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New Cases of Universality Theorem for Gravitational Theories*

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Abstract: The "Universality Theorem" for gravity shows that $f(R)$ theories (in their metric-affine formulation) in vacuum are dynamically equivalent to vacuum Einstein equations with suitable cosmological constants. This holds true for a generic (i.e. except sporadic degenerate cases) analytic function $f(R)$ and standard gravity without cosmological constant is reproduced if f is the identity function (i.e. $f(R) = R$). The theorem is here extended introducing in dimension 4 a 1-parameter family of invariants ${}^{\beta}R$ inspired by the Barbero-Immirzi formulation of GR (which in the Euclidean sector includes also selfdual formulation). It will be proven that $f({}^{\beta}R)$ theories so defined are dynamically equivalent to the corresponding metric–affine $f(R)$ theory. In particular for the function $f(R) = R$ the standard equivalence between GR and Holst Lagrangian is obtained.

1. Introduction

It is well-known that for almost any analytic function $f(R)$ the metric-affine theory with Lagrangian $L_f(g, j^1\Gamma) = \sqrt{g} f(R(g, j^1\Gamma))$ is dynamically equivalent to standard GR with a suitably quantized cosmological constant (encoded by the function f); see [**1**] and [**2**]. Hereafter $j^1\Gamma$ refers to the fact that the Lagrangian depends on the connection Γ and its first derivatives.

The original universality theorem was established in vacuum in [**1**]. In fact, matter coupling produces in $f(R)$ theories new effects with respect to Einstein equations, since a new metric conformal to the original one can be defined (see [**3**]). The theory in these new variables is quite similar to standard GR though with the addition of an effective energy momentum tensor; see [**2**]. Such new effects have been recently investigated aiming to find a specific $f(R)$ theory able to model dark energy and dark matter phenomenology; see e.g. [**4**], [**5**], [**6**], [**7**].

Models based on $f(R)$ theories have been recently used also to semi-classically account for the quantum effect of standard LQG theories; see [**8**].

The metric-affine formulation has been recently criticized. The most serious criticism is based on a theorem which shows that, under specific state equation hypotheses, internal and external solutions for a (stationary, spherically symmetric) polytropic star do not match on the boundary and produce singularities on the star surface; see [**9**]. However, as it often happens when precise no-go theorems are formulated, it has been later argued that the hypotheses of this theorem are in fact physically unreasonable; see [**10**]. In this particular case it has been shown that (at least for specific examples of f) the singularity depends on matter densities far lower than the one that is reasonably expected. We shall not discuss further these aspects here.

Another criticism to $f(R)$ theories (this time in purely-metric formulation; see [**2**]) is based

* This paper is published despite the effects of the Italian law 133/08 (`http://groups.google.it/group/scienceaction`). This law drastically reduces public funds to public Italian universities, which is particularly dangerous for free scientific research, and it will prevent young researchers from getting a position, either temporary or tenured, in Italy. The authors are protesting against this law to obtain its cancellation.

on fixing the variation of first derivatives of field on the boundary; see [**11**]. This criticism seems to be based on a physically and also mathematically questionable method discussed [**12**], that is meaningful only in standard GR; in standard purely-metric GR, one can in fact subtract a suitable boundary term from the action so that fixing first derivatives of the metric on the boundary (as it is prescribed by Calculus of Variations; [**13**]) is not necessary any longer provided one accepts to modify the Hilbert Lagrangian by adding suitable *ad-hoc* boundary counterterms. The same unconventional procedure, however, cannot be carried over for a generic $f(R)$. We argue that one cannot consider this a problem; see [**14**] for a detailed criticism of the method. In standard (higher order) variational calculus all derivatives of variations of fields are fixed up to one order less with respect to the effective order of the theory. The fact that first derivatives do not need to be fixed in GR, which is a second order gravitational theory when formulated in terms of the Hilbert Lagrangian, is just a consequence of the well-known fact that GR is in fact degenerate and can be formulated even covariantly as an equivalent first order theory; see [**15**]. Moreover, not fixing higher order variations causes unreasonable results in many cases, but we shall not discuss these aspects here, either; see [**14**].

We shall here establish the dynamical equivalence between $f(R)$ theory *à la* Palatini and its Barbero-Immirzi formulation. The special case for $f(R) = R$ is already well-known; see [**16**], [**17**], [**18**], [**19**], [**20**]. The extension to a generic $f(R)$ is new, to the best of our knowledge.

2. Notation and Holst Formulation of GR

Holst formulation of GR is the classical basis of LQG formulation in terms of the Barbero-Immirzi connection.

Let us consider a 4 dimensional (paracompact, connected, orientable) manifold M which allows metrics in signature $\eta = (3, 1)$ and global spin structures (i.e. with zero $1st$ and $2nd$ Stiefel-Whitney classes). Let P be a principal bundle over M with $\mathrm{SO}(\eta)$ as structure group. Notation follows [**21**].

