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Online Expectation Maximization based algorithms for inference in Hidden Markov Models

Sylvain Le Corff*[†] and Gersende Fort[†]

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This is a supplementary material to the paper [8].

It contains technical discussions and/or results adapted from published papers: in section 2, we show that geometrically ergodic Markov chains satisfy the assumptions H???? and H??; in sections 3 and 4, we provide results - useful for the proofs of some theorems in [8] - which are close to existing results in the literature.

It also contains in Section 5, additional plots for the numerical analyses in [8, Section 3].

To make this supplement paper as self-contained as possible, we decided to rewrite in Section 1 the model and the main definitions introduced in [8].

1 Assumptions and Model

Let $\mathbf{Y} = \{Y_t\}_{t \in \mathbb{Z}}$ be the observation process defined on $(\Omega, \mathbb{P}_*, \mathcal{F})$ and taking values in $\mathbb{Y}^{\mathbb{Z}}$ where \mathbb{Y} is a general space endowed with a countably generated σ -field $\mathcal{B}(\mathbb{Y})$.

A HMM model parameterized by θ , for θ in a set $\Theta \subseteq \mathbb{R}^{d_\theta}$, is fitted to the observations: consider a family of transition kernels $\{m_\theta(x, x') d\lambda(x')\}_{\theta \in \Theta}$ onto $\mathbb{X} \times \mathcal{B}(\mathbb{X})$ where \mathbb{X} is a general state-space equipped with a countably generated σ -field $\mathcal{B}(\mathbb{X})$, and λ is a bounded non-negative measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Let $\{g_\theta(x, y) d\nu(y)\}_{\theta \in \Theta}$ be a family of transition kernels on $(\mathbb{X} \times \mathcal{B}(\mathbb{Y}))$, where ν is a measure on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$.

It is assumed:

- H1** (a) There exist continuous functions $\phi : \Theta \rightarrow \mathbb{R}$, $\psi : \Theta \rightarrow \mathbb{R}^d$ and $S : \mathbb{X} \times \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}^d$ s.t.

$$\log m_\theta(x, x') + \log g_\theta(x', y) = \phi(\theta) + \langle S(x, x', y), \psi(\theta) \rangle ,$$

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where $\langle \cdot, \cdot \rangle$ denotes the scalar product on \mathbb{R}^d .

- (b) There exists an open subset \mathcal{S} of \mathbb{R}^d that contains the convex hull of $S(\mathbb{X} \times \mathbb{X} \times \mathbb{Y})$.
- (c) There exists a continuous function $\bar{\theta} : \mathcal{S} \rightarrow \Theta$ s.t. for any $s \in \mathcal{S}$,

$$\bar{\theta}(s) = \operatorname{argmax}_{\theta \in \Theta} \{ \phi(\theta) + \langle s, \psi(\theta) \rangle \} .$$

H2 There exist σ_- and σ_+ s.t. for any $(x, x') \in \mathbb{X}^2$ and any $\theta \in \Theta$, $0 < \sigma_- \leq m_\theta(x, x') \leq \sigma_+$. Set $\rho \stackrel{\text{def}}{=} 1 - (\sigma_-/\sigma_+)$.

H3-(γ) $\mathbb{E}_* [\sup_{x, x' \in \mathbb{X}^2} |S(x, x', \mathbf{Y}_0)|^\gamma] < +\infty$.

H4 (a) Under \mathbb{P}_* , \mathbf{Y} is a stationary sequence.

(b) The shift operator is ergodic with respect to \mathbb{P}_* .

(c) $\mathbb{E}_* [|\log b_-(\mathbf{Y}_0)| + |\log b_+(\mathbf{Y}_0)|] < +\infty$ where

$$b_-(y) \stackrel{\text{def}}{=} \inf_{\theta \in \Theta} \int g_\theta(x, y) \lambda(dx) , \quad (1)$$

$$b_+(y) \stackrel{\text{def}}{=} \sup_{\theta \in \Theta} \int g_\theta(x, y) \lambda(dx) . \quad (2)$$

For any sequence of r.v. $Z \stackrel{\text{def}}{=} \{Z_t\}_{t \in \mathbb{Z}}$ on $(\Omega, \tilde{\mathbb{P}}, \mathcal{F})$, let

$$\mathcal{F}_k^Z \stackrel{\text{def}}{=} \sigma(\{Z_u\}_{u \leq k}) \quad \text{and} \quad \mathcal{G}_k^Z \stackrel{\text{def}}{=} \sigma(\{Z_u\}_{u \geq k})$$

be σ -fields associated to Z . We also define the mixing coefficients by, see [4],

$$\beta^Z(n) = \sup_{u \in \mathbb{Z}} \sup_{B \in \mathcal{G}_{u+n}^Z} |\tilde{\mathbb{P}}(B|\mathcal{F}_u^Z) - \tilde{\mathbb{P}}(B)| , \forall n \geq 0 . \quad (3)$$

H5 There exist $C \in [0, 1)$ and $\beta \in (0, 1)$ s.t. for any $n \geq 0$, $\beta^{\mathbf{Y}}(n) \leq C\beta^n$, where $\beta^{\mathbf{Y}}$ is defined in (3).

H6-(γ) The block size sequence $\{\tau_n\}_{n \geq 1}$ satisfies $\sum_{k \geq 0} \tau_k^{-\gamma/2} < \infty$.

Define for any $\theta \in \Theta$,

$$\bar{S}(\theta) \stackrel{\text{def}}{=} \mathbb{E}_* [\mathbb{E}_\theta [S(X_{-1}, X_0, \mathbf{Y}_0) | \mathbf{Y}]] . \quad (4)$$

$$R(\theta) \stackrel{\text{def}}{=} \bar{\theta}(\bar{S}(\theta)) . \quad (5)$$

$$G(s) \stackrel{\text{def}}{=} \bar{S}(\bar{\theta}(s)) , \quad \forall s \in \mathcal{S} , \quad (6)$$

where $\bar{\theta}$ is given by H1(c).

H7 (a) \bar{S} and $\bar{\theta}$ are twice continuously differentiable on Θ and \mathcal{S} .

(b) There exists $0 < \gamma < 1$ s.t. $\operatorname{sp}(\nabla_s(\bar{S} \circ \bar{\theta})_{s=\bar{S}(\theta_*)}) \leq \gamma$ where sp denotes the spectral norm.

Note that under H7, $\text{sp}(\Gamma) < \gamma$, where $\Gamma \stackrel{\text{def}}{=} \nabla G(s_*)$ and $s_* = \bar{S}(\theta_*)$. Set

$$T_n \stackrel{\text{def}}{=} \sum_{i=1}^n \tau_i, \quad T_0 \stackrel{\text{def}}{=} 0.$$

- H8** (a) $\{\tau_{n+1}/\tau_n\}_{n \geq 0}$ converges to q and $\gamma q < 1$.
(b) $\limsup_n \sum_{k=1}^n \left\{ \left| \frac{\tau_{k+1}}{\tau_k} - q \right| \sqrt{\tau_k} + \log \tau_k \right\} / \sqrt{T_n} < \infty$.

2 Checking H4(b) and H5

2.1 Checking H4(b)

The following discussion has been suggested by R. Douc¹ and E. Moulines². The authors would like to thank them for the fruitful discussions.

Assumption H4(b) can be easily proved when \mathbf{Y} is a positive recurrent and ψ -irreducible Markov chain. Assume first that \mathbf{Y} is a one-sided Markov chain $\{\mathbf{Y}_t\}_{t \geq 0}$ with invariant probability π defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\pi)$. We may choose $\Omega \stackrel{\text{def}}{=} \mathbb{Y}^{\mathbb{N}}$ and $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{B}(\mathbb{Y})^{\otimes \mathbb{N}}$, \mathbf{Y} being the canonical process. To prove H4(b), we show that \mathbf{Y} is mixing (see [4, Chapter 13]) i.e.

$$\lim_{k \rightarrow +\infty} \mathbb{P}_\pi \{ \mathbf{Y} \in A, \vartheta^k(\mathbf{Y}) \in B \} = \mathbb{P}_\pi \{ \mathbf{Y} \in A \} \mathbb{P}_\pi \{ \mathbf{Y} \in B \}, \forall A, B \in \mathcal{B}(\mathbb{Y})^{\otimes \mathbb{N}}. \quad (7)$$

This is sufficient to prove (7) when A is a cylinder, i.e. when there exists $p \in \mathbb{N}$ such that $A = \{ \omega \stackrel{\text{def}}{=} \{w_i\}_{i \geq 0} \in \mathbb{Y}^{\mathbb{N}}; (w_{i_1}, \dots, w_{i_p}) \in H \}$, where $H \in \mathcal{B}(\mathbb{Y})^{\otimes p}$ and (i_1, \dots, i_p) is a p -tuple of distinct non-negative integers. For all sufficiently large k , by the Markov property,

$$\begin{aligned} \mathbb{P}_\pi \{ \mathbf{Y} \in A, \vartheta^k(\mathbf{Y}) \in B \} &= \mathbb{E}_\pi \left[\mathbf{1}_A(\mathbf{Y}) \mathbf{1}_B(\vartheta^k(\mathbf{Y})) \right] \\ &= \mathbb{E}_\pi \left[\mathbf{1}_A(\mathbf{Y}) \mathbb{E}_{\mathbf{Y}_{i_p}} \left[\mathbb{E}_{\mathbf{Y}_{k-i_p}} \left[\mathbf{1}_B(\mathbf{Y}) \right] \right] \right]. \end{aligned}$$

Under the stated assumption on \mathbf{Y} , we can choose \mathbb{Y}_0 such that $\pi(\mathbb{Y}_0) = 1$ and for any $x \in \mathbb{Y}_0$, $\mathbb{E}_x \left[\mathbb{E}_{\mathbf{Y}_{k-i_p}} \left[\mathbf{1}_B(\mathbf{Y}) \right] \right] \xrightarrow{k \rightarrow +\infty} \mathbb{P}_\pi \{ B \}$. The proof is then concluded by the dominated convergence theorem. This result implies that if \mathbf{Y} is a two sided Markov chain $\{\mathbf{Y}_t\}_{t \in \mathbb{Z}}$, with a positive recurrent and ψ -irreducible transition kernel, then \mathbf{Y} is ergodic. Indeed, \mathbb{P}_π may be extended on $(\mathbb{Y}^{\otimes \mathbb{Z}}, \mathcal{B}(\mathbb{Y})^{\otimes \mathbb{Z}})$ where $\mathcal{B}(\mathbb{Y})^{\otimes \mathbb{Z}}$ is generated by $\cup_{i < j \in \mathbb{Z}} \mathcal{F}_{i,j}$, with, for any integers $i < j$, $\mathcal{F}_{i,j} \stackrel{\text{def}}{=} \sigma(\{\mathbf{Y}_k; i \leq k \leq j\})$. For any integers i, j and k s.t. $i < j$ and any $A, B \in \mathcal{F}_{i,j}$,

$$\mathbb{P}_\pi \{ A \cap \vartheta^{-k}(B) \} = \mathbb{P}_\pi \{ \vartheta^{-i}(A) \cap \vartheta^{-k}(\vartheta^{-i}(B)) \},$$

where $\vartheta^{-i}(A), \vartheta^{-i}(B) \in \sigma(\{\mathbf{Y}_k; k \geq 0\})$. Then, we can conclude as above.

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2.2 Checking H5

- 1) Combining [11, Chapter 9] and recent results on the control of the ergodicity of Markov chains by coupling technique, see [6], it can be proved that H5 holds for ψ -irreducible, aperiodic and geometrically ergodic Markov chains.
- 2) Upon noting that, for all $n \geq 0$, $\beta^{\mathbf{Y}}(n) \leq \beta^{(\mathbf{X}, \mathbf{Y})}(n)$, we can similarly prove that H4(b) (and H5) hold when \mathbf{Y} is the observation process of a joint ψ -irreducible and aperiodic Markov chain (\mathbf{X}, \mathbf{Y}) (e.g. (\mathbf{X}, \mathbf{Y}) is a hidden Markov model). In this case, irreducibility, aperiodicity and geometric ergodicity have to be established for the Markov transition kernel Q of the $\mathbb{X} \times \mathbb{Y}$ valued Markov chain $\{(X_t, Y_t)\}_{t \in \mathbb{Z}}$.

