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► **To cite this version:**

A Blasi, N Maggiore, N Magnoli, S Storace. MaxwellChernSimons theory with a boundary. *Classical and Quantum Gravity*, 2010, 27 (16), pp.165018. 10.1088/0264-9381/27/16/165018 . hal-00616259

**HAL Id: hal-00616259**

**<https://hal.science/hal-00616259>**

Submitted on 21 Aug 2011

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GEF-TH-01/2010

## Maxwell–Chern–Simons Theory With Boundary

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**Abstract** : The Maxwell–Chern–Simons (MCS) theory with planar boundary is considered. The boundary is introduced according to Symanzik’s basic principles of locality and separability. A method of investigation is proposed, which, avoiding the straight computation of correlators, is appealing for situations where the computation of propagators, modified by the boundary,

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becomes quite complex. For MCS theory, the outcome is that a unique solution exists, in the form of chiral conserved currents, satisfying a Kač–Moody algebra, whose central charge does not depend on the Maxwell term.

**Keywords:** Quantum Field Theory , Boundary Conditions, Boundary Algebra

**PACS Nos:** 03.70.+k Theory of Quantized Fields, 11.15.Wx Topologically massive gauge theories 11.15.Yc Chern–Simons gauge theory

# 1 Introduction

In the last decades, there has been vast interest in the role of the space boundary in Quantum Field Theory (QFT) [1, 2, 3]. The main reason is that the presence of a boundary changes the properties of a physical system, originating phenomena like the Casimir Effect [4] or the edge states in the Fractional Quantum Hall Effect [5].

In some cases, the boundary even represents the center of the investigation, as in the case of black holes [6], where the horizon of events can be indeed regarded as a boundary. Boundary effects play an important role also in critical phenomena [7].

In particular, the effect of a boundary is extremely interesting in topological field theories. Indeed, topological theories are known to have no local observables except when the base manifold has a boundary [2, 3]. Our attention will be actually focused on the topological, three dimensional Chern–Simons (CS) model [8, 9].

The introduction of a boundary in CS theory has been investigated in previous works [2, 3], with the result that the local observables arising from the presence of the boundary are two-dimensional conserved chiral currents generating the Kač-Moody algebra [10] of the Wess-Zumino-Witten model [11]. In other words, the boundary establishes a strong connection between CS theory and conformal field theories [12].

An analogous result has been obtained for another topological field theory, namely the topological BF model [13] in three dimensions, for which it has been shown as well the existence of chiral currents living on the boundary and satisfying a Kač-Moody algebra with central extension [14].

The existence of a Kač-Moody algebra with central extension in the CS model with boundary has led to interesting applications due to the connection of the CS theory with several physical systems.

In fact, following the equivalence between CS theory and (2+1) dimensional gravity theory with a cosmological constant [15, 16], it has been stressed that the algebra plays a crucial role in understanding the statistical origin of the entropy of a black hole [6]. It is thus possible to use the algebra to compute the BTZ black hole (negative cosmological constant) entropy [17] and the Kerr-de Sitter space (positive cosmological constant) entropy [18].

In condensed matter physics, the abelian CS model provides an effective low energy theory for the Fractional Quantum Hall Effect [5]. Further, when a boundary is taken into account, the Kač-Moody algebra describes the boundary chiral currents which are indeed observed on the edge of the Hall bar (edge states) [5].

On the other hand, the coupling of the CS model to other theories also gives rise to interesting results. For instance, it can be coupled to fermion or boson fields to attach magnetic flux to charge density, thus providing an explicit realization of *anyons*, *i.e.* particles living only in systems with two spatial dimensions and satisfying a fractional statistics [19].

Among the others, one of the most striking properties of the CS term is that, when added to the three dimensional Yang-Mills action, it originates a *topologically massive gauge theory* [20].

Its abelian version, the Maxwell–Chern–Simons (MCS) theory, when coupled with fermions defines a three dimensional modified electrodynamics, in which the “photons” are massive and have a single state of helicity [20]. The Casimir effect for topologically massive electrodynamics could provide, in principle, a way to “measure” the topological mass of the “photons” [21].

Moreover, the addition of a Maxwell term to the effective low energy theory for the Quantum Hall Effect allows the description of the gap between the ground state and the bulk elementary excitations [5].

The Maxwell theory coupled to the CS action on a manifold with boundary has a further application in (2+1) dimensional quantum gravity: the so called Einstein–Maxwell–Chern–Simons theory [22]. In this framework, the boundary can be regarded as the horizon of the black hole solutions, and the gauge field coupled to gravity describes a topologically massive electromagnetic field which provides the black hole with an electric charge, which, coming from the CS term, is “kind of” *topological*. The result is thus called “charged black hole”.

On the other hand, the Maxwell model is not topological, and therefore the question if its addition spoils the chiral current algebra of the CS theory arises naturally: the aim of this paper is actually to discuss this issue.

To reach this task, we first must face a further problem, *i.e.* the method. Indeed, the inclusion of a boundary in field theory is a highly non trivial task if one wishes to preserve locality and power counting, the most basic ingredients of QFT.

In 1981 K. Symanzik [1] addressed this question: his key idea was to add to the bulk action a local boundary term which modifies the propagators of the fields in such a way that nothing propagates from one side of the boundary to the other. He called this property “separability” and showed that it requires the realization of a well identified class of boundary conditions that can be implemented by a local bilinear interaction.

These ideas strongly inspired the authors of [23], [24] and [25], who used a closely related approach to compute the chiral current algebra living on the boundary of the three-dimensional CS model.

Indeed, the authors of [23] added to the action local boundary terms compatible with power counting, using a covariant gauge fixing. In [24, 25], on the other hand, a regularization-free procedure was followed: the equations of motion, rather than the action, were modified by appropriate boundary terms, and a non-covariant axial gauge was preferred.

The main reason for the latter choice was that the main advantage of a covariant gauge, *i.e.* Lorentz invariance, already fails due to the presence of the boundary. On the other hand, the axial choice does not completely fix the gauge [26], and a residual gauge invariance exists, implying the existence of a Ward identity which plays a crucial role since, when restricted to the boundary, it might generate a chiral current algebra, as we shall see.

In both these works, the explicit computation of the propagators of the theory seems to be necessary. However, this step could be quite difficult in other theories, like MCS, and another way of investigation is worthy, which avoids the explicit computation of the Green functions of the theory.

This is precisely the choice that we make. Our approach is actually more similar to that used in [24, 25], since we focus on the effect of the boundary on the equations of motion rather than on the action, and we adopt a noncovariant axial gauge rather than a covariant one.

The basic idea is that, after modifying the equations of motion by means of boundary terms satisfying general basic requirements, we integrate them in proximity of the boundary and use Symanzik's idea of separability to *determine*, rather than *impose*, the boundary conditions on the propagators, that can be expressed as boundary conditions on the fields [1]. These, in turn, have an effect on the boundary breaking term of the residual Ward identity that generates the Kač-Moody algebra living on the boundary.

For these reasons, our method is suitable also for a different kind of investigation. Indeed, the approach in [27] and [28] is good to describe just a portion of space: there is not a *beyond the boundary*. This is not a problem for the description of systems like electrodynamics on a disk [29] or a Hall bar in the framework of the Quantum Hall Effect [5], in which one is interested in the dynamics of the internal system only.

However, the description of what there is beyond the boundary is fundamental in the physics of *defects* [30], and our approach is actually suitable for this task. In the study of semiconductors [31], for instance, it is important to be able to describe local interactions between the particles and the imperfections of the material, approximated by  $\delta$ -type interactions.

On the other hand, topological defects are gathering more and more importance also in the framework of astroparticle physics, where they are proposed as an explanation for the formation of cosmic structures and for the genera-

tion of extremely high-energy cosmic rays [32, 33].

Even considering an impenetrable boundary, we keep both the left and the right side of spacetime. Indeed, we can fix the parameters of our general description in such a way to decouple the opposite sides of spacetime completely, and therefore in principle we are able to examine just one of the two sides, if we wish. In a certain way, this is actually what we will do by imposing the conservation of the bulk discrete symmetry (2.11) which we called parity.

In other words, according to Symanzik’s approach, the boundary is *defined* by the decoupling condition, which prevents correlations between points belonging to opposite sides of the boundary. A defect could be treated by using the same approach described in this paper, by simply relaxing the decoupling condition. In this way, the resulting modified theory describes interactions between a bulk and a  $\delta$ -type insertion in the action, *i.e.* a defect, as it is done in [34].

We organized this paper as follows. In Section 2 we reconsider the abelian CS theory with boundary, and we show that our method leads again to the correct results found in [23] and [24] for both the boundary conditions and the residual Ward identity, and we illustrate how the Ward identity generates the Kač-Moody algebra.

