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The EPRL intertwiners and corrected partition function

Wojciech Kamiński, Marcin Kisielowski, Jerzy Lewandowski

*Instytut Fizyki Teoretycznej,
Uniwersytet Warszawski, ul. Hoża 69,
00-681 Warszawa (Warsaw), Polska (Poland)*

Abstract Do the $SU(2)$ intertwiners parametrize the space of the EPRL solutions to the simplicity constraint? What is a complete form of the partition function written in terms of this parametrization? We prove that the EPRL map is injective in the general n -valent vertex case for the Barbero-Immirzi parameter less than 1. We find, however, that the EPRL map is not isometric. In the consequence, a partition function can be defined either using the EPRL intertwiners Hilbert product or the $SU(2)$ intertwiners Hilbert product. We use the EPRL one and derive a new, complete formula for the partition function. Next, we view it in terms of the $SU(2)$ intertwiners. The result, however, goes beyond the $SU(2)$ spin-foam models framework and the original EPRL proposal.

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I. INTRODUCTION

The main technical ingredient of the spin-foam models of 4-dimensional gravity is so called quantum simplicity constraint. Imposing suitably defined constraint on the domain of the (discrete) path integral turns the $SU(2)\times SU(2)$ (or $SL(2,C)$) BF theory into the spin-foam model of the Euclidean (respectively, Lorentzian) gravity [1]. The formulation of the simplicity constraint believed to be correct, or at least fitting gravity the best among the known approaches [5], is the one derived by Engle, Pereira, Rovelli, Livine (EPRL) [1] (and independently derived by Freidel and Krasnov [3]). The solutions to the EPRL simplicity constraint are EPRL $SU(2)\times SU(2)$ intertwiners. They are defined by the EPRL transformation, which maps each $SU(2)$ intertwiner into a EPRL solution of the simplicity constraint. An attempt is made in the literature [1] to parametrize the space of the EPRL solutions by the $SU(2)$ intertwiners. The vertex amplitude and the partition function of the EPRL model seem to be written in terms of that parametrization. The questions we raise and answer in this paper are:

- Is the EPRL map injective, doesn't it kill any $SU(2)$ intertwiner?
- Is the EPRL map isometric, does it preserve the scalar product between the $SU(2)$ intertwiners?
- If not, what is a form of a partition function derived from the $SO(4)$ intertwiner Hilbert product written directly in terms of the $SU(2)$ intertwiners, the preimages of the EPRL map?

We prove the EPRL map is injective in the general, n valent vertex case and for the Barbero-Immirzi parameter $|\gamma| < 1$. The proof in the $|\gamma| > 1$ has already been provided in [4]. Hence, there are as many $SU(2)\times SU(2)$ EPRL intertwiners as there are the $SU(2)$ intertwiners. Owing to this result the $SU(2)$ intertwiners indeed can be used to parametrize the space of the EPRL $SU(2)\times SU(2)$ intertwiners. However, we find the EPRL map is not isometric. In consequence, there are two inequivalent definitions of the partition function. One possibility is to use a basis in the EPRL intertwiners space orthonormal with respect to the $SO(4)$ representations. And this is what we do in this paper. A second possibility, is to use the basis obtained as the image of an orthonormal basis of the $SU(2)$ intertwiners under the EPRL map. The partition function derived in [1] corresponds to the second choice, whereas the first one is ignored therein. The goal of this part of our paper is pointing out the first possibility and deriving the corresponding partition function. After the derivation, we compare our partition function with that of EPRL on a possibly simple example. We conjecture, that the difference converges to zero for large spins.

To make the paper intelligible we start presentation of the new results with the derivation of the partition function in Section II 4. The final formula for our proposal for the partition function

for the EPRL model is presented in Section II 5. The lack of the isometricity of the EPRL map is illustrated on specific examples in Section II 6. Finally, the proof of the injectivity of the EPRL map takes all the Section III B.

This work is written in terms of the EPRL framework [1] combined with our previous paper [4] on the EPRL model.

II. OUR PROPOSAL FOR A PARTITION FUNCTION OF THE EPRL MODEL

1. Partition functions for the spin-foam models of 4-gravity: definition

Consider an oriented 2-complex κ whose faces (2-cells) are labeled by ρ with the irreducible representations of $G = \text{SU}(2) \times \text{SU}(2)$,

$$\kappa^{(2)} \ni f \mapsto \rho(f),$$

and denote by $\mathcal{H}(f)$ the corresponding Hilbert space. For every edge (1-cell) e we have the set/set of incoming/outgoing faces, that is the faces which contain e and whose orientation agrees/disagrees with the orientation of e . We use them to define the Hilbert space

$$\mathcal{H}(e) = \bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^*. \quad (2.1)$$

The extra data we use, is a subspace

$$\mathcal{H}(e)^{\text{SIMPLE}} \subset \mathcal{H}(e) \quad (2.2)$$

defined by some constraints called the quantum simplicity constraints. In this paper, starting from section below, we will be considering the subspace proposed by Engle-Pereira-Rovelli-Livine. For the time being $\mathcal{H}(e)^{\text{SIMPLE}}$ is any subspace of the space of invariants of the representation $\bigotimes_{f \text{ incoming}} \rho(f) \otimes \bigotimes_{f' \text{ outgoing}} \rho(f')^*$,

$$\mathcal{H}(e)^{\text{SIMPLE}} \subset \text{Inv}_{\text{SU}(2) \times \text{SU}(2)} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^* \right). \quad (2.3)$$

(The subspace $\mathcal{H}_e^{\text{SIMPLE}}$ may be trivial for generic representations $\rho(f)$ and $\rho(f')$. Typically the simplicity constraints constrain also the representations themselves.) To every edge we assign the operator of the orthogonal projection onto $\mathcal{H}(e)^{\text{SIMPLE}}$,

$$P_e^{\text{SIMPLE}} : \bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^* \rightarrow \bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^* \quad (2.4)$$