3 Detailed proofs of [8]

Recall the following definition from [8]: for a distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, positive integers T, τ and $\theta \in \Theta$, set

$$\bar{S}_\tau^{\chi, T}(\theta, \mathbf{Y}) \stackrel{\text{def}}{=} \frac{1}{\tau} \sum_{t=T+1}^{T+\tau} \Phi_{\theta, t, T+\tau}^{\chi, T}(S, \mathbf{Y}) , \quad (8)$$

where S is the function given by H1(a) and

$$\begin{aligned} & \Phi_{\theta, s, t}^{\chi, r}(S, \mathbf{y}) \\ \stackrel{\text{def}}{=} & \frac{\int \chi(dx_r) \{ \prod_{i=r}^{t-1} m_\theta(x_i, x_{i+1}) g_\theta(x_{i+1}, y_{i+1}) \} S(x_{s-1}, x_s, y_s) d\lambda(x_{r+1:t})}{\int \chi(dx_r) \{ \prod_{i=r}^{t-1} m_\theta(x_i, x_{i+1}) g_\theta(x_{i+1}, y_{i+1}) \} d\lambda(x_{r+1:t})} . \quad (9) \end{aligned}$$

3.1 Proof of [8, Theorem 4.4]

We check the assumptions of [7, Proposition 9] and [7, Proposition 11] with $T(\theta) \stackrel{\text{def}}{=} R(\theta)$ (see (5)), $F_n(\theta) \stackrel{\text{def}}{=} \bar{\theta} \left(\bar{S}_{\tau_{n+1}}^{\chi, T_n}(\theta_n, \mathbf{Y}) \right)$ and $\mathcal{L} \stackrel{\text{def}}{=} \{\theta \in \Theta; R(\theta) = \theta\}$. We start by checking the conditions of [7, Proposition 11]. Under the stated assumptions, (a) holds. For (c) we prove that for any compact subset $\mathcal{K} \subset \Theta$,

$$\left| W \circ R(\theta_n) - W \circ \bar{\theta}(\bar{S}_{\tau_{n+1}}^{\chi, T_n}(\theta_n, \mathbf{Y})) \right| \mathbf{1}_{\theta_n \in \mathcal{K}} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}_* - \text{a.s.} \quad (10)$$

By [8, Theorem 4.1], the function \bar{S} given by (4) is continuous on Θ and then $\bar{S}(\mathcal{K}) \stackrel{\text{def}}{=} \{s \in \mathcal{S}; \exists \theta \in \mathcal{K}, s = \bar{S}(\theta)\}$ is compact and, for any $\delta > 0$ (small enough), we can define the compact subset $\bar{S}(\mathcal{K}, \delta) \stackrel{\text{def}}{=} \{s \in \mathbb{R}^d; d(s, \bar{S}(\mathcal{K})) \leq \delta\}$ of \mathcal{S} , where $d(s, \bar{S}(\mathcal{K})) \stackrel{\text{def}}{=} \inf_{s' \in \bar{S}(\mathcal{K})} |s - s'|$. Let $\delta > 0$ (small enough) and $\varepsilon > 0$. Since $W \circ \bar{\theta}$ is continuous (see H1(c) and [8, Proposition 4.2]) and $\bar{S}(\mathcal{K}, \delta)$ is compact, $W \circ \bar{\theta}$ is uniformly continuous on $\bar{S}(\mathcal{K}, \delta)$ and there exists $\eta > 0$ s.t.,

$$\forall x, y \in \bar{S}(\mathcal{K}, \delta) , \quad |x - y| \leq \eta \Rightarrow |W \circ \bar{\theta}(x) - W \circ \bar{\theta}(y)| \leq \varepsilon . \quad (11)$$

Set $\alpha \stackrel{\text{def}}{=} \delta \wedge \eta$ and $\Delta S_n \stackrel{\text{def}}{=} |\bar{S}(\theta_n) - \bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})| \mathbf{1}_{\theta_n \in \mathcal{K}}$. We write,

$$\begin{aligned} & \mathbb{P}_\star \left\{ \left| \mathbb{W} \circ \bar{\theta}(\bar{S}(\theta_n)) - \mathbb{W} \circ \bar{\theta}(\bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})) \right| \mathbf{1}_{\theta_n \in \mathcal{K}} \geq \varepsilon \right\} \\ &= \mathbb{P}_\star \left\{ \left| \mathbb{W} \circ \bar{\theta}(\bar{S}(\theta_n)) - \mathbb{W} \circ \bar{\theta}(\bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})) \right| \mathbf{1}_{\theta_n \in \mathcal{K}} \geq \varepsilon; \Delta S_n > \delta \right\} \\ &\quad + \mathbb{P}_\star \left\{ \left| \mathbb{W} \circ \bar{\theta}(\bar{S}(\theta_n)) - \mathbb{W} \circ \bar{\theta}(\bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})) \right| \mathbf{1}_{\theta_n \in \mathcal{K}} \geq \varepsilon; \Delta S_n \leq \delta \right\} \\ &\leq \mathbb{P}_\star \{ \Delta S_n > \delta \} + \mathbb{P}_\star \{ \Delta S_n > \eta \} \leq 2\mathbb{P}_\star \{ \Delta S_n > \alpha \} . \end{aligned}$$

By the Markov inequality and [8, Proposition 6.5], since $2 < \bar{p}_1 < \bar{p}_2$, there exists a constant C s.t.

$$\begin{aligned} & \mathbb{P}_\star \left\{ \left| \mathbb{W} \circ \bar{\theta}(\bar{S}(\theta_n)) - \mathbb{W} \circ \bar{\theta}(\bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})) \right| \mathbf{1}_{\theta_n \in \mathcal{K}} \geq \varepsilon \right\} \\ &\leq \frac{2}{\alpha^{\bar{p}_1}} \mathbb{E}_\star \left[|\bar{S}(\theta_n) - \bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})|^{\bar{p}_1} \right] \leq C \left[\frac{1}{\tau_{n+1}} \right]^{\bar{p}_1/2} . \end{aligned}$$

(10) follows from H6-(\bar{p}_1) and the Borel-Cantelli lemma. The proof of the condition (b) follows the same lines. By [7, Proposition 11], this implies that $\limsup_n p_n < +\infty$ \mathbb{P}_\star -a.s and that $\{\theta_n\}_{n \geq 0}$ is a compact sequence \mathbb{P}_\star -a.s. For the other statements, we apply [7, Proposition 9]. $\mathcal{L} \cap \mathcal{K}$ is compact since \mathcal{L} is closed and \mathcal{K} is compact. We now prove that for any compact subset $\mathcal{K} \subset \Theta$,

$$|\mathbb{W}(\theta_{n+1}) - \mathbb{W} \circ \mathbb{R}(\theta_n)| \mathbf{1}_{\theta_n \in \mathcal{K}} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \mathbb{P}_\star - \text{a.s.} \quad (12)$$

Since $\limsup_n p_n < +\infty$ \mathbb{P}_\star -a.s, it is sufficient to prove this convergence on the set $\{\omega \in \Omega; \limsup_n p_n(\omega) < +\infty\}$. For any ω s.t. $\limsup_n p_n(\omega) < +\infty$, there exists (a random) n_0 s.t., for any $n \geq n_0$, $p_n(\omega) = p_{n+1}(\omega)$ and then $\theta_{n+1}(\omega) = \theta_{n+1/2}(\omega)$, see [8, Eq.(6)]. Therefore, (12) follows from (10).

3.2 Proof of [8, Proposition 6.6]

Proposition 3.1 shows that we can address equivalently the convergence of the statistics $\{\bar{S}_{\tau_{n+1}}^{X, T_n}(\theta_n, \mathbf{Y})\}_{n \geq 0}$ to some fixed point of G and the convergence of the sequence $\{\theta_n\}_{n \geq 0}$ to some fixed point of R .

Proposition 3.1. *Assume H1, H2, H3-(\bar{p}_2), H4(a-b), H5 and H6-(\bar{p}_1) for some $2 < \bar{p}_1 < \bar{p}_2$.*

(i) *Let $\theta_\star \in \mathcal{L}$. Set $s_\star \stackrel{\text{def}}{=} \bar{S}(\theta_\star) = G(s_\star)$. Then \mathbb{P}_\star -a.s,*

$$\lim_{n \rightarrow +\infty} \left| \bar{S}_{\tau_n}^{X, T_{n-1}}(\theta_{n-1}, \mathbf{Y}) - s_\star \right| \mathbf{1}_{\lim_n \theta_n = \theta_\star} = 0 .$$

(ii) *Let $s_\star \in \mathcal{S}$ s.t. $G(s_\star) = s_\star$. Set $\theta_\star \stackrel{\text{def}}{=} \bar{\theta}(s_\star) = R(\theta_\star)$. Then \mathbb{P}_\star -a.s,*

$$\lim_{n \rightarrow +\infty} |\theta_n - \theta_\star| \mathbf{1}_{\lim_n \bar{S}_{\tau_n}^{X, T_{n-1}}(\theta_{n-1}, \mathbf{Y}) = s_\star} = 0 .$$

Proof. Let \bar{S} be given by (4). By [8, Proposition 6.5] and H6-(\bar{p}_1),

$$\lim_n \left(\bar{S}_{\tau_{n+1}}^{\chi, T_n}(\theta_n, \mathbf{Y}) - \bar{S}(\theta_n) \right) = 0 \quad \mathbb{P}_\star - \text{a.s.}$$

By [8, Theorem 4.1], \bar{S} is continuous. Hence,

$$\lim_n \left| \bar{S}_{\tau_{n+1}}^{\chi, T_n}(\theta_n, \mathbf{Y}) - \bar{S}(\theta_\star) \right| \mathbf{1}_{\lim_n \theta_n = \theta_\star} = 0 \quad \mathbb{P}_\star - \text{a.s.}$$

and the proof of (i) follows. Since $\bar{\theta}$ is continuous, (ii) follows. \square

We start with rewriting some definitions and assumptions introduced in [8]. Define the sequences $S_n, \mu_n, \rho_n, n \geq 0$ by

$$S_0 \stackrel{\text{def}}{=} \bar{S}_{\tau_1}^{\chi, 0}(\theta_0, \mathbf{Y}) \quad \text{and} \quad S_n \stackrel{\text{def}}{=} \bar{S}_{\tau_{n+1}}^{\chi, T_n}(\theta_n, \mathbf{Y}), \quad \forall n \geq 0, \quad (13)$$

where $\bar{S}_\tau^{\chi, T}$ is given by (8); $\mu_0 = 0, \rho_0 = S_0 - s_\star$ and

$$\mu_n \stackrel{\text{def}}{=} \Gamma \mu_{n-1} + e_n, \quad \rho_n \stackrel{\text{def}}{=} S_n - s_\star - \mu_n, \quad n \geq 1, \quad (14)$$

where,

$$e_n \stackrel{\text{def}}{=} S_n - \bar{S}(\theta_n), \quad n \geq 1, \quad (15)$$

and \bar{S} is given by (4).

(proof) Let $p \in (2, \bar{p}_2)$. By (14), for all $n \geq 1$, $\mu_n = \sum_{k=0}^{n-1} \Gamma^k e_{n-k}$. By H7 and the Minkowski inequality, for all $n \geq 1$, $\|\mu_n\|_{\star, p} \leq \sum_{k=0}^{n-1} \gamma^k \|e_{n-k}\|_{\star, p}$. By (13) and [8, Proposition 6.5], there exists a constant C s.t. for any $n \geq 1$,

$$\|\mu_n\|_{\star, p} \leq C \sum_{k=0}^{n-1} \gamma^k \sqrt{\frac{1}{\tau_{n+1-k}}}.$$

By [9, Result 178, p. 39] and H8(a) (upon noting that $q \geq 1$ so that $\sqrt{q}\gamma < 1$), this yields $\sqrt{\tau_n} \mu_n = \mathcal{O}_{L^p}(1)$.

By H7, using a Taylor expansion with integral form of the remainder term,

$$\begin{aligned} & G(S_{n-1}) - G(s_\star) - \Gamma(S_{n-1} - s_\star) \\ &= \sum_{i,j=1}^d (S_{n-1,i} - s_{\star,i})(S_{n-1,j} - s_{\star,j}) R_{n-1}(i, j) \\ &= \sum_{i,j=1}^d (\mu_{n-1,i} + \rho_{n-1,i})(\mu_{n-1,j} + \rho_{n-1,j}) R_{n-1}(i, j), \end{aligned}$$

where $x_{n,i}$ denotes the i -th component of $x_n \in \mathbb{R}^d$ and

$$R_n(i, j) \stackrel{\text{def}}{=} \int_0^1 (1-t) \frac{\partial^2 G}{\partial s_i \partial s_j}(s_\star + t(S_n - s_\star)) dt, \quad n \in \mathbb{N}, 1 \leq i, j \leq d.$$

Observe that under H7, $\limsup_n |R_n| \mathbf{1}_{\lim_n \theta_n = \theta_*} < \infty$ w.p.1. Define for $n \geq 1$ and $k \leq n$,

$$H_n \stackrel{\text{def}}{=} \sum_{i=1}^d (2\mu_{n,i} + \rho_{n,i}) R_n(i, \cdot), \quad r_n \stackrel{\text{def}}{=} \sum_{i,j=1}^d R_n(i, j) \mu_{n,i} \mu_{n,j}, \quad (16)$$

$$\delta_n \stackrel{\text{def}}{=} \bar{S}(\theta_n) - G(S_{n-1}), \quad \psi(n, k) \stackrel{\text{def}}{=} (\Gamma + H_n) \cdots (\Gamma + H_k) \quad (17)$$

with the convention $\psi(n, n+1) \stackrel{\text{def}}{=} \text{Id}$. By (14),

$$\rho_n = \psi(n-1, 0) \rho_0 + \sum_{k=0}^{n-1} \psi(n-1, k+1) r_k + \sum_{k=1}^n \psi(n-1, k) \delta_k. \quad (18)$$

Since $\sqrt{\tau_n} \mu_n = \mathcal{O}_{L_p}(1)$, H6-(\bar{p}_1) implies that $\mu_n \xrightarrow[n \rightarrow +\infty]{} 0$ \mathbb{P}_* -a.s. Then, by (14), $\rho_n \mathbf{1}_{\lim_n S_n = s_*} \xrightarrow[n \rightarrow +\infty]{} 0$ \mathbb{P}_* -a.s and by (16) $\lim_{n \rightarrow +\infty} |H_n| \mathbf{1}_{\lim_n S_n = s_*} = 0$ \mathbb{P}_* -a.s. Let $\tilde{\gamma} \in (\gamma, q^{-1})$, where γ is given by H7 and q by H8(a). Since $\lim_{n \rightarrow +\infty} |H_n| \mathbf{1}_{\lim_n S_n = s_*} = 0$, there exists a \mathbb{P}_* -a.s finite random variable Z_1 s.t., for all $0 \leq k \leq n-1$,

$$|\psi(n-1, k)| \mathbf{1}_{\lim_n S_n = s_*} \leq \tilde{\gamma}^{n-k} Z_1 \mathbf{1}_{\lim_n S_n = s_*}. \quad (19)$$

Therefore, $|\psi(n-1, 0) \rho_0| \mathbf{1}_{\lim_n S_n = s_*} \leq \tilde{\gamma}^n Z_1 |\rho_0|$ \mathbb{P}_* -a.s, and, by H3-(\bar{p}_2), (9), (8) and (13) $\mathbb{E}_* [|\rho_0|^{\bar{p}_2}] < +\infty$ which implies that $\rho_0 < +\infty$ \mathbb{P}_* -a.s. Since $q\tilde{\gamma} < 1$, the first term in the RHS of (18) is $\tau_n^{-1} \mathcal{O}_{L_p}(1) \mathcal{O}_{\text{a.s.}}(1)$.