Although the method illustrated in Section 2 cannot be entirely found elsewhere already, the main original results can be found in Section 3, where we introduce the Maxwell term in the CS theory. We follow the same scheme of Section 2 to investigate whether, also in this case, conserved chiral currents exist which satisfy some kind of algebra, like in the pure CS theory. We stress again that the answer to this question is not at all to be taken for granted, due to the non-topological character of the bulk theory.

Our concluding remarks summarize our results and draw some conclusions, with some further suggestions of possible applications and extensions.

## 2 “Boundarization”: the pure Chern–Simons case

### 2.1 The model

In this Section we recall some known results concerning CS theory with planar boundary [23, 24, 25]. This will give us the opportunity to illustrate the method we shall follow in what really matters us, that is the MCS theory with boundary, in order to investigate the boundary physics without calculating explicitly the correlators of the theory in presence of the boundary.

In euclidean space, the abelian CS action reads :

$$S_{cs} = -\frac{k}{2} \int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho . \quad (2.1)$$

It will be convenient to work in euclidean light-cone coordinates, defined as:

$$\begin{aligned} u &= x_2 \\ z &= \frac{1}{\sqrt{2}}(x_1 - ix_0) \\ \bar{z} &= \frac{1}{\sqrt{2}}(x_1 + ix_0) , \end{aligned} \quad (2.2)$$

which induce similar definitions in the space of the fields:

$$\begin{aligned} A_u &= A_2 \\ A &= \frac{1}{\sqrt{2}}(A_1 + iA_0) \quad . \\ \bar{A} &= \frac{1}{\sqrt{2}}(A_1 - iA_0) \end{aligned} \quad (2.3)$$

The CS action then reads

$$S_{cs} = -k \int dudzd\bar{z} \left( \bar{A} \partial_u A + A_u \partial \bar{A} - A_u \bar{\partial} A \right) . \quad (2.4)$$

The gauge fixing term is :

$$S_{gf} = - \int dudzd\bar{z} A_u b , \quad (2.5)$$

which corresponds to the axial gauge choice

$$A_u = 0 . \quad (2.6)$$

The complete action is

$$\Sigma_{CS} = S_{cs} + S_{gf} + S_{ext} , \quad (2.7)$$

where in  $S_{ext}$  we coupled, as usual, external sources  $J_\Phi$  to the quantum fields  $\Phi = A_u, A, \bar{A}, b$  :

$$S_{ext} = - \int dudzd\bar{z} \sum_{\Phi} J_\Phi \Phi . \quad (2.8)$$

Besides the canonical mass dimensions, the fields appearing in the action  $\Sigma_{CS}$  are assigned an additional quantum number, called *helicity*, which encodes the two-dimensional Lorentz invariance of the theory on the planes  $u = constant$ .

The dimensions and helicities of the fields and sources are summarized in Table 1 :

Table 1 : CS Quantum Numbers

	$A_u$	$A$	$\bar{A}$	$b$	$J_u$	$J$	$\bar{J}$	$J_b$	$\partial_u$	$\partial$	$\bar{\partial}$	$u$	$z$	$\bar{z}$
dim	1	1	1	2	2	2	2	1	1	1	1	-1	-1	-1
hel	0	1	-1	0	0	1	-1	0	0	1	-1	0	-1	+1

Moreover, as it is well known, the axial gauge choice (2.6) does not completely fix the gauge: a residual local gauge invariance remains, which is expressed by the local Ward identity

$$\partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_u b = 0 , \quad (2.9)$$



which, once integrated, gives

$$\int du (\partial \bar{J} + \bar{\partial} J) = 0 . \quad (2.10)$$

Finally, the set of symmetries of the action  $\Sigma_{CS}$  is completed by the discrete transformation involving at the same time coordinates and fields:

$$\begin{aligned} z &\leftrightarrow \bar{z} \\ u &\rightarrow -u \\ A &\leftrightarrow \bar{A} \\ A_u &\rightarrow -A_u \\ b &\rightarrow -b \\ J &\leftrightarrow \bar{J} \\ J_u &\rightarrow -J_u \\ J_b &\rightarrow -J_b , \end{aligned} \quad (2.11)$$

which we will refer to by using the term “*parity*”.

The field equations of motion are

$$k (\bar{\partial} A_u - \partial_u \bar{A}) + \bar{J} = 0 \quad (2.12)$$

$$k (\partial_u A - \bar{\partial} A_u) + J = 0 \quad (2.13)$$

$$k \left( \partial \bar{A} - \bar{\partial} A + \frac{1}{k} b \right) + J_u = 0 \quad (2.14)$$

$$A_u + J_b = 0 . \quad (2.15)$$

## 2.2 The boundary

We now want to introduce a boundary in the theory, and we choose the planar surface  $u = 0$ .

The effect of the boundary is to break the original equations of motion, in a way which must respect the following basic constraints.

**Locality:** The boundary contribution must be local. This means that all possible breaking terms have the form

$$\delta^{(n)}(u) X(z, \bar{z}, u) , \quad (2.16)$$

where  $\delta^{(n)}(u)$  is the  $n$ -order derivative of the Dirac delta function with respect to its argument, and  $X(z, \bar{z}, u)$  is a local functional.

**Separability:** This constraint, called also *decoupling condition*, refers to Symanzik’s original idea [1] according to which the  $n$ -point Green functions which involve two fields computed in points belonging to opposite sides of space must vanish. In particular, the propagators of the theory must satisfy :

$$uu' < 0 \Rightarrow \Delta_{AB}(x - x') = \langle T (\varphi_A(x) \varphi_B(x')) \rangle = 0 , \quad (2.17)$$

where

$$x = (z, \bar{z}, u) , \quad (2.18)$$

and  $\varphi_A(x)$  is a generic field of the theory. This property is satisfied by propagators which split according to

$$\Delta_{AB}(x - x') = \theta_+ \Delta_{AB+}(x - x') + \theta_- \Delta_{AB-}(x - x') , \quad (2.19)$$

where  $\Delta_{AB+}(x - x')$  and  $\Delta_{AB-}(x - x')$  are respectively the propagators for the right and the left side of spacetime,

$$\theta_{\pm} = \theta(\pm u)\theta(\pm u') \quad (2.20)$$

and  $\theta(x)$  is the step function, defined as usual.

The relation (2.19) is referred to as “*decoupling condition*”, which induces the decomposition of the generating functional  $\mathcal{W}$  of connected Green functions according to

$$\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^- , \quad (2.21)$$

where  $\mathcal{W}^+$  and  $\mathcal{W}^-$  are the generators of the connected Green functions for the right side and the left side of spacetime, respectively.

**Linearity:** Finally, we require that the boundary contributions to the equations of motion be linear in the fields because, in general, the symmetries of the classical action, if only linearly broken at the classical level, nonetheless remain exact symmetries of the quantum action [35]. Moreover, we are considering a free field theory and therefore nonlinear terms in the field must not occur in the equations of motion.

The most general broken equations of motion which satisfy all the above requirements are

$$k \left( \bar{\partial} A_u - \partial_u \bar{A} \right) + \bar{J} = \delta(u) (c_1^+ \bar{A}_+ + c_1^- \bar{A}_-) \quad (2.22)$$

$$k \left( \partial_u A - \bar{\partial} A_u \right) + J = \delta(u) (c_2^+ A_+ + c_2^- A_-) \quad (2.23)$$

$$k \left( \partial \bar{A} - \bar{\partial} A + \frac{1}{k} b \right) + J_u = \delta(u) (c_3^+ A_{u+} + c_3^- A_{u-}) \quad (2.24)$$

$$A_u + J_b = 0 , \quad (2.25)$$

where  $c_i^{\pm}$  are constant parameters, and  $A_{\pm}(Z)$ ,  $\bar{A}_{\pm}(Z)$  and  $A_{u\pm}(Z)$  are the boundary fields on the right, respectively on the left, of the boundary:

$$A_{\pm}(Z) = \lim_{u \rightarrow 0^{\pm}} A(x) \quad (2.26)$$

$$\bar{A}_{\pm}(Z) = \lim_{u \rightarrow 0^{\pm}} \bar{A}(x) \quad (2.27)$$

$$A_{u\pm}(Z) = \lim_{u \rightarrow 0^{\pm}} A_u(x) , \quad (2.28)$$

with  $Z = (z, \bar{z})$ . Under parity (2.11), the boundary fields transform according to