Let us denote by J^1P the first jet prolongation of P; there is a standard right action of $\mathrm{SO}(\eta)$ on J^1P, namely the prolongation of the canonical right action $R_g : P \to P$ on P; see [**21**]. The bundle $C(P) = J^1P/\mathrm{SO}(\eta)$ admits a global family of local coordinates (x^μ, Γ^{ab}_μ) and (global) sections are by definition (global) $\mathrm{SO}(\eta)$-connections on P.

Let $\lambda : (\mathrm{GL}(4) \times \mathrm{SO}(\eta)) \times \mathrm{GL}(4) \to \mathrm{GL}(4) : (J, \ell, e) \mapsto J \cdot e \cdot \ell^{-1}$ be the natural action of the group $\mathrm{GL}(4)\times\mathrm{SO}(\eta)$ on the manifold $\mathrm{GL}(4)$. The associated bundle $e(P) = (L(M)\times P)\times_\lambda \mathrm{GL}(4)$ has coordinates (x^μ, e^μ_a) and (global) sections, that always exist under our hypotheses on M, are by definition *(global) frames.*

Here *frame* is meant in a non-standard sense. A global frame for us is not a global section of the frame bundle $L(M)$, that may not exist, but rather a global section of $e(P)$. A global frame is here a family of local sections of $L(M)$ defined on an open covering of M and such that transition functions are valued in $\mathrm{SO}(\eta)$ rather than in $\{\mathbb{I}\}$ as required for global sections of $L(M)$.
This is done since generic manifolds do not allow global sections of $L(M)$; the topological condition for this is physically too strong in general, while $e(P)$ always allows global frames in the above hypotheses; see [**21**].
Moreover, the standard covariant derivative of sections of $e(P)$ is the covariant derivative introduced *ad hoc* and used in literature for frames, which is not the standard covariant derivative for sections

of $L(M)$; see **[21]**.

Vielbein e^a_μ are frame inverses; vielbein induce a metric on M

$$g_{\mu\nu} = e^a_\mu \, \eta_{ab} \, e^b_\nu \tag{2.1}$$

where η^{ab} is the standard diagonal matrix of signature $\eta = (3,1)$. Hereafter Greek indices are moved up and down by the metric $g_{\mu\nu}$ while Latin indices are moved by η_{ab}.

Let us now define the curvature 2-form

$$R^{ab} = \tfrac{1}{2} R^{ab}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{2.2}$$

where $R^{ab}{}_{\mu\nu}(j^1\Gamma)$ is the curvature tensor of Γ^{ab}_μ, the local representative of an arbitrary global section Γ of $C(P)$. The *vielbein form* is the 1-form

$$e^a = e^a_\mu \, dx^\mu \tag{2.3}$$

For the Holst formulation (see **[18]**, **[20]**, **[22]**) let us set $\mathcal{C} = C(P) \times_M e(P)$ for the configuration bundle; the Lagrangian is:

$$L_H = \tfrac{1}{4\kappa} \left(R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} - \tfrac{2}{\gamma} R^{ab} \wedge e_a \wedge e_b \right) \tag{2.4}$$

where κ and γ are constants. For later covanience this Lagrangian can be written as

$$\begin{aligned} L_H =& \tfrac{1}{8\kappa} \left(R^{ab}{}_{\mu\nu} e^c_\rho e^d_\sigma \epsilon_{abcd} - \tfrac{2}{\gamma} R^{ab}{}_{\mu\nu} e_{a\rho} e_{b\sigma} \right) \epsilon^{\mu\nu\rho\sigma} \, ds = \\ =& \tfrac{e}{2\kappa} \left(R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b - \tfrac{1}{2\gamma} R^{ab}{}_{\mu\nu} e^\mu_c e^\nu_d \epsilon^{cd\,\cdot\,\cdot}{}_{ab} \right) \, ds = \tfrac{e}{2\kappa} \, {}^\beta R \, ds \end{aligned} \tag{2.5}$$

where ds is the standard local volume form induced by coordinates, e is the (module of) the determinant of the vielbein e^a_μ and we set $\beta = -\frac{1}{2\gamma}$ and

$${}^\beta R = R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b + \beta R^{ab}{}_{\mu\nu} e^{c\mu} e^{d\nu} \epsilon_{cdab} \tag{2.6}$$

Let us also set $R := R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b$ and $R^b_\nu := R^{ab}{}_{\mu\nu} e^\mu_a$ which are functions of $(e, j^1\Gamma)$.

Let us stress that the additional term $R^{ab}{}_{\mu\nu} e^{c\mu} e^{d\nu} \epsilon_{cdab}$ in ${}^\beta R$ is peculiar of the Palatini formulation; if we assumed the connection to be metric from the beginning, the curvature would be the Riemann tensor of a metric and the additional term would vanish identically due to the symmetry properties of Riemann tensors, in particular the first Bianchi identity $R^\alpha{}_{[\beta\mu\nu]} = 0$. One obtains the same result also using no frames, but allowing torsion; it is sufficient to use the correction $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}(j^1\Gamma)$ which does not vanish identically if torsion is allowed. Here we have chosen to use frames in view of possible applications to LQG or possible applications with spinor couplings.

