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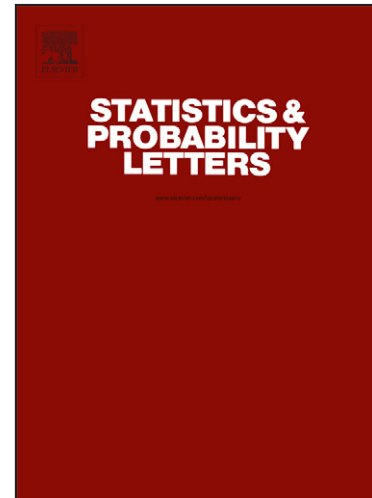
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# Central limit theorem of random quadratics forms involving random matrices

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## Abstract

Let  $s_i = \frac{1}{\sqrt{N}}(v_{i,1}, \dots, v_{i,N})^T$  and  $S = (s_1, s_2, \dots, s_K)$  where random variables  $\{v_{ij}, i, j = 1, 2, \dots, \}$  are i.i.d. with  $Ev_{11}^4 < \infty$ . The central limit theorem of the random quadratic forms  $s_1^T (SS^T)^i s_1$  is established, which arises from the application in wireless communications.

## 1 Introduction

Let random variables  $\{v_{ij}, i, j = 1, \dots, \}$  be independent and identically distributed (i.i.d.) with  $Ev_{11} = 0$  and  $Ev_{11}^2 = 1$ . Define  $s_i = \frac{1}{\sqrt{N}}(v_{i,1}, \dots, v_{i,N})^T$ ,  $i = 1, \dots, K$ ,  $S_1 = (s_2, s_3, \dots, s_K)$ ,  $S = (s_1, s_2, \dots, s_K)$ . Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  be the eigenvalues of  $\frac{N}{K}S_1S_1^T$

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and  $F_N$  be the empirical spectral distribution of  $\frac{N}{K}S_1S_1^T$ , i.e.,

$$F_N(x) = \frac{\sum_{i=1}^N I(\lambda_i \leq x)}{N}.$$

It is well known that if  $Ev_{11}^2 < \infty$  and  $\lim_{N \rightarrow \infty} \frac{K}{N} = y > 0$ , then

$$F_N(x) \xrightarrow{a.s.} F_y(x),$$

where  $F_y(x)$  denotes the limiting spectral distribution. There are many literatures about the convergence of  $F_N(x)$  (see Bai (1999) and the references given there).

Lately, the theory of large dimensional random matrices has been a popular tool in the research of the asymptotic problem of code division multiple-access (CDMA) communication systems. For example, one may refer to the work of Tse and Hanly (1999) and Tse and Zeitouni (2000). The application in the communication also puts forward challenges to the large dimensional matrix theory. For instance, to analyze the large system performance of the signal-to-interference (SIR) of some kind of receiver, the random quadratic forms involving random matrix,  $s_1^t(S_1S_1^T)^i s_1$  and  $s_1^t(SS^T)^i s_1$ , were introduced by Trichard, et al. (2002). However, only the convergence in probability for a special random sequence was given in their work. As we know, the central limit theorem is more powerful than the convergence in probability. Even in wireless communications, central limit theorem can also be used to characterize the fluctuation of the performance of SIR around the asymptotic value. The main object of this paper is to prove that  $s_1^t(SS^T)^i s_1$  is asymptotic normal under the assumption  $Ev_{11}^4 < \infty$ . Here we would like to emphasize that the random vector  $s_1$  is not independent of the random matrix  $SS^T$ .

## 2 Main results

For convenience, some notations are firstly introduced. Let

$$\alpha_i^{(N)} = s_1^T(S_1S_1^T)^i s_1, \beta_i^{(N)} = s_1^T(SS^T)^i s_1, \hat{X}_i = y^i \int_{(1-\sqrt{y})^2}^{(1+\sqrt{y})^2} t^i dW^0(F_y(t))$$

$$y_N = \frac{K}{N}, \phi_i(y_N) = y_N^i \int x^i dF_{y_N}(x), X_i = \hat{X}_i + \frac{\xi}{\sqrt{2}} y^i \int x^i dF_y(x)$$

where  $F_y(t)$  is the limiting spectral distribution,  $W^0(t)$  is a Brownian bridge independent of  $\xi$ , which is  $N(0, Ev_{11}^4 - 1)$ . Here  $F_{y_N}(x)$  is obtained from  $F_y(x)$  by replacing  $y$  with  $y_N$  and other analogues below such as  $\phi_i(y_N)$  have similar meanings. Moreover,  $i \geq 1$  is required in the definition of  $X_i$ . Our main results are as below.

