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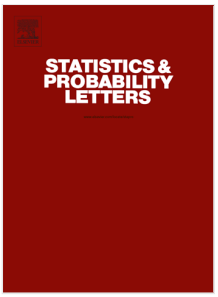

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Note on integer-valued bilinear time series models

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Summary. This note reconsiders the nonnegative integer-valued bilinear processes introduced by Doukhan, Latour, and Oraichi (2006). Using a hidden Markov argument, we extend their result of the existence of a stationary solution for the INBL(1,0,1,1) process to the class of superdiagonal INBL(p, q, m, n) models. Our approach also yields improved parameter restrictions for several moment conditions compared to the ones in Doukhan, Latour, and Oraichi (2006).

Keywords: count data, integer-valued time series, bilinear model

1. Introduction

In many sciences one encounters nonnegative discrete valued time series, often as counts of events or objects at consecutive points in time. Especially in economics and medicine many interesting variables are (nonnegative) integer-valued. For example: the number of transactions in IBM during each minute, the number of patients in a hospital at the end of the day, the number of claims an insurance company receives during each day, the number of epileptic seizures a patient suffers each day, etcetera. Until the mid seventies modeling discrete valued time series did not attract much attention since most traditional representations of dependence become either impossible or impractical. The last two decades there were many developments in the literature on integer-valued time series; see McKenzie (2003) for a detailed review; for example, INteger-valued Moving Average processes (Al-Osh and Alzaid (1991)), INteger-valued AutoRegressive processes (Al-Osh and Alzaid (1987), Al-Osh and Alzaid (1990), and Du and Li (1991)), and Generalized INteger-valued AutoRegressive processes (Latour (1998)) are common choices. Doukhan et al. (2006) introduced the class of nonnegative INteger-valued BiLinear time series, which contains the three aforementioned classes. Higher order processes turn out to be highly releveant see (Latour (1998)) for the class of GINAR models. For extensions to regression based count data time series models see, for example, Fahrmeir and Tutz (2001) and Kedem and Fokianos (2002).

Recall the definition of an INBL(p, q, m, n) process: let $(X_{-1}, \ldots, X_{-p\vee m}, \varepsilon_{-1}, \ldots, \varepsilon_{-q\vee n})$ be the (nonnegative) integer-valued initial points generated by some probability distribution ν, and define X_t, $t = 0, 1, 2, \ldots$, recursively by,

$$X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \sum_{j=1}^{q} \beta_j \circ \varepsilon_{t-j} + \sum_{k=1}^{m}\sum_{\ell=1}^{n} \gamma_{k,\ell} \circ (X_{t-k}\varepsilon_{t-\ell}) + \varepsilon_t, \tag{1}$$

where $(\varepsilon_t)_{t\geq 0}$ is a collection of i.i.d. nonnegative integer-valued variables with distribution G with mean $\mu_G \in [0, \infty]$ and variance $\sigma_G^2 \in [0, \infty]$, and where the Steutel-van Harn operators

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$\eta\circ$ applied to some nonnegative integer-valued random variable Z are defined as

$$\eta \circ Z = \sum_{s=1}^{Z} U_s^{(\eta)},$$

where $(U_s^{(\eta)})_{s\geq 1}$ is a collection of i.i.d. nonnegative integer-valued variables with distribution function F_η with mean η.* In proofs we suppose, without loss of generality, $p = q = m = n$ since we may introduce additional lags in (1) with $\eta = 0$ or equivalently $F_\eta\{0\} = 1$. The class of GINAR(p) processes arises by taking $q = m = n = 0$. Specializing further to the situation where the F_η-distributions are Bernoulli, one gets the class of INAR(p) models. Similarly INMA(q) processes are obtained when $p = m = n = 0$.

In this note we consider stationarity and existence of moments of superdiagonal INBL(p, q, m, n) processes with $\gamma_{k,\ell} = 0$ if $k < \ell$. Using Markov chain techniques we obtain the existence and uniqueness of a strictly stationary solution of (1) when $\sum_i \alpha_i + \mu_G \sum_{k,\ell} \gamma_{k,\ell} < 1$ (see Theorem 2.1). Theorem 2.2 provides sufficient parameter restrictions ensuring the existence of (higher-order) moments under the stationary distribution. Along completely different lines our results generalize Sections 2 and 3 in Doukhan et al. (2006) who restricted attention to the INBL$(1, 0, 1, 1)$ process. Moreover our parameter restrictions in Theorem 2.2 are less severe. Estimation of superdiagonal INBL processes can be performed along the same lines as in Doukhan et al. (2006); The details are beyond the scope of this note.