$$A_{\pm} \leftrightarrow \bar{A}_{\mp} \quad (2.29)$$

$$A_{u\pm} \rightarrow -A_{u\mp} . \quad (2.30)$$

Imposing the parity constraint (2.11) on the broken equations of motion leads to

$$c_1^\pm = c_2^\mp \quad (2.31)$$

$$c_3^+ = -c_3^- = c_3 . \quad (2.32)$$

In addition, the broken equations of motion must satisfy what in [24] has been called *compatibility*, which is nothing else than the commutation property which equations of motion in general obey. This algebraic constraint further reduces the number of parameters appearing in the r.h.s. of the boundary equations of motion :

$$c_1^+ = c_1^- = c . \quad (2.33)$$

Finally, the equations of motion, broken by the presence of the planar boundary  $u = 0$ , which respect parity and compatibility are

$$k \left( \bar{\partial} A_u - \partial_u \bar{A} \right) + \bar{J} = c \delta(u) (\bar{A}_+ + \bar{A}_-) \quad (2.34)$$

$$k \left( \partial_u A - \bar{\partial} A_u \right) + J = c \delta(u) (A_+ + A_-) \quad (2.35)$$

$$k \left( \partial \bar{A} - \bar{\partial} A + \frac{1}{k} b \right) + J_u = c_3 \delta(u) (A_{u+} - A_{u-}) \quad (2.36)$$

$$A_u + J_b = 0 . \quad (2.37)$$

### 2.3 The boundary conditions

In QFT with boundary, a crucial issue is represented by the boundary condition for the quantum fields. This problem is often solved by imposing “by hand” boundary conditions and see what happens. In Symanzik’s approach, the only constraint is the decoupling condition, which concerns the Green functions, as we discussed previously. The boundary conditions on the fields, rather than being imposed, are read, once the basic separability condition on propagators is realized.

The price to pay is, of course, the need of an explicit computation of the propagators, which, in presence of a boundary, is not at all an easy task, usually.

For CS theory the problem has been solved in [23] and in [24], by two quite different approaches, and, from the expression of the propagators on the boundary, the boundary conditions on the fields –which turn out to be of the Dirichlet type – are inferred. But, for more complicated situations, like the one this paper is devoted to, this program might be out of reach.

Here, we present an easy way out. That is, we claim to be able to carry out Symanzik’s program for QFTs with boundary, and find out the boundary physics, without explicitly computing the Green functions.

In this Section we are treating the pure CS case, which is known, just to illustrate the idea.

Definitely more interesting is the adoption of our simple, but powerful, method, to the MCS theory, what will be done in the next Section.

After setting the sources to zero, let us integrate (2.34) and (2.35) with respect to  $u$  from  $-\varepsilon$  to  $\varepsilon$ . Taking into account also (2.37), we get

$$(k - c) \bar{A}_- = (k + c) \bar{A}_+ \quad (2.38)$$

$$(k + c) A_- = (k - c) A_+ . \quad (2.39)$$

Due to the decoupling condition, each side of the above identities must vanish separately:

$$(k - c) \bar{A}_- = 0 \quad (2.40)$$

$$(k + c) \bar{A}_+ = 0 \quad (2.41)$$

$$(k + c) A_- = 0 \quad (2.42)$$

$$(k - c) A_+ = 0 . \quad (2.43)$$

The nontrivial solutions are given by

$$c = k \Rightarrow \bar{A}_+ = 0 = A_- \quad (2.44)$$

$$c = -k \Rightarrow \bar{A}_- = 0 = A_+ . \quad (2.45)$$

This is exactly the same result previously found in [24, 25, 36]: the fields obey *Dirichlet* boundary conditions on both sides of the dividing plane  $u = 0$ .

The curious reader is invited to compare how the same result has been obtained in [23] and [24], where, like we did, the boundary conditions are not put “by hand” in the theory but derived from the Symanzik’s decoupling condition.

It is clear that the choices (2.44) and (2.45) essentially describe the same physics, since they are related by the parity transformation (2.11). Therefore, in what follows we choose the solution with  $c = k$ , keeping in mind that a kind of “mirror solution” exists.

The equations of motion thus become

$$k (\bar{\partial} A_u - \partial_u \bar{A}) + \bar{J} = k \delta(u) \bar{A}_- \quad (2.46)$$

$$k (\partial_u A - \partial A_u) + J = k \delta(u) A_+ \quad (2.47)$$

$$k \left( \partial \bar{A} - \bar{\partial} A + \frac{1}{k} b \right) + J_u = c_3 \delta(u) (A_{u+} - A_{u-}) \quad (2.48)$$

$$A_u + J_b = 0 . \quad (2.49)$$

Correspondingly, the local Ward identity (2.9) acquires a boundary breaking:

$$\partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_u b = k \delta(u) (\bar{\partial} A_+ + \partial \bar{A}_-) + \partial_u [c_3 \delta(u) (A_{u+} - A_{u-})] , \quad (2.50)$$

which yields the integrated Ward identity

$$\frac{1}{k} \int du (\partial \bar{J} + \bar{\partial} J) = \bar{\partial} A_+ + \partial \bar{A}_- . \quad (2.51)$$

## 2.4 The boundary algebra

In this Section we recall how the Ward identity (2.51) implies the existence of a Kač-Moody algebra of conserved chiral currents on the boundary.

We can rewrite the Ward identity in a functional way as

$$\frac{1}{k} \int du [\partial \bar{J}(x) + \bar{\partial} J(x)] = \bar{\partial} \left. \frac{\delta \mathcal{W}_+}{\delta \bar{J}(x)} \right|_{u=0^+} + \partial \left. \frac{\delta \mathcal{W}_-}{\delta J(x)} \right|_{u=0^-} . \quad (2.52)$$

We then differentiate with respect to  $\bar{J}(x')$ , with  $x'$  lying on the right side of space next to the boundary, and then set the sources to zero. We thus get

$$\begin{aligned} \frac{1}{k} \int du \partial \delta^3(x - x') &= \left( \bar{\partial} \left. \frac{\delta^2 \mathcal{W}_+}{\delta \bar{J}(x') \delta \bar{J}(x)} \right|_{u', u=0^+} \right) \Big|_{J_\Phi=0} \\ &+ \left( \partial \left. \frac{\delta^2 \mathcal{W}_-}{\delta \bar{J}(x') \delta J(x)} \right|_{u'=0^+, u=0^-} \right) \Big|_{J_\Phi=0} . \end{aligned} \quad (2.53)$$

Recalling that  $\mathcal{W}_\pm$  are the generators of the connected Green functions for the  $u > 0$  and  $u < 0$  sides of spacetime respectively, the right-hand side of this expression involves propagators, the second of which vanishes due to the decoupling condition. Therefore, we are left with

$$\frac{1}{k} \partial \delta^2(Z - Z') = \bar{\partial} \langle T(A_+(Z') A_+(Z)) \rangle . \quad (2.54)$$

Keeping in mind the definition of the time-ordering operator  $T$ , we have to specify the role of time in light-cone variables. This issue has already been extensively studied in the literature [37]. The outcome is that a possibility is to identify the light-cone variable  $\bar{z}$  with time.

We can now explicitly compute the right-hand side of (2.54):

$$\begin{aligned} &\bar{\partial} \langle T(A_+(Z') A_+(Z)) \rangle = \\ &= \bar{\partial} \langle \theta(\bar{z}' - \bar{z}) A_+(Z') A_+(Z) + \theta(\bar{z} - \bar{z}') A_+(Z) A_+(Z') \rangle \\ &= - \langle \delta(\bar{z}' - \bar{z}) A_+(Z') A_+(Z) + \theta(\bar{z}' - \bar{z}) A_+(Z') \bar{\partial} A_+(Z) \rangle \\ &\quad + \langle \delta(\bar{z} - \bar{z}') A_+(Z) A_+(Z') + \theta(\bar{z} - \bar{z}') \bar{\partial} A_+(Z) A_+(Z') \rangle \\ &= \langle \theta(\bar{z}' - \bar{z}) A_+(Z') \bar{\partial} A_+(Z) + \theta(\bar{z} - \bar{z}') \bar{\partial} A_+(Z) A_+(Z') \rangle \\ &\quad + \delta(\bar{z} - \bar{z}') \langle [A_+(Z), A_+(Z')] \rangle . \end{aligned} \quad (2.55)$$

On the other hand, after setting the sources to zero, the Ward identity (2.51) and the decoupling condition yield the *chirality condition* [23, 24, 25]:

$$0 = \bar{\partial} A_+ \Rightarrow A_+ = A_+(z) \quad (2.56)$$

$$0 = \partial \bar{A}_- \Rightarrow \bar{A}_- = \bar{A}_-(\bar{z}) \quad (2.57)$$

Substituting (2.56) into (2.55) and then into (2.54) we get

$$\frac{1}{k}\partial\delta^2(Z-Z') = \frac{1}{k}\delta(\bar{z}-\bar{z}')\langle\partial\delta(z-z')\rangle = \delta(\bar{z}-\bar{z}')\langle[A_+(z), A_+(z')]\rangle, \quad (2.58)$$

which finally yields the commutation relation

$$[A_+(z), A_+(z')] = \frac{1}{k}\partial\delta(z-z'). \quad (2.59)$$

This is the abelian counterpart of the Kač-Moody algebra [10] of the Wess-Zumino-Witten model [11] generated by the chiral currents found in [3, 23, 24]. Under this respect, the coefficient  $\frac{1}{k}$  can be seen as the central charge of the Kač-Moody algebra. Note that the parity symmetry (2.11) implies the mirror algebra

$$[\bar{A}_-(\bar{z}), \bar{A}_-(\bar{z}')] = \frac{1}{k}\bar{\partial}\delta(\bar{z}-\bar{z}') \quad (2.60)$$

on the opposite side of the boundary.