**Theorem 2.1** *Assume that  $\{v_{ij}\}$  are independent and identically distributed (i.i.d.) for all  $i$  and  $j$  with  $Ev_{11} = 0$ ,  $Ev_{11}^2 = 1$  and  $Ev_{11}^4 < \infty$ . In addition, suppose  $\lim_{N \rightarrow \infty} \frac{K}{N} = y > 0$ . Then for  $i \geq 1$*

$$\sqrt{\frac{N}{2}}(\beta_i^{(N)} - \beta_i(y_N)) \xrightarrow{D} Y_i$$

where

$$\begin{aligned} Y_i = & X_i + \phi_0(y)Y_{i-1} + \beta_{i-1}(y)X_0 + \phi_1(y)Y_{i-2} + \beta_{i-2}(y)X_1 \\ & + \cdots + \phi_{i-1}(y)Y_0 + \beta_0(y)X_{i-1} \end{aligned}$$

and

$$\begin{aligned} \beta_i(y_N) = & \phi_i(y_N) + \phi_0(y_N)\beta_{i-1}(y_N) + \phi_1(y_N)\beta_{i-2}(y_N) \\ & + \cdots + \phi_{i-2}(y_N)\beta_1(y_N) + \phi_{i-1}(y_N)\beta_0(y_N), \end{aligned}$$

with  $Y_0 = X_0 = \frac{\xi}{\sqrt{2}}$ ,  $\phi_0(y_N) = \phi_0(y) = \beta_0(y_N) = \beta_0(y) = 1$  and the corresponding term being zero if some subscript is negative.

Further, we have a multidimensional version of central limit theorem.

**Theorem 2.2** *If the conditions of Theorem 2.1 are satisfied, then*

$$\left\{ \sqrt{\frac{N}{2}}(\beta_i^{(N)} - \beta_i(y_N)) \right\}_{i=1}^{\infty} \xrightarrow{D} \{Y_i\}_{i=1}^{\infty},$$

where  $D$  denotes weak convergence on  $R^\infty$ .

### 3 The proof of theorem

To simplify the notations, the constant  $c$  may denote different values at different occasions in what follows. Before proceeding, we first need some lemmas.

**Lemma 3.1** *Let  $g_N(s_1)$  be a function of the random vector  $s_1$  and  $\xi_N$  be a random variable. Suppose that for  $u \in C(F), v \in C(G)$ ,*

$$\lim_{N \rightarrow \infty} P(\xi_N \leq u | s_1) = F(u), \quad a.s., \quad \lim_{N \rightarrow \infty} P(g_N(s_1) \leq v) = G(v)$$

where  $F(u)$  and  $G(v)$  both are the non-random distribution functions,  $C(F)$  and  $C(G)$  denote the sets of the continuity points of  $F(u)$  and  $G(v)$ , respectively. Then

$$\lim_{N \rightarrow \infty} P(\xi_N \leq u, g_N(s_1) \leq v) = F(u)G(v).$$

**Proof.** It is observed that

$$P(\xi_N \leq u, g_N(s_1) \leq v) = \int I(g_N(s_1) \leq v) P(\xi_N \leq u | s_1) dP.$$

It follows, with the help of the Fatou lemma, that

$$\begin{aligned} & \limsup_{N \rightarrow \infty} |P(\xi_N \leq u, g_N(s_1) \leq v) - F(u)P(g_N(s_1) \leq v)| \\ & \leq \limsup_{N \rightarrow \infty} \left| \int I(g_N(s_1) \leq v) [P(\xi_N \leq u | s_1) - F(u)] dP \right| \\ & \leq \int \limsup_{N \rightarrow \infty} |P(\xi_N \leq u | s_1) - F(u)| dP \\ & = 0. \end{aligned}$$

On the other hand

$$\int I(g_N(s_1) \leq v) F(u) dP \rightarrow F(u)G(v).$$

Thus the proof is complete.

Based on Theorem 4.1 of Silverstein (1990), we next establish a lemma concerning multidimensional central limit theorem.