We now consider the second term in the RHS of (18). From equation (19),

$$\left| \sum_{k=0}^{n-1} \psi(n-1, k+1) r_k \right| \mathbf{1}_{\lim_n S_n = s_*} \leq Z_1 \sum_{k=0}^{n-1} \tilde{\gamma}^{n-k-1} |r_k| \mathbf{1}_{\lim_n S_n = s_*} \quad \mathbb{P}_* \text{-a.s.}$$

By (16) and H7, there exists a \mathbb{P}_* -a.s finite random variable Z_2 s.t.

$$|r_k| \mathbf{1}_{\lim_n S_n = s_*} \leq Z_2 \sum_{i,j=1}^d \mu_{k,i} \mu_{k,j}, \quad \mathbb{P}_* \text{-a.s.}$$

In addition, since $\sqrt{\tau_n} \mu_n = \mathcal{O}_{L_p}(1)$, there exists a constant C s.t.

$$\left\| \sum_{k=0}^{n-1} \tilde{\gamma}^{n-k-1} \sum_{i,j=1}^d \mu_{k,i} \mu_{k,j} \right\|_{\star, p/2} \leq C \sum_{k=0}^{n-1} \frac{\tilde{\gamma}^{n-k-1}}{\tau_k}.$$

Applying again [9, Result 178, p. 39] yields that the second term in the RHS of (18) is $\tau_n^{-1} \mathcal{O}_{\text{a.s.}}(1) \mathcal{O}_{L_{p/2}}(1)$. We finally consider the third term in the RHS of (18). By H1(c), on the set $\{\omega \in \Omega; \lim_{n \rightarrow +\infty} S_n(\omega) = s_*\}$ we have $\lim_{n \rightarrow +\infty} \bar{\theta}(S_n(\omega)) = \bar{\theta}(s_*)$. Hence, for any $\omega \in \Omega_*$, the set $\{\bar{\theta}(S_n(\omega))\}_{n \geq 0}$ is compact and $\theta_{n+1}(\omega) =$

$\theta_{n+1/2}(\omega)$ for all large n . By (6) and (4), there exists a random integer $n_0(\omega)$ s.t. for all $n \geq n_0(\omega)$, $\delta_n(\omega) = 0$. Then, there exists a \mathbb{P}_\star – a.s.-finite random variable Z_3 s.t. for all $n \geq 1$,

$$\left| \sum_{k=1}^n \psi(n-1, k) \delta_k \mathbf{1}_{\lim_n S_n = s_\star} \right| \leq \tilde{\gamma}^n Z_3 .$$

Since under H7 $\lim_n \tau_n \tilde{\gamma}_n = 0$, this implies that the third term in the RHS of (18) is $\tau_n^{-1} o_{\text{a.s.}}(1)$.

4 General results on HMM

In this section, we derive results on the forgetting properties of HMM (Section 4.1), on their applications to bivariate smoothing distributions (Section 4.2), on the asymptotic behavior of the normalized log-likelihood (Section 4.3) and on the normalized score (Section 4.4).

We consider a HMM with kernels $m(x, x') d\lambda(x')$ onto $\mathbb{X} \times \mathcal{B}(\mathbb{X})$ and $g(x, y) d\nu(y)$ on $(\mathbb{X} \times \mathcal{B}(\mathbb{Y}))$. \mathbb{X} is a general state-space equipped with a countably generated σ -field $\mathcal{B}(\mathbb{X})$, and λ is a bounded non-negative measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$; ν is a measure on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$.

For any initial distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, any $r < s \leq t$ and any sequence $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, define the probability measure $\Phi_{s,t}^{\chi,r}(\cdot, \mathbf{y})$ by

$$\begin{aligned} & \Phi_{s,t}^{\chi,r}(h, \mathbf{y}) \\ \stackrel{\text{def}}{=} & \frac{\int \chi(dx_r) \{ \prod_{i=r}^{t-1} m(x_i, x_{i+1}) g(x_{i+1}, y_{i+1}) \} h(x_{s-1}, x_s, y_s) d\lambda(x_{r+1:t})}{\int \chi(dx_r) \{ \prod_{i=r}^{t-1} m(x_i, x_{i+1}) g(x_{i+1}, y_{i+1}) \} d\lambda(x_{r+1:t})} , \end{aligned} \quad (20)$$

for any bounded function h .

For any $s \in \mathbb{Z}$ and any $A \in \mathcal{B}(\mathbb{X})$, define

$$L_s(x, A) \stackrel{\text{def}}{=} \int m(x, x') g(x', y_{s+1}) \mathbb{1}_A(x') \lambda(dx') , \quad (21)$$

and, for any $s \leq t$ denote by $L_{\theta, s:t}$ the composition of the kernels defined by

$$L_{s:s} \stackrel{\text{def}}{=} L_s , \quad L_{s:u+1}(x, A) \stackrel{\text{def}}{=} \int L_{s:u}(x, dx') L_{u+1}(x', A) .$$

By convention, $L_{s:s-1}$ is the identity kernel: $L_{s:s-1}(x, A) = \delta_x(A)$. For any sequence $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ and any function $h : \mathbb{X}^2 \times \mathbb{Y} \rightarrow \mathbb{R}$, denote by h_s the function on $\mathbb{X}^2 \rightarrow \mathbb{R}$ given by

$$h_s(x, x') \stackrel{\text{def}}{=} h(x, x', y_s) . \quad (22)$$

With these notations, equation (20) becomes

$$\Phi_{s,t}^{\chi,r}(h, \mathbf{y}) = \frac{\int \chi(dx_r) L_{r:s-2}(x_s, dx_{s-1}) h_s(x_{s-1}, x_s) L_{s:t-1}(x_s, \mathbb{X})}{\int \chi(dx_r) L_{r:t-1}(x_r, \mathbb{X})} . \quad (23)$$

4.1 Forward and Backward forgetting

For any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, any probability distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and for any integers such that $r \leq s < t$, let us define two Markov kernels on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ by

$$F_{s,t}(x, A) \stackrel{\text{def}}{=} \frac{\int L_s(x, dx_{s+1}) \mathbb{1}_A(x_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X})}{L_{s:t-1}(x, \mathbb{X})}, \quad (24)$$

$$B_s^{\chi,r}(x, A) \stackrel{\text{def}}{=} \frac{\int \phi_{s|r:s}^{\chi,r}(dx_s) \mathbb{1}_A(x_s) m(x_s, x)}{\int \phi_{s|r:s}^{\chi,r}(dx_s) m(x_s, x)}, \quad (25)$$

where

$$\phi_{s|r:s}^{\chi,r}(A) \stackrel{\text{def}}{=} \frac{\int \chi(dx_r) L_{r:s-1}(x_r, dx_s) \mathbb{1}_A(x_s)}{\int \chi(dx_r) L_{r:s-1}(x_r, \mathbb{X})}.$$

Finally, the Dobrushin coefficient of a Markov kernel $F : (\mathbb{X}, \mathcal{B}(\mathbb{X})) \rightarrow [0, 1]$ is defined by:

$$\delta(F) \stackrel{\text{def}}{=} \frac{1}{2} \sup_{(x, x') \in \mathbb{X}^2} \|F(x, \cdot) - F(x', \cdot)\|_{\text{TV}}.$$

Lemma 4.1. *Assume that there exist positive numbers σ_-, σ_+ such that $\sigma_- \leq m(x, x') \leq \sigma_+$ for any $x, x' \in \mathbb{X}$. Then for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, $\delta(F_{s,t}) \leq \rho$ and $\delta(B_s^{\chi,r}) \leq \rho$ where $\rho \stackrel{\text{def}}{=} \sigma_- / \sigma_+$.*

Proof. Let r, s, t be such that $r \leq s < t$. Under the stated assumptions,

$$\begin{aligned} & \int L_s(x_s, dx_{s+1}) \mathbb{1}_A(x_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X}) \\ & \geq \sigma_- \int g(x_{s+1}, y_{s+1}) \mathbb{1}_A(x_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X}) \lambda(dx_{s+1}) \end{aligned}$$

and

$$L_{s:t-1}(x_s, \mathbb{X}) \leq \sigma_+ \int g(x_{s+1}, y_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X}) \lambda(dx_{s+1}).$$

This yields to

$$F_{s,t}(x_s, A) \geq \frac{\sigma_-}{\sigma_+} \frac{\int g(x_{s+1}, y_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X}) \mathbb{1}_A(x_{s+1}) \lambda(dx_{s+1})}{\int g(x_{s+1}, y_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathbb{X}) \lambda(dx_{s+1})}.$$

Similarly, the assumption implies

$$B_s^{\chi,r}(x_{s+1}, A) \geq \frac{\sigma_-}{\sigma_+} \phi_{s|r:s}^{\chi,r}(A),$$

which gives the upper bound for the Dobrushin coefficients, see [3, Lemma 4.3.13]. \square

Lemma 4.2. *Assume that there exist positive numbers σ_-, σ_+ such that $\sigma_- \leq m(x, x') \leq \sigma_+$ for any $x, x' \in \mathbb{X}$. Let $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$.*

(i) for any bounded function h , any probability distributions χ and $\tilde{\chi}$ and any integers $r \leq s \leq t$

$$\left| \frac{\int \chi(\mathrm{d}x_r) L_{r:s-1}(x_r, \mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathbb{X})}{\int \chi(\mathrm{d}x_r) L_{r:t-1}(x_r, \mathbb{X})} - \frac{\int \tilde{\chi}(\mathrm{d}x_r) L_{r:s-1}(x_r, \mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathbb{X})}{\int \tilde{\chi}(\mathrm{d}x_r) L_{r:t-1}(x_r, \mathbb{X})} \right| \leq \rho^{s-r} \operatorname{osc}(h), \quad (26)$$

(ii) for any bounded function h , for any non-negative functions f and \tilde{f} and any integers $r \leq s \leq t$

$$\left| \frac{\int \chi(\mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) f(x_t)}{\int \chi(\mathrm{d}x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) f(x_t)} - \frac{\int \chi(\mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) \tilde{f}(x_t)}{\int \chi(\mathrm{d}x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) \tilde{f}(x_t)} \right| \leq \rho^{t-s} \operatorname{osc}(h). \quad (27)$$

Proof of (i). See [3, Proposition 4.3.23].