We refer to **[18]** or **[22]** for the equivalence between the Lagrangian (2.5) and standard GR; here it will follow from the universality result proved below, in the particular case $f({}^\beta R) = {}^\beta R$.

Before proceeding to consider these further extended gravitational theories let us first briefly review the standard results about metric-affine $f(R)$ theories. The metric-affine formulation of $f(R)$ models is described by the following Lagrangian:

$$L_f(g, j^1\Gamma, j^1\phi) = \sqrt{g} f(R) + \mathcal{L}_{mat}(g, j^1\phi) \tag{2.7}$$

where Γ is now a (torsionless) linear connection and ϕ is a set of matter fields; the matter Lagrangian does not depend on Γ (though it could depend on the Levi-Civita connection of the metric g).

The variation of the Lagrangian L_f with respect to the metric g and the connection Γ is:

$$\begin{aligned}\delta L_f =&\sqrt{g}\left(-\tfrac{1}{2}fg_{\mu\nu}\delta g^{\mu\nu}+f'\left(\delta g^{\mu\nu}R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}\right)\right)-\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu}=\\ =&\sqrt{g}\left(-\tfrac{1}{2}fg_{\mu\nu}\delta g^{\mu\nu}+f'\left(\delta g^{\mu\nu}R_{\mu\nu}+2g^{\mu\nu}\nabla_\lambda\delta u^\lambda_{\mu\nu}\right)\right)-\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu}=\\ =&\sqrt{g}\left(f'R_{\mu\nu}-\tfrac{1}{2}fg_{\mu\nu}-T_{\mu\nu}\right)\delta g^{\mu\nu}-2\nabla_\lambda\left(\sqrt{g}f'g^{\mu\nu}\right)\delta u^\lambda_{\mu\nu}+\nabla_\lambda\left(2\sqrt{g}f'g^{\mu\nu}\delta u^\lambda_{\mu\nu}\right)\end{aligned} \tag{2.8}$$

where f and $f'=\frac{df}{dR}$ are understood to be evaluated at R and we set $u^\lambda_{\mu\nu}=\Gamma^\lambda_{\mu\nu}-\delta^\lambda_{(\mu}\Gamma^\alpha_{\nu)\alpha}$ and $-\sqrt{g}T_{\mu\nu}=\frac{\delta L_{mat}}{\delta g^{\mu\nu}}$. One should also consider variations with respect to matter fields, which account for matter field equations; we are not interested here in that.

As usual the last boundary term in the variation vanishes because of boundary conditions ($\delta u^\alpha_{\beta\mu}=0$ since $\delta\Gamma^\alpha_{\beta\mu}=0$) and field equations are

$$\begin{cases} f'R_{(\mu\nu)}-\frac{1}{2}fg_{\mu\nu}=T_{\mu\nu}\\ \nabla_\lambda\left(\sqrt{g}f'g^{\mu\nu}\right)=0\end{cases} \tag{2.9}$$

By tracing the first equation with g, one obtains the *master equation* $f'(R)R-2f(R)=T$, being $T=g^{\mu\nu}T_{\mu\nu}$; except in degenerate cases this can be solved for $R(T)$.

Let us now set $\sigma=\mathrm{sign}(f')$ and define a new metric $\tilde{g}_{\mu\nu}=|f'|g_{\mu\nu}$; accordingly one has $\sigma\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}=\sqrt{g}f'g^{\mu\nu}$. This can be used in the second field equation which implies that $\Gamma^\lambda_{\mu\nu}$ are nothing but the Christoffel symbols of the metric $\tilde{g}$.

We can replace this information back into the first field equation and obtain

$$f'\,\tilde{R}_{\mu\nu}-\tfrac{1}{2}fg_{\mu\nu}=T_{\mu\nu} \tag{2.10}$$

where $\tilde{R}_{\mu\nu}$ is now the Ricci tensor of the metric $\tilde{g}$. Let us stress that f' and f are still evaluated along the Ricci scalar R of the original metric g.

This equation can be recasted as an Einstein equation for the conformal metric $\tilde{g}$

$$\tilde{G}_{\mu\nu}:=\tilde{R}_{\mu\nu}-\tfrac{1}{2}\tilde{R}\,\tilde{g}_{\mu\nu}=\tfrac{1}{f'}\left(T_{\mu\nu}-\tfrac{1}{2}\left((f')^2\tilde{R}-f\right)g_{\mu\nu}\right)=\tfrac{1}{f'}T_{\mu\nu}+T^{(g)}_{\mu\nu} \tag{2.11}$$

with an additional effective source $T^{(g)}_{\mu\nu}=-\frac{1}{2f'}\left((f')^2\tilde{R}-f\right)g_{\mu\nu}$. The conservation laws for matter follow from Bianchi identities of $\tilde{g}$ and read as follows

$$\tilde{\nabla}_\mu\left(\tfrac{1}{f'}T^{\mu\nu}+(T^{(g)})^{\mu\nu}\right)=0 \tag{2.12}$$

where $\tilde{\nabla}$ is the covariant derivative with respect to $\tilde{g}$.