Our index notation is as follows (we drop ‘SIMPLE’ for simplicity)

$$(P_e v)^{A \dots B \dots} = P_{e A' \dots B' \dots}^{A \dots B' \dots} v^{A' \dots B' \dots} \quad (2.5)$$

where the upper/lower indices of any vector $v \in \bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^*$ correspond to incoming/outgoing faces. In the operator P_e , for each face containing e , there are two indices, an upper and a lower one corresponding to the Hilbert space $\mathcal{H}(f)$. If f is incoming (outgoing), then we assign the corresponding lower/upper index of P_e to the beginning/end (end/beginning) of the edge. That rule is illustrated on Fig. 1. Now, for every pair of edges e and e' , which belong to a same face f , and share a vertex v , there is defined the natural contraction at v of the corresponding vertex of P_e with the corresponding vertex of $P_{e'}$. The contraction defines the following trace,

$$\bigotimes_{e \in \kappa^{(1)}} P_e^{\text{SIMPLE}} \mapsto \text{Tr} \left(\bigotimes_{e \in \kappa^{(1)}} P_e^{\text{SIMPLE}} \right). \quad (2.6)$$

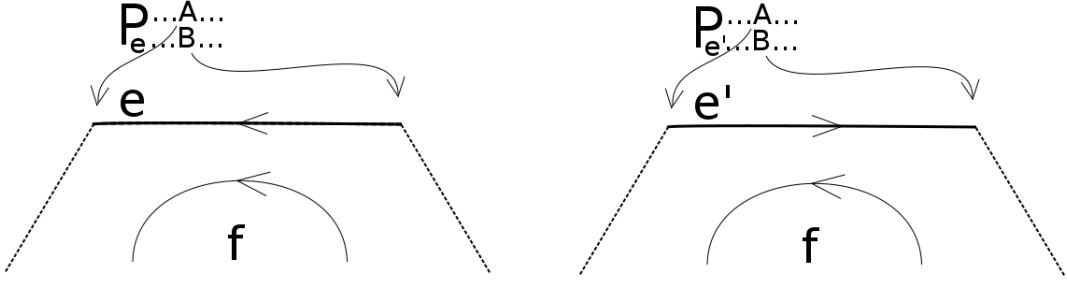


FIG. 1: According to this rule, given an edge e (e') contained in incoming (outgoing) face f , the indices of P_e ($P_{e'}$) corresponding to $\mathcal{H}(f)$ are assigned to the beginning and, respectively, to the end of the edge. The oriented arc only marks the orientation of the polygonal face f .

Define partition function $Z(\kappa, \rho)$ to be the following number:

$$Z(\kappa, \rho) := \prod_{f \in \kappa^{(2)}} d(f) \text{Tr} \left(\bigotimes_{e \in \kappa^{(1)}} P_e^{\text{SIMPLE}} \right) \mathcal{A}(\text{boundary}), \quad d(f) := \dim \mathcal{H}(f) \quad (2.7)$$

where $\mathcal{A}(\text{boundary})$ is a factor that depends only on the boundary of (κ, ρ) , and we derive it elsewhere.

2. Partition functions for the spin-foam models of 4-gravity: the amplitude form

The partition function is usually rewritten in the spin-foam amplitude form [6–8]. For that purpose one needs an orthonormal basis in each Hilbert space $\mathcal{H}(e)^{\text{SIMPLE}}$; denote its elements by $\iota_{e,\alpha} \in \mathcal{H}(e)^{\text{SIMPLE}}$, $\alpha = 1, 2, \dots, n(e)$. Then

$$P_e^{\text{SIMPLE}} = \sum_{\alpha=1}^{n_e} \iota_{e,\alpha} \otimes \iota_{e,\alpha}^\dagger \quad (2.8)$$

where by ‘ \dagger ’, for every Hilbert space \mathcal{H} we denote the duality map

$$\mathcal{H} \ni v \mapsto v^\dagger \in \mathcal{H}^*$$

defined by the Hilbert scalar product. In the Dirac notation

$$\iota_{e,\alpha} = |e, \alpha\rangle, \quad \text{and} \quad \iota_{e,\alpha}^\dagger = \langle e, \alpha|.$$

Substituting the right hand side of (2.8) for each P_e^{SIMPLE} in (2.7), one writes the partition function in terms of the vertex amplitudes in the following way:

- For each edge of κ choose an element of the corresponding orthonormal basis; denote this assignment by

$$\iota : e \rightarrow \iota_{e,\alpha_e} \quad (2.9)$$

- At each vertex $v \in \kappa^{(0)}$:

- take $\iota_{e_1,\alpha_{e_1}}, \dots, \iota_{e_m,\alpha_{e_m}}$ where e_1, \dots, e_m are the incoming edges
- take $\iota_{e'_1,\alpha_{e'_1}}^\dagger, \dots, \iota_{e'_{m'},\alpha_{e'_{m'}}}^\dagger$ where $e'_1, \dots, e'_{m'}$ are the outgoing edges

– define the vertex amplitude

$$A_v(\iota) := \text{Tr} \left(\iota_{e_1, \alpha_1} \otimes \dots \otimes \iota_{e_m, \alpha_m} \otimes \iota_{e'_1, \alpha'_1}^\dagger \otimes \dots \otimes \iota_{e'_{m'}, \alpha'_{m'}}^\dagger \right) \quad (2.10)$$

where ‘Tr’ stands for the contraction (2.6) and can be defined by the evaluation of the spin-networks corresponding to the vertices (see [4]).

- to each face f assign the face amplitude $d(f)$.

With this data, with the vertex amplitudes and face amplitudes, the partition function takes the famous form

$$Z(\kappa, \rho) = \prod_{f \in \kappa^{(2)}} d(f) \sum_{\iota} \prod_{v \in \kappa^{(0)}} A_v(\iota) \mathcal{A}(\text{boundary}), \quad (2.11)$$

The result is independent of the choice of the orthonormal basis of each $\mathcal{H}_e^{\text{SIMPLE}}$.