2. Results

First we discuss the existence of a stationary solution. The insightful proof of Theorem 2.1 in Doukhan et al. (2006) gives an explicit construction of approximating processes for the INBL$(1, 0, 1, 1)$ process. For the general superdiagonal INBL(p, q, m, n) case we provide a compact proof using Markov chain techniques, yielding also an alternative proof for the INBL$(1, 0, 1, 1)$ process.

Let $F * G$ denote the convolution of the probability measures F and G and introduce the process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t = (X_{t-1}, \ldots, X_{t-p\vee m}, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q\vee n}). \tag{2}$$

Notice that $Z_0 \sim \nu$. It is easy to see that Z is a Markov chain. For $t \geq 0$ and $z_t = (x_{t-1}, \ldots, x_{t-p\vee m}, e_{t-1}, \ldots, e_{t-q\vee n})$ we have

$$\begin{aligned}
&\mathbb{P}(Z_{t+1} = z_{t+1} \mid Z_t = z_t) \\
&\quad = G\{e\} \left(*_{i=1}^{p} (*_{s=1}^{x_{t-i}} F_{\alpha_i}) *_{j=1}^{q} (*_{s=1}^{e_{t-j}} F_{\beta_j}) *_{k=1}^{m} *_{\ell=1}^{n} (*_{s=1}^{x_{t-k} e_{t-\ell}} F_{\gamma_{k,\ell}})\right) \{z - e\},
\end{aligned}$$

*Implicitly it is understood that distinct Steutel-van Harn operators are independent and also that Steutel-van Harn operators applied to different random variables are independent; see also Du and Li (1991).

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if $z_{t+1} = (z, x_{t-1}, \ldots, x_{t+1-p\vee m}, e, e_{t-1}, \ldots, e_{t+1-q\vee n})$ for some e, z, and with $\mathbb{P}(Z_{t+1} = z_{t+1} \mid Z_t = z_t) = 0$ otherwise. From (1) it is immediate that, for $t \geq 0$,

$$\mathbb{E}[X_t \mid Z_t] = \mu_G + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j} + \sum_{k=1}^{m}\sum_{\ell=1}^{n} \gamma_{k,\ell} X_{t-k}\varepsilon_{t-\ell} \in [0, \infty]. \tag{3}$$

The following proposition provides sufficient conditions for Z to be an irreducible aperiodic Markov chain. The proof is immediate from the definition of the state space.

Proposition 2.1 *Let Z be the time series defined in (2) linked to the INBL process X defined in (1) and assume, for all relevant i, j, k, ℓ,*

$$0 < G\{0\}, \quad 0 < F_{\alpha_i}\{0\}, \quad 0 < F_{\beta_j}\{0\}, \quad 0 < F_{\gamma_{k,\ell}}\{0\}. \tag{4}$$

Then Z is an irreducible aperiodic Markov chain on the state space

$$\mathcal{S} = \{z \mid \exists t \geq 1 : \mathbb{P}(Z_t = z \mid Z_0 = \underline{0}) > 0\}.$$

Remark Condition (4) allows the process X to arrive at 0 and therefore (4) seems to be harmless in real-life applications. Condition (4) can be easily verified from a sufficiently long data series just by checking whether the process has hit zero. At the cost of lengthy derivations, condition (4) might be weakened but this is outside the scope of this short note.

Theorem 2.1 *Let X be the superdiagonal INBL process as defined in (1) with $\gamma_{k,\ell} = 0$ if $k < \ell$ and assume (4), $\sigma_G^2 < \infty$, $\sum_{j=1}^{q} \beta_j < \infty$, and*

$$\sum_{i=1}^{p} \alpha_i + \mu_G \sum_{k=1}^{m}\sum_{\ell=1}^{n} \gamma_{k,\ell} < 1. \tag{5}$$

Then there exists a unique initial distribution ν^* such that X is strictly stationary under $\mathbb{P}_{\nu^*}$. Furthermore the first moment of X is finite under $\mathbb{P}_{\nu^*}$.