Moreover, from (2.56) and (2.44), we get

$$\bar{\partial}A_+ + \partial\bar{A}_+ = 0, \quad (2.61)$$

which is the conservation relation for the planar boundary field, written in light-cone coordinates [25]. Indeed, recalling (2.3), we can come back to the euclidean components of the gauge field:

$$\begin{aligned} A_+ &= \frac{1}{\sqrt{2}}(A_{1+} + iA_{0+}) \\ \bar{A}_+ &= \frac{1}{\sqrt{2}}(A_{1+} - iA_{0+}). \end{aligned} \quad (2.62)$$

The relation (2.61) then takes the form

$$\partial_1 A_{1+} + \partial_0 A_{0+} = 0, \quad (2.63)$$

which is easily identified with a continuity relation involving a density  $A_{0+}$  and a current  $A_{1+}$ .

Furthermore, from (2.44) and (2.56) follows

$$\partial_1 A_{0+} - i\partial_0 A_{0+} = 0, \quad (2.64)$$

which, compared to (2.63), implies the identification of  $A_{0+}$  and  $A_{1+}$

$$A_{1+} = iA_{0+}. \quad (2.65)$$

Consequently, the algebra (2.59) can be written in terms of the density  $A_{0+}$ :

$$[A_{0+}(z), A_{0+}(z')] = -\frac{1}{2k}\partial\delta(z-z'). \quad (2.66)$$

Summarizing, in this Section we recovered the general result [2, 3] that a topological field theory acquires local observables only when a boundary is introduced. In the CS case, the observables are conserved chiral currents living on the boundary and satisfying a Kač-Moody algebra, whose central charge is the inverse of the CS coupling constant, as discussed in [23, 24, 25]. Our approach led us to the same results previously obtained in [23, 24, 25], but, we stress once again, with the remarkable difference that we have used an algebraic method which has allowed to avoid the explicit computation of the propagators of the theory. This will be extremely useful in cases, like that described in the next Section, in which the explicit computation of correlators is not that easy.

### 3 One step beyond: adding a Maxwell term

#### 3.1 The model

In this Section we study the effect of a planar boundary on the MCS theory. We stress that adding a Maxwell term to the CS theory is not a mere extension, in particular for what concerns the boundary physics. The Maxwell term, indeed, breaks the topological character of the theory, and, hence, all properties which are peculiar to topological QFTs in principle do not hold anymore. The Maxwell term introduces local degrees of freedom, which are absent in topological CS theory, and what happens when a boundary is introduced is far from being obvious.

From the physical point of view, the MCS theory describes a massive spin 1 particle with a single state of helicity, and which can be coupled with fermions to define a modified electrodynamics of fermions interacting with each other and with topologically massive “photons” [20]. The introduction of local degrees of freedom renders the MCS theory particularly relevant both in solid state physics (Quantum Hall Effect [5]) and (black holes [6]), as explained in the Introduction.

For this reason, it is interesting to investigate whether

1. chiral
2. conserved currents exist, which
3. satisfy an algebra, and of which type, and,
4. if the Kač-Moody structure is preserved, how sensitive is the central charge to the Maxwell term.

The MCS action, in euclidean space, is

$$S_{MCS} = S_{CS} + S_M , \tag{3.1}$$

where  $S_{CS}$  is the CS action (2.1),  $S_M$  is the Maxwell action

$$+\frac{\alpha}{4} \int d^3x F_{\mu\nu} F_{\mu\nu} , \quad (3.2)$$

and  $F_{\mu\nu}(x)$  is the field strength.

Notice that the theory actually depends only on one parameter only, that we identify with  $k$ , which represents the topological mass. In fact, the parameter  $\alpha$  in front of the Maxwell term can be reabsorbed by a redefinition of the gauge field  $A_\mu(x)$ . Nevertheless, we prefer to keep it in order to be able, at a later step, to switch off the Maxwell term and make contact with the pure CS case. Another good reason to keep  $\alpha$ , is that in the nonabelian extension it is not possible to reabsorb it, and therefore in the general case it is a real coupling constant.

The Maxwell action in light-cone coordinates (2.3), is :

$$S_M = -\frac{i\alpha}{2} \int dudzdz\bar{z} \left[ \bar{\partial}A\bar{\partial}A + \partial\bar{A}\partial\bar{A} - 2\partial\bar{A}\bar{\partial}A + 2\partial_u A \bar{\partial}A_u + 2\partial_u \bar{A} \partial A_u - 2\partial_u \bar{A} \partial_u A - 2\partial A_u \bar{\partial}A_u \right] . \quad (3.3)$$

Recalling that the CS theory in euclidean light-cone coordinates is given by  $S_{CS}$  (2.4), that the gauge fixing term for the axial gauge  $A_u = 0$  is  $S_{gf}$  (2.5), and that the external sources are coupled to quantum fields through  $S_{ext}$  (2.8), the complete action for the theory that we now consider is

$$\Sigma_{MCS} = S_{CS} + S_M + S_{gf} + S_{ext} . \quad (3.4)$$

The presence of the Maxwell term modifies the canonical mass dimensions of fields and sources. Table 2 summarizes dimensions and helicities of MCS theory. Notice also that the CS coupling constant  $k$  acquires a mass dimension  $[k] = 1$ .

Table 2 : MCS Quantum Numbers

	$A_u$	$A$	$\bar{A}$	$b$	$J_u$	$J$	$\bar{J}$	$J_b$
dim	1/2	1/2	1/2	5/2	5/2	5/2	5/2	1/2
hel	0	1	-1	0	0	1	-1	0

The local, and integrated, Ward identities (2.9), (2.10), expressing the residual gauge invariance, are left unchanged, and also the discrete parity symmetry (2.11) still holds for MCS theory.

The equations of motion induced by the action  $\Sigma_{MCS}$  (3.4) are

$$k(\bar{\partial}A_u - \partial_u \bar{A}) + i\alpha[\bar{\partial}(\partial\bar{A} - \bar{\partial}A) + \partial_u(\partial_u \bar{A} - \bar{\partial}A_u)] + \bar{J} = 0 \quad (3.5)$$

$$k(\partial_u A - \partial A_u) + i\alpha[\partial(\bar{\partial}A - \partial\bar{A}) + \partial_u(\partial_u A - \partial A_u)] + J = 0 \quad (3.6)$$

$$k(\partial\bar{A} - \bar{\partial}A) + i\alpha[2\partial\bar{\partial}A_u - \partial_u(\bar{\partial}A + \partial\bar{A})] + b + J_u = 0 \quad (3.7)$$

$$A_u + J_b = 0 . \quad (3.8)$$



### 3.2 The boundary

We now introduce the planar boundary  $u = 0$ . The decoupling condition imposes that the generating functional of the connected Green functions  $\mathcal{W}$  can be written as

$$\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_- \quad (3.9)$$

and, consequently, the propagators of the theory take the form (2.19).