**Lemma 3.2** *Under the assumptions of Theorem 2.1, the following holds:*

$$\begin{aligned} & \sqrt{\frac{N}{2}} \left( \alpha_i^{(N)} - \phi_i(y_N), \dots, \alpha_1^{(N)} - \phi_1(y_N), \alpha_0^{(N)} - 1, (s_1^T s_1)^2 - 1 \right) \\ & \xrightarrow{D} (X_i, \dots, X_1, \eta_1, \eta_2) \end{aligned}$$

where  $\eta_1 = \frac{\xi}{\sqrt{2}}$ ,  $\eta_2 = \sqrt{2}\xi$  and  $D$  denotes weak convergence.

**Proof.** For any real numbers  $t_{-1}, t_0, t_1, \dots, t_i$  and any positive integer  $i$ , write

$$\begin{aligned} & \sqrt{\frac{N}{2}} \left( t_i \alpha_i^{(N)} + \dots + t_1 \alpha_1^{(N)} + t_0 \alpha_0^{(N)} + t_{-1} (s_1^T s_1)^2 \right. \\ & \quad \left. - t_i \phi_i(y_N) - \dots - t_1 \phi_1(y_N) - t_0 - t_{-1} \right) \\ & \triangleq I_N^{(1)} + I_N^{(2)} \end{aligned}$$

where

$$\begin{aligned} I_N^{(1)} &= \sqrt{\frac{N}{2}} \left[ t_i \left( \frac{\alpha_i^{(N)}}{\|s_1\|^2} - \frac{1}{N} \text{tr}(S_1 S_1^T)^i \right) \|s_1\|^2 + t_i \frac{1}{N} \text{tr}(S_1 S_1^T)^i (\|s_1\|^2 - 1) \right. \\ & \quad + \dots + t_1 \left( \frac{\alpha_1^{(N)}}{\|s_1\|^2} - \frac{1}{N} \text{tr}(S_1 S_1^T) \right) \|s_1\|^2 + t_1 \frac{1}{N} \text{tr}(S_1 S_1^T) (\|s_1\|^2 - 1) \\ & \quad \left. + t_0 (\alpha_0^{(N)} - 1) + t_{-1} ((s_1^T s_1)^2 - 1) \right] \end{aligned}$$

and

$$I_N^{(2)} = \sqrt{\frac{N}{2}} \left( \frac{1}{N} t_i \text{tr}(S_1 S_1^T)^i + \dots + \frac{1}{N} t_1 \text{tr}(S_1 S_1^T) - t_i \phi_i(y_N) - \dots - t_1 \phi_1(y_N) \right),$$

where  $\|\cdot\|$  denotes the Euclidean norm.

We first consider the term  $I_N^{(1)}$ . Define  $I_N^{(3)} = \sqrt{N}(s_1^T s_1 - 1)$ ,

$$\begin{aligned} f(u) &= \left( \frac{t_i}{\sqrt{2}} y^i \int x^i dF_y(x) + \frac{t_{i-1}}{\sqrt{2}} y^{i-1} \int x^{i-1} dF_y(x) \right. \\ & \quad \left. + \dots + \frac{t_1}{\sqrt{2}} y \int x dF_y(x) + \frac{t_0}{\sqrt{2}} \right) u + \frac{t_{-1}}{\sqrt{2}} u^2, \end{aligned}$$

and

$$\xi_N = \sqrt{\frac{N}{2}} \left[ t_i \left( \frac{\alpha_i^{(N)}}{\|s_1\|^2} - \frac{1}{N} \text{tr}(S_1 S_1^T)^i \right) \|s_1\|^2 + \dots + t_1 \left( \frac{\alpha_1^{(N)}}{\|s_1\|^2} - \frac{1}{N} \text{tr}(S_1 S_1^T) \right) \|s_1\|^2 \right]$$

By the classical central limit theorem,

$$I_N^{(3)} \xrightarrow{D} \xi,$$

Here  $\xi$  is defined in the second part. This, together with Theorem 2.5.2 of Chen (1999), implies that

$$\sqrt{N}(f(\|s_1\|^2) - f(1)) \xrightarrow{D} a\xi \quad (3.1)$$

where

$$\begin{aligned} a &= \frac{t_i}{\sqrt{2}} y^i \int x^i dF_y(x) + \frac{t_{i-1}}{\sqrt{2}} y^{i-1} \int x^{i-1} dF_y(x) \\ &\quad + \cdots + \frac{t_1}{\sqrt{2}} y \int x dF_y(x) + \frac{t_0}{\sqrt{2}} + \sqrt{2}t_{-1}. \end{aligned}$$