Proof. We prove that there exists a unique initial distribution ν^* such that Z is stationary under $\mathbb{P}_{\nu^*}$, which yields the result. Since Z is irreducible and aperiodic on $\mathcal{S}$ (and since, for countable Markov chains, finite sets are petite) it suffices to prove, see Theorem 15.0.1 in Meyn and Tweedie (1994), that there exists a mapping $V : z \mapsto [1, \infty)$ such that the following Foster-Lyapunov drift criterium holds: there exists $\delta > 0$ and a finite set such that for all Z_t outside this finite set we have

$$W(Z_t) = (1 - \delta)V(Z_t) - \mathbb{E}[V(Z_{t+1}) \mid Z_t] \geq 0. \tag{6}$$

Take, for notational simplicity, $p = q = m = n$. To verify (6) choose $\delta > 0$ sufficiently small such that

$$\sum_i \alpha_i + \mu_G \sum_{k\geq\ell} \gamma_{k,\ell} < (1 - \delta)^n - n\delta - \frac{1}{2}n(n + 1)\mu_G\delta,$$

and define $V : z \mapsto [1, \infty)$ by

$$V(Z_t) = 1 + \sum_i \tilde{\alpha}_i X_{t-i} + \sum_j \tilde{\beta}_j \varepsilon_{t-j} + \sum_{k\geq\ell} \tilde{\gamma}_{k,\ell} X_{t-k}\varepsilon_{t-\ell},$$

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where the nonnegative tilde parameters are recursively defined from $\tilde{\alpha}_{n+1} = \tilde{\beta}_{n+1} = \tilde{\gamma}_{n+1,\ell} = 0$ and

$$\begin{aligned}(1-\delta)\tilde{\alpha}_i &= \tilde{\alpha}_{i+1} + \mu_G\tilde{\gamma}_{i+1,1} + \alpha_i + \delta,\\ (1-\delta)\tilde{\beta}_j &= \tilde{\beta}_{j+1} + \beta_j + \delta,\\ (1-\delta)\tilde{\gamma}_{k,\ell} &= \tilde{\gamma}_{k+1,\ell+1} + \gamma_{k,\ell} + \delta.\end{aligned}$$

Note, by the choice of δ,

$$\theta \equiv \tilde{\alpha}_1 + \mu_G\tilde{\gamma}_{1,1} = \sum_i (\alpha_i + \delta)/(1-\delta)^i + \mu_G \sum_{k\geq\ell} (\gamma_{k,\ell} + \delta)/(1-\delta)^k < 1.$$

Hence we obtain, using (1), (3), and the independence of ε_t and $X_t - \varepsilon_t$,

$$\begin{aligned}W(Z_t) = & -\delta - \mu_G(\tilde{\alpha}_1 + \tilde{\beta}_1) - (\mu_G^2 + \sigma_G^2)\tilde{\gamma}_{1,1}\\ & + \sum_i [(1-\delta)\tilde{\alpha}_i - (\tilde{\alpha}_1 + \mu_G\tilde{\gamma}_{1,1})\alpha_i - \tilde{\alpha}_{i+1} - \mu_G\tilde{\gamma}_{i+1,1}]X_{t-i}\\ & + \sum_j [(1-\delta)\tilde{\beta}_j - (\tilde{\alpha}_1 + \mu_G\tilde{\gamma}_{1,1})\beta_j - \tilde{\beta}_{j+1}]\varepsilon_{t-j}\\ & + \sum_{k\geq\ell} [(1-\delta)\tilde{\gamma}_{k,\ell} - (\tilde{\alpha}_1 + \mu_G\tilde{\gamma}_{1,1})\gamma_{k,\ell} - \tilde{\gamma}_{k+1,\ell+1}]X_{t-k}\varepsilon_{t-\ell}\\ = & -\delta - \mu_G(\tilde{\alpha}_1 + \tilde{\beta}_1) - (\mu_G^2 + \sigma_G^2)\tilde{\gamma}_{1,1} + (1-\theta)\times\\ & \left[\sum_i (\alpha_i + \frac{\delta}{1-\theta})X_{t-i} + \sum_j (\beta_j + \frac{\delta}{1-\theta})\varepsilon_{t-j} + \sum_{k\geq\ell} (\gamma_{k,\ell} + \frac{\delta}{1-\theta})X_{t-k}\varepsilon_{t-\ell}\right].\end{aligned}$$

Conclude $W(Z_t) \geq 0$ outside a finite set. Hence the drift condition (6) holds, which concludes the proof of the stationarity. The existence of the first moment under ν^* is immediate from Theorem 2.2 below. □

Remark For $p = m = n = 1$ and $q = 0$, (5) reduces to the condition of Theorem 2.1 in Doukhan et al. (2006).