3. The EPRL map

Now we turn to the EPRL intertwiners. For every edge $e \in \kappa^{(1)}$

$$\mathcal{H}(e)^{\text{SIMPLE}} = \mathcal{H}(e)^{\text{EPRL}}, \quad P(e)^{\text{SIMPLE}} = P(e)^{\text{EPRL}}. \quad (2.12)$$

The definition of $\mathcal{H}(e)^{\text{EPRL}}$ uses a fixed number $\gamma \in \mathbb{R}$ called Barbero-Immirzi parameter. The Hilbert space $\mathcal{H}(e)^{\text{EPRL}}$ can be non-empty only if the 2-complex κ is labeled by EPRL representations. A representation $\rho = (\rho_{j^-}, \rho_{j^+})$ of $\text{SU}(2) \times \text{SU}(2)$, where $j^\pm \in \frac{1}{2}\mathbb{N}$ define the $\text{SU}(2)$ representations in the usual way, is an EPRL representation, provided there is $k \in \frac{1}{2}\mathbb{N}$ such that

$$j^\pm = \frac{|1 \pm \gamma|}{2} k. \quad (2.13)$$

Therefore, we will be considering here labellings of the faces of the 2-complex κ with EPRL representations

$$f \mapsto \rho(f) = (\rho_{j^-(f)}, \rho_{j^+(f)}), \quad (2.14)$$

each of which is defined in the Hilbert space

$$\mathcal{H}(f) = \mathcal{H}_{j^-(f)} \otimes \mathcal{H}_{j^+(f)}.$$

Each labeling defines also a labeling with $\text{SU}(2)$ representations given by (2.13),

$$f \mapsto \rho_{k(f)}, \quad (2.15)$$

defined in the Hilbert space $\mathcal{H}_{k(f)}$. Given an edge e and the corresponding Hilbert space

$$\mathcal{H}(e) = \bigotimes_{f \text{ incoming}} \mathcal{H}_{j^-(f)} \otimes \mathcal{H}_{j^+(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{j^-(f')}^* \otimes \mathcal{H}_{j^+(f')}^*, \quad (2.16)$$

the natural isometric embeddings

$$C : \mathcal{H}_k \rightarrow \mathcal{H}_{j^-} \otimes \mathcal{H}_{j^+} \quad (2.17)$$

and the orthogonal projection operator

$$P : \mathcal{H}(e) \rightarrow \mathcal{H}(e) \quad (2.18)$$

onto the subspace $\text{Inv}_{\text{SU}(2) \times \text{SU}(2)}(\mathcal{H}(e))$ defines the natural map, the EPRL map:

$$\begin{aligned} \iota^{\text{EPRL}} : \text{Inv}_{\text{SU}(2)} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}_{k(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k(f')}^* \right) \\ \rightarrow \bigotimes_{f \text{ incoming}} \mathcal{H}_{j^-(f)} \otimes \mathcal{H}_{j^+(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{j^-(f')}^* \otimes \mathcal{H}_{j^+(f')}^* \end{aligned}$$

Its image is promoted to the Hilbert space (2.2),

$$\mathcal{H}_e^{\text{EPRL}} := \iota^{\text{EPRL}} \left(\text{Inv}_{\text{SU}(2)} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}_{k(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k(f')}^* \right) \right). \quad (2.19)$$

4. The problem with the EPRL intertwiners

All the EPRL intertwiners can be constructed from the $SU(2)$ intertwiners by using the EPRL map. The point is, that one has to be more careful while doing that. First, one has to make sure that the map ι^{EPRL} is injective. If not, then Hilbert space of the $SU(2) \times SU(2)$ EPRL intertwiners is smaller than the corresponding space of the $SU(2)$ intertwiners and we should know how big it is. For $\gamma \geq 1$, the injectivity was proved in [4]. In the next section we present a proof of the injectivity for $|\gamma| < 1$. Secondly, one should check whether or not the map ι^{EPRL} is isometric. Given an orthonormal basis $\mathcal{I}_{e,1}, \dots, \mathcal{I}_{e,n_e}$ of the Hilbert space $\text{Inv}_{SU(2)} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}_{k(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k(f')}^* \right)$, we have a corresponding basis $\iota^{\text{EPRL}}(\mathcal{I}_{e,1}), \dots, \iota^{\text{EPRL}}(\mathcal{I}_{e,n_e})$ of the corresponding Hilbert space $\mathcal{H}_e^{\text{EPRL}}$. The question is, whether or not the latter basis is also orthonormal. We show in Section II 6, this is not the case. The direct procedure would be to orthonormalize the basis. We propose, however, a simpler solution.

5. A solution

An intelligent way, is to go back to the formula (2.7) for the partition function and repeat the step leading to (2.11) with each projection $P_e^{\text{SIMPLE}} = P_e^{\text{EPRL}}$ written in terms of the corresponding basis $\iota^{\text{EPRL}}(\mathcal{I}_{e,1}), \dots, \iota^{\text{EPRL}}(\mathcal{I}_{e,n_e})$. The suitable formula reads

$$P_e^{\text{EPRL}} = \sum_{a,b=1}^{n_e} h_e^{a\bar{b}} \iota^{\text{EPRL}}(\mathcal{I}_{e,a}) \otimes \iota^{\text{EPRL}}(\mathcal{I}_{e,b})^\dagger \quad (2.20)$$

where $h_e^{a\bar{b}}$, $a, b = 1, \dots, n_e$ define the inverse matrix to the matrix

$$h_{e,\bar{b}a} := (\iota^{\text{EPRL}}(\mathcal{I}_{e,b}) | \iota^{\text{EPRL}}(\mathcal{I}_{e,a})) \quad (2.21)$$

given by the Hilbert product $(\cdot | \cdot)$ in the Hilbert space (2.16). (In the Dirac notation, $P_e^{\text{EPRL}} = h_e^{a\bar{b}} |e, a\rangle \langle e, b|$.)