Appealing to the strong law of large numbers gives

$$\lim_{N \rightarrow \infty} \|s_1\|^2 = 1, \quad \lim_{N \rightarrow \infty} \sum_{i=1}^N \left( \frac{\frac{1}{\sqrt{N}} v_{1i}}{\|s_1\|} \right)^4 = 0 \quad \text{a.s.}$$

It follows that

$$\begin{aligned} &\lim_{N \rightarrow \infty} P(\xi_N \leq u | s_1) \\ &= P(t_i \hat{X}_i + t_{i-1} \hat{X}_{i-1} + \cdots + t_1 \hat{X}_1 \leq u) \quad \text{a.s.}, \end{aligned} \quad (3.2)$$

where we make use of Theorem 4.1 of Silverstein (1990), Theorem 4.1 of Billingsley (1968) and the fact that  $s_1$  is independent of  $S_1 S_1^T$ .

Combining (3.1), (3.2) with Lemma 3.1, we obtain

$$\begin{aligned} &\lim_{N \rightarrow \infty} P(\xi_N \leq u, \sqrt{N}(f(\|s_1\|^2) - f(1)) \leq v) \\ &= P(t_i \hat{X}_i + t_{i-1} \hat{X}_{i-1} + \cdots + t_1 \hat{X}_1 \leq u, a\xi \leq v) \end{aligned}$$

Hence, using Theorem 4.1, Corollary 1 to Theorem 5.1 of Billingsley (1968) and the fact that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{tr}(S_1 S_1^T)^i = y^i \int x^i dF_y(x) \quad \text{a.s.} \quad (3.3)$$

(One may see Jonsson (1982) or Yin (1986)), we have

$$I_N^{(1)} \xrightarrow{D} t_i X_i + t_{i-1} X_{i-1} + \cdots + t_1 X_1 + t_0 \eta_1 + t_{-1} \eta_2.$$



Finally, it suffices to show that

$$I_N^{(2)} \xrightarrow{i.p.} 0.$$

However this is immediately given by Theorem 1.1 of Bai and Silverstein (2004). Thus we are done according to Crammer-Wold method.

From now on, we begin with the proof of Theorem 2.1. Recall the definitions of  $\beta_i^{(N)}$ , we know that the present difficulty is that  $s_1$  is not independent of  $SS^T$ . To overcome it, we mainly employ the induction.

**Proof of Theorem 2.1.** First, one can easily derive the following equality:

$$\beta_i^{(N)} = \alpha_i^{(N)} + \alpha_0^{(N)} \beta_{i-1}^{(N)} + \alpha_1^{(N)} \beta_{i-2}^{(N)} + \cdots + \alpha_{i-2}^{(N)} \beta_1^{(N)} + \alpha_{i-1}^{(N)} \beta_0^{(N)}, \quad (3.4)$$

where  $\alpha_i^{(N)} = \beta_i^{(N)} = 0$ , if  $i < 0$  and  $\alpha_0^{(N)} = \beta_0^{(N)} = s_1^T s_1$ .

We proceed by induction on  $i$ . By Lemma 3.2, for  $i = 1$ , we have

$$\sqrt{\frac{N}{2}}(\beta_1^{(N)} - \beta_1(y_N)) = \sqrt{\frac{N}{2}}(\alpha_1^{(N)} - \phi_1(y_N) + (s_1^T s_1)^2 - 1) \xrightarrow{D} Y_1 \triangleq X_1 + \sqrt{2}\xi.$$

Here, we should mention a key fact that the asymptotic distribution of the random variable  $\beta_1^{(N)}$  is derived by the asymptotic distribution of the linear combination of the random variables  $\alpha_1^{(N)}$  and  $(s_1^T s_1)^2$ .

For  $i \leq m$ , we need make the following two inductive hypothesis :

a): Theorem 2.1 is true for  $i \leq m$ ,

b): For  $i \leq m$ , the asymptotic distribution of the random variable  $\beta_i^{(N)}$  can be derived by the asymptotic distribution of the linear combination of the random variables  $\{\alpha_j^{(N)}, 0 \leq j \leq i\}$  and  $(s_1^T s_1)^2$ . In other words, there exist constant number  $t_0, t_1, \dots, t_i, t_{-1}$  such that

$$\begin{aligned} \sqrt{\frac{N}{2}}(\beta_i^{(N)} - \beta_i(y_N)) &= \sqrt{\frac{N}{2}}(t_i \alpha_i^{(N)} + \cdots + t_0 \alpha_0^{(N)} + t_{-1} (s_1^T s_1)^2 \\ &\quad - t_i \phi_i(y_N) - \cdots - t_0 - t_{-1}) \xrightarrow{i.p.} 0 \end{aligned} \quad (3.5)$$