The next theorem gives a sufficient condition for the existence of higher order moments of X_t under $\mathbb{P}_{\nu^*}$.

Theorem 2.2 *Let X be the superdiagonal INBL process as defined in (1) with $\gamma_{k,\ell} = 0$ if $k < \ell$. Suppose, for some $K \in \mathbb{N}$, the existence of the K-th order moments of $F_{\alpha_i}, F_{\beta_j}$, and $F_{\gamma_{k,\ell}}$, and assume (4), $\mathbb{E}_G\varepsilon_0^{2K} < \infty$, $\sum_{j=1}^q \beta_j < \infty$, and*

$$\sum_{i=1}^p \alpha_i + \left(\mathbb{E}_G\varepsilon_0^K\right)^{1/K} \sum_{k=1}^m \sum_{\ell=1}^n \gamma_{k,\ell} < 1. \tag{7}$$

Then, with ν^* the stationary distribution of Theorem 2.2,

$$\mathbb{E}_{\nu^*} X_0^K < \infty.$$

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Proof. Recall from Proposition 2.1 that Z is an irreducible aperiodic Markov chain on $\mathcal{S}$ and $\underline{0} \in \mathcal{S}$. Hence, since Z has stationary distribution ν^*, we have $Z_t \xrightarrow{d} \nu^*$, under $\mathbb{P}_{\delta_{\underline{0}}}$ as $t \to \infty$. Thus the convergence also holds for marginals. By the Portmanteau theorem, and nonnegativity of X, this implies for $L \in \mathbb{N}$,

$$\mathbb{E}_{\nu^*} X_0^{L-1} \leq \liminf_{t\to\infty} \mathbb{E}_{\delta_{\underline{0}}} X_t^{L-1} \leq \sup_{0\leq t<\infty} \mathbb{E}_{\delta_{\underline{0}}} X_t^{L-1}. \tag{8}$$

To prove that the right hand side of (8) is bounded for $L \leq K+1$, we use induction. Clearly the statement holds true for $L = 1$. To show the boundedness for $L = 2$ we make some preliminary remarks. Denote, for notational convenience, $\mathbb{E}_{\delta_{\underline{0}}}$ by $\mathbb{E}_0$ and let, for $P \geq 1$, $\|Z\|_P = \left(\mathbb{E}_0|Z|^P\right)^{1/P}$ denote the $\mathrm{L}_P(\mathbb{P}_{\delta_{\underline{0}}})$ norm. Recall $\|\cdot\|_P \leq \|\cdot\|_Q$ for $1 \leq P \leq Q$. Hence (7) implies that $\sum_i \alpha_i + \|\varepsilon_0\|_v \sum_{k,\ell} \gamma_{k,\ell} < 1$ for all $1 \leq v \leq K$. Since, for $t \leq s$, $X_t - \varepsilon_t$ and ε_s are independent, we have, for $P \geq 1$,

$$\|X_t\varepsilon_s\|_P \leq \|(X_t - \varepsilon_t)\varepsilon_s\|_P + \|\varepsilon_t\varepsilon_s\|_P \leq \|\varepsilon_0\|_P\|X_t\|_P + \|\varepsilon_0^2\|_P. \tag{9}$$

Using (3), (9) with $P = 1$, the zero starting values, and (7), the boundedness for $L = 2$ is obtained from,

$$\begin{aligned}
\mathbb{E}_0 X_t &= \mu_G + \sum_i \alpha_i \mathbb{E}_0 X_{t-i} + \sum_j \beta_j \mu_G + \sum_{k\geq\ell} \gamma_{k,\ell} \mathbb{E}_0 X_{t-k}\varepsilon_{t-\ell} \\
&\leq \mu_G + \mu_G \sum_j \beta_j + (\mu_G^2 + \sigma_G^2) \sum_{k\geq\ell} \gamma_{k,\ell} + \sum_i \alpha_i \mathbb{E}_0 X_{t-i} + \mu_G \sum_{k\geq\ell} \gamma_{k,\ell} \mathbb{E}_0 X_{t-k} \\
&\leq \left[\mu_G + \mu_G \sum_j \beta_j + (\mu_G^2 + \sigma_G^2) \sum_{k\geq\ell} \gamma_{k,\ell}\right] \sum_{s=0}^{\infty} \left(\sum_i \alpha_i + \mu_G \sum_{k\geq\ell} \gamma_{k,\ell}\right)^s < \infty.
\end{aligned}$$

To complete the induction, we assume that the right hand side of (8) is bounded for some $2 \leq L \leq K$ and we prove that this implies $\sup_{0\leq t<\infty} \mathbb{E}_{\delta_{\underline{0}}} X_t^L < \infty$. We use the following result (see, for example, Dharmadhikari et al. (1968)).