Now, we are in a position to write the resulting spin-foam amplitude formula for the partition function. It is assigned to a fixed 2-complex κ and a fixed labeling of the faces by the EPRL representations

$$f \mapsto \rho(f) = (\rho_{j_f^-}, \rho_{j_f^+}). \quad (2.22)$$

The labeling is accompanied by the corresponding labeling with the $SU(2)$ i

$$f \mapsto \rho_{k_f}, \quad (2.23)$$

according to (2.13). For every edge $e \in \kappa^{(1)}$, in addition to the Hilbert space $\mathcal{H}_e^{\text{EPRL}} \subset \text{Inv}_{SU(2) \times SU(2)}(\mathcal{H}_e)$, we also have its preimage, the Hilbert space

$$\text{Inv} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}_{(k_f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k_{f'}}^* \right). \quad (2.24)$$

Therein, we fix an orthonormal basis

$$\mathcal{I}_{e,a}, \quad a = 1, 2, \dots, n_e. \quad (2.25)$$

To define the partition function we proceed as follows:

- assign to every edge of κ a pair of elements of the basis,

$$\mathcal{II} : e \mapsto (\mathcal{I}_{e,a_e}, \mathcal{I}_{e,b_e}^\dagger), \quad (2.26)$$

more specifically, \mathcal{I}_{e,a_e} is assigned to the end point and $\mathcal{I}_{e,b_e}^\dagger$ to the beginning point of e , and we denote the assignment by the double symbol \mathcal{II} ;

- define for every edge an edge amplitude to be

$$h_e(\mathcal{II}) := h^{b_e \bar{a}_e}$$

- to every vertex v of κ assign the vertex amplitude with the trace defined by Fig.1, (2.6) and (2.20)

$$A_v(\mathcal{II}) := \text{Tr} \left(\bigotimes_{e \text{ incoming}} \iota^{\text{EPRL}}(\mathcal{I}_{e, a_e}) \otimes \bigotimes_{e' \text{ outgoing}} \iota^{\text{EPRL}}(\mathcal{I}_{e', b_{e'}})^\dagger \right)$$

- to every face f assign the amplitude d_f

Finally, the spin-foam amplitude formula for the partition function reads

$$Z(\kappa, \rho) = \prod_f d_f \sum_{\mathcal{II}} \prod_e h_e(\mathcal{II}) \prod_v A_v(\mathcal{II}) \mathcal{A}(\text{boundary}). \quad (2.27)$$

The matrix (2.21) can be written in terms of the EPRL fusion coefficients,

$$\iota^{\text{EPRL}}(\mathcal{I}_{e, a}) =: f_{e_a}^{c^- c^+} \iota_{e, c^-} \otimes \iota_{e, c^+}, \quad (2.28)$$

defined by the decomposition into an orthonormal basis $\iota_{e, c^-} \otimes \iota_{e, c^+}$ in

$$\text{Inv}_{SU(2) \times SU(2)} \left(\bigotimes_{f \text{ incoming}} \mathcal{H}_{j^-(f)} \otimes \mathcal{H}_{j^+(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{j^-(f')}^* \otimes \mathcal{H}_{j^+(f')}^* \right).$$

Then, we have

$$h_{e, \bar{a}b} = \sum_{c^+, c^-} \overline{f_{e_a}^{c^- c^+}} f_{e_b}^{c^- c^+}. \quad (2.29)$$

We are in a position now, to compare our partition function with that of [1]. The partition function of [1] is given by replacing the matrix $h_{e, \bar{a}b}$ in (2.27) with δ_{ab} . The example below gives quantitative idea about the difference between the two possible definitions of partition function for the EPRL model.

6. Example of the edge amplitude $h^{\bar{b}a}$ showing that the EPRL map is not isometry

We will show in this section, that the EPRL map is not isometric. We calculate the edge amplitude defined in the previous section in a simple example, and see that its matrix is not proportional to the identity matrix, or even not diagonal. Consider an edge at which exactly four faces meet. Assume the orientation of each face is opposite to the orientation of the edge. We have an intertwiner $\mathcal{I} \in \text{Inv}_{SU(2)}(\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \otimes \mathcal{H}_{k_3} \otimes \mathcal{H}_{k_4})$ assigned to the end point of this edge. We choose a basis $|k_i m_i\rangle$ (the eigenvector of the third component of angular momentum operator with eigenvalue m_i) in each space \mathcal{H}_{k_i} , $i \in \{1, \dots, 4\}$. We choose a real basis of the space $\text{Inv}_{SU(2)}(\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \otimes \mathcal{H}_{k_3} \otimes \mathcal{H}_{k_4})$ in the following form [9]:

$$(\mathcal{I}_a)_{k_1 m_1 k_2 m_2 k_3 m_3 k_4 m_4} = \sqrt{2a+1} \sum_{m=-a}^a \sum_{m'=-a}^a (-1)^{a+m} \begin{pmatrix} k_1 & k_2 & a \\ m_1 & m_2 & m \end{pmatrix} \delta_{m, -m'} \begin{pmatrix} k_3 & k_4 & a \\ m_3 & m_4 & m' \end{pmatrix}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3j-Symbol, $\delta_{m, m'}$ is the Kronecker Delta.

Let $\iota_{a^+} \otimes \iota_{a^-}$ be the basis of the space $\text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_4^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_4^-})$. The intertwiner $\iota^{\text{EPRL}}(\mathcal{I}_a)$ expressed in this basis takes the following form (we skip in this section the subscript e indicating the dependence on edge):

$$\iota^{\text{EPRL}}(\mathcal{I}_a) = f_a^{a^+ a^-} \iota_{a^+} \otimes \iota_{a^-}$$

where $f_a^{a^+a^-}$ are real and are known as fusion coefficients [1]. The tensor h_{ab} could be expressed in terms of them:

$$h_{ab} = (\iota^{\text{EPRL}}(\mathcal{I}_a) | \iota^{\text{EPRL}}(\mathcal{I}_b)) = \sum_{a^+a^-} f_a^{a^+a^-} f_b^{a^+a^-}$$

As an example we give the result of the calculation of the h_{ab} matrix for $\gamma = \frac{1}{2}$, $j_1 = 2$, $j_2 = 4$, $j_3 = 4$, $j_4 = 2$; $a, b \in \{2, \dots, 6\}$):

$$\begin{pmatrix} \frac{53723}{175616} & -\frac{2265\sqrt{\frac{5}{7}}}{50176} & \frac{5093\sqrt{5}}{1053696} & -\frac{3\sqrt{55}}{25088} & 0 \\ -\frac{2265\sqrt{\frac{5}{7}}}{50176} & \frac{117853}{501760} & -\frac{12805}{301056\sqrt{7}} & \frac{45\sqrt{\frac{11}{7}}}{7168} & -\frac{3\sqrt{\frac{13}{7}}}{8960} \\ \frac{5093\sqrt{5}}{1053696} & -\frac{12805}{301056\sqrt{7}} & \frac{741949}{3512320} & -\frac{781\sqrt{11}}{752640} & \frac{5\sqrt{13}}{5376} \\ -\frac{3\sqrt{55}}{25088} & \frac{45\sqrt{\frac{11}{7}}}{7168} & -\frac{781\sqrt{11}}{752640} & \frac{583}{2560} & 0 \\ 0 & -\frac{3\sqrt{\frac{13}{7}}}{8960} & \frac{5\sqrt{13}}{5376} & 0 & \frac{13}{40} \end{pmatrix}$$