Proof of (ii) When $s = t$, then (ii) is equal to

$$\left| \frac{\int \chi(\mathrm{d}x_t) h(x_t) f(x_t)}{\int \chi(\mathrm{d}x_t) f(x_t)} - \frac{\int \chi(\mathrm{d}x_t) h(x_t) \tilde{f}(x_t)}{\int \chi(\mathrm{d}x_t) \tilde{f}(x_t)} \right|.$$

This is of the form $(\eta - \tilde{\eta}) h$ where η and $\tilde{\eta}$ are probability distributions on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then,

$$|(\eta - \tilde{\eta}) h| \leq \frac{1}{2} \|\eta - \tilde{\eta}\|_{\mathrm{TV}} \operatorname{osc}(h) \leq \operatorname{osc}(h).$$

Let $s < t$. By definition of the backward smoothing kernel, see (25),

$$\mathbb{B}_s^{\chi, s}(x_{s+1}, A) = \frac{\int \chi(\mathrm{d}x_s) 1_A(x_s) m(x_s, x_{s+1})}{\int \chi(\mathrm{d}x_s) m(x_s, x_{s+1})}.$$

Therefore,

$$\begin{aligned} & \int \chi(\mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) f(x_t) \\ &= \int \chi(\mathrm{d}x_s) L_s(x_s, \mathrm{d}x_{s+1}) \mathbb{B}_s^{\chi, s}(x_{s+1}) L_{s+1:t-1}(x_{s+1}, \mathrm{d}x_t) f(x_t). \end{aligned}$$

By repeated application of the backward smoothing kernel we have

$$\int \chi(\mathrm{d}x_s) h(x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) f(x_t) = \int \chi(\mathrm{d}x_s) L_{s:t-1}(x_s, \mathrm{d}x_t) \mathbb{B}_{t-1:s}^{\chi, s} h(x_t) f(x_t),$$

where we denote by $B_{t-1:s}^{\chi,s}$ the composition of the kernels defined by induction for $s \leq u$

$$B_{s:s}^{\chi,s} \stackrel{\text{def}}{=} B_s^{\chi,s}, \quad B_{u:s}^{\chi,s}(x, A) \stackrel{\text{def}}{=} \int B_u^{\chi,s}(x, dx') B_{u-1:s}^{\chi,s}(x', A).$$

Finally, by definition of $\phi_{t|s:t}^{\chi,s}$

$$\begin{aligned} & \left| \frac{\int \chi(dx_s) h(x_s) L_{s:t-1}(x_s, dx_t) f(x_t)}{\int \chi(dx_s) L_{s:t-1}(x_s, dx_t) f(x_t)} - \frac{\int \chi(dx_s) h(x_s) L_{s:t-1}(x_s, dx_t) \tilde{f}(x_t)}{\int \chi(dx_s) L_{s:t-1}(x_s, dx_t) \tilde{f}(x_t)} \right| \\ &= \left| \frac{\phi_{t|s:t}^{\chi,s} [(B_{t-1:s}^{\chi,s} h) f]}{\phi_{t|s:t}^{\chi,s} [f]} - \frac{\phi_{t|s:t}^{\chi,s} [(B_{t-1:s}^{\chi,s} h) \tilde{f}]}{\phi_{t|s:t}^{\chi,s} [\tilde{f}]} \right|. \end{aligned}$$

This is of the form $(\eta - \tilde{\eta}) B_{t-1:s}^{\chi,s} h$ where η and $\tilde{\eta}$ are probability distributions on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. The proof of the second statement is completed upon noting that

$$\begin{aligned} |\eta B_{t-1:s}^{\chi,s} h - \tilde{\eta} B_{t-1:s}^{\chi,s} h| &\leq \frac{1}{2} \|\eta - \tilde{\eta}\|_{\text{TV}} \text{osc}(B_{t-1:s}^{\chi,s} h) \\ &\leq \frac{1}{2} \|\mu - \tilde{\mu}\|_{\text{TV}} \delta(B_{t-1:s}^{\chi,s}) \text{osc}(h) \leq \rho^{t-s} \text{osc}(h), \end{aligned}$$

where we used Lemma 4.1 in the last inequality. \square

4.2 Bivariate smoothing distribution

In this Section, the kernel m and g may depend on a parameter $\theta \in \Theta$. The parameter θ is then introduced in the notation for a better clarity.

Proposition 4.3. *Assume H2. Let $\chi, \tilde{\chi}$ be two distributions on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. For any measurable function $h : \mathbb{X}^2 \times \mathbb{Y} \rightarrow \mathbb{R}^d$ and any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\sup_{x,x'} |h|(x, x', y_s) < +\infty$ for any $s \in \mathbb{Z}$*

(i) *For any $r < s \leq t$ and any $\ell_1, \ell_2 \geq 1$,*

$$\sup_{\theta \in \Theta} \left| \Phi_{\theta,s,t}^{\tilde{\chi},r}(h, \mathbf{y}) - \Phi_{\theta,s,t+\ell_2}^{\chi,r-\ell_1}(h, \mathbf{y}) \right| \leq (\rho^{s-1-r} + \rho^{t-s}) \text{osc}(h(\cdot, \cdot, y_s)). \quad (28)$$

(ii) *For any $\theta \in \Theta$, there exists a function $\mathbf{y} \mapsto \Phi_{\theta}(h, \mathbf{y})$ s.t. for any distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and any $r < s \leq t$*

$$\sup_{\theta \in \Theta} \left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) - \Phi_{\theta}(h, \vartheta^s \mathbf{y}) \right| \leq (\rho^{s-1-r} + \rho^{t-s}) \text{osc}(h(\cdot, \cdot, y_s)). \quad (29)$$

Remark 4.4. (a) If $\chi = \tilde{\chi}$, $\ell_1 = 0$ and $\ell_2 \geq 1$, (28) becomes

$$\sup_{\theta \in \Theta} \left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) - \Phi_{\theta,s,t+\ell_2}^{\chi,r}(h, \mathbf{y}) \right| \leq \rho^{t-s} \text{osc}(h_s).$$

(b) if $\ell_2 = 0$ and $\ell_1 \geq 1$, (28) becomes

$$\sup_{\theta \in \Theta} \left| \Phi_{\theta,s,t}^{\tilde{\chi},r}(h, \mathbf{y}) - \Phi_{\theta,s,t}^{\chi,r-\ell_1}(h, \mathbf{y}) \right| \leq \rho^{s-1-r} \text{osc}(h_s).$$

Proof. We will use the shorthand h_s for $h_s(x, x') \stackrel{\text{def}}{=} h(x, x', y_s)$.

(i) Let r, s, t such that $r < s \leq t$, $\ell_1, \ell_2 \geq 1$, and $\theta \in \Theta$. Define the distribution $\chi_{\theta,r-\ell_1:r}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ by

$$\chi_{\theta,r-\ell_1:r}(A) \stackrel{\text{def}}{=} \frac{\int \chi(\text{d}x_{r-\ell_1}) L_{\theta,r-\ell_1:r-1}(x_{r-\ell_1}, \text{d}x_r) 1_A(x_r)}{\int \chi(\text{d}x_{r-\ell_1}) L_{\theta,r-\ell_1:r-1}(x_{r-\ell_1}, \mathbb{X})}, \quad \forall A \in \mathcal{B}(\mathbb{X}).$$

We write $\left| \Phi_{\theta,s,t}^{\tilde{\chi},r}(h, \mathbf{y}) - \Phi_{\theta,s,t+\ell_2}^{\chi,r-\ell_1}(h, \mathbf{y}) \right| \leq \tilde{T}_1 + \tilde{T}_2$ where, by using (9),

$$\begin{aligned} \tilde{T}_1 \stackrel{\text{def}}{=} & \left| \frac{\int \tilde{\chi}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) h_s(x_{s-1}, x_s) L_{\theta,s-1}(x_{s-1}, \text{d}x_s) L_{\theta,s:t-1}(x_s, \mathbb{X})}{\int \tilde{\chi}_r(\text{d}x_r) L_{\theta,r:t-1}(x_r, \mathbb{X})} \right. \\ & \left. - \frac{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) h_s(x_{s-1}, x_s) L_{\theta,s-1}(x_{s-1}, \text{d}x_s) L_{\theta,s:t-1}(x_s, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:t-1}(x_r, \mathbb{X})} \right|, \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_2 \stackrel{\text{def}}{=} & \left| \frac{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) h_s(x_{s-1}, x_s) L_{\theta,s-1}(x_{s-1}, \text{d}x_s) L_{\theta,s:t-1}(x_s, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:t-1}(x_r, \mathbb{X})} \right. \\ & \left. - \frac{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) h_s(x_{s-1}, x_s) L_{\theta,s-1}(x_{s-1}, \text{d}x_s) L_{\theta,s:t+\ell_2-1}(x_s, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:t+\ell_2-1}(x_r, \mathbb{X})} \right|. \end{aligned}$$

Set $\bar{h}_{s,t} : x \mapsto \int F_{\theta,s-1,t}(x, \text{d}x_s) h_s(x, x_s)$ where $F_{\theta,s-1,t}$ is the forward smoothing kernel (see e.g. (24)). Then,

$$\begin{aligned} \tilde{T}_1 = & \left| \frac{\int \tilde{\chi}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) \bar{h}_{s,t}(x_{s-1}) L_{\theta,s-1:t-1}(x_{s-1}, \mathbb{X})}{\int \tilde{\chi}_r(\text{d}x_r) L_{\theta,r:t-1}(x_r, \mathbb{X})} \right. \\ & \left. - \frac{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:s-2}(x_r, \text{d}x_{s-1}) \bar{h}_{s,t}(x_{s-1}) L_{\theta,s-1:t-1}(x_{s-1}, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:r}(\text{d}x_r) L_{\theta,r:t-1}(x_r, \mathbb{X})} \right|. \end{aligned}$$

By Lemma 4.2(i),

$$\tilde{T}_1 \leq \rho^{s-1-r} \text{osc}(\bar{h}_{s,t}) \leq 2\rho^{s-1-r} \sup_{x \in \mathbb{X}} |\bar{h}_{s,t}(x)| \leq 2\rho^{s-1-r} \sup_{(x,x') \in \mathbb{X}^2} |h_s(x, x')|.$$

Set $\tilde{h}_s : x \mapsto \int \text{B}_{\theta,s-1}^{\chi_{\theta,r-\ell_1:s-1},s-1}(x, \text{d}x_{s-1}) h_s(x_{s-1}, x)$, where $\text{B}_{\theta,s-1}^{\chi_{\theta,r-\ell_1:s-1},s-1}$ is the backward smoothing kernel (see (25)). Then,

$$\begin{aligned} \tilde{T}_2 = & \left| \frac{\int \chi_{\theta,r-\ell_1:s}(\text{d}x_s) \tilde{h}_s(x_s) L_{\theta,s:t-1}(x_s, \text{d}x_t) L_{\theta,t:t+\ell_2-1}(x_t, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:s}(\text{d}x_s) L_{\theta,s:t-1}(x_s, \text{d}x_t) L_{\theta,t:t+\ell_2-1}(x_t, \mathbb{X})} \right. \\ & \left. - \frac{\int \chi_{\theta,r-\ell_1:s}(\text{d}x_s) \tilde{h}_s(x_s) L_{\theta,s:t-1}(x_s, \mathbb{X})}{\int \chi_{\theta,r-\ell_1:s}(\text{d}x_s) L_{\theta,s:t-1}(x_s, \mathbb{X})} \right|. \end{aligned}$$

Then, by Lemma 4.2(ii),

$$\tilde{T}_2 \leq \rho^{t-s} \text{osc}(\tilde{h}_s) \leq 2\rho^{t-s} \sup_{x \in \mathbb{X}} |\tilde{h}_s(x)| \leq 2\rho^{t-s} \sup_{(x,x') \in \mathbb{X}^2} |h_s(x, x')|.$$

The proof is concluded upon noting that, for any constant c ,

$$\text{osc}(h) = 2 \inf_{c \in \mathbb{R}} \left\{ \sup_{(x,x') \in \mathbb{X}^2} |h_s(x, x') - c| \right\}.$$

(ii) By (28), for any increasing sequence of non negative integers $(r_\ell)_{\ell \geq 0}$, $(t_\ell)_{\ell \geq 0}$ s.t. $\lim r_\ell = \lim t_\ell = +\infty$, the sequence $\{\Phi_{\theta,0,t_\ell}^{\chi, -r_\ell}(h, \mathbf{y})\}_{\ell \geq 0}$ is a Cauchy sequence uniformly in θ and χ . Then, there exists a limit $\Phi_\theta(h, \mathbf{y})$ s.t.

$$\lim_{\ell \rightarrow +\infty} \sup_{\chi} \sup_{\theta \in \Theta} \left| \Phi_{\theta,0,t_\ell}^{\chi, -r_\ell}(h, \mathbf{y}) - \Phi_\theta(h, \mathbf{y}) \right| = 0. \quad (30)$$

We write, for any $r < s \leq t$ and any $\ell \geq 1$

$$\begin{aligned} & \left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) - \Phi_\theta(h, \vartheta^s \mathbf{y}) \right| \\ & \leq \left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) - \Phi_{\theta,s,t+\ell}^{\chi,r-\ell}(h, \mathbf{y}) \right| + \left| \Phi_{\theta,s,t+\ell}^{\chi,r-\ell}(h, \mathbf{y}) - \Phi_\theta(h, \vartheta^s \mathbf{y}) \right|. \end{aligned}$$

Since $\Phi_{\theta,s,t+\ell}^{\chi,r-\ell}(h, \mathbf{y}) = \Phi_{\theta,0,t+\ell-s}^{\chi,r-\ell-s}(h, \vartheta^s \mathbf{y})$, Proposition 4.3(i) yields

$$\begin{aligned} \left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) - \Phi_\theta(h, \vartheta^s \mathbf{y}) \right| & \leq (\rho^{s-r-1} + \rho^{t-s}) \text{osc}(h_s) \\ & \quad + \left| \Phi_{\theta,0,t+\ell-s}^{\chi,r-\ell-s}(h, \vartheta^s \mathbf{y}) - \Phi_\theta(h, \vartheta^s \mathbf{y}) \right|. \end{aligned}$$

The proof is concluded by (30). \square

Lemma 4.5 is a consequence resp. of (9) and Proposition 4.3(ii).