Let us stress that the effective energy-momentum tensor $T^{(g)}_{\mu\nu}$, being *effective*, does not need to obey other separated physical energy conditions which are usually required for matter energy-momentun tensors. In this sense one can hope to choose f so that the effective energy-momentum tensor mimick dark energy/matter without needing to introduce extra exotic matter fields. Of course, the freedom in choosing matter fields to model the dark side of the universe is transformed into the freedom in choosing the function f.

The standard universality result has been proved in the metric-affine formulation (2.7); it holds in vacuum (i.e. $T_{\mu\nu}=0$ and, more generally, when the trace T vanishes); the master equation reads thence as

$$f'R-2f=0 \tag{2.13}$$

which, except few particular cases and the degenerate case $f = R^2$ that makes it an identity, has isolated zeros; by choosing ρ_0 to be one of the simple zeroes then one must have $R = \rho_0$ on-shell and the field equation can be recasted as

$$f'\left(\tilde{R}_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}\right) = 0 \quad \Rightarrow \tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = -\tfrac{1}{4|f'|}\rho_0\tilde{g}_{\mu\nu} = \Lambda\tilde{g}_{\mu\nu} \tag{2.14}$$

which are in fact Einstein equations with cosmological constant $\Lambda = -\frac{1}{4|f'|}\rho_0$. Multiple zeroes are discussed in [**1**].

3. Universality Theorem

Let us now consider the Lagrangian

$$L = ef({}^{\beta}R) + L_{mat}(e, j^1\phi) \tag{3.1}$$

where $e = \det(e^a_\mu)$ denotes the determinant of the frame, f is an analytic function such that the so-called *master equation* ${}^{\beta}Rf'({}^{\beta}R) - 2f({}^{\beta}R) = T$ can be solved for ${}^{\beta}R(T)$ as in Section 2. Here ϕ is a set of matter fields; the important issue is the hypothesis that the matter Lagrangian is independent of Γ^{ab}_μ (while it could depend on the spin connection $\omega^{ab}_\mu(e)$ induced by the frame).

The variation of the matter Lagrangian defines the matter energy-momentum tensor

$$-2eT^a_\mu = \frac{\delta L_{mat}}{\delta e^\mu_a} \tag{3.2}$$

Let us start computing the variation of the Lagrangian (3.1) with respect to the frame and the connection:

$$\begin{aligned}\delta L =& e\Big(- fe^a_\mu\delta e^\mu_a + f'\Big(2\nabla_\mu\delta\Gamma^{ab}_\nu e^\mu_a e^\nu_b + 2R^b{}_\nu\delta e^\nu_b + \\ &+ 2\beta\nabla_\mu\delta\Gamma^{ab}_\nu e^{c\mu}e^{d\nu}\epsilon_{cdab} + 2\beta R^{ab}{}_{\mu\nu}e^{c\mu}\delta e^{d\nu}\epsilon_{cdab}\Big) - 2T^a_\mu\delta e^\mu_a\Big) = \\ =& 2e\left(f'R^a{}_\mu - \tfrac{1}{2}fe^a_\mu + \beta f'R^{cd}{}_{\mu\nu}e^\nu_b\epsilon_{cd}{}^{ab}_{\cdot\cdot} - T^a_\mu\right)\delta e^\mu_a + \\ &- 2\nabla_\mu\left(ef'e^\mu_c e^\nu_d\right)\left(\delta^c_{[a}\delta^d_{b]} + \beta\epsilon^{cd}_{\cdot\cdot ab}\right)\delta\Gamma^{ab}_\nu + 2\nabla_\mu\left(ef'e^\mu_c e^\nu_d\left(\delta^c_{[a}\delta^d_{b]} + \beta\epsilon^{cd}_{\cdot\cdot ab}\right)\delta\Gamma^{ab}_\nu\right)\end{aligned} \tag{3.3}$$

As usual the boundary term vanishes because of the boundary conditions $\delta\Gamma^{ab}_\mu = 0$ and field equations are

$$\begin{cases} f'R^a{}_\mu - \tfrac{1}{2}fe^a_\mu + \beta f'R^{cd}{}_{\mu\nu}e^\nu_b\epsilon_{cd}{}^{ab}_{\cdot\cdot} = T^a_\mu \\ \nabla_\mu\left(ef'e^\mu_c e^\nu_d\right)\left(\delta^c_{[a}\delta^d_{b]} + \beta\epsilon^{cd}_{\cdot\cdot ab}\right) = 0 \end{cases} \tag{3.4}$$

Let us set $\Phi^{cd}_{ab} := \delta^c_{[a}\delta^d_{b]} + \beta\epsilon^{cd}_{\cdot\cdot ab}$; being it skew in both the upper and lower pair it defines an endomorphisms $\Phi : \Lambda^2 \to \Lambda^2$ in the space of skew 2-tensors. It can be proved (e.g. by explicit computation by Maple) that it is invertible. One can easily check that its inverse is

$$\left(\Phi^{-1}\right)^{ab}_{cd} = (1 + 4\beta^2)\left(\delta^a_{[c}\delta^b_{d]} - \beta\epsilon^{ab}_{\cdot\cdot cd}\right) \tag{3.5}$$

Here we are discussing Lorentzian sector. In Euclidean signature the special cases $\beta = \pm\frac{1}{2}$ must be dealt with separately. They correspond to (anti)selfdual cases in which the map Φ^{cd}_{ab} is degenerate. Accordingly, the analysis needs extra care though it leads to similar results.