We used the analytic expression for the fusion coefficient presented in [9]. Clearly this matrix is nondiagonal. It shows that the EPRL map is not isometric. The edge amplitude $h_e^{b\bar{a}}$ is given by the inverse matrix:

$$\begin{pmatrix} \frac{46976713}{14064543} & \frac{31728718\sqrt{\frac{7}{5}}}{70322715} & -\frac{75194882}{257849955\sqrt{5}} & -\frac{3865813}{70322715\sqrt{55}} & \frac{13066606}{773549865\sqrt{65}} \\ \frac{31728718\sqrt{\frac{7}{5}}}{70322715} & \frac{7682388364}{1758067875} & \frac{67078172\sqrt{7}}{586022625} & -\frac{318127222\sqrt{\frac{7}{11}}}{1758067875} & \frac{7212044\sqrt{\frac{7}{13}}}{1758067875} \\ -\frac{75194882}{257849955\sqrt{5}} & \frac{67078172\sqrt{7}}{586022625} & \frac{112636131412}{23636245875} & \frac{1305090458}{6446248875\sqrt{11}} & -\frac{12462294236}{70908737625\sqrt{13}} \\ -\frac{3865813}{70322715\sqrt{55}} & -\frac{318127222\sqrt{\frac{7}{11}}}{1758067875} & \frac{1305090458}{6446248875\sqrt{11}} & \frac{85031744497}{19338746625} & -\frac{192524374}{19338746625\sqrt{143}} \\ \frac{13066606}{773549865\sqrt{65}} & \frac{7212044\sqrt{\frac{7}{13}}}{1758067875} & -\frac{12462294236}{70908737625\sqrt{13}} & -\frac{192524374}{19338746625\sqrt{143}} & \frac{8510451083428}{2765440767375} \end{pmatrix}$$

III. INJECTIVITY OF THE MAP $\mathcal{I} \mapsto \iota^{\text{EPRL}}(\mathcal{I})$

This part of the paper is devoted to the injectivity of EPRL intertwiner. More explicitly, we will prove the result stated in 1.

A. Statement of the result

We assume that $\gamma \in \mathbb{R}$ and $|\gamma| < 1$. Suppose that

$$(k_1, \dots, k_n) \in \frac{1}{2}\mathbb{N}$$

$$\forall_i j_i^\pm = \frac{1 \pm \gamma}{2} k_i \in \frac{1}{2}\mathbb{N}.$$

We consider the EPRL map

$$\iota^{\text{EPRL}}: \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-}) \quad (3.1)$$

$$\iota^{\text{EPRL}}(\mathcal{I})_{j_1^+ A_1 \dots j_n^+ A_n j_1^- B_1 \dots j_n^- B_n} = \mathcal{I}_{k_1 C_1 \dots k_n C_n} C_{j_1^+ D_1 j_1^- E_1}^{k_1 C_1} \dots C_{j_n^+ D_n j_n^- E_n}^{k_n C_n} P_{j_1^+ A_1 \dots j_n^+ A_n}^{j_1^+ D_1 \dots j_n^+ D_n} P_{j_1^- B_1 \dots j_n^- B_n}^{j_1^- E_1 \dots j_n^- E_n}$$

with P standing for the orthogonal projections onto the subspaces of the $\text{SU}(2)$ invariants of the Hilbert spaces $\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}$, and respectively, $\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-}$.

Now we can state our result.

Theorem 1. For any sequence $(k_1, \dots, k_n) \in \frac{1}{2}\mathbb{N}$ such that $\text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$ is nontrivial the map

$$\iota^{\text{EPRL}}: \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-})$$

is injective.

B. Proof of the theorem

In order to make the proof transparent, we divide it into subsections. In subsection III B 1 some auxiliary definitions are introduced. We state also an inductive hypothesis, that will be proved in subsection III B 5. The injectivity of EPRL map follows from that result. The main technical tool of the proof is placed in subsection III B 3, where the map γ is defined.

1. Auxiliary definitions

Let us introduce some notations

Definition 2. For $x \in \mathbb{R}$ we define

- $[x]_+$ as the only half integer number in the interval $(x - \frac{1}{4}, x + \frac{1}{4}]$
- $[x]_-$ as the only half integer number in the interval $[x - \frac{1}{4}, x + \frac{1}{4})$

and

Definition 3. A sequence of half natural numbers (k_1, \dots, k_n) satisfies triangle inequality if

$$\forall_i k_i \leq \sum_{j \neq i} k_j.$$

One can define map ι under condition

Con n : Sequences of half natural numbers (k_1, \dots, k_n) and $(j_1^\pm, \dots, j_n^\pm)$ are such that

- (k_1, \dots, k_n) satisfies triangle inequality,
- $j_i^+ + j_i^- = k_i$ for $i = 1, \dots, n$,
- $j_i^\pm = \frac{1 \pm \gamma}{2} k_i$ for $i \neq 1$ and

$$j_1^+ = \left[\frac{1 + \gamma}{2} k_1 \right]_+, \quad j_1^- = \left[\frac{1 - \gamma}{2} k_1 \right]_-$$

or

$$j_1^+ = \left[\frac{1 + \gamma}{2} k_1 \right]_-, \quad j_1^- = \left[\frac{1 - \gamma}{2} k_1 \right]_+.$$

Let us define

$$\iota_{k_1 \dots k_n}: \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-})$$

$$\iota_{\dots}(\mathcal{I})_{j_1^+ A_1 \dots j_n^+ A_n j_1^- B_1 \dots j_n^- B_n} = \mathcal{I}_{k_1 C_1 \dots k_n C_n} C_{j_1^+ D_1 j_1^- E_1}^{k_1 C_1} \dots C_{j_n^+ D_n j_n^- E_n}^{k_n C_n} P_{j_1^+ A_1 \dots j_n^+ A_n}^{j_1^+ D_1 \dots j_n^+ D_n} P_{j_1^- B_1 \dots j_n^- B_n}^{j_1^- E_1 \dots j_n^- E_n}$$

with P standing for projections onto invariant subspaces. We will use the letter $\iota_{k_1 \dots k_n}$ for all sequences (k_1, \dots, k_n) , $(j_1^\pm, \dots, j_n^\pm)$ if it do not cause any misunderstanding.