Lemma 4.5. *Assume H2. Let $r < s \leq t$ be integers, $\theta \in \Theta$ and $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, and $h : \mathbb{X}^2 \times \mathbb{Y} \rightarrow \mathbb{R}^d$ s.t. for any $s \in \mathbb{Z}$, $\sup_{x,x'} |h|(x, x', y_s) < \infty$. Then*

$$\left| \Phi_{\theta,s,t}^{\chi,r}(h, \mathbf{y}) \right| \leq \sup_{(x,x') \in \mathbb{X}^2} |h(x, x', y_s)|, \quad \left| \Phi_\theta(h, \vartheta^s \mathbf{y}) \right| \leq \sup_{(x,x') \in \mathbb{X}^2} |h(x, x', y_s)|.$$

4.3 Limiting normalized log-likelihood

Define for any $r \leq s$,

$$\delta_{\theta,s}^{\chi,r}(\mathbf{y}) \stackrel{\text{def}}{=} \ell_{\theta,s+1}^{\chi,r}(\mathbf{y}) - \ell_{\theta,s}^{\chi,r}(\mathbf{y}), \quad (31)$$

where $\ell_{\theta,s+1}^{\chi,r}(\mathbf{y})$ is defined by

$$\ell_{\theta,s+1}^{\chi,r}(\mathbf{Y}) \stackrel{\text{def}}{=} \log \left(\int \chi(dx_r) \prod_{u=r+1}^{s+1} m_\theta(x_{u-1}, x_u) g_\theta(x_u, \mathbf{Y}_u) \lambda(dx_{r+1}) \cdots \lambda(dx_{s+1}) \right). \quad (32)$$

For any $T > 0$ and any probability distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, we thus have

$$\ell_{\theta, T}^{\chi, 0}(\mathbf{y}) = \sum_{s=0}^{T-1} \left(\ell_{\theta, s+1}^{\chi, 0}(\mathbf{y}) - \ell_{\theta, s}^{\chi, 0}(\mathbf{y}) \right) = \sum_{s=0}^{T-1} \delta_{\theta, s}^{\chi, 0}(\mathbf{y}). \quad (33)$$

It is established in Lemma 4.6 that for any $\theta \in \Theta$, $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, $s \geq 0$ and any initial distribution χ , the sequence $\{\delta_{\theta, s}^{\chi, s-r}(\mathbf{y})\}_{r \geq 0}$ is a Cauchy sequence and its limit does not depend upon χ . Regularity conditions on this limit are given in Lemmas 4.7 and 4.8. Finally, Theorem 4.9 shows that for any θ , $\lim_T T^{-1} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y})$ exists w.p.1. and this limit is a (deterministic) continuous function in θ .

Lemma 4.6. *Assume H2.*

(i) *For any $\ell, r, s \geq 0$, any initial distributions χ, χ' on \mathbb{X} and any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$*

$$\sup_{\theta \in \Theta} \left| \delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) - \delta_{\theta, s}^{\chi', s-r-\ell}(\mathbf{y}) \right| \leq \frac{2}{1-\rho} \rho^r.$$

(ii) *For any $\theta \in \Theta$, there exists a function $\mathbf{y} \mapsto \delta_{\theta}(\mathbf{y})$ such that for any initial distribution χ , any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ and any $r, s \geq 0$,*

$$\sup_{\theta \in \Theta} \left| \delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) - \delta_{\theta}(\vartheta^s \circ \mathbf{y}) \right| \leq \frac{2}{1-\rho} \rho^r.$$

Proof. *Proof of (i).* Let $s \geq 0$ and r and r' be such that $r' > r$. By (31) and (32), we have $|\delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) - \delta_{\theta, s}^{\chi', s-r'}(\mathbf{y})| = |\log \alpha - \log \beta|$ where

$$\begin{aligned} \alpha &\stackrel{\text{def}}{=} \frac{\int \chi(dx_{s-r}) \prod_{i=s-r+1}^{s+1} m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i) \lambda(dx_i)}{\int \chi(dx_{s-r}) \prod_{i=s-r+1}^s m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i) \lambda(dx_i)}, \\ \beta &\stackrel{\text{def}}{=} \frac{\int \chi'(dx_{s-r'}) \prod_{i=s-r'+1}^{s+1} m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i) \lambda(dx_i)}{\int \chi'(dx_{s-r'}) \prod_{i=s-r'+1}^s m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i) \lambda(dx_i)}. \end{aligned} \quad (34)$$

We prove that

$$\alpha \wedge \beta \geq \sigma_- \int g_{\theta}(x_{s+1}, y_{s+1}) \lambda(dx_{s+1}), \quad (35)$$

$$|\alpha - \beta| \leq 2\rho^r \sigma_+ \int g_{\theta}(x_{s+1}, y_{s+1}) \lambda(dx_{s+1}), \quad (36)$$

and the proof is concluded since $|\log \alpha - \log \beta| \leq |\alpha - \beta| / (\alpha \wedge \beta)$.

The minorization on α and β is a consequence of H2 upon noting that α and β are of the form $\int \mu(dx_s) m_{\theta}(x_s, x_{s+1}) g_{\theta}(x_{s+1}, y_{s+1}) \lambda(dx_{s+1})$ for some probability measure μ . The upper bound on $|\alpha - \beta|$ is a consequence of Lemma 4.2(i) applied with

$$\tilde{\chi}(dx_{s-r}) \leftarrow \int_{\mathbb{X}^{r'-r}} \chi'(dx_{s-r'}) \left\{ \prod_{i=s-r'}^{s-r-1} g_{\theta}(x_i, y_i) m_{\theta}(x_i, x_{i+1}) \right\} \lambda(dx_{s-r'+1:s-r})$$

and $h(u) \leftarrow \int g_\theta(x_{s+1}, y_{s+1}) m_\theta(u, x_{s+1}) \lambda(dx_{s+1})$.

Proof of (ii). By (i), for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, the sequence $\{\delta_{\theta,0}^{\chi,-r}(\mathbf{y})\}_{r \geq 0}$ is a Cauchy sequence uniformly in θ : there exists a limit denoted by $\delta_\theta(\mathbf{y})$ - which does not depend upon χ - such that

$$\lim_{r \rightarrow +\infty} \sup_{\theta \in \Theta} \left| \delta_{\theta,0}^{\chi,-r}(\mathbf{y}) - \delta_\theta(\mathbf{y}) \right| = 0. \quad (37)$$

We write for $r \leq r'$

$$\left| \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \delta_\theta(\vartheta^s \circ \mathbf{y}) \right| \leq \left| \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \delta_{\theta,s}^{\chi,s-r'}(\mathbf{y}) \right| + \left| \delta_{\theta,s}^{\chi,s-r'}(\mathbf{y}) - \delta_\theta(\vartheta^s \circ \mathbf{y}) \right|.$$

Observe that by definition, $\delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) = \delta_{\theta,0}^{\chi,-r}(\vartheta^s \circ \mathbf{y})$. This property, combined with Lemma 4.6(i), yield

$$\sup_{\theta \in \Theta} \left| \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \delta_\theta(\vartheta^s \circ \mathbf{y}) \right| \leq \frac{2}{1-\rho} \rho^r + \sup_{\theta \in \Theta} \left| \delta_{\theta,0}^{\chi,-r'}(\vartheta^s \circ \mathbf{y}) - \delta_\theta(\vartheta^s \circ \mathbf{y}) \right|.$$

When $r' \rightarrow +\infty$, the second term in the rhs tends to zero by (37) - for fixed \mathbf{y}, s and χ -. This concludes the proof. \square

Lemma 4.7. *Assume H2. For any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ and $s \geq 0$,*

$$\sup_{r \geq 0} \sup_{\theta \in \Theta} \left| \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) \right| \leq |\log \sigma_+ b_+(\mathbf{y}_{s+1})| + |\log \sigma_- b_-(\mathbf{y}_{s+1})|,$$

and, for any $r \geq 0$,

$$\sup_{\theta \in \Theta} |\delta_\theta(\mathbf{y})| \leq \frac{2}{(1-\rho)} \rho^r + |\log \sigma_+ b_+(\mathbf{y}_1)| + |\log \sigma_- b_-(\mathbf{y}_1)|,$$

where b_+ and b_- are defined by (1).

Proof. For any $0 < m \leq A/B \leq M$, $|\log(A/B)| \leq |\log M| + |\log m|$. Note that by definition, $\delta_{\theta,s}^{\chi,0}(\mathbf{y})$ is of the form $\log(A/B)$ and under H4(c), $\sigma_- b_-(y_{s+1}) \leq A/B \leq \sigma_+ b_+(y_{s+1})$. The second upper bound is a consequence of Lemma 4.6(ii). \square

Lemma 4.8. *Assume H1, H2, H4(a) and H4(c). Then, $\theta \mapsto \mathbb{E}_*[\delta_\theta(\mathbf{Y})]$ is continuous on Θ . If in addition Θ is compact,*

$$\lim_{\eta \rightarrow 0} \mathbb{E}_* \left[\sup_{\{\theta, \theta' \in \Theta; |\theta - \theta'| < \eta\}} |\delta_\theta(\mathbf{Y}) - \delta_{\theta'}(\mathbf{Y})| \right] = 0 \quad \mathbb{P}_* - \text{a.s.} \quad (38)$$

Proof. By the dominated convergence theorem, Lemma 4.7 and H4(c), $\theta \mapsto \mathbb{E}_*[\delta_\theta(\mathbf{Y})]$ is continuous if $\theta \mapsto \delta_\theta(\mathbf{y})$ is continuous for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$. Let $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$. By Lemma 4.6(ii), $\lim_{r \rightarrow +\infty} \sup_{\theta \in \Theta} |\delta_{\theta,0}^{\chi,-r}(\mathbf{y}) - \delta_\theta(\mathbf{y})| = 0$. Therefore, $\theta \mapsto \delta_\theta(\mathbf{y})$ is continuous provided for any $r \geq 0$, $\theta \mapsto \delta_{\theta,0}^{\chi,-r}(\mathbf{y})$ is continuous (for fixed \mathbf{y} and

χ). By definition of $\delta_{\theta,0}^{\chi,-r}(\mathbf{y})$, see (31), it is sufficient to prove that $\theta \mapsto \ell_{\theta,s}^{\chi,-r}(\mathbf{y})$ is continuous for $s \in \{0,1\}$. By definition of $\ell_{\theta,s}^{\chi,-r}(\mathbf{y})$, see (32),

$$\ell_{\theta,s}^{\chi,-r}(\mathbf{y}) = \log \int \chi(dx_{-r}) \prod_{i=-r+1}^s m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i) \lambda(dx_i).$$

Under H1(a), $\theta \mapsto \prod_{i=-r+1}^s m_{\theta}(x_{i-1}, x_i) g_{\theta}(x_i, y_i)$ is continuous on Θ , for any $x_{-r:s}$ and \mathbf{y} . In addition, under H1, for any $\theta \in \Theta$,

$$\begin{aligned} & \left| \prod_{i=-r+1}^s m_{\theta}(x_i, x_{i+1}) g_{\theta}(x_{i+1}, \mathbf{y}_{i+1}) \right| \\ &= \exp \left((s+r)\phi(\theta) + \left\langle \psi(\theta), \sum_{i=-r+1}^s S(x_i, x_{i+1}, \mathbf{y}_{i+1}) \right\rangle \right). \end{aligned}$$

Let \mathcal{K} be a compact subset of Θ . Since by H1 ϕ and ψ are continuous, there exist constants C_1 and C_2 such that,

$$\begin{aligned} & \sup_{\theta \in \mathcal{K}} \left| \prod_{i=-r+1}^s m_{\theta}(x_i, x_{i+1}) g_{\theta}(x_{i+1}, \mathbf{y}_{i+1}) \right| \\ & \leq C_1 \exp \left(C_2 \sum_{i=-r+1}^s \sup_{x, x'} |S(x, x', \mathbf{y}_{i+1})| \right). \end{aligned}$$

Since the measure $\chi(dx_{-r}) \prod_{i=-r+1}^s \lambda(dx_i)$ is finite, the dominated convergence theorem now implies that $\ell_{\theta,s}^{\chi,-r}(\mathbf{y})$ is continuous on Θ .

For the proof of (38), let us apply the dominated convergence theorem again. Since Θ is compact, for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$, $\theta \mapsto \delta_{\theta}(\mathbf{y})$ is uniformly continuous and $\lim_{\eta \rightarrow 0} \sup_{|\theta - \theta'| < \eta} |\delta_{\theta}(\mathbf{y}) - \delta_{\theta'}(\mathbf{y})| = 0$. In addition, we have by Lemma 4.7

$$\begin{aligned} & \sup_{\{\theta, \theta' \in \Theta; |\theta - \theta'| < \eta\}} |\delta_{\theta}(\mathbf{y}) - \delta_{\theta'}(\mathbf{y})| \\ & \leq 2 \sup_{\theta \in \Theta} |\delta_{\theta}(\mathbf{y})| \leq \frac{4}{(1-\rho)} + 2 \{|\log \sigma_+ b_+(y_1)| + |\log \sigma_- b_-(y_1)|\}. \end{aligned}$$

Under H4(a) and H4(c), this upper bound is \mathbb{P}_{\star} -integrable. This concludes the proof. \square

Theorem 4.9. *Assume H1, H2, H4. Define the function $c_{\star} : \Theta \rightarrow \mathbb{R}$ by $c_{\star}(\theta) \stackrel{\text{def}}{=} \mathbb{E}_{\star}[\delta_{\theta}(\mathbf{Y})]$, where $\delta_{\theta}(\mathbf{y})$ is defined in Lemma 4.7.*

(i) *The function $\theta \mapsto c_{\star}(\theta)$ is continuous on Θ .*

(ii) *For any initial distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$*

$$\left| \frac{1}{T} \ell_{\theta,T}^{\chi,0}(\mathbf{Y}) - c_{\star}(\theta) \right| \xrightarrow{T \rightarrow +\infty} 0 \quad \mathbb{P}_{\star} - \text{a.s.} \quad (39)$$

where $\ell_{\theta,T}^{\chi,0}(\mathbf{Y})$ is defined in (32).