The second field equation is then

$$\nabla_\mu \left(e f' e^\mu_{[c} e^\nu_{d]} \right) = 0 \tag{3.6}$$

Let us now define a new frame

$$\tilde{e}^a_\mu = \sqrt{|f'|} e^a_\mu \tag{3.7}$$

We shall systematically denote by a tilde the quantities computed in the new frame.

The second field equation is then

$$\nabla_\mu \left(\tilde{e} \tilde{e}^\mu_{[c} \tilde{e}^\nu_{d]} \right) = 0 \tag{3.8}$$

which is the same equation obtained in the standard frame-affine framework; it implies that $\Gamma^{ab}_\mu \equiv \tilde{\omega}^{ab}_\mu$ coincides with the spin connection induced by the frame $\tilde{e}^\mu_a$.

Equation (3.8) can be recasted as:

$$\nabla_\mu \left(\tilde{e} \tilde{e}^\mu_c \tilde{e}^\nu_d \epsilon^{abcd} \right) = 0 \qquad \Rightarrow \nabla_\mu \left(\tilde{e}^a_\alpha \tilde{e}^b_\beta \epsilon^{\alpha\beta\mu\nu} \right) = 0 \qquad \Rightarrow \nabla_{[\mu} \left(\tilde{e}^a_\alpha \tilde{e}^b_{\beta]} \right) = 0 \tag{3.9}$$

Let us define $k^{ab}{}_c := \left(\Gamma^{ab}_\mu - \tilde{\omega}^{ab}_\mu \right) \tilde{e}^\mu_c$; this is the difference between the dynamical connection Γ and the spin connection induced by the frame $\tilde{e}$. By construction $k^{abc} = -k^{bac}$. By considering the covariant derivative of the frame we have

$$\tilde{e}^\mu_c \nabla_\mu \tilde{e}^a_\nu = \tilde{e}^\mu_c \left(d_\mu \tilde{e}^a_\nu - \tilde{\Gamma}^\lambda_{\nu\mu} \tilde{e}^a_\lambda + \Gamma^a{}_{b\mu} \tilde{e}^b_\nu \right) = \tilde{e}^\mu_c \left(\tilde{\nabla}_\mu \tilde{e}^a_\nu \right) + k^a{}_{\cdot bc} \tilde{e}^b_\nu = k^a{}_{\cdot bc} \tilde{e}^b_\nu \tag{3.10}$$

Hence equation (3.9) can be recasted as

$$\begin{aligned} &\nabla_{[\mu} \tilde{e}^{[a}_\alpha \tilde{e}^{b]}_{\beta]} = 0 \quad \iff k^a{}_{cd} \tilde{e}^c_{[\alpha} \tilde{e}^d_\mu \tilde{e}^b_{\beta]} - k^b{}_{cd} \tilde{e}^c_{[\alpha} \tilde{e}^d_\mu \tilde{e}^a_{\beta]} = 0 \\ &\iff k^a{}_{eg} \delta^b_f + k^a{}_{fe} \delta^b_g + k^a{}_{gf} \delta^b_e = k^b{}_{eg} \delta^a_f + k^b{}_{fe} \delta^a_g + k^b{}_{gf} \delta^a_e \\ &\Rightarrow 3k^a{}_{eg} + 2k^a{}_{ge} = k^b{}_{gb} \delta^a_e \quad \Rightarrow 2k^a{}_{ga} = 4k^a{}_{ga} \quad \Rightarrow k^a{}_{ga} = 0 \end{aligned} \tag{3.11}$$

and substituting back into the original equation we obtain

$$3k^a{}_{eg} + 2k^a{}_{ge} = 0 \quad \Rightarrow k^a{}_{eg} = -2k^a{}_{(ge)} \quad \Rightarrow k^a{}_{[eg]} = 0 \tag{3.12}$$

Finally, we have the following lemma:

Lemma: if $k_{abc} = -k_{bac}$ and $k_{abc} = k_{acb}$ then $k_{abc} = 0$

Proof: Let us simply notice that

$$k_{abc} = -k_{bac} = -k_{bca} = k_{cba} = k_{cab} = -k_{acb} = -k_{abc} \tag{3.13}$$

from which the thesis follows. ■

Hence since $k^{ab}{}_\mu = 0$ then $\Gamma^{ab}_\mu = \tilde{\omega}^{ab}_\mu$.

