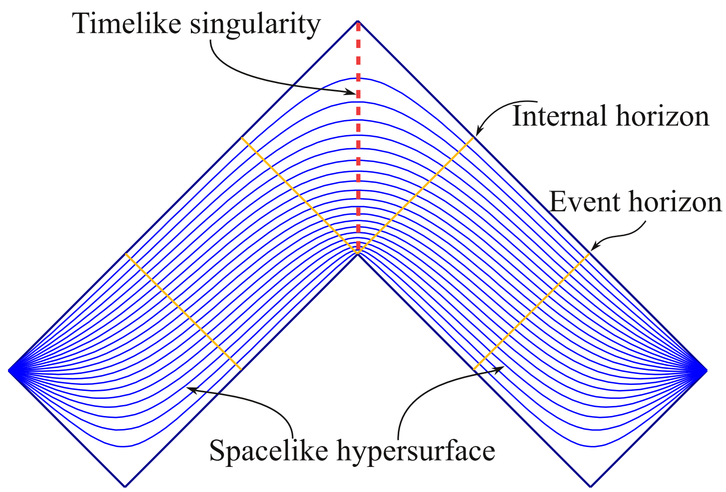

FIGURE 5. Space-like foliation of the Reissner-Nordström solution.

Both these solutions ignore the other universes appearing when the equations are analytically extended to their maximum domain in Penrose coordinates. Our purpose is to provide a spacelike foliation which allows well-posed Cauchy problems for a spacetime. It is debatable if we really need, from physical viewpoint, to include the infinite number of other universes appearing in such maximal extensions. Our foliation takes the outside and the inside of the black hole and foliates it as required to solve this problem. Taking the maximally extended solution would not allow us to do this, because it contains Cauchy horizons, beyond which the evolution equations cannot be evolved causally ([9], p. 159, [15], p. 317).

Also, we shall see that, by limiting to one universe in these solutions, instead of taking the maximal extensions, we can combine them with non-singular solutions and have black holes with a beginning and an

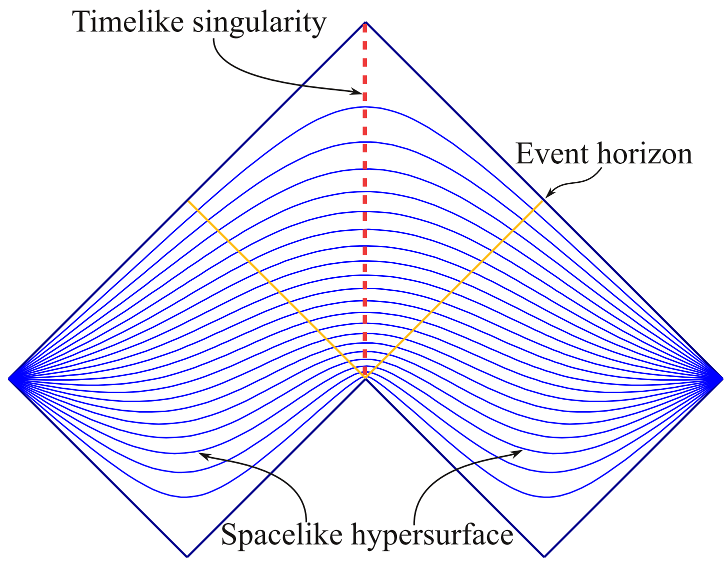

FIGURE 6. Space-like foliation of the extremal Reissner-Nordström solution.

end. Such black holes seem more plausible from physical viewpoint, although we will not exclude the primordial and eternal black holes.

The rotating neutral black holes (Kerr), and the rotating electrically charged black holes (Kerr-Newman) don't have spherical symmetry, they are axisymmetric. Their Penrose diagrams are similar with those for Reissner-Nordström black holes (Figure 4), in coordinates (t, r), but there are some major differences. In Boyer-Lindquist coordinates the singularity appears to be a ring, and in its neighborhood the coordinate φ is timelike. The curves defined by $(t = \text{ const.}, r = \text{ const.}, \vartheta = \text{ const.})$ are closed timelike curves (*cf. e.g.* [9], §5.6, [15], p. 315). This is an artifact of using spherical coordinates. The coordinate φ is cyclical and spacelike far from the singularity, but when it crosses the coordinate singularity it becomes timelike. There is no reason to maintain it to be cyclical, because it no longer represents the axial symmetry of the black hole. Maintaining the cyclicity of φ when it becomes timelike leads to the cyclicity of the singularity and the timelike curves. Interpreting it to be non-cyclical for these regions, breaks the closed timelike curves and the singularity, as it is visible in its Penrose coordinates. We therefore foliate the rotating black holes as in the case of the charged ones.