The presence of the boundary induces in the equations of motion boundary terms that must respect all the constraints of locality, linearity in the quantum fields, power counting, conserved quantum numbers and analyticity in the parameters. The most general broken equations of motion satisfying all the requirements are

$$\begin{aligned} & k(\bar{\partial}A_u - \partial_u\bar{A}) + i\alpha[\bar{\partial}(\partial\bar{A} - \bar{\partial}A) + \partial_u(\partial_u\bar{A} - \bar{\partial}A_u)] + \bar{J} = \\ & = \delta(u) \left[ \alpha_1^+ \bar{A}_+ + \alpha_1^- \bar{A}_- + \alpha_2^+ (\partial_u\bar{A})_+ + \alpha_2^- (\partial_u\bar{A})_- \right] \\ & + \delta'(u) \left[ \alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_- \right] + \delta(u) \left( \alpha_6^+ \bar{\partial}A_{u+} + \alpha_6^- \bar{\partial}A_{u-} \right) \end{aligned} \quad (3.10)$$

$$\begin{aligned} & k(\partial_u A - \partial A_u) + i\alpha[\partial(\bar{\partial}A - \partial\bar{A}) + \partial_u(\partial_u A - \partial A_u)] + J = \\ & = \delta(u) \left[ \gamma_1^+ A_+ + \gamma_1^- A_- + \gamma_2^+ (\partial_u A)_+ + \gamma_2^- (\partial_u A)_- \right] \\ & + \delta'(u) \left[ \gamma_3^+ A_+ + \gamma_3^- A_- \right] + \delta(u) \left( \gamma_4^+ \partial A_{u+} + \gamma_4^- \partial A_{u-} \right) \end{aligned} \quad (3.11)$$

$$\begin{aligned} & k(\partial\bar{A} - \bar{\partial}A) + i\alpha[2\partial\bar{\partial}A_u - \partial_u(\bar{\partial}A + \partial\bar{A})] + b + J_u = \\ & = \delta(u) \left[ \alpha_4^+ \bar{\partial}A_+ + \alpha_4^- \bar{\partial}A_- + \alpha_5^+ \partial\bar{A}_+ + \alpha_5^- \partial\bar{A}_- \right] \\ & + \delta(u) \left( \alpha_8^+ A_{u+} + \alpha_8^- A_{u-} \right) + \delta(u) \left[ \alpha_9^+ (\partial_u A_u)_+ + \alpha_9^- (\partial_u A_u)_- \right] \\ & + \delta'(u) \left( \alpha_{10}^+ A_{u+} + \alpha_{10}^- A_{u-} \right) \end{aligned} \quad (3.12)$$

$$A_u + J_b = 0 . \quad (3.13)$$

In the above equations,  $\alpha_i^\pm$  and  $\gamma_i^\pm$  are constant parameters. Notice that the lower dimension of the gauge field, with respect to the CS case, allows the presence of differentiated delta functions  $\delta'(u)$  in the right-hand sides of the broken equations of motion.

Again, we have used the boundary fields on the left, respectively on the right of the boundary:

$$\bar{A}_\pm(Z) = \lim_{u \rightarrow 0^\pm} \bar{A}(x) \quad (3.14)$$

$$A_\pm(Z) = \lim_{u \rightarrow 0^\pm} A(x) \quad (3.15)$$

$$A_{u\pm}(Z) = \lim_{u \rightarrow 0^\pm} A_u(x) \quad (3.16)$$

$$(\partial_u \bar{A})_\pm(Z) = \lim_{u \rightarrow 0^\pm} \partial_u \bar{A}(x) \quad (3.17)$$

$$(\partial_u A)_\pm(Z) = \lim_{u \rightarrow 0^\pm} \partial_u A(x) \quad (3.18)$$

$$(\partial_u A_u)_\pm(Z) = \lim_{u \rightarrow 0^\pm} \partial_u A_u(x) , \quad (3.19)$$

where  $Z = (z, \bar{z})$ .

Under the parity symmetry (2.11), the boundary fields transform according to

$$\begin{aligned} A_\pm &\leftrightarrow \bar{A}_\mp \\ A_{u\pm} &\rightarrow -A_{u\mp} \\ (\partial_u A)_\pm &\leftrightarrow -(\partial_u \bar{A})_\mp \\ (\partial_u A_u)_\pm &\leftrightarrow (\partial_u A_u)_\mp . \end{aligned} \quad (3.20)$$

We now proceed by imposing the constraints that we discussed in the previous Section.

The parity constraint yields the following conditions:

$$\alpha_1^\pm = \gamma_1^\mp \quad (3.21)$$

$$\alpha_2^\pm = -\gamma_2^\mp \quad (3.22)$$

$$\alpha_3^\pm = -\gamma_3^\mp \quad (3.23)$$

$$\alpha_6^\pm = -\gamma_4^\mp \quad (3.24)$$

$$\alpha_4^\pm = -\alpha_5^\mp \quad (3.25)$$

$$\alpha_8^+ = \alpha_8^- \quad (3.26)$$

$$\alpha_9^+ = -\alpha_9^- \quad (3.27)$$

$$\alpha_{10}^+ = -\alpha_{10}^- . \quad (3.28)$$

We now consider the algebraic constraint of ‘‘compatibility’’ between equations of motion. We already said in the previous Section that this simply corresponds to requiring that the equations of motion, even if broken by boundary terms, commute with each other.

After a little algebra, and recalling some basic properties of the  $\delta$ -functions [38], we get

$$\alpha_1^+ = \alpha_1^- = \alpha_1 \quad (3.29)$$

$$\alpha_2^\pm = \alpha_3^\mp \quad (3.30)$$

$$\alpha_6^+ = -\alpha_4^+ \quad (3.31)$$

$$\alpha_6^- = \alpha_5^+ \quad (3.32)$$

$$\alpha_9^+ = -\alpha_{10}^+ . \quad (3.33)$$

The broken equations of motion thus become

$$k(\bar{\partial}A_u - \partial_u \bar{A}) + i\alpha[\bar{\partial}(\partial \bar{A} - \bar{\partial}A) + \partial_u(\partial_u \bar{A} - \bar{\partial}A_u)] + \bar{J} =$$

$$\begin{aligned}
&= \delta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \right] \\
&+ \delta'(u) \left[ \alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_- \right] - \delta(u) \left( \alpha_4^+ \bar{\partial} A_{u+} - \alpha_5^+ \bar{\partial} A_{u-} \right) \quad (3.34)
\end{aligned}$$

$$\begin{aligned}
&k(\partial_u A - \partial A_u) + i\alpha[\partial(\bar{\partial} A - \partial \bar{A}) + \partial_u(\partial_u A - \partial A_u)] + J = \\
&= \delta(u) \left[ \alpha_1 (A_+ + A_-) - \alpha_3^+ (\partial_u A)_+ - \alpha_3^- (\partial_u A)_- \right] \\
&- \delta'(u) \left[ \alpha_3^- A_+ + \alpha_3^+ A_- \right] - \delta(u) \left( \alpha_5^+ \partial A_{u+} - \alpha_4^+ \partial A_{u-} \right) \quad (3.35)
\end{aligned}$$

$$\begin{aligned}
&k(\partial \bar{A} - \bar{\partial} A) + i\alpha[2\partial \bar{\partial} A_u - \partial_u(\bar{\partial} A + \partial \bar{A})] + b + J_u = \\
&= \delta(u) \left[ \alpha_4^+ (\bar{\partial} A_+ - \partial \bar{A}_-) + \alpha_5^+ (\partial \bar{A}_+ - \bar{\partial} A_-) \right] \\
&+ \delta(u) \left[ \alpha_8^+ (A_{u+} + A_{u-}) - \alpha_{10}^+ [(\partial_u A_u)_+ - (\partial_u A_u)_-] \right] \\
&+ \delta'(u) \alpha_{10}^+ (A_{u+} - A_{u-}) \quad (3.36)
\end{aligned}$$

$$A_u + J_b = 0. \quad (3.37)$$

Correspondingly, the local Ward identity (2.9) acquires a boundary breaking:

$$\begin{aligned}
&\partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_u b = \delta(u) \left[ \alpha_1 (\partial \bar{A}_+ + \partial \bar{A}_- + \bar{\partial} A_+ + \bar{\partial} A_-) \right] \\
&+ \delta(u) \left\{ \alpha_3^- \left[ \partial (\partial_u \bar{A})_+ - \bar{\partial} (\partial_u A)_- \right] + \alpha_3^+ \left[ \partial (\partial_u \bar{A})_- - \bar{\partial} (\partial_u A)_+ \right] \right\} \\
&- \delta(u) \left[ (\alpha_4^+ + \alpha_5^+) \partial \bar{\partial} A_{u+} - (\alpha_4^+ + \alpha_5^+) \partial \bar{\partial} A_{u-} \right] \\
&+ \delta'(u) \left[ \alpha_3^+ (\partial \bar{A}_+ - \bar{\partial} A_-) + \alpha_3^- (\partial \bar{A}_- - \bar{\partial} A_+) \right] \\
&+ \delta'(u) \left[ \alpha_4^+ (\bar{\partial} A_+ - \partial \bar{A}_-) + \alpha_5^+ (\partial \bar{A}_+ - \bar{\partial} A_-) \right] \\
&+ \delta'(u) \left[ \alpha_8^+ (A_{u+} + A_{u-}) - \alpha_{10}^+ [(\partial_u A_u)_+ - (\partial_u A_u)_-] \right] \\
&+ \delta''(u) \alpha_{10}^+ (A_{u+} - A_{u-}), \quad (3.38)
\end{aligned}$$

which, once integrated and taking into account the axial gauge condition  $A_u = 0$ , yields the residual Ward identity

$$\begin{aligned}
&\int du (\partial \bar{J} + \bar{\partial} J) = \alpha_1 (\partial \bar{A}_+ + \partial \bar{A}_- + \bar{\partial} A_+ + \bar{\partial} A_-) + \\
&+ \alpha_3^- \left[ \partial (\partial_u \bar{A})_+ - \bar{\partial} (\partial_u A)_- \right] + \alpha_3^+ \left[ \partial (\partial_u \bar{A})_- - \bar{\partial} (\partial_u A)_+ \right]. \quad (3.39)
\end{aligned}$$

In the previous Section, we have seen that, in CS theory with boundary, it is the integrated residual Ward identity (3.39) which implies the existence of an algebra of chiral conserved currents on the boundary. Our aim will now be to study the change of the boundary conditions induced by the additional Maxwell term on the fields, and the consequences on the algebra: indeed, it is not obvious that there will be still an algebra of boundary currents.