This piece of information can be used to express the curvature tensor in terms of the Riemann tensor of the metric $\tilde{g}_{\mu\nu}$ induced by the frame $\tilde{e}^\mu_a$:

$$\begin{aligned}
&R^{ab}{}_{\mu\nu} = \tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{g}^{\lambda\beta}\tilde{e}^a_\alpha\tilde{e}^b_\beta \\
&R^a{}_\mu = R^{ab}{}_{\mu\nu}e^\nu_b = \sqrt{|f'|}\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{g}^{\lambda\beta}\tilde{e}^a_\alpha\tilde{e}^b_\beta\tilde{e}^\nu_b = \sqrt{|f'|}\tilde{g}^{\alpha\beta}\tilde{R}_{\beta\mu}\tilde{e}^a_\alpha = \epsilon f'\tilde{g}^{\alpha\beta}\tilde{R}_{\beta\mu}e^a_\alpha = \tilde{R}^\alpha_{\cdot\,\mu}e^a_\alpha \\
&R = R^a{}_\mu e^\mu_a = \epsilon f'\tilde{R}
\end{aligned} \tag{3.14}$$

(Let us stress that in our notation Greek indices are moved by the metric g and not by $\tilde{g}$; hence $\tilde{R}^\alpha_{\cdot\,\mu}$ means $\tilde{R}^\alpha_{\cdot\,\mu} = g^{\alpha\beta}\tilde{R}_{\beta\mu}$, not $\tilde{g}^{\alpha\beta}\tilde{R}_{\beta\mu}$ as usual. This is a consequence of having two metrics around. One should specify which metric is involved at any step!)

Before being ready to manipulate the first field equation we have to prove that the extra term in equation (3.4) vanishes:

$$\begin{aligned}
R^{cd}{}_{\mu\nu}e^\nu_b\epsilon_{cd}{}^{ab}_{\cdot\,\cdot} =& \sqrt{|f'|}\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{g}^{\lambda\beta}\tilde{e}^c_\alpha\tilde{e}^d_\beta\tilde{e}^\nu_b\epsilon_{cd}{}^{ab}_{\cdot\,\cdot} = \sqrt{|f'|}\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{g}^{\lambda\beta}\tilde{e}^c_\alpha\tilde{e}^d_\beta\tilde{e}^b_\sigma\tilde{g}^{\sigma\nu}\epsilon_{cdeb}\eta^{ae} = \\
=& \sqrt{|f'|}\tilde{e}\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{e}^a_\gamma\epsilon_{\alpha\beta\rho\sigma}\tilde{g}^{\lambda\beta}\tilde{g}^{\gamma\rho}\tilde{g}^{\sigma\nu} = -\sqrt{|f'|}(\tilde{e})^{-1}\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{e}^a_\gamma\tilde{g}_{\alpha\beta}\epsilon^{\beta\lambda\gamma\nu} = \\
=& -\sqrt{|f'|}(\tilde{e})^{-1}\tilde{g}_{\mu\alpha}\tilde{e}^a_\gamma\,\tilde{R}^\alpha{}_{[\nu\beta\lambda]}\epsilon^{\beta\lambda\gamma\nu} = 0
\end{aligned} \tag{3.15}$$

and that the scalar ${}^\beta R$ coindices with the Ricci scalar R of the metric g induced by the original frame e^μ_a:

$$\begin{aligned}
{}^\beta R =& R + \beta R^{ab}{}_{\mu\nu}e^{c\mu}e^{d\nu}\epsilon_{cdab} = R + \epsilon\beta f'\tilde{R}^\alpha{}_{\lambda\mu\nu}\tilde{g}^{\lambda\beta}\tilde{e}^a_\alpha\tilde{e}^b_\beta\tilde{e}^c_\rho\tilde{e}^d_\sigma\epsilon_{cdab}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma} = \\
=& R + \epsilon\beta\tilde{e}f'\tilde{R}^\alpha{}_{\lambda\mu\nu}\epsilon_{\rho\sigma\alpha\beta}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}\tilde{g}^{\lambda\beta} = R - \epsilon\beta(\tilde{e})^{-1}f'\tilde{R}^\alpha{}_{[\lambda\mu\nu]}\tilde{g}_{\alpha\beta}\epsilon^{\mu\nu\beta\lambda} = R
\end{aligned} \tag{3.16}$$