2.3. **Non-stationary black holes.** The stationary solutions are much idealized. In reality, a black hole doesn't necessarily exist from the beginning of the universe, it may have a beginning at a finite time. Also, it may evaporate completely at a finite time.

For example, a spacelike foliation of a non-rotating and electrically neutral black hole which formed at a finite time, and continues to exist forever, is represented in Figure 7.

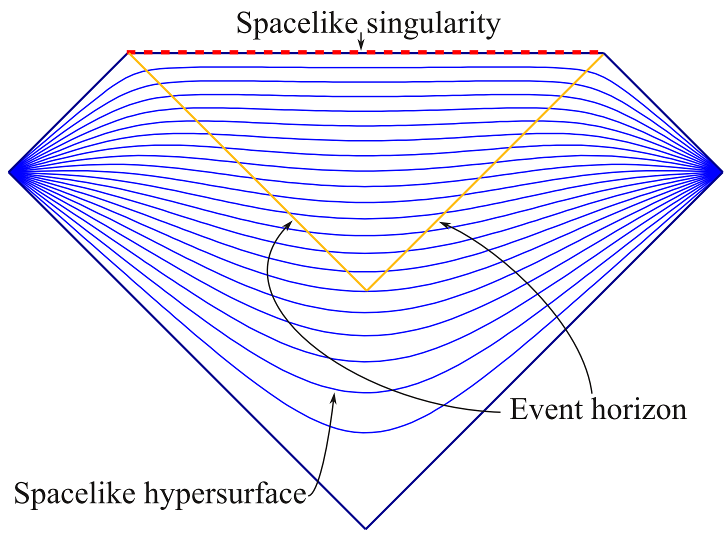

FIGURE 7. The spacelike foliation for a non-rotating and electrically neutral black hole formed at a finite time, which continues to exist forever.

The prevertices of the Schwarz-Christoffel mapping (5) whose image is represented in Figure 7 are given by the set

$$(-\infty, 0, +\infty, a+i, -a+i)\,, \tag{12}$$

where $0 < a$ is a positive real number. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right). \tag{13}$$

We see that, because the typical spacelike hypersurface in the foliation in Figure 3 is diffeomorphic with the space $\mathbb{R}^3$ of the Minkowski spacetime $\mathbb{R}^3 \times \mathbb{R}$, the topology doesn't change because of the occurrence of a neutral non-rotating black hole. In fact, the typical spacelike hypersurface of the foliation of a charged and rotating black hole also has the same topology as $\mathbb{R}^3$, as we have seen. So they can appear and evaporate as well in an $\mathbb{R}^3$ space, without disrupting the topology. This condition is required to have a good time evolution.

If the non-rotating and electrically neutral black hole is primordial (exists from the beginning of the universe), but evaporates completely after a finite time, the spacelike foliation is represented in Figure 8. The prevertices and the angles are identical to those for Figure 6.