Lemma 2.1 *If $Z_1, \ldots, Z_n$ are i.i.d. variables with mean zero and a finite k-th moment, $k \geq 2$, then we have the bound*

$$\mathbb{E}\left|\sum_{s=1}^n Z_s\right|^k \leq C_k n^{k/2} \mathbb{E}|Z_1|^k,$$

where the constant $C_k > 0$ only depends on k (and not on the distribution of Z_1).

Using that the Steutel-van Harn operator $\eta \circ Z$, conditional on Z, follows a $*_{s=1}^{Z} F_\eta$ distribution, Lemma 2.1 implies the following inequality for $L \geq 2$,

$$\mathbb{E}|\eta \circ Z - \eta Z|^L = \mathbb{E}\mathbb{E}\left[|\eta \circ Z - \eta Z|^L \mid Z\right] \leq C_L \mathbb{E} Z^{L/2} \mathbb{E}|U^{(\eta)} - \eta|^L \leq C_{L,\eta} \|Z\|_{L-1}^{L/2}. \tag{10}$$

We have for $t \geq 0$, using (9),

$$\|X_t\|_L \leq \|X_t - \sum_i \alpha_i X_{t-i} - \sum_j \beta_j \varepsilon_{t-j} - \sum_{k\geq\ell} \gamma_{k,\ell} X_{t-k}\varepsilon_{t-\ell}\|_L$$

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$$
\begin{aligned}
& + \|\sum_i \alpha_i X_{t-i} + \sum_j \beta_j \varepsilon_{t-j} + \sum_{k\geq \ell} \gamma_{k,\ell} X_{t-k}\varepsilon_{t-\ell}\|_L \\
\leq \quad & \sum_i \|\alpha_i \circ X_{t-i} - \alpha_i X_{t-i}\|_L + \sum_j \|\beta_j \circ \varepsilon_{t-j} - \beta_j \varepsilon_{t-j}\|_L \\
& + \sum_{k\geq \ell} \|\gamma_{k,\ell} \circ (X_{t-k}\varepsilon_{t-\ell}) - \gamma_{k,\ell} X_{t-k}\varepsilon_{t-\ell}\|_L + \|\varepsilon_t\|_L \\
& + \sum_i \alpha_i \|X_{t-i}\|_L + \sum_j \beta_j \|\varepsilon_{t-j}\|_L + \sum_{k\geq \ell} \gamma_{k,\ell}(\|\varepsilon_0\|_L \|X_{t-k}\|_L + \|\varepsilon_0^2\|_L) \\
\leq \quad & M + \sum_i \alpha_i \|X_{t-i}\|_L + \|\varepsilon_0\|_L \sum_{k\geq \ell} \gamma_{k,\ell} \|X_{t-k}\|_L,
\end{aligned}
$$

where the appropriately chosen constant $M < \infty$, not depending on t, can be obtained from the assumed moment conditions and the induction hypothesis applied to (10) and (9). Condition (7) completes the induction argument just as for the case $L = 2$,

$$
\|X_t\|_L \leq M \sum_{s=0}^{\infty} \left(\sum_i \alpha_i + \|\varepsilon_0\|_L \sum_{k\geq \ell} \gamma_{k,\ell} \right)^s .
$$

This completes the proof. □

Remark For the INBL(1,0,1,1) process Doukhan et al. (2006) showed that the condition $\|U^{(\alpha)}\|_K + \|\varepsilon_0\|_K \|U^{(\gamma)}\|_K < 1$ suffices to obtain the existence of the K-th moment of X_t under $\mathbb{P}_{\nu^*}$. Condition (7) is weaker since, for $K \geq 1$, $\eta = \|U^{(\eta)}\|_1 \leq \|U^{(\eta)}\|_K$.

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