(iii) If in addition Θ is compact, for any initial distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$

$$\sup_{\theta \in \Theta} \left| \frac{1}{T} \ell_{\theta,T}^{\chi,0}(\mathbf{Y}) - c_*(\theta) \right| \xrightarrow{T \rightarrow +\infty} 0 \quad \mathbb{P}_* - \text{a.s.} \quad (40)$$

Proof. (i) is proved in Lemma 4.8.

(ii) By (33), for any $T > 0$, we have, for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$:

$$\begin{aligned} \frac{1}{T} \ell_{\theta,T}^{\chi,0}(\mathbf{y}) &= \frac{1}{T} \sum_{s=0}^{T-1} \delta_{\theta,s}^{\chi,0}(\mathbf{y}) \\ &= \frac{1}{T} \sum_{s=0}^{T-1} \left(\delta_{\theta,s}^{\chi,0}(\mathbf{y}) - \delta_{\theta}(\vartheta^s \circ \mathbf{y}) \right) + \frac{1}{T} \sum_{s=0}^{T-1} \delta_{\theta}(\vartheta^s \circ \mathbf{y}) . \end{aligned}$$

By Lemma 4.6(ii), for any $0 \leq s \leq T-1$, $\left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right| \leq 2 \frac{\rho^s}{1-\rho}$. Since $\rho \in (0, 1)$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{s=0}^{T-1} \left(\delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right) = 0 \quad \mathbb{P}_* - \text{a.s.}$$

By Lemma 4.7

$$\mathbb{E}_* [\delta_{\theta}(\mathbf{Y})] \leq \frac{2}{(1-\rho)} + \mathbb{E}_* [|\log \sigma_+ b_+(\mathbf{Y}_1)| + |\log \sigma_- b_-(\mathbf{Y}_1)|] ,$$

and the rhs is finite under assumption H4(c). By H4(a-b), the ergodic theorem, see [1, Theorem 24.1, p.314], concludes the proof.

(iii) Since Θ is compact, (40) holds if for any $\varepsilon > 0$, any $\theta' \in \Theta$, there exists $\eta > 0$ such that

$$\lim_{T \rightarrow +\infty} \sup_{\{\theta; |\theta - \theta'| < \eta\} \cap \Theta} \left| T^{-1} \ell_{\theta,T}^{\chi,0}(\mathbf{Y}) - T^{-1} \ell_{\theta',T}^{\chi,0}(\mathbf{Y}) \right| \leq \varepsilon , \quad \mathbb{P}_* - \text{a.s.} \quad (41)$$

Let $\varepsilon > 0$ and $\theta' \in \Theta$. Choose $\eta > 0$ such that

$$\mathbb{E}_* \left[\sup_{\{\theta \in \Theta; |\theta - \theta'| < \eta\}} |\delta_{\theta}(\mathbf{Y}) - \delta_{\theta'}(\mathbf{Y})| \right] \leq \varepsilon ; \quad (42)$$

such an η exists by Lemma 4.8. By (33), we have, for any $\theta \in \Theta$ such that $|\theta - \theta'| < \eta$

$$\left| \frac{1}{T} \ell_{\theta,T}^{\chi,0}(\mathbf{Y}) - \frac{1}{T} \ell_{\theta',T}^{\chi,0}(\mathbf{Y}) \right| \leq \frac{1}{T} \sum_{s=0}^{T-1} \left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta',s}^{\chi,0}(\mathbf{Y}) \right| . \quad (43)$$

In addition, by Lemma 4.6(ii)

$$\begin{aligned} & \sum_{s=0}^{T-1} \left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta',s}^{\chi,0}(\mathbf{Y}) \right| \\ & \leq 2 \sum_{s=0}^{T-1} \sup_{\theta \in \Theta} \left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right| + \sum_{s=0}^{T-1} \left| \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) - \delta_{\theta'}(\vartheta^s \circ \mathbf{Y}) \right| \\ & \leq \frac{4}{(1-\rho)^2} + \sum_{s=0}^{T-1} \Xi(\vartheta^s \circ \mathbf{Y}) \end{aligned}$$

where $\Xi(\mathbf{y}) \stackrel{\text{def}}{=} \sup_{\{\theta \in \Theta; |\theta - \theta'| < \eta\}} |\delta_{\theta}(\mathbf{y}) - \delta_{\theta'}(\mathbf{y})|$. This implies that

$$\lim_{T \rightarrow +\infty} \sup_{\{\theta \in \Theta; |\theta - \theta'| < \eta\}} \frac{1}{T} \sum_{s=0}^{T-1} \left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta',s}^{\chi,0}(\mathbf{Y}) \right| \leq \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{s=0}^{T-1} \Xi(\vartheta^s \circ \mathbf{Y}).$$

Under H4(a-b), the ergodic theorem implies that the rhs converges \mathbb{P}_{\star} a.s to $\mathbb{E}_{\star}[\Xi(\mathbf{Y})]$, see [1, p.314]. Then, using again (42),

$$\lim_{T \rightarrow +\infty} \sup_{\{\theta \in \Theta; |\theta - \theta'| < \eta\}} \frac{1}{T} \sum_{s=0}^{T-1} \left| \delta_{\theta,s}^{\chi,0}(\mathbf{Y}) - \delta_{\theta',s}^{\chi,0}(\mathbf{Y}) \right| \leq \varepsilon \quad \mathbb{P}_{\star} - \text{a.s.}$$

Then, (41) holds and this concludes the proof. \square

4.4 Limit of the normalized score

This section is devoted to the proof of the \mathbb{P}_{\star} -a.s convergence of the normalized score $T^{-1} \nabla_{\theta} \ell_{\theta,T}^{\chi,0}(\mathbf{Y})$ to $\nabla_{\theta} c_{\star}(\theta)$. This result is established under additional assumptions on the model.

- S1** (a) For any $y \in \mathbb{Y}$ and for all $(x, x') \in \mathbb{X}^2$, $\theta \mapsto g_{\theta}(x, y)$ and $\theta \mapsto m_{\theta}(x, x')$ are continuously differentiable on Θ .
- (b) We assume that $\mathbb{E}_{\star}[\phi(\mathbf{Y}_0)] < +\infty$ where

$$\phi(y) \stackrel{\text{def}}{=} \sup_{\theta \in \Theta} \sup_{(x, x') \in \mathbb{X}^2} |\nabla_{\theta} \log m_{\theta}(x, x') + \nabla_{\theta} \log g_{\theta}(x', y)|. \quad (44)$$

Lemma 4.10. *Assume S1. For any initial distribution χ , any integers $s, r \geq 0$ and any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\phi(\mathbf{y}_u) < +\infty$ for any $u \in \mathbb{Z}$, the function $\theta \mapsto \ell_{\theta,s}^{\chi,s-r}(\mathbf{y})$ is continuously differentiable on Θ and*

$$\nabla_{\theta} \ell_{\theta,s}^{\chi,s-r}(\mathbf{y}) = \sum_{u=s-r}^s \Phi_{\theta,u,s}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}),$$

where Υ_{θ} is the function defined on $\mathbb{X}^2 \times \mathbb{Y}$ by

$$\Upsilon_{\theta} : (x, x', y) \mapsto \nabla_{\theta} \log \{m_{\theta}(x, x')g_{\theta}(x', y)\}.$$

Proof. Under S1, the dominated convergence theorem implies that the function $\theta \mapsto \ell_{\theta,s}^{\chi,s-r}(\mathbf{y})$ is continuously differentiable and its derivative is obtained by permutation of the gradient and integral operators. \square

Lemma 4.11. *Assume H2 and S1.*

- (i) *There exists a function $\xi : \mathbb{Y}^{\mathbb{Z}} \rightarrow \mathbb{R}_+$ such that for any $s \geq 0$ and any $r, r' \geq s$, any initial distribution χ, χ' on \mathbb{X} and any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\phi(\mathbf{y}_u) < +\infty$ for any $u \in \mathbb{Z}$,*

$$\sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta,s}^{\chi',s-r'}(\mathbf{y}) \right| \leq \frac{16\rho^{-1/4}}{1-\rho} \rho^{(r' \wedge r)/4} \xi(\mathbf{y}),$$

where

$$\xi(\mathbf{y}) \stackrel{\text{def}}{=} \sum_{u \in \mathbb{Z}} \phi(\mathbf{y}_u) \rho^{|u|/4}. \quad (45)$$

- (ii) *For any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ satisfying $\xi(\mathbf{y}) < +\infty$, the function $\theta \mapsto \delta_{\theta}(\mathbf{y})$ given by Lemma 4.6(ii) is continuously differentiable on Θ ; and, for any $\theta \in \Theta$, any initial distribution χ and any integers $r \geq s \geq 0$,*

$$\sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta}(\mathbf{y} \circ \vartheta^s) \right| \leq \frac{16\rho^{-1/4}}{1-\rho} \rho^{r/4} \xi(\mathbf{y}).$$

Proof. (i) By definition of $\delta_{\theta,s}^{\chi,s-r}(\mathbf{y})$, see (31) and Lemma 4.10,

$$\begin{aligned} & \nabla_{\theta} \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta,s}^{\chi',s-r'}(\mathbf{y}) \\ &= \nabla_{\theta} \ell_{\theta,s+1}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \ell_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \ell_{\theta,s+1}^{\chi',s-r'}(\mathbf{y}) + \nabla_{\theta} \ell_{\theta,s}^{\chi',s-r'}(\mathbf{y}) \\ &= \sum_{u=s-r}^s \left(\Phi_{\theta,u,s+1}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,u,s}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) \right) \\ &\quad - \sum_{u=s-r'}^s \left(\Phi_{\theta,u,s+1}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,u,s}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}) \right) \\ &\quad + \Phi_{\theta,s+1,s+1}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,s+1,s+1}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}). \end{aligned}$$

We can assume without loss of generality that $r' \leq r$ so that

$$\begin{aligned} & \nabla_{\theta} \delta_{\theta,s}^{\chi,s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta,s}^{\chi',s-r'}(\mathbf{y}) \\ &= \sum_{u=s-r}^{s-r'-1} \left\{ \Phi_{\theta,u,s+1}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,u,s}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) \right\} + \Phi_{\theta,s+1,s+1}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,s+1,s+1}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}) \\ &\quad + \sum_{u=s-r'}^s \left\{ \Phi_{\theta,u,s+1}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,u,s}^{\chi,s-r-1}(\Upsilon_{\theta}, \mathbf{y}) - \Phi_{\theta,u,s+1}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}) + \Phi_{\theta,u,s}^{\chi',s-r'-1}(\Upsilon_{\theta}, \mathbf{y}) \right\}. \end{aligned}$$

Under H2 and S1, Remark 4.4 can be applied and for any $s - r \leq u \leq s - r' - 1$,

$$\left| \Phi_{\theta, u, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) \right| \leq 2\rho^{s-u}\phi(\mathbf{y}_u),$$

where $\phi_u(\mathbf{y})$ is defined in (44). Similarly, by Remark 4.4

$$\left| \Phi_{\theta, s+1, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, s+1, s+1}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) \right| \leq 2\rho^{r'+1}\phi(\mathbf{y}_{s+1}).$$

For any $s - r' \leq u \leq s$, by Remark 4.4,

$$\begin{aligned} & \left| \Phi_{\theta, u, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s+1}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) + \Phi_{\theta, u, s}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) \right| \\ & \leq \left| \Phi_{\theta, u, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s+1}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) \right| + \left| \Phi_{\theta, u, s}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) \right| \\ & \leq 4\rho^{u+r'-s}\phi(\mathbf{y}_u) \end{aligned}$$

and by Remark 4.4,

$$\begin{aligned} & \left| \Phi_{\theta, u, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s+1}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) + \Phi_{\theta, u, s}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) \right| \\ & \leq \left| \Phi_{\theta, u, s+1}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi, s-r-1}(\Upsilon_\theta, \mathbf{y}) \right| + \left| \Phi_{\theta, u, s+1}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) - \Phi_{\theta, u, s}^{\chi', s-r'-1}(\Upsilon_\theta, \mathbf{y}) \right| \\ & \leq 4\rho^{s-u}\phi(\mathbf{y}_u). \end{aligned}$$

Hence,

$$\left| \nabla_\theta \delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) - \nabla_\theta \delta_{\theta, s}^{\chi', s-r'}(\mathbf{y}) \right| \leq 2 \sum_{u=s-r}^{s-r'-1} \rho^{s-u}\phi(\mathbf{y}_u) + 4 \sum_{u=s-r'}^{s+1} \left(\rho^{u+r'-s} \wedge \rho^{s-u} \right) \phi(\mathbf{y}_u).$$

Furthermore,

$$\begin{aligned} & \sum_{u=s-r'}^{s+1} \phi(\mathbf{y}_u) \left(\rho^{u+r'-s} \wedge \rho^{s-u} \right) \\ & \leq \sum_{s-r' \leq u \leq \lfloor s-r'/2 \rfloor} \rho^{s-u}\phi(\mathbf{y}_u) + \sum_{u \geq \lfloor s-r'/2 \rfloor} \rho^{u+r'-s}\phi(\mathbf{y}_u) \\ & \leq \rho^{r'/2} \sum_{u \in \mathbb{Z}} \phi(\mathbf{y}_u) \rho^{|u|/4} \dots \\ & \quad \times \left(\sum_{u \leq \lfloor s-r'/2 \rfloor} \rho^{s-u-r'/2-|u|/4} + \sum_{\lfloor s-r'/2 \rfloor + 1 \leq u \leq s+1} \rho^{u+r'/2-s-|u|/4} \right) \\ & \leq 2 \frac{\rho^{(r'-1)/4}}{1-\rho} \sum_{u \in \mathbb{Z}} \phi(\mathbf{y}_u) \rho^{|u|/4}, \end{aligned}$$

where we used that $\sup_{s-r' \leq u \leq \lfloor s-r'/2 \rfloor} |u| \leq r'$ and $\sup_{\lfloor s-r'/2 \rfloor + 1 \leq u \leq s+1} |u| \leq r' + 1$. Moreover, upon noting that $-u/2 + (s+1)/2 \leq s-u-r'/2$ when

$$u \leq s - r' - 1,$$

$$\begin{aligned} \sum_{u=s-r}^{s-r'-1} \phi(\mathbf{y}_u) \rho^{s-u} &\leq \rho^{r'/2} \sum_{u=s-r}^{s-r'-1} \phi(\mathbf{y}_u) \rho^{s-u-r'/2} \\ &\leq \rho^{r'/2} \sum_{u=s-r}^{s-r'-1} \phi(\mathbf{y}_u) \rho^{-u/2+(s+1)/2} \\ &\leq \rho^{r'/2} \rho^{(s+1)/2} \sum_{u=s-r}^{s-r'-1} \phi(\mathbf{y}_u) \rho^{|u|/2}, \end{aligned}$$

where we used that $s - r' - 1 \leq 0$ in the last inequality.