### 3.3 The boundary conditions I: the set up

We now infer the boundary conditions on the fields, following the method illustrated in the previous Section for the pure CS case.

The first step is to set the sources to zero and integrate the broken equations of motion between the two sides of the boundary, thus getting the conditions

$$\begin{aligned} & k(\bar{A}_- - \bar{A}_+) + i\alpha \int_{-\varepsilon}^{\varepsilon} du \bar{\partial}(\partial\bar{A} - \bar{\partial}A) - i\alpha \left[ (\partial_u \bar{A})_- - (\partial_u \bar{A})_+ \right] = \\ & = \alpha_1(\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \end{aligned} \quad (3.40)$$

$$\begin{aligned} & k(A_+ - A_-) + i\alpha \int_{-\varepsilon}^{\varepsilon} du \partial(\bar{\partial}A - \partial\bar{A}) - i\alpha \left[ (\partial_u A)_- - (\partial_u A)_+ \right] = \\ & = \alpha_1(A_+ + A_-) - \alpha_3^+ (\partial_u A)_+ - \alpha_3^- (\partial_u A)_- \end{aligned} \quad (3.41)$$

$$\begin{aligned} & k \int_{-\varepsilon}^{\varepsilon} du (\partial\bar{A} - \bar{\partial}A) - i\alpha (\bar{\partial}A_+ - \bar{\partial}A_- + \partial\bar{A}_+ - \partial\bar{A}_-) + \int_{-\varepsilon}^{\varepsilon} du b = \\ & \alpha_4^+ (\bar{\partial}A_+ - \partial\bar{A}_-) + \alpha_5^+ (\partial\bar{A}_+ - \bar{\partial}A_-) , \end{aligned} \quad (3.42)$$

where we have also used the axial gauge condition  $A_u = 0$ . Notice that (3.41) can be obtained from the first one by means of the parity transformation (2.11). Therefore, we can consider only (3.40) and then apply parity to the resulting conditions to get the analogous ones generated by (3.41). Taking the limit  $\varepsilon \rightarrow 0$  in (3.40) and (3.42) we get the two conditions

$$\begin{aligned} & (\alpha_1 + k)\bar{A}_+ + (\alpha_3^- - i\alpha) (\partial_u \bar{A})_+ = \\ & = -(\alpha_1 - k)\bar{A}_- - (\alpha_3^+ + i\alpha) (\partial_u \bar{A})_- \end{aligned} \quad (3.43)$$

$$\begin{aligned} & (\alpha_4^+ + i\alpha) \partial\bar{A}_- + (\alpha_5^+ + i\alpha) \bar{\partial}A_- = \\ & = (\alpha_4^+ + i\alpha) \bar{\partial}A_+ + (\alpha_5^+ + i\alpha) \partial\bar{A}_+ . \end{aligned} \quad (3.44)$$

Let us focus on (3.44) first. As usual, separability splits it into

$$(\alpha_4^+ + i\alpha) \bar{\partial}A_+ + (\alpha_5^+ + i\alpha) \partial\bar{A}_+ = 0 \quad (3.45)$$

$$(\alpha_4^+ + i\alpha) \partial\bar{A}_- + (\alpha_5^+ + i\alpha) \bar{\partial}A_- = 0 . \quad (3.46)$$

Since the parameters  $\alpha_4^+$  and  $\alpha_5^+$  do not appear in the Ward identity (3.39), the boundary algebra does not depend on them, and hence we can solve (3.45) and (3.46) by choosing :

$$\alpha_4^+ = \alpha_5^+ \neq -i\alpha , \quad (3.47)$$

so that (3.45) and (3.46) yield

$$\bar{\partial}A_+ + \partial\bar{A}_+ = 0 \quad (3.48)$$

$$\bar{\partial}A_- + \partial\bar{A}_- = 0 , \quad (3.49)$$

which, as in the pure CS case, can be interpreted as conservation relations for currents living on the right, respectively on the left of the boundary.

On the other hand, the request for separability splits (3.43) into

$$(\alpha_1 + k) \bar{A}_+ + (\alpha_3^- - i\alpha) (\partial_u \bar{A})_+ = 0 \quad (3.50)$$

$$(\alpha_1 - k) \bar{A}_- + (\alpha_3^+ + i\alpha) (\partial_u \bar{A})_- = 0 . \quad (3.51)$$

Using the parity transformation (2.11) on these conditions we get the analogous ones generated by (3.41):

$$(\alpha_1 + k) A_- - (\alpha_3^- - i\alpha) (\partial_u A)_- = 0 \quad (3.52)$$

$$(\alpha_1 - k) A_+ - (\alpha_3^+ + i\alpha) (\partial_u A)_+ = 0 . \quad (3.53)$$

Since the broken equations of motion contain second-order derivatives, the second step towards the determination of the boundary conditions on the quantum fields, consists into integrating them twice.

Let us set the sources to zero and then integrate (3.34):

$$\begin{aligned} & \int_{-\infty}^u du' \left\{ k(\bar{\partial} A_u - \partial_u \bar{A}) + i\alpha [\bar{\partial}(\partial \bar{A} - \bar{\partial} A) + \partial_u(\partial_u \bar{A} - \bar{\partial} A_u)] + \bar{J} \right\} = \\ & = \int_{-\infty}^u du' \delta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \right] \\ & + \int_{-\infty}^u du' \delta'(u) (\alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_-) ; \end{aligned} \quad (3.54)$$

from this we will get also the conditions generated by (3.35) by means of the parity transformation (2.11), like we have done above. From (3.54) we get

$$\begin{aligned} & -k\bar{A} + i\alpha \int_{-\infty}^u du' \bar{\partial}(\partial \bar{A} - \bar{\partial} A) + i\alpha \partial_u \bar{A} = \\ & = \theta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \right] \\ & + \delta(u) (\alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_-) , \end{aligned} \quad (3.55)$$

where we have assumed that the fields and their derivatives vanish at infinity. We then integrate (3.55) between the two sides of the (infinitesimal) boundary:

$$\begin{aligned} & - \int_{-\varepsilon}^{\varepsilon} du k\bar{A} + i\alpha \int_{-\varepsilon}^{\varepsilon} du \int_{-\infty}^u du' \bar{\partial}(\partial \bar{A} - \bar{\partial} A) + i\alpha \int_{-\varepsilon}^{\varepsilon} du \partial_u \bar{A} = \\ & = \int_{-\varepsilon}^{\varepsilon} du \theta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \right] \\ & + \int_{-\varepsilon}^{\varepsilon} du \delta(u) (\alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_-) ; \end{aligned} \quad (3.56)$$

taking the limit  $\varepsilon \rightarrow 0$  we finally get

$$-i\alpha\bar{A}_- + i\alpha\bar{A}_+ = \alpha_3^+\bar{A}_+ + \alpha_3^-\bar{A}_- , \quad (3.57)$$

which can be rearranged as

$$-(i\alpha + \alpha_3^-)\bar{A}_- = (-i\alpha + \alpha_3^+)\bar{A}_+ . \quad (3.58)$$

The request for separability again splits this condition into two separate ones for the opposite sides of the boundary:

$$(i\alpha + \alpha_3^-)\bar{A}_- = 0 \quad (3.59)$$

$$(-i\alpha + \alpha_3^+)\bar{A}_+ = 0 ; \quad (3.60)$$

Applying the parity transformation (2.11) to these conditions we get those generated by (3.35)

$$(i\alpha + \alpha_3^-)A_+ = 0 \quad (3.61)$$

$$(-i\alpha + \alpha_3^+)A_- = 0 . \quad (3.62)$$

At the end, we have obtained a set of eight boundary conditions involving both the fields and their first derivatives, which must be simultaneously satisfied.