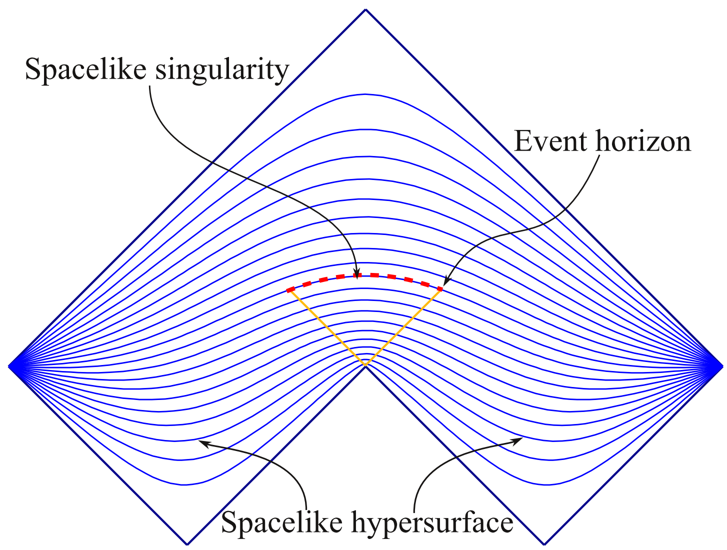

FIGURE 8. The spacelike foliation for a non-rotating and electrically neutral primordial black hole, which evaporates after a finite time.

If this non-rotating and electrically neutral black hole is not primordial and it evaporates completely after a finite time, then at large distances the spacetime is very close to the flat spacetime of Minkowski, being assymptotically flat. Consequently, a spacelike foliation looks like that in Figure 9.

The prevertices whose image is represented in Figure 9 are

$$(-\infty, 0, +\infty, i)\,, \tag{14}$$

and the angles are

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \tag{15}$$

Let's see now what happens if the singularity is timelike, as in the case of the charged and/or rotating black holes. If the black hole is primordial and evaporates, the corresponding spacelike foliation is represented in Figure 10.

The prevertices and the angles are the same as those for the Figure 6.

If the black hole is not primordial and does not evaporate, the spacelike foliation is represented in Figure 11. The prevertices and the angles are again those from equations (14), respectively (15).

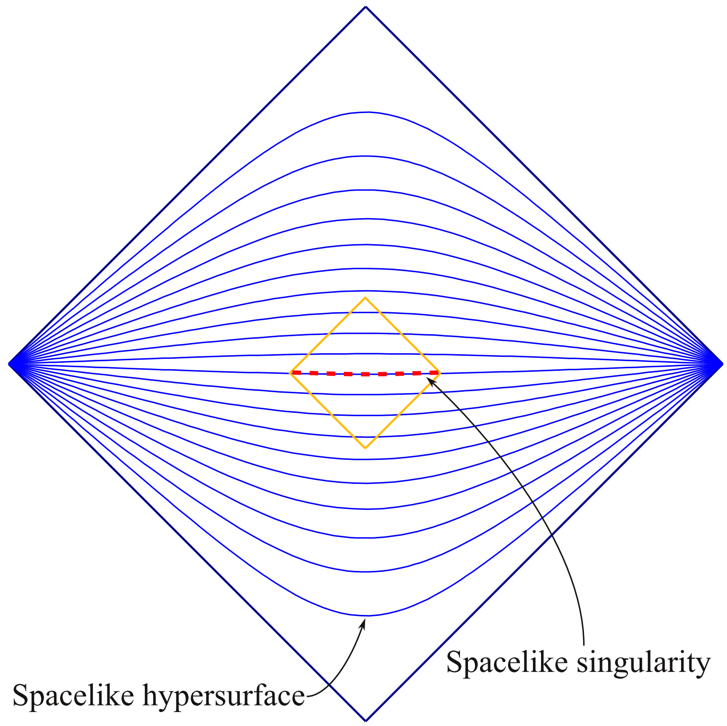

FIGURE 9. The spacelike foliation for a non-rotating and electrically neutral black hole formed at a finite time, which evaporates after a finite time.

The same prevertices and angles are used to construct the spacelike foliation for a non-primordial evaporating black hole, represented in Figure 12.

3. Time evolution in spacetimes with singularities

The typical black hole solutions presented above can now be combined in more complex spacetimes, since they don't disrupt the topology and the conformal structure.

The conformal structure is given equivalently by the lightcones. This structure is involved in the Cauchy development in General Relativity. Namely, each point in spacetime is affected by what happens in its past lightcone.