We will base our prove on the following inductive hypothesis:

Hyp n : Suppose that (k_1, \dots, k_n) and $(j_1^\pm, \dots, j_n^\pm)$ satisfy condition **Con n** and that $\mathcal{I} \in \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$. Then, there exists

$$\phi \in \text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-})$$

such that $\langle \iota_{k_1 \dots k_n}(\mathcal{I}), \phi \rangle \neq 0$.

This in fact proves the injectivity.

2. Useful inequalities

Both $[x]_{\pm}$ are increasing functions and satisfy ($x, y \in \mathbb{R}, j \in \frac{1}{2}\mathbb{N}$)

- a) $[x + j] = [x] + j$ and $[j]_{\pm} = j$,
- b) if $x > y$ then $[x]_{-} \geq [y]_{+}$ and if $x \geq y$ then $[x]_{+} \geq [y]_{-}$
- c) if $x + y \in \mathbb{Z}$ then $[x]_{+} + [y]_{-} = x + y$
- d) if $x + y \geq j$ then $[x]_{+} + [y]_{-} \geq j$

In order to prove the last point, we notice that $[x]_{+} > x - \frac{1}{4}$ and $[y]_{-} \geq y - \frac{1}{4}$ so $[x]_{+} + [y]_{-} > x + y - \frac{1}{2} \geq j - \frac{1}{2}$ but as j is an half integer number $[x]_{+} + [y]_{-} \geq j$.

Lemma 4. *Suppose that (k, l, j) satisfies triangle inequality and that $\frac{1+\gamma}{2}k \in \frac{1}{2}\mathbb{N}$ then*

- both triples $\left(\frac{1+\gamma}{2}k, [\frac{1+\gamma}{2}l]_{\pm}, [\frac{1+\gamma}{2}j]_{\pm}\right)$ satisfy triangle inequalities if $k + l = j$ or $k + j = l$
- both triples $\left(\frac{1+\gamma}{2}k, [\frac{1+\gamma}{2}l]_{\pm}, [\frac{1+\gamma}{2}j]_{\mp}\right)$ satisfy triangle inequalities if $k + l > j$ and $k + j > l$.

Proof. In the first case suppose that $k + l = j$ holds, then $\frac{1+\gamma}{2}k + [\frac{1+\gamma}{2}l]_{\pm} = [\frac{1+\gamma}{2}j]_{\pm}$ that proves triangle inequality.

In the second case, we restrict our attention to $\left(\frac{1+\gamma}{2}k, [\frac{1+\gamma}{2}l]_{+}, [\frac{1+\gamma}{2}j]_{-}\right)$. We have

- $\frac{1+\gamma}{2}k + [\frac{1+\gamma}{2}j]_{-} \geq [\frac{1+\gamma}{2}l]_{+}$ because $\frac{1+\gamma}{2}k + \frac{1+\gamma}{2}l > \frac{1+\gamma}{2}j$,
- $\frac{1+\gamma}{2}k + [\frac{1+\gamma}{2}l]_{+} \geq [\frac{1+\gamma}{2}j]_{-}$ because $\frac{1+\gamma}{2}k + \frac{1+\gamma}{2}j \geq \frac{1+\gamma}{2}l$,
- $\frac{1+\gamma}{2}k \leq [\frac{1+\gamma}{2}l]_{+} + [\frac{1+\gamma}{2}j]_{-}$ from the property d) listed above.

The case of $\left(\frac{1+\gamma}{2}k, [\frac{1+\gamma}{2}l]_{-}, [\frac{1+\gamma}{2}j]_{+}\right)$ is analogous. □

Lemma 5. *Suppose that (k_1, \dots, k_n) satisfies triangle inequality and that $\frac{1+\gamma}{2}k_i \in \frac{1}{2}\mathbb{N}$ for $i = 2, \dots, n$, then $\left([\frac{1+\gamma}{2}k_1]_{\pm}, \frac{1+\gamma}{2}k_2, \dots, \frac{1+\gamma}{2}k_n\right)$ also satisfy triangle inequalities.*

Proof. That follows from the monotonicity of functions $[x]_{\pm}$ and the fact that in the inequality

$$\frac{1+\gamma}{2}k_i \leq \sum_{j \neq i} \frac{1+\gamma}{2}k_j$$

all terms but one are half-integer. □

Lemma 6. *Suppose that (k_1, \dots, k_n) satisfies triangle inequality then*

$$\text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$$

is nontrivial.

Proof. We will find an k_{α} such that both (k_{α}, k_1, k_2) and $(k_{\alpha}, k_3, \dots, k_n)$ satisfy triangle inequalities. By induction there would be

$$0 \neq \phi \in \text{Inv}(\mathcal{H}_{k_{\alpha}} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n}),$$

and then

$$0 \neq C_{k_1 A_1 k_2 A_2}^{k_{\alpha} A} \phi_{k_{\alpha} A k_3 A_3 \dots k_n A_n}$$

proves nontriviality. Now we extract conditions on k_α from triangle inequalities (we assume for simplicity that $k_1 \geq k_2$)

$$\begin{aligned} k_1 + k_2 &\geq k_\alpha \geq k_1 - k_2 \\ \sum_{i \geq 3} k_i &\geq k_\alpha \geq k_i - \sum_{j \neq i, j \geq 3} k_j, \quad i \geq 3 \end{aligned}$$

For the existence of such k_α we need only to show that

$$\begin{aligned} k_1 + k_2 &\geq k_i - \sum_{j \neq i, j \geq 3} k_j, \quad i \geq 3 \\ \sum_{i \geq 3} k_i &\geq k_1 - k_2 \end{aligned}$$

but these are exactly conditions for (k_1, \dots, k_n) to satisfy triangle inequality. \square

3. Important maps

Every $\mathcal{I} \in \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$ may be uniquely written as

$$\mathcal{I}_{k_1 A_1 k_2 A_2 \dots k_n A_n} = \sum_{k_\alpha} C_{k_1 A_1 k_2 A_2}^{k_\alpha A_\alpha} \mathcal{I}_{k_\alpha A_\alpha k_3 A_3 \dots k_n A_n}^{k_\alpha}$$

where $\mathcal{I}^{k_\alpha} \in \text{Inv}(\mathcal{H}_{k_\alpha} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n})$. Summation is taken over such k_α that (k_α, k_1, k_2) and $(k_\alpha, k_3, \dots, k_n)$ satisfy triangle inequality.