Suppose  $i = m + 1$ . It follows from (3.4) that

$$\begin{aligned}
 & \sqrt{\frac{N}{2}}(\beta_{m+1}^{(N)} - \beta_{m+1}(y_N)) \\
 = & \sqrt{\frac{N}{2}}\left(\alpha_{m+1}^{(N)} - \phi_{m+1}(y_N) + \alpha_0^{(N)}\beta_m^{(N)} - \beta_m(y_N) \right. \\
 & \left. + \alpha_1^{(N)}\beta_{m-1}^{(N)} - \phi_1(y_N)\beta_{m-1}(y_N) + \cdots + \alpha_m^{(N)}\beta_0^{(N)} - \phi_m(y_N)\right) \\
 = & \sqrt{\frac{N}{2}}\left(\alpha_{m+1}^{(N)} - \phi_{m+1}(y_N) + \alpha_0^{(N)}(\beta_m^{(N)} - \beta_m(y_N)) + \beta_m(y_N)(\alpha_0^{(N)} - 1) \right. \\
 & \left. + \alpha_1^{(N)}(\beta_{m-1}^{(N)} - \beta_{m-1}(y_N)) + \beta_{m-1}(y_N)(\alpha_1^{(N)} - \phi_1(y_N)) \right. \\
 & \left. + \cdots + \alpha_m^{(N)}(\beta_0^{(N)} - 1) + \alpha_m^{(N)} - \phi_m(y_N)\right). \tag{3.6}
 \end{aligned}$$

Define

$$\begin{aligned}
 \eta_N = & \sqrt{\frac{N}{2}}\left(\alpha_{m+1}^{(N)} - \phi_{m+1}(y_N) + (\beta_m^{(N)} - \beta_m(y_N)) + \beta_m(y_N)(\alpha_0^{(N)} - 1) \right. \\
 & \left. + \phi_1(y_N)(\beta_{m-1}^{(N)} - \beta_{m-1}(y_N)) + \beta_{m-1}(y_N)(\alpha_1^{(N)} - \phi_1(y_N)) \right. \\
 & \left. + \cdots + \phi_m(y_N)(\beta_0^{(N)} - 1) + \alpha_m^{(N)} - \phi_m(y_N)\right).
 \end{aligned}$$

By Lemma 3.2 and Theorem 4.1 of Billingsley (1968) and inductive hypothesis b), we see that the asymptotic distribution of  $\eta_N$  can be obtained by the asymptotic distribution of the linear combination of the random variables  $\{\alpha_j^{(N)}, 0 \leq j \leq m+1\}$  and  $(s_1^T s_1)^2$ . Thus, applying Lemma 3.2, Theorem 4.1 of Billingsley (1968), inductive hypothesis a) and (3.5), we conclude that

$$\eta_N \xrightarrow{D} Y_{m+1}. \tag{3.7}$$

On the other hand, by Lemma 2.7 of Bai and Silverstein (1998), we have

$$\begin{aligned}
 E(\alpha_m^{(N)} - \frac{1}{N}tr(S_1 S_1^T)^m)^2 &= E \left[ E \left( (\alpha_m^{(N)} - \frac{1}{N}tr(S_1 S_1^T)^m)^2 | S_1 S_1^T \right) \right] \\
 &\leq \frac{c}{N^2} E X_{11}^4 E tr(S_1 S_1^T)^{2m},
 \end{aligned}$$

which implies that

$$\alpha_m^{(N)} \xrightarrow{i.p.} \phi_m(y_N), \tag{3.8}$$

where we use (3.3) and the fact

$$\frac{1}{N} \text{Etr}(S_1 S_1^T)^{2m} \rightarrow y^{2m} \int x^{2m} dF_y(x),$$

as  $N \rightarrow \infty$ . Based on (3.6)-(3.8), via Theorem 4.1 of Billingsley (1968), we have

$$\sqrt{\frac{N}{2}}(\beta_{m+1}^{(N)} - \beta_{m+1}(y_N)) \xrightarrow{D} Y_{m+1}. \quad (3.9)$$

By the above argument, one see easily that inductive hypothesis b) is also true for  $i = m + 1$ . Hence the proof is complete.

**Proof of Theorem 2.2:** This Theorem follows easily upon Cramer-Wold method and the argument of Theorem 2.1. Details are omitted.

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