Summarizing:

$$(\alpha_1 + k)\bar{A}_+ + (\alpha_3^- - i\alpha)(\partial_u\bar{A})_+ = 0 \quad (3.63)$$

$$(\alpha_1 - k)\bar{A}_- + (\alpha_3^+ + i\alpha)(\partial_u\bar{A})_- = 0 \quad (3.64)$$

$$(\alpha_1 + k)A_- - (\alpha_3^- - i\alpha)(\partial_u A)_- = 0 \quad (3.65)$$

$$(\alpha_1 - k)A_+ - (\alpha_3^+ + i\alpha)(\partial_u A)_+ = 0 \quad (3.66)$$

$$(i\alpha + \alpha_3^-)\bar{A}_- = 0 \quad (3.67)$$

$$(-i\alpha + \alpha_3^+)\bar{A}_+ = 0 \quad (3.68)$$

$$(i\alpha + \alpha_3^-)A_+ = 0 \quad (3.69)$$

$$(-i\alpha + \alpha_3^+)A_- = 0 , \quad (3.70)$$

which must be considered together with the conservation condition (3.47):

$$\alpha_4^+ = \alpha_5^+ \neq -i\alpha \quad (3.71)$$

$$\bar{\partial}A_+ + \partial\bar{A}_+ = 0 \quad (3.72)$$

$$\bar{\partial}A_- + \partial\bar{A}_- = 0 . \quad (3.73)$$

We are now left with the task of solving the above constraints, thus identifying all the possible choices for the parameters and their consequences on the boundary fields and the residual Ward identity.

### 3.4 The boundary conditions II: solution(s) and algebra(s)

The first natural request is to recover the CS result in the limit  $\alpha \rightarrow 0$ . This implies a dependence of the parameter  $\alpha_1$  on  $\alpha$  of the type

$$\alpha_1 = \pm k (1 + 2f(\alpha)) , \quad \lim_{\alpha \rightarrow 0} f(\alpha) = 0 \quad (3.74)$$

but, as we will show later, the case  $f(\alpha) \neq 0$  is incompatible with the conditions (3.72) and (3.73) expressing the conservation condition. Therefore, the choice

$$\alpha_1 = \pm k \quad (3.75)$$

is compulsory. Moreover, the case  $\alpha_1 = -k$  can be obtained from  $\alpha_1 = k$  by means of the changes

$$\bar{A}_\pm \rightarrow \bar{A}_\mp \quad (3.76)$$

$$A_\pm \rightarrow A_\mp \quad (3.77)$$

$$(\partial_u \bar{A})_\pm \rightarrow (\partial_u \bar{A})_\mp \quad (3.78)$$

$$(\partial_u A)_\pm \rightarrow (\partial_u A)_\mp \quad (3.79)$$

$$k \rightarrow -k \quad (3.80)$$

$$\alpha_3^\pm \rightarrow -\alpha_3^\mp , \quad (3.81)$$

as it can be directly checked by comparison with (3.63)-(3.70). Therefore, we can focus on the eight cases with  $\alpha_1 = k$ . Reminding that the boundary algebra is determined by the residual Ward identity (3.39), for each solution we write the boundary conditions on the fields and the corresponding Ward identity:

1.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = -i\alpha \\ \alpha_3^- = +i\alpha \end{cases} \Rightarrow A_\pm = \bar{A}_\pm = 0$$

$$\begin{aligned} \int du (\partial \bar{J} + \bar{\partial} J) &= i\alpha \partial \left[ (\partial_u \bar{A})_+ - (\partial_u \bar{A})_- \right] \\ &+ i\alpha \bar{\partial} \left[ (\partial_u A)_+ - (\partial_u A)_- \right] ; \end{aligned} \quad (3.82)$$

2.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = -i\alpha \\ \alpha_3^- = -i\alpha \end{cases} \Rightarrow \begin{cases} \bar{A}_+ = 0 = A_- \\ (\partial_u \bar{A})_+ = 0 = (\partial_u A)_- \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = k (\partial \bar{A}_- + \bar{\partial} A_+) + i\alpha \left[ \bar{\partial} (\partial_u A)_+ - \partial (\partial_u \bar{A})_- \right] ; \quad (3.83)$$

3.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = +i\alpha \\ \alpha_3^- = -i\alpha \end{cases} \Rightarrow \begin{cases} (\partial_u A)_+ = (\partial_u \bar{A})_- = 0 \\ k\bar{A}_+ - i\alpha (\partial_u \bar{A})_+ = 0 \\ kA_- + i\alpha (\partial_u A)_- = 0 \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = k (\partial \bar{A}_- + \bar{\partial} A_+) ; \quad (3.84)$$

4.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = -i\alpha \\ \alpha_3^- \neq \mp i\alpha \end{cases} \Rightarrow \begin{cases} \bar{A}_\pm = 0 = A_\pm \\ (\partial_u \bar{A})_+ = 0 = (\partial_u A)_- \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = i\alpha [\bar{\partial} (\partial_u A)_+ - \partial (\partial_u \bar{A})_-] ; \quad (3.85)$$

5.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ \neq -i\alpha \\ \alpha_3^- = +i\alpha \end{cases} \Rightarrow \begin{cases} \bar{A}_\pm = 0 = A_\pm \\ (\partial_u A)_+ = 0 = (\partial_u \bar{A})_- \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = i\alpha [\partial (\partial_u \bar{A})_+ - \bar{\partial} (\partial_u A)_-] ; \quad (3.86)$$

6.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = +i\alpha \\ \alpha_3^- \neq \mp i\alpha \end{cases} \Rightarrow \begin{cases} A_+ = \bar{A}_- = (\partial_u A)_+ = (\partial_u \bar{A})_- = 0 \\ 2k\bar{A}_+ + (-i\alpha + \alpha_3^-) (\partial_u \bar{A})_+ = 0 \\ 2kA_- - (-i\alpha + \alpha_3^-) (\partial_u A)_- = 0 \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = k \frac{i\alpha + \alpha_3^-}{i\alpha - \alpha_3^-} (\partial \bar{A}_+ + \bar{\partial} A_-) ; \quad (3.87)$$

7.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ \neq \mp i\alpha \\ \alpha_3^- = -i\alpha \end{cases} \Rightarrow \begin{cases} \bar{A}_+ = A_- = 0 \\ (\partial_u \bar{A})_\pm = (\partial_u A)_\pm = 0 \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = k (\partial \bar{A}_- + \bar{\partial} A_+) ; \quad (3.88)$$

8.

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ \neq \mp i\alpha \\ \alpha_3^- \neq \mp i\alpha \end{cases} \Rightarrow \begin{cases} A_\pm = \bar{A}_\pm = 0 \\ (\partial_u A)_\pm = (\partial_u \bar{A})_\pm = 0 \end{cases}$$

$$\int du (\partial \bar{J} + \bar{\partial} J) = 0 . \quad (3.89)$$



However, not all these solutions of the system (3.63)-(3.70) are acceptable. Indeed, we must still keep in mind that taking the limit  $\alpha \rightarrow 0$  we must recover the result found for the CS theory.

This condition eliminates the 1., 4., 5. and 8. Furthermore, the conservation relations (3.72) and (3.73), together with the boundary conditions, imply that the Ward identity for the cases 6.) and 7.) becomes

$$\int du (\partial\bar{J} + \bar{\partial}J) = 0 , \quad (3.90)$$

while for the case 2.) it takes the form

$$\int du (\partial\bar{J} + \bar{\partial}J) = +i\alpha \left[ \bar{\partial}(\partial_u A)_+ - \partial(\partial_u \bar{A})_- \right] . \quad (3.91)$$

Recalling the form (2.51) of the Ward identity for pure CS theory with boundary, these cases do not have the correct limit  $\alpha \rightarrow 0$  either, and therefore they are forbidden as well.

At the end, we are left only with the case 3.), which corresponds to *Neumann* and *Robin* boundary conditions

$$\begin{cases} \alpha_1 = k \\ \alpha_3^+ = +i\alpha \\ \alpha_3^- = -i\alpha \\ \alpha_4^+ = \alpha_5^+ \neq -ik \end{cases} \quad (3.92)$$

which yields on the rhs of the boundary

$$\begin{cases} (\partial_u A)_+ = 0 \\ k\bar{A}_+ - i\alpha (\partial_u \bar{A})_+ = 0 \\ \bar{\partial}A_+ + \partial\bar{A}_+ = 0 \end{cases} \quad (3.93)$$

$$\int du (\partial\bar{J} + \bar{\partial}J) = k (\partial\bar{A}_- + \bar{\partial}A_+) . \quad (3.94)$$

The Ward identity (3.94) is the same as in the CS case, and it implies both the chirality

$$\bar{\partial}A_+ = 0 \quad (3.95)$$

and the boundary Kač-Moody algebra

$$[A_+(z), A_+(z')] = \frac{1}{k} \partial\delta(z - z') \quad (3.96)$$

involving the conserved chiral current  $A_+(z)$ .