Hence,

$$\sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,s}^{\chi, s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta,s}^{\chi', s-r'}(\mathbf{y}) \right| \leq \frac{16}{1-\rho} \rho^{(r'-1)/4} \sum_{u \in \mathbb{Z}} \phi(\mathbf{y}_u) \rho^{|u|/4}. \quad (46)$$

(ii) Let $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\xi(\mathbf{y}) < +\infty$. Then for any $u \in \mathbb{Z}$, $\phi(\mathbf{y}_u) < +\infty$. By Lemma 4.10 and Eq. (31), the functions $\{\theta \mapsto \delta_{\theta,0}^{\chi, -r}(\mathbf{y})\}_{r \geq 0}$ are C^1 functions on Θ . By (i), there exists a function $\theta \mapsto \tilde{\delta}_{\theta}(\mathbf{y})$ such that

$$\lim_{r \rightarrow +\infty} \sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,0}^{\chi, -r}(\mathbf{y}) - \tilde{\delta}_{\theta}(\mathbf{y}) \right| = 0.$$

Furthermore, by Lemma 4.6,

$$\lim_{r \rightarrow +\infty} \sup_{\theta \in \Theta} \left| \delta_{\theta,0}^{\chi, -r}(\mathbf{y}) - \delta_{\theta}(\mathbf{y}) \right| = 0.$$

Then, $\theta \mapsto \delta_{\theta}(\mathbf{y})$ is C^1 on Θ and for any $\theta \in \Theta$, $\tilde{\delta}_{\theta}(\mathbf{y}) = \nabla_{\theta} \delta_{\theta}(\mathbf{y})$.

We thus proved that for any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\xi(\mathbf{y}) < +\infty$ and for any initial distribution χ ,

$$\lim_{r \rightarrow +\infty} \sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,0}^{\chi, -r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta}(\mathbf{y}) \right| = 0. \quad (47)$$

Observe that by definition, $\nabla_{\theta} \delta_{\theta,s}^{\chi, s-r}(\mathbf{y}) = \nabla_{\theta} \delta_{\theta,0}^{\chi, -r}(\vartheta^s \circ \mathbf{y})$. This property, combined with Lemma 4.11(i), yields

$$\begin{aligned} \sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,s}^{\chi, s-r}(\mathbf{y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{y}) \right| \\ \leq \frac{16\rho^{-1/4}}{1-\rho} \rho^{r/4} \xi(\mathbf{y}) + \sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta,0}^{\chi, -r'}(\vartheta^s \circ \mathbf{y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{y}) \right|. \end{aligned}$$

Since $\xi(\vartheta^s \circ \mathbf{y}) < +\infty$, when $r' \rightarrow +\infty$, the second term tends to zero by (47) - for fixed \mathbf{y} , s and χ -. This concludes the proof. \square

Lemma 4.12. (i) Assume S1. For any $\mathbf{y} \in \mathbb{Y}^{\mathbb{Z}}$ such that $\phi(\mathbf{y}_u) < +\infty$ for any $u \in \mathbb{Z}$, for any integers $r, s \geq 0$,

$$\sup_{\theta \in \Theta} \left| \nabla_{\theta} \delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) \right| \leq 2 \sum_{u=s-r}^{s+1} \phi(\mathbf{y}_u).$$

(ii) Assume H2 and S1. Then, for any $y \in \mathbb{Y}^{\mathbb{Z}}$ such that $\xi(\mathbf{y}) < +\infty$ and for any $r \geq 0$,

$$\sup_{\theta \in \Theta} |\nabla_{\theta} \delta_{\theta}(\mathbf{y})| \leq 2 \sum_{u=-r}^1 \phi(\mathbf{y}_u) + \frac{16\rho^{-1/4}}{1-\rho} \xi(\mathbf{y}) \rho^{r/4},$$

where $\xi(\mathbf{y})$ is defined in Lemma 4.11.

Proof. (i) By (31) and Lemma 4.10,

$$\begin{aligned} & \left| \nabla_{\theta} \delta_{\theta, s}^{\chi, s-r}(\mathbf{y}) \right| = \left| \nabla_{\theta} \ell_{\theta, s+1}^{\chi, s-r}(\mathbf{y}) - \nabla_{\theta} \ell_{\theta, s}^{\chi, s-r}(\mathbf{y}) \right| \\ & \leq 2 \sum_{u=s-r}^{s+1} \left| \frac{\int \chi(\mathrm{d}x_{s-r}) L_{\theta, s-r:u-1}(x_{s-r}, \mathrm{d}x_u) \nabla_{\theta} \log [m_{\theta}(x_{u-1}, x_u) g_{\theta}(x_u, \mathbf{y}_u)] L_{\theta, u:s-1}(x_u, \mathbb{X})}{\int \chi(\mathrm{d}x_{s-r}) L_{\theta, s-r:s-1}(x_{s-r}, \mathbb{X})} \right|. \end{aligned}$$

The proof is concluded upon noting that for any $s-r \leq u \leq s+1$,

$$\left| \frac{\int \chi(\mathrm{d}x_{s-r}) g_{\theta}(x_{s-r}, y_{s-r}) L_{\theta, s-r:u-1}(x_{s-r}, \mathrm{d}x_u) \nabla_{\theta} \log g_{\theta}(x_u, \mathbf{y}_u) L_{\theta, u:s-1}(x_u, \mathbb{X})}{\int \chi(\mathrm{d}x_{s-r}) g_{\theta}(x_{s-r}, \mathbf{y}_{s-r}) L_{\theta, s-r:s-1}(x_{s-r}, \mathbb{X})} \right|$$

is upper bounded by $\phi(\mathbf{y}_u)$.

(ii) is a consequence of Lemma 4.11(ii) and Lemma 4.12(i). \square

Theorem 4.13. Assume H2, H4(a-b) and S1.

(i) For any $T \geq 0$ and any distribution χ on \mathbb{X} , the functions $\theta \mapsto \ell_{\theta, T}^{\chi, 0}(\mathbf{Y})$ and $\theta \mapsto c_{\star}(\theta)$ are continuously differentiable \mathbb{P}_{\star} -a.s.

(ii) For any initial distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$,

$$\frac{1}{T} \nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) \xrightarrow{T \rightarrow +\infty} \nabla_{\theta} c_{\star}(\theta) \quad \mathbb{P}_{\star} - a.s. \quad (48)$$

Proof. By (33) and Lemma 4.10, for any \mathbf{y} such that $\phi(\mathbf{y}_u) < +\infty$ for any $u \in \mathbb{Z}$, $\ell_{\theta, T}^{\chi, 0}(\mathbf{y})$ and $\delta_{\theta, s}^{\chi, 0}(\mathbf{y})$ are continuously differentiable and (33) implies

$$\nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{y}) = \sum_{s=0}^{T-1} \nabla_{\theta} \delta_{\theta, s}^{\chi, 0}(\mathbf{y}).$$

This decomposition leads to

$$\frac{1}{T} \nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) = \frac{1}{T} \sum_{s=0}^{T-1} \left(\nabla_{\theta} \delta_{\theta, s}^{\chi, 0}(\mathbf{Y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right) + \frac{1}{T} \sum_{s=0}^{T-1} \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) . \quad (49)$$

Consider the first term of the rhs of (49). Since \mathbf{Y} is a stationary process, assumption S1(b) implies that $\mathbb{E}_{\star} [\xi(\mathbf{Y})] < +\infty$, where ξ is defined by (45). Then, $\xi(\mathbf{Y}) < +\infty \quad \mathbb{P}_{\star}$ a.s and by Lemma 4.11(ii), for any $0 \leq s \leq T-1$,

$$\left| \nabla_{\theta} \delta_{\theta, s}^{\chi, 0}(\mathbf{Y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right| \leq \xi(\mathbf{Y}) \frac{16\rho^{-1/4}}{1-\rho} \rho^{s/4} .$$

Therefore

$$\frac{1}{T} \sum_{s=0}^{T-1} \left| \nabla_{\theta} \delta_{\theta, s}^{\chi, 0}(\mathbf{Y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right| \leq \frac{1}{T} \xi(\mathbf{Y}) \frac{16\rho^{-1/4}}{1-\rho} \frac{1}{1-\rho^{1/4}} ,$$

and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{s=0}^{T-1} \left(\nabla_{\theta} \delta_{\theta, s}^{\chi, 0}(\mathbf{Y}) - \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) \right) = 0 \quad \mathbb{P}_{\star} \text{ a.s.}$$

Finally, consider the second term of the rhs of (49). By Lemma 4.12 (applied with $r = 1$), $\mathbb{E}_{\star} [|\nabla_{\theta} \delta_{\theta}(\mathbf{Y})|] < +\infty$. Under H4(b), the ergodic theorem (see [1, Theorem 24.1, p.314]) states that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{s=0}^{T-1} \nabla_{\theta} \delta_{\theta}(\vartheta^s \circ \mathbf{Y}) = \mathbb{E}_{\star} [\nabla_{\theta} \delta_{\theta}(\mathbf{Y})] \quad \mathbb{P}_{\star} \text{ a.s.}$$

Then, by (49) and the above discussion,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) = \mathbb{E}_{\star} [\nabla_{\theta} \delta_{\theta}(\mathbf{Y})] \quad \mathbb{P}_{\star} \text{ a.s.}$$

By Lemma 4.12, applied with $r = 0$,

$$\sup_{\theta \in \Theta} |\nabla_{\theta} \delta_{\theta}(\mathbf{Y})| \leq 2[\phi(Y_0) + \phi(Y_1)] + \xi(\mathbf{Y}) \rho^{1/2} ,$$

and the rhs is integrable under the stated assumptions. Therefore, by the dominated convergence theorem, $\mathbb{E}_{\star} [\nabla_{\theta} \delta_{\theta}(\mathbf{Y})] = \nabla_{\theta} \mathbb{E}_{\star} [\delta_{\theta}(\mathbf{Y})] = \nabla_{\theta} c_{\star}(\theta)$. This concludes the proof. \square

4.5 Contrast and the limit set \mathcal{L}

Theorem 4.14. *Assume H2, H3-(1), H4(a-b) and S1. Then, $\theta \in \mathcal{L}$ if and only if $\nabla_{\theta} c_{\star}(\theta) = 0$ where $c_{\star}(\theta) = \lim_{T \rightarrow +\infty} T^{-1} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) \quad \mathbb{P}_{\star}$ - a.s for any initial distribution χ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$.*

Proof. For any initial distribution χ , all $\theta \in \Theta$ and all $T > 0$,

$$\frac{1}{T} \nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) = \frac{1}{T} \int \nabla_{\theta} \log p_{\theta}^{\chi}(x_{0:T}, \mathbf{Y}_{1:T}) \frac{p_{\theta}^{\chi}(x_{0:T}, \mathbf{Y}_{1:T})}{\int p_{\theta}^{\chi}(z_{0:T}, \mathbf{Y}_{1:T}) \lambda(dz_{0:T})} \lambda(dx_{0:T}),$$

where p_{θ}^{χ} is defined in [8, Eq. (21)]. Under Assumption H1(a)

$$\frac{1}{T} \nabla_{\theta} \log p_{\theta}^{\chi}(x_{0:T}, \mathbf{Y}_{1:T}) = \nabla_{\theta} \phi(\theta) + \nabla_{\theta} \psi^T(\theta) \left\{ \frac{1}{T} \sum_{t=1}^T S(x_{t-1}, x_t, \mathbf{Y}_t) \right\},$$

and then, by definition of $\Phi_{\theta, t, T}^{\chi, 0}(S, \mathbf{Y})$ (see (20)),

$$\frac{1}{T} \nabla_{\theta} \ell_{\theta, T}^{\chi, 0}(\mathbf{Y}) = \nabla_{\theta} \phi(\theta) + \nabla_{\theta} \psi^T(\theta) \left\{ \frac{1}{T} \sum_{t=1}^T \Phi_{\theta, t, T}^{\chi, 0}(S, \mathbf{Y}) \right\}. \quad (50)$$

Under the stated assumptions, [8, Theorem 4.1] and Theorem 4.13 can be applied. Therefore, (50) becomes, as $T \rightarrow +\infty$,

$$\nabla_{\theta} c_{\star}(\theta) = \nabla_{\theta} \phi(\theta) + \nabla_{\theta} \psi^T(\theta) \{ \mathbb{E}_{\star} [\mathbb{E}_{\theta} [S(X_{-1}, X_0, \mathbf{Y}_0) | \mathbf{Y}]] \}.$$

The proof follows upon noting that by definition of $\bar{\theta}$, the unique solution to the equation $\nabla_{\theta} \phi(\tau) + \nabla_{\theta} \psi^T(\tau) \{ \mathbb{E}_{\star} [\mathbb{E}_{\theta} [S(X_{-1}, X_0, \mathbf{Y}_0) | \mathbf{Y}]] \} = 0$ is $\tau = \mathbf{R}(\theta)$. \square

5 Additional experiments

In this section, we provide additional plots for the applications studied in [8, Section 3].