The first field equation in (3.4) can be now recasted as follows

$$\left(\epsilon(f')^2\tilde{g}^{\alpha\beta}\tilde{R}_{\beta\mu} - \tfrac{1}{2}f\delta^\alpha_\mu\right)e^a_\alpha = T^a_\mu \tag{3.17}$$

$$\left(f'\tilde{R}^\alpha_{\cdot\,\mu} - \tfrac{1}{2}f\delta^\alpha_\mu\right) = T^a_\mu e^\alpha_a =: T^\alpha_{\cdot\,\mu} \tag{3.18}$$

$$f'\tilde{R}_{\mu\nu} - \tfrac{1}{2}fg_{\mu\nu} =: T_{\mu\nu} \tag{3.19}$$

which coincides with the equation in standard metric-affine $f(R)$-theories (see equation (2.10)). In fact, f' and f are evaluated at the Ricci scalar R of the original metric g and if the matter Lagrangian depends on the frame through its associated metric (as assumed otherwise there is no metric affine formulation to compare with) one has

$$-2eT^a_\mu = \frac{\delta L_{mat}}{\delta e^\mu_a} = 2\frac{\delta L_{mat}}{\delta g_{\mu\nu}}e^{a\nu} = -2\sqrt{g}T_{\mu\nu}e^{a\nu} \quad \Rightarrow T_{\mu\nu} = T^a_\mu e_{a\nu} \tag{3.20}$$

We stress that equivalence holds both in vacuum and in the presence of matter.

4. Conservation Laws

We shall here compute and discuss conservation laws for $f({}^{\beta}R)$ theories, following [**21**]. We shall use the formalism introduced in [**23**] to which we refer for motivations. The case of standard Holst action has been already discussed in [**24**].

The Lagrangian (3.1) is gauge-natural (see [**21**]); any generator $\Xi = \xi^{\mu}\partial_{\mu} + \xi^{ab}\sigma_{ab}$ of automorphisms on P is thence a symmetry. Here σ_{ab} is a right invariant pointwise basis of vertical vector fields on P. Accordingly, (in vacuum) the following Noether current is conserved

$$\mathcal{E} = e\left(2f' e^{\mu}_{c} e^{\nu}_{d} \Phi^{cd}_{ab} \pounds_{\Xi}\Gamma^{ab}_{\nu} - \xi^{\mu} f\right) ds_{\mu} \tag{4.1}$$

where ds_{μ} is the local basis of 3-forms on M induced by coordinates and $\pounds_{\Xi}\Gamma^{ab}_{\mu} = \xi^{\nu} R^{ab}{}_{\nu\mu} + \nabla_{\mu}\xi^{ab}_{(v)}$ denotes the Lie derivative and we set $\xi^{ab}_{(v)} = \xi^{ab} + \xi^{\mu}\Gamma^{ab}_{\mu}$ for the vertical part of the symmetry generator. Hence we have

$$\begin{aligned}
\mathcal{E} =& e\left(2f' e^{\mu}_{c} e^{\nu}_{d} \Phi^{cd}_{ab}\left(\xi^{\lambda} R^{ab}{}_{\lambda\nu} + \nabla_{\mu}\xi^{ab}_{(v)}\right) - f\xi^{\mu}\right) ds_{\mu} = \\
=& e\left(\xi^{\lambda}\left(2f' e^{\mu}_{c} e^{\nu}_{d} R^{cd}{}_{\lambda\nu} + 2\beta f' e^{\mu}_{c} e^{\nu}_{d} \epsilon^{cd}_{\cdot\cdot ab} R^{ab}{}_{\lambda\nu} - \delta^{\mu}_{\lambda} f\right) + 2f' e^{\mu}_{c} e^{\nu}_{d} \Phi^{cd}_{ab} \nabla_{\mu}\xi^{ab}_{(v)}\right) ds_{\mu} = \\
=& \left(2e\xi^{\lambda}\left(f' R^{c}{}_{\lambda} - \tfrac{1}{2} f e^{c}_{\lambda} + \beta f' e^{\nu}_{d} \epsilon^{cd}_{\cdot\cdot ab} R^{ab}{}_{\nu\lambda}\right) e^{\mu}_{c} - 2\nabla_{\mu}\left(e f' e^{\mu}_{c} e^{\nu}_{d}\right) \Phi^{cd}_{ab}\xi^{ab}_{(v)}\right) ds_{\mu} + \\
&+ \mathrm{Div}\left(2e f' e^{\mu}_{c} e^{\nu}_{d} \Phi^{cd}_{ab}\xi^{ab}_{(v)} ds_{\mu\nu}\right)
\end{aligned} \tag{4.2}$$

where $ds_{\mu\nu}$ is the local basis of 2-forms on M induced by coordinates.

Let us define the reduced current $\tilde{\mathcal{E}}$ and the superpotential $\mathcal{U}$

$$\begin{cases}
\tilde{\mathcal{E}} = \left(2e\xi^{\lambda}\left(f' R^{c}{}_{\lambda} - \tfrac{1}{2} f e^{c}_{\lambda} + \beta f' e^{\nu}_{d} \epsilon^{cd}_{\cdot\cdot ab} R^{ab}{}_{\nu\lambda}\right) e^{\mu}_{c} - 2\nabla_{\mu}\left(e f' e^{\mu}_{c} e^{\nu}_{d}\right) \Phi^{cd}_{ab}\xi^{ab}_{(v)}\right) ds_{\mu} \\
\mathcal{U} = 2f'\, e e^{\mu}_{c} e^{\nu}_{d}\, \Phi^{cd}_{ab}\xi^{ab}_{(v)}\, ds_{\mu\nu} = 2\epsilon\, \tilde{e}\tilde{e}^{\mu}_{c}\tilde{e}^{\nu}_{d}\, \Phi^{cd}_{ab}\xi^{ab}_{(v)}\, ds_{\mu\nu}
\end{cases} \tag{4.3}$$

Accordingly, the Noether current is

$$\mathcal{E} = \tilde{\mathcal{E}} + Div\left(\mathcal{U}\right) \tag{4.4}$$

Notice that the reduced current vanishes on-shell (see equations (3.4)). Conserved quantities are then generated by integrating the superpotential.