In our formulation the singularities are repaired by a conformal transformation (multiplication of the metric tensor with a scalar field), and the topology of the spacelike hypersurface is preserved by the time evolution. The conformal structure is identical, in all cases examined,

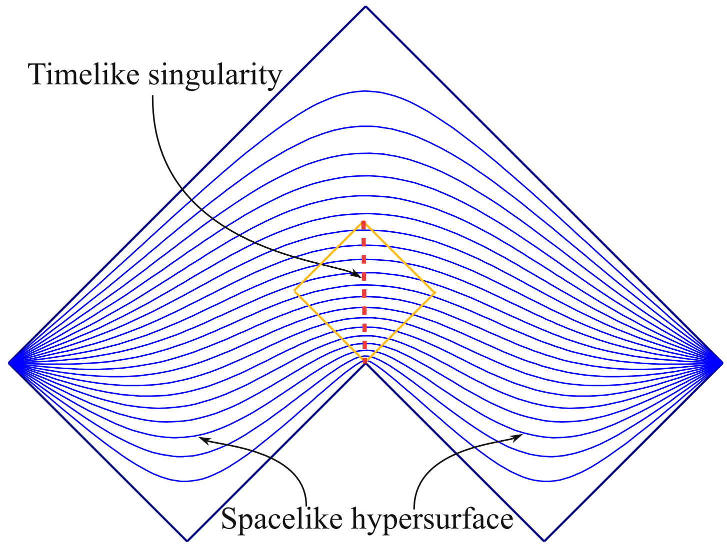

FIGURE 10. Primordial evaporating black hole with timelike singularity.

to the conformal structure of non-singular spacetimes which have well-posed Cauchy problems. Namely, they are all conformally equivalent to the Minkowski spacetime. This is evident from the conformal equivalence of all solutions with the strip $\mathcal{S}$ from (4), via a Schwarz-Christoffel mapping of the form (5). Therefore, the property of global hyperbolicity of the Minkowski spacetime transfers to our solutions as well, ensuring the well-posedness of the Cauchy problem.

This analysis shows that, even in the presence of singularities, the physical laws don't necessarily break down. We can make

(1) an appropriate choice of the conformal factor in Einstein's equations with singularities, so that the equations are non-singular (see §1)
(2) an appropriate foliation of the spacetime into spacelike hypersurfaces
(3) an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved.

These choices, which are not determined by the non-singular General Relativity, ensure us that the time evolution is not disrupted, and the Cauchy data, hence the information, is preserved. Also, this allows the unitarity to be restored, because there is no loss in the information, and a pure state can no longer become mixed by this mechanism

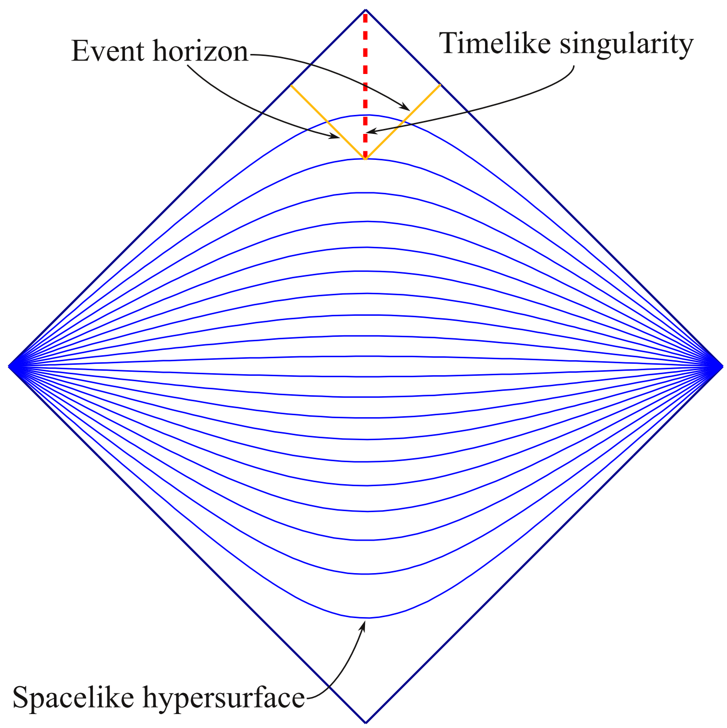

FIGURE 11. Non-primordial non-evaporating black hole with timelike singularity.

4. Singular General Relativity

There are many actions we can take when encountering singularities in the solutions of Einstein's equation. There is no need to say about them "here be dragons". We can explore them.