This gives us decomposition of $\text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$ into orthogonal subspaces

$$\oplus_\alpha \mathcal{H}_\alpha,$$

where each \mathcal{H}_α is isomorphic to $\text{Inv}(\mathcal{H}_{k_\alpha} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n})$. Let us define maps which assign these partial isometries

$$Q_{k_\alpha} : \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{k_\alpha} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n}), \quad Q_{k_\alpha} \mathcal{I} = \mathcal{I}^{k_\alpha}$$

Adjoint to them are embeddings $Q_{k_\alpha}^*$.

$$Q_{k_\alpha}^* (\mathcal{I}^{k_\alpha})_{k_1 A_1 k_2 A_2 \dots k_n A_n} = C_{k_1 A_1 k_2 A_2}^{k_\alpha A_\alpha} \mathcal{I}_{k_\alpha A_\alpha k_3 A_3 \dots k_n A_n}^{k_\alpha}$$

These maps are also well defined in a case that α does not occur in the decomposition $\oplus \mathcal{H}_\alpha$ but (k_α, k_1, k_2) satisfies triangle inequalities. Then the space $\text{Inv}(\mathcal{H}_{k_\alpha} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n})$ is trivial and the maps Q_{k_α} and $Q_{k_\alpha}^*$ too.

Let us fix (k_1, \dots, k_n) and $(j_1^\pm, \dots, j_n^\pm)$ satisfying triangle inequalities and such that $j_i^+ + j_i^- = k_i$.

Lemma 7. *Suppose k_α, j_α^\pm are such that $j_\alpha^+ + j_\alpha^- = k_\alpha$ and (k_α, k_1, k_2) and $(j_\alpha^\pm, j_1^\pm, j_2^\pm)$ satisfy triangle inequalities. Then there exists an operator*

$$\begin{aligned} G_{k_\alpha j_\alpha^+ j_\alpha^-} &: \text{Inv}(\mathcal{H}_{j_\alpha^+} \otimes \mathcal{H}_{j_3^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_\alpha^-} \otimes \mathcal{H}_{j_3^-} \otimes \dots \otimes \mathcal{H}_{j_n^-}) \rightarrow \\ &\rightarrow \text{Inv}(\mathcal{H}_{j_1^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_1^-} \otimes \dots \otimes \mathcal{H}_{j_n^-}) \end{aligned}$$

such that for all $\mathcal{I} \in \text{Inv}(\mathcal{H}_{k_\beta} \otimes \mathcal{H}_{k_3} \otimes \dots \otimes \mathcal{H}_{k_n})$ and $\phi \in \text{Inv}(\mathcal{H}_{j_\alpha^+} \otimes \mathcal{H}_{j_3^+} \otimes \dots \otimes \mathcal{H}_{j_n^+}) \otimes \text{Inv}(\mathcal{H}_{j_\alpha^-} \otimes \mathcal{H}_{j_3^-} \otimes \dots \otimes \mathcal{H}_{j_n^-})$

$$\langle \iota_{k_1 \dots k_n} Q_{k_\beta}^* \mathcal{I}, G_{k_\alpha j_\alpha^+ j_\alpha^-} \phi \rangle = \begin{cases} \langle \iota_{k_\alpha k_3 \dots k_n} \mathcal{I}, \phi \rangle, & k_\beta = k_\alpha \\ 0, & k_\beta > k_\alpha \\ *, & k_\beta < k_\alpha \end{cases}$$

Proof. We define $G_{k_\alpha j_\alpha^\pm}$ as

$$G_{k_\alpha j_\alpha^\pm}(\phi)_{j_1^+ A_1 \dots j_n^+ A_n j_1^- B_1 \dots j_n^- B_n} = \beta C_{j_1^+ A_1 j_2^+ A_2}^{j_\alpha^+ A_\alpha} C_{j_1^- A_1 j_2^- A_2}^{j_\alpha^- B_\alpha} \phi_{j_\alpha^+ A_\alpha j_3^+ A_3 \dots j_\alpha^- B_\alpha j_3^- B_3 \dots},$$

with β nonzero constant to be defined later. Let us compute

$$\langle \iota_{k_1 \dots k_n} Q_{k_\beta}^* \mathcal{I}, G_{k_\alpha j_\alpha^\pm} \phi \rangle$$

In the definition of ι one can skip projection because both ϕ and $G_{k_\alpha j_\alpha^\pm} \phi$ are invariants. Let us write explicitly $\langle \iota_{k_1 \dots k_n} Q_{k_\beta}^* \mathcal{I}, G_{k_\alpha j_\alpha^\pm} \phi \rangle$. We have

$$\begin{aligned} \mathcal{I}_{k_\beta A \dots k_n A_n} C_{k_1 A_1 k_2 A_2}^{k_\beta A} C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} \dots C_{j_n^+ B_n j_n^- C_n}^{k_n A_n} \beta C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2} \phi_{j_\alpha^+ B j_3^+ B_3 \dots j_\alpha^- C j_3^- C_3 \dots} = \\ = \beta C_{k_1 A_1 k_2 A_2}^{k_\beta A} C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{j_2^+ B_2 j_2^- C_2}^{k_2 A_2} C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2} \\ \mathcal{I}_{k_\beta A \dots k_n A_n} C_{j_3^+ B_3 j_3^- C_3}^{k_3 A_3} \dots C_{j_n^+ B_n j_n^- C_n}^{k_n A_n} \phi_{j_\alpha^+ B j_3^+ B_3 \dots j_\alpha^- C j_3^- C_3 \dots}. \end{aligned}$$