As usual in gauge–natural theories which are equivalent to a natural theory the correspondence among conservation laws is established by means of the so-called Kosmann lift $\xi^{ab}_{(v)} = \tilde{e}^{a}_{\alpha}\tilde{e}^{b\beta}\tilde{\nabla}_{\beta}\xi^{\alpha}$; see [**25**]. Then the superpotential is:

$$\begin{aligned}
\mathcal{U} =& 2\epsilon\tilde{e}\left(\tilde{e}^{\mu}_{c}\tilde{e}^{\nu}_{d}\Phi^{cd}_{ab}\tilde{e}^{a}_{\alpha}\tilde{e}^{b\beta}\tilde{\nabla}_{\beta}\xi^{\alpha}\right) ds_{\mu\nu} = \\
=& 2\epsilon\tilde{e}\left(\tilde{\nabla}^{\nu}\xi^{\mu} + \beta\tilde{e}^{\mu}_{c}\tilde{e}^{\nu}_{d}\epsilon^{cd}_{\cdot\cdot ab}\tilde{e}^{a}_{\alpha}\tilde{e}^{b\beta}\tilde{\nabla}_{\beta}\xi^{\alpha}\right) ds_{\mu\nu} = \\
=& 2\epsilon\tilde{e}\left(\tilde{\nabla}^{\nu}\xi^{\mu} + \beta\tilde{e}^{\mu}_{c}\tilde{e}^{\nu}_{d}\tilde{e}^{\alpha}_{a}\tilde{e}^{\beta}_{b}\epsilon^{cdba}\tilde{\nabla}_{\beta}\xi_{\alpha}\right) ds_{\mu\nu} = \\
=& 2\epsilon\tilde{e}\tilde{\nabla}^{\nu}\xi^{\mu}\, ds_{\mu\nu} + 2\epsilon\beta\epsilon^{\mu\nu\beta\alpha}\tilde{\nabla}_{\beta}\xi_{\alpha}\, ds_{\mu\nu} = \\
=& 2\epsilon\tilde{e}\tilde{\nabla}^{\nu}\xi^{\mu}\, ds_{\mu\nu} + \mathrm{Div}\left(\tfrac{2}{3}\epsilon\beta\epsilon^{\mu\nu\beta\alpha}\xi_{\alpha}\, ds_{\mu\nu\beta}\right)
\end{aligned} \tag{4.5}$$

which differs from the Komar superpotential computed for the frame $\tilde{e}$ by a pure divergence which does not contribute when integrated along closed regions and hence it does not contribute to conserved quantities.

Let us stress that one is forced to choose the Kosmann lift along the new frame $\tilde{e}$ in order to obtain a correspondence with the conservation laws of metric-affine formulation.

5. Conclusions and Perspectives

The result we obtained can be considered from two points of view. From the point of view of classical gravitational theories, it is interesting to have an equivalent formulation of the usual extended theories $f(R)$ *à la* Palatini. Using frames allows coupling to spinors; one has just to extend the structure group from $\mathrm{SO}(\eta)$ to $\mathrm{Spin}(\eta)$. It is also interesting to know that the universality theorem extends further to $f({}^{\beta}R)$ theories. Moreover we obtained a non-trivial correspondence among conservation laws. This correspondence selects $\tilde{e}$ as preferred frame with respect to e.

From the point of view of Quantum Gravity these models allow a direct approach to quantization *à la* LQG of all $f(R)$ theories; see [**26**]. Classically these models are known to produce modified dynamics for gravitational physics, in particular in Cosmology. Even not considering the issue of whether $f(R)$ theories better describe physics than standard GR, it is interesting from the theoretical viewpoint to explore the dynamics of such an infinite family of models. This could improve the understanding of the emergence of the classical dynamics from the quantum world. For example it would be interesting to explicitly see whether the formalism developed in these years is able to catch the classical difference of dynamics of these models when compared with standard GR. This is particularly relevant in Cosmology where the comparison could help in better understanding the relation between LQC and LQG; see [**27**].

All $f({}^{\beta}R)$ are gauge–natural theories. As such the gauge and diffeomorphism constraint should be unchanged. Accordingly nothing should change in defining the Volume and Area operators together with their quantizations. Hence the modified dynamics should appear in Hamiltonian constraint. It will be interesting, at least from a theoretical viewpoint, to test the proposals for quantization techniques against therse extended models; see [**28**].

A forthcoming paper will be devoded to study the Wheeler-DeWitt equation for the extended models introduced here.

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