This article presents and approach of the problems posed by the singularities. The solution is obtained by combining an idea from [13], with appropriate choices of the topology of the singularities, and of the spacelike foliations.

We can do even more. The quantities obtained by multiplying Einstein's equations have no evident geometrical meaning. Einstein's equation is an identity between Einstein's tensor, which is defined by the curvature, and the stress-energy tensor of the matter fields. By multiplying them with some functions, so that the partial differential equations expressed in a coordinate system become non-singular, we can extend them at the singularities. But the quantities involved now seem to have no meaning, unlike the original terms. This suggests that it may be useful to develop the Semi-Riemannian Geometry to deal with such situations when the metric is degenerate or singular.

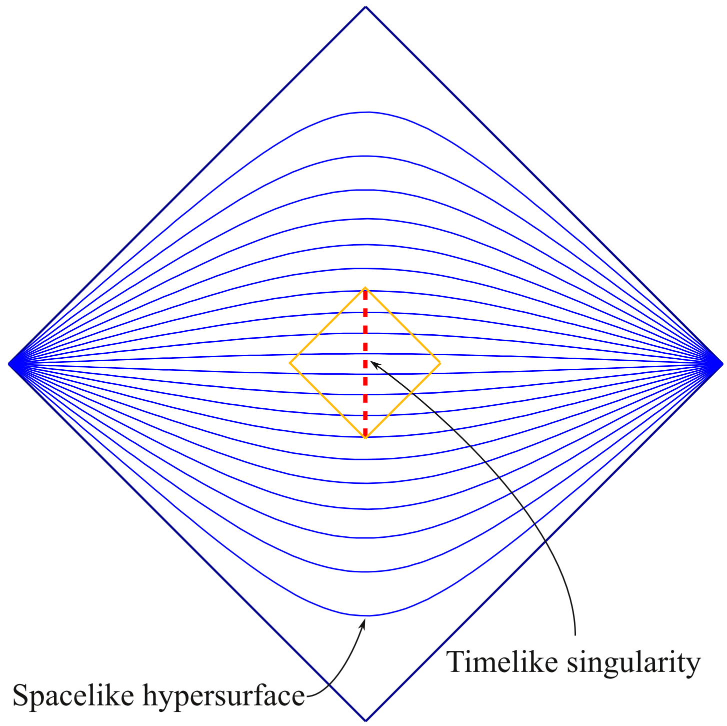

Figure 12. Non-primordial evaporating black hole with timelike singularity.

We already explored the case when the metric is allowed to pass from being non-degenerate to being degenerate in [16, 17]. We showed there that we can still have invariants like the familiar Riemann curvature and like the covariant derivative, if the metric satisfies some reasonable conditions. This allowed us to write a densitized version of Einstein's equation, which remains smooth at the singularities due to the degeneracy of the metric, although the curvature operator $R^a{}_{bcd}$, the Einstein tensor $R_{ab} - \frac{1}{2}g^{ab}R_{ab}$, and the Kretschmann scalar $R_{abcd}R^{abcd}$ become singular.

As we have seen from the black hole solutions analyzed in this article, the singularities are in general due not only to the degeneracy of some components of the metric tensor, but also because other components diverge. At the present moment we don't have geometric invariants which can work at those points. This is an open problem which would be nice to be solved, but it doesn't affect the conclusions of this article.

5. Conclusions

This article shows how one can make

(1) an appropriate choice of the conformal factor in Einstein's equations with singularities, so that the equations are non-singular (see §1)
(2) an appropriate foliation of the spacetime into spacelike hypersurfaces
(3) an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved,

so that the time evolution in the presence of singularities is restored. The examples given here showed this explicitly for the neutral and charged, rotating or non-rotating, primordial or not, evaporating or not black holes. These models have been proved here to be conformally equivalent to the Minkowski spacetime, inheriting therefore from the latter the global hyperbolicity, hence the well-posedness of the Cauchy problem. Consequently, the Cauchy data is preserved, and the information loss is avoided. This allows the construction of Quantum Field Theories in such curved spacetimes ([4], p. 9), and the unitarity is restored.

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