We need only to show that

$$\beta C_{k_1 A_1 k_2 A_2}^{k_\beta A} C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{j_2^+ B_2 j_2^- C_2}^{k_2 A_2} C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2} = \begin{cases} C_{j_\alpha^+ B j_\alpha^- C}^{k_\beta A}, & k_\beta = j_\alpha^+ + j_\alpha^- \\ 0 & k_\beta > j_\alpha^+ + j_\alpha^-. \end{cases}$$

The second equality is obvious because there exists no intertwiner if $k_\beta > j_\alpha^+ + j_\alpha^-$. The first will be proved in the next subsection III B 4¹. \square

4. Relation among intertwiners

We know that $C_{k_1 A_1 k_2 A_2}^{k_\beta A} C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{j_2^+ B_2 j_2^- C_2}^{k_2 A_2} C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2}$ is proportional to $C_{j_\alpha^+ B j_\alpha^- C}^{k_\beta A}$. In order to prove that the factor of proportionality is nonzero we will show that

$$C_{k_1 A_1 k_2 A_2}^{k_\alpha A} C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{j_2^+ B_2 j_2^- C_2}^{k_2 A_2} C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2} C_{k_\alpha A}^{j_\alpha^+ B j_\alpha^- C} \neq 0$$

and that would be β^{-1} . In fact it is enough to show that the intertwiner

$$C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{j_2^+ B_2 j_2^- C_2}^{k_2 A_2} C_{j_\alpha^+ B}^{j_1^+ B_1 j_2^+ B_2} C_{j_\alpha^- C}^{j_1^- C_1 j_2^- C_2} C_{k_\alpha A}^{j_\alpha^+ B j_\alpha^- C}$$

is not equal zero or equivalently the same for

$$C_{j_1^+ B_1 j_1^- C_1}^{k_1 A_1} C_{k_2 A_2}^{j_2^+ B_2 j_2^- C_2} C_{j_2^+ B_2 j_2^- C_2}^{j_1^+ B_1} C_{j_2^- C_2 j_\alpha^- C}^{j_1^- C_1} C_{k_\alpha A}^{j_\alpha^+ B j_\alpha^- C}$$

We only sketch the proof. First of all, we remind some facts about intertwiners and diagrammatic notation.

Let P^k stands for projection onto symmetric subspace in $\mathcal{H}_{1/2}^{\otimes 2k}$ equivalent to \mathcal{H}_k (k is a half natural number).

$$P^k : \mathcal{H}_{1/2}^{\otimes 2k} \rightarrow \mathcal{H}_{1/2}^{\otimes 2k}.$$

In this subsection we regard \mathcal{H}_k as this subspace of $\mathcal{H}_{1/2}^{\otimes 2k}$. Let us also denote the canonical map $\epsilon : \mathbb{C} \mapsto \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$.

¹ Although it seems to be standard, we include it for a sake of completeness.

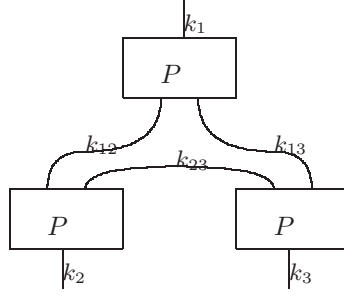


FIG. 2: An intertwiner *proportional* to $C_{k_2 k_3}^{k_1}$, $k_{12} = k_1 + k_2 - k_3$ and etc.

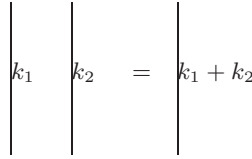


FIG. 3: An equality between $\mathcal{H}_{1/2}^{2k_1} \otimes \mathcal{H}_{1/2}^{2k_2}$ and $\mathcal{H}_{1/2}^{2k_1+2k_2}$.

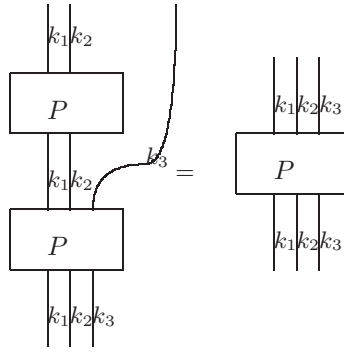


FIG. 4: An equality $P^{k_1+k_2+k_3} \circ P^{k_1+k_2} \otimes \mathbb{I} = P^{k_1+k_2+k_3}$.

The intertwiner $C^{k_1 k_2 k_3}: \mathbb{C} \rightarrow \mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \otimes \mathcal{H}_{k_3}$ is *proportional* to $P^{k_1} \otimes P^{k_2} \otimes P^{k_3} e^{\otimes 2k_1+2k_2+2k_3}$.

In the diagrammatic language this can be depicted as on figure 2. We skip the index k in P^k on the diagrams for notations' brevity. The line with symbol k denotes $\mathcal{H}_{1/2}^{\otimes 2k}$.

We have to notice important properties, that in diagrammatic language is shown on figures 3 and 4.

Our intertwiner can be written as shown on the figure 5.

Now using properties mentioned earlier we see that one can merge j_{ij}^+ with j_{ij}^- into k_{ij} and that intertwiner is equal to intertwiner shown on the figure 2 and is *nonzero*.

5. Inductive steps

Induction starts with $n = 1$. In this case $k_1 = 0$ and so also $j_1^\pm = 0$. The map $\iota_0 = C_{00}^0: \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{C}$ is the identity.

Suppose now, that we have just proved **Hyp** $n - 1$.

In the decomposition of given $\mathcal{I} \in \text{Inv}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_n})$ into subspaces \mathcal{H}_α we choose minimal k_α such that $Q_{k_\alpha} \mathcal{I}$ is nonzero. We know, by lemmas 4 and 5 that for either

$$(j_\alpha^+, j_\alpha^-) = \left(\left[\frac{1+\gamma}{2} k_\alpha \right]_+, \left[\frac{1-\gamma}{2} k_\alpha \right]_- \right)$$

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