The corresponding solution on the lhs of the boundary can be obtained by parity, and read

$$\begin{cases} (\partial_u \bar{A})_- = 0 \\ kA_- + i\alpha (\partial_u A)_- = 0 \\ \bar{\partial}A_- + \partial\bar{A}_- = 0 \end{cases} \quad (3.97)$$

$$\partial\bar{A}_- = 0 \quad (3.98)$$

$$[\bar{A}_-(\bar{z}), \bar{A}_-(\bar{z}')] = \frac{1}{k}\bar{\partial}\delta(\bar{z} - \bar{z}') \quad (3.99)$$

involving the conserved chiral current  $\bar{A}_-(\bar{z})$ .

This is a remarkable result, since we have found not only that the boundary terms are uniquely fixed, but also that there is still a Kač-Moody algebra, which is *the same* as in the CS case. In other words, the Maxwell term does not affect the boundary physics of the light-cone currents.

This result is in agreement with the outcomes of [27] and [28], where different approaches have been adopted, and a disk instead of a plane has been considered.

More recent approaches, like [39] just define one boundary (say on the right) and formulate the local boundary interaction by means of a lagrangian or an algebraic way. The introduction of two boundaries or defects/impurities can be done as in [40].

The claim of [27] and [28] is that the boundary is characterized by Kač-Moody algebras of conserved charges with the same structure of the CS model, thus concluding that the algebra is a consequence of the CS term rather than of the nature of the entire model. We refer to our conclusions for a more precise comparison.

We conclude this Chapter by illustrating the reason why the parameter  $\alpha_1$  must be set to the value  $k$ . As we previously pointed out, the request that the limit  $\alpha \rightarrow 0$  lead to the CS result implies (3.74). In such a case, it can be checked that the only acceptable solution of the system (3.63)-(3.70) is

$$\left\{ \begin{array}{l} \alpha_1 = k(1 + 2f(\alpha)) \\ \alpha_3^+ = +i\alpha \\ \alpha_3^- = -i\alpha \\ \alpha_4^+ = \alpha_5^+ \neq -i\alpha \end{array} \right. \Rightarrow \left\{ \begin{array}{l} k(1 + f(\alpha))\bar{A}_+ - i\alpha(\partial_u\bar{A})_+ = 0 \\ k(1 + f(\alpha))A_- + i\alpha(\partial_u A)_- = 0 \\ kf(\alpha)A_+ - i\alpha(\partial_u A)_+ = 0 \\ kf(\alpha)\bar{A}_- + i\alpha(\partial_u\bar{A})_- = 0 \end{array} \right. ,$$

which corresponds to the broken residual Ward identity

$$\int du (\partial\bar{J} + \bar{\partial}J) = kf(\alpha) (\partial\bar{A}_+ + \bar{\partial}A_-) + k(1 + f(\alpha)) (\partial\bar{A}_- + \bar{\partial}A_+) . \quad (3.100)$$

Equation (3.100), with the external sources set to zero, together with separability, implies the relations

$$kf(\alpha)\partial\bar{A}_+ + k(1 + f(\alpha))\bar{\partial}A_+ = 0 \quad (3.101)$$

$$kf(\alpha)\bar{\partial}A_- + k(1 + f(\alpha))\partial\bar{A}_- = 0 , \quad (3.102)$$

which are incompatible with the existence of conserved quantities as expressed by (3.72) and (3.73) unless  $f(\alpha) = 0$ . Notice that this argument

shows also that the request that the boundary currents be conserved automatically implies their chirality.

## 4 Conclusions

In this paper the three dimensional Maxwell-Chern-Simons (MCS) theory with planar boundary has been considered, following the pioneering work of Symanzik [1].

We adopted a method similar to the one introduced in [24], which avoids the problem of regularizing the boundary action, which is necessarily  $\delta$ -function dependent, but we modified it in a significant manner. In fact, one of the main steps towards the realization of the Symanzik's constraint of separability, which implements the introduction of a boundary in quantum field theory, is the direct computation of the propagators of the theory, taking into account, of course, the boundary interactions. While this is quite feasible in CS theory, in other more complicated, and physically more relevant cases – of which the MCS model is an example – this could be a much more difficult task. In fact, we have been able to reach the same goals, as in the simpler case of pure CS theory, without a direct computation of the propagators of the theory.

As already pointed out, we stress that the addition of the Maxwell term to the CS action spoils its topological character, and thus it is not at all granted that the same results can be obtained.

Indeed, we found that, under certain conditions,

- 1. chiral conserved currents living on the boundary exist,**
- 2. which satisfy, like in the CS case, a Kač-Moody algebra, whose central charge coincides with the inverse of the CS coupling constant.**

In particular, we have found that there are many possible boundary terms compatible with Symanzik's constraint of separability, each associated with different boundary conditions on the fields. However, the requirement to recover the CS case in the limit of vanishing Maxwell term, uniquely sets the boundary parameters to values which imply *Neumann and Robin boundary conditions* on the fields on both sides of the boundary. In this situation, the same residual Ward identity of the CS case holds, thus implying the results 1. and 2. above mentioned.

This result leads us to conclude that the boundary physics is independent of the Maxwell term which breaks the topological character of the CS model, even though the Maxwell boundary contributions to the equations of motion are nontrivial. This is a remarkable result, since it suggests that the existence

of a boundary Kač-Moody algebra of conserved chiral currents depends only on the presence of a CS term in the theory, rather than on the particular theory itself.

We point out that our main results hold also for the nonabelian model, namely the Yang-Mills-Chern-Simons action. Indeed, the conservation of boundary currents, their chirality and the central charge of their Kač-Moody algebra are entirely determined by the quadratic part of the action only, and therefore are shared with the nonabelian extension, which affects the vertices of the theory.

Our result, concerning the existence of a Kač-Moody boundary algebra with the same structure of the CS model, somehow agrees with Deser's recent statement that, for spin one and in three spacetime dimensions, "*everything is CS*" [41]. In Ref. [41], in fact, he shows how a Yang-Mills-Chern-Simons (YMCS) action can be rewritten in the form of a pure CS action in terms of a new field.

But, even in the earlier work [42], three dimensional Yang-Mills gauge theories in the presence of the Chern-Simons action were shown to be generated by the pure topological Chern-Simons term through nonlinear covariant redefinitions of the gauge field. And indeed these claims are supported by our results, which do not depend on the presence of a bulk Maxwell term.

Another point of contact is with the outcome of [22] for the (2+1) dimensional black hole coupled with electrodynamics, where it is shown that the black hole coupled to the pure CS theory does not change configuration when a Maxwell term is "*turned on*".

In the framework of the Fractional Quantum Hall Effect, our result justifies the fact that an additional Maxwell term to the pure CS effective low energy model describes the gap of the elementary bulk excitations, without affecting the properties of the edge states [5].

Finally, we are in agreement also with [27] and [28], where the nonabelian YMCS theory defined on a manifold with boundary is considered in the Weyl gauge. In Ref. [27], a (2+1) dimensional disk is considered, and the Dirac's procedure is followed. The boundary Kač-Moody algebra is then obtained as the projection of the Dirac's brackets on the boundary of the disk.

In [28] the discussion is instead carried out in terms of the Gauss law generators, and the algebra in terms of the fields is just a consequence. Nevertheless, they both find that also in the YMCS theory there is a boundary algebra, which is the same as that of the CS model.

However, our approach is not only simpler, but also stays in a more general framework. In fact, in both [27] and [28] the boundary conditions are chosen *a priori*, provided that they satisfy some general requirements, while we find all possible boundary conditions compatible with the simple and fundamental requests of locality and separability, without requiring any other constraint.

Moreover, in our description the boundary is the most general one compatible with very general principles - like locality, power counting and helicity conservation - and with the algebraic structure of the theory.

We conclude our discussion by indicating other possible further developments.

In fact, the three dimensional CS theory is not the only possible topological field theory. The other important Schwarz type topological field theory [8] is represented by the BF models [13], which, contrarily to the CS theory, which is intrinsically three dimensional, exist in an arbitrary number of dimensions.

In three dimensions, the BF model with boundary has been studied in [14], where a richer algebraic structure than the CS case has been found, always of the Kač-Moody type.

A natural possible extension could therefore be the study of the effect of a  $(d-1)$  dimensional boundary in the  $d$ -dimensional BF theory. The investigation of the possible boundary algebra in these cases is a challenge.

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