5.1 Linear Gaussian model

Figure 1 illustrates the fact that the convergence properties of the BOEM do not depend on the initial distribution χ used in each block. Data are sampled using $\phi = 0.97$, $\sigma_u^2 = 0.6$ and $\sigma_v^2 = 1$. All runs are started with $\phi = 0.1$, $\sigma_u^2 = 1$ and $\sigma_v^2 = 2$. Figure 1 displays the estimation of ϕ by the averaged BOEM algorithm with $\tau_n \sim n$ and $\tau_n \sim n^{1.5}$, over 100 independent Monte Carlo runs as a function of the number of blocks. We consider first the case when χ is the stationary distribution of the hidden process i.e. $\chi \equiv \mathcal{N}(0, (1 - \phi^2)^{-1} \sigma_u^2)$, and the case when χ is the filtering distribution obtained at the end of the previous block, computed with the Kalman filter. The estimation error is similar for both initialization schemes, even when ϕ is close to 1 and for any choice of $\{\tau_n\}_{n \geq 1}$.

The theoretical analysis of BOEM says that a sufficient condition for convergence is the increasing size of the blocks. On Figure 2, we compare different strategies for the definition of $\tau_n \stackrel{\text{def}}{=} T_n - T_{n-1}$. A slowly increasing sequence $\{\tau_n\}_{n \geq 0}$ is compared to different strategies using the same number of observations within each block. We consider the Linear Gaussian model:

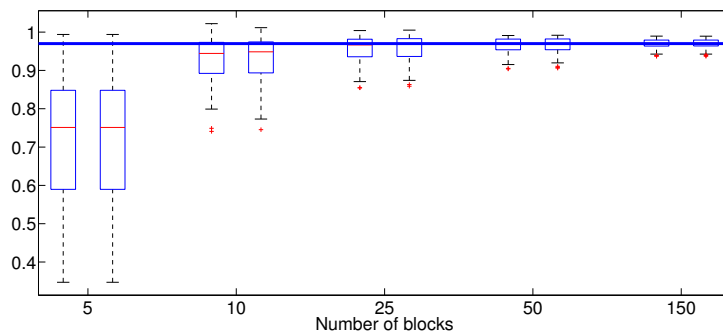
$$X_{t+1} = \phi X_t + \sigma_u U_t, \quad Y_t = X_t + \sigma_v V_t,$$

where $X_0 \sim \mathcal{N}(0, \sigma_u^2(1 - \phi^2)^{-1})$, $\{U_t\}_{t \geq 0}, \{V_t\}_{t \geq 0}$ are i.i.d. standard Gaussian r.v., independent from X_0 . Data are sampled using $\phi = 0.9$, $\sigma_u^2 = 0.6$ and $\sigma_v^2 = 1$. All runs are started with $\phi = 0.1$, $\sigma_u^2 = 1$ and $\sigma_v^2 = 2$. Figure 2 shows the estimation of ϕ over 100 independent Monte Carlo runs (same conclusions could be drawn for σ_u^2 and σ_v^2). For each choice of $\{\tau_n\}_{n \geq 0}$, the median and first and last quartiles of the estimation are represented as a function of the number of observations.

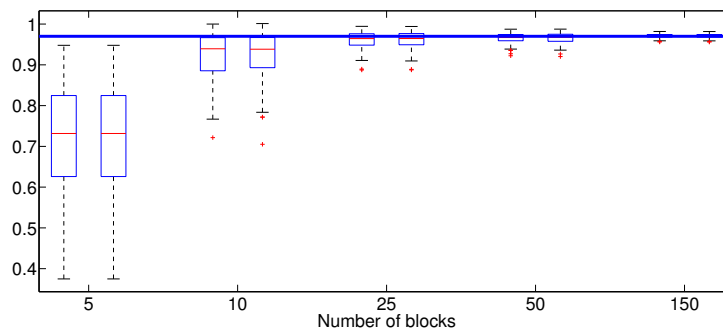
We observe that BOEM does not converge when the block size sequence is constant and small: as shown in Figure 2, if the number of observations is too small ($\tau_n = 25$), the algorithm is a poor approximation of the *limiting EM* recursion and does not converge. With greater block sizes ($\tau_n = 100$ or $\tau_n = 350$), the algorithm converges but the convergence is slower because it is initialized far from the true value and many observations are needed to get several estimations. BOEM with slowly increasing block sizes has a better behavior since many estimations are produced at the beginning and, once the estimates are closer to the true value, the bigger block sizes reduce the variance of the estimation.

Moreover, our convergence rates are given up to a multiplicative constant : the theory says that $\sum_n \tau_n^{-\gamma/2} < \infty$ where γ is related to the ergodic behavior of the HMM (see assumptions H6).

Even if the sequence is chosen to increase at a polynomial rate, we can have $\tau_n \sim c n^\alpha$ ($\alpha > 1$) with a constant c such that the first blocks are quite small



(a) $\tau_n \sim n$



(b) $\tau_n \sim n^{1.5}$

Figure 1: Estimation of ϕ after 5, 10, 25, 50 and 150 blocks, with two different initialization schemes: the stationary distribution (left) and the filtering distribution at the end of the previous block (right). The boxplots are computed with 100 Monte Carlo runs.

to allow a sufficiently large number of updates of the parameters $\{\theta_n, n \geq 1\}$. During a (deterministic) "burn-in" period, the first blocks can even be of a fixed length before beginning the "increasing" procedure.

5.2 Finite state-space HMM

Observations are sampled using $d = 6$, $v = 0.5$, $x_i = i, \forall i \in \{1, \dots, d\}$ and the true transition matrix is given by

$$m = \begin{pmatrix} 0.5 & 0.05 & 0.1 & 0.15 & 0.15 & 0.05 \\ 0.2 & 0.35 & 0.1 & 0.15 & 0.05 & 0.15 \\ 0.1 & 0.1 & 0.6 & 0.05 & 0.05 & 0.1 \\ 0.02 & 0.03 & 0.1 & 0.7 & 0.1 & 0.05 \\ 0.1 & 0.05 & 0.13 & 0.02 & 0.6 & 0.1 \\ 0.1 & 0.1 & 0.13 & 0.12 & 0.1 & 0.45 \end{pmatrix}.$$

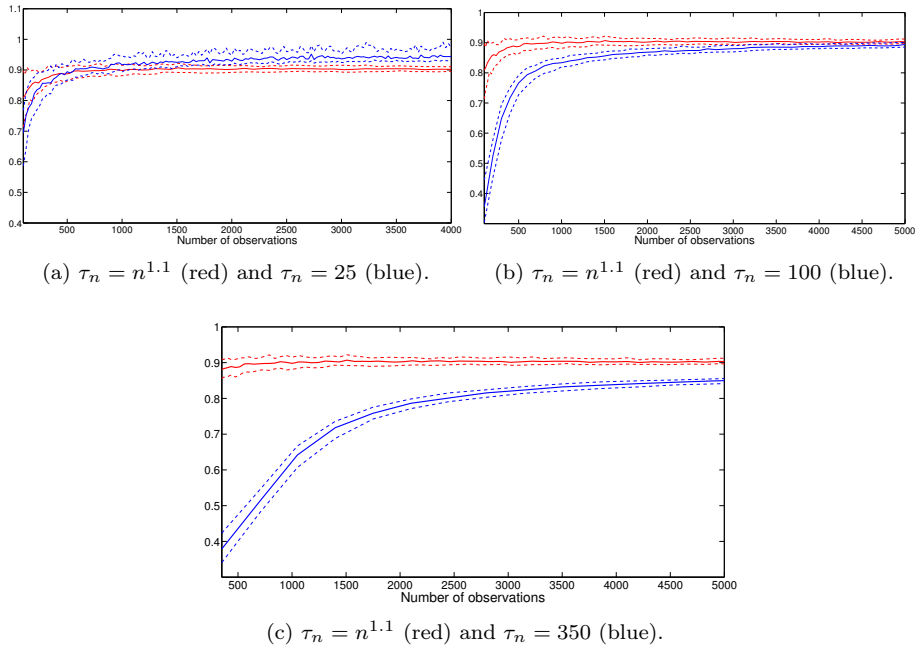


Figure 2: Estimation of ϕ with different block size schemes: the median (bold line) and the first and last quartiles (dotted line) are shown for $\tau_n = n^{1.1}$ (red), $\tau_n = 100$ (black) and $\tau_n = 350$ (purple). The quantities are computed with 100 Monte Carlo runs.

5.2.1 Comparison to an online EM based procedure

In this case, we want to estimate the states $\{x_1, \dots, x_d\}$. All the runs are started from $v = 2$ and from the initial states $\{-1; 0; .5; 2; 3; 4\}$. The experiment is the same as the one in [8, Section 3.2.1]. The averaged BOEM is compared to an online EM procedure (see [2]) combined with Polyak-Ruppert averaging (see [10]). This online EM based algorithm follows a stochastic approximation update and depends on a step-size sequence $\{\gamma_n\}_{n \geq 0}$ which is chosen in the same way as in [8, Section 3.2.1]. Figure 3 displays the empirical median and first and last quartiles for the estimation of x_2 with both averaged algorithms as a function of the number of observations. These estimates are obtained over 100 independent Monte Carlo runs with $\tau_n = n^{1.1}$ and $\gamma_n = n^{-0.53}$. Both algorithms converge to the true value $x_2 = 2$ and these plots confirm the similar behavior of BOEM and the online EM of [2].

5.2.2 Comparison to a recursive maximum likelihood procedure

In the numerical applications below, we give supplementary graphs to compare the convergence of the averaged BOEM with the convergence of the Polyak-

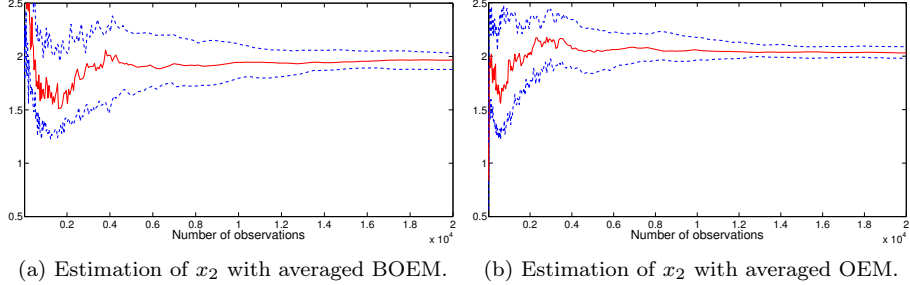


Figure 3: Estimation of x_2 using the averaged online EM and averaged BOEM. Each plot displays the empirical median (bold line) and the first and last quartiles (dotted lines) over 100 independent Monte Carlo runs with $\tau_n = n^{1.1}$ and $\gamma_n = n^{-0.53}$. The first ten observations are omitted for a better visibility.

Ruppert averaged RML procedure. The experiment is the same as the one in [8, Section 3.2.2]. Figure 4 and 5 displays the empirical median and first and last quartiles of the estimation of v and $m(1, 2)$ over 100 independent Monte Carlo runs. Both algorithms have a similar behavior for the estimation of these parameters.

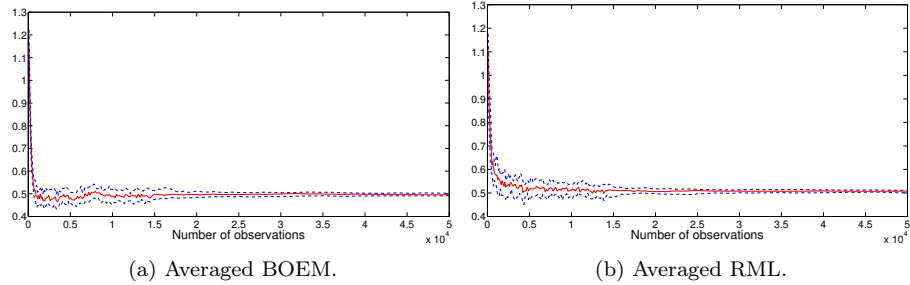


Figure 4: Empirical median (bold line) and first and last quartiles (dotted line) for the estimation of v using the averaged RML algorithm (right) and the averaged BOEM algorithm (left). The true values is $v = 0.5$ and the averaging procedure is started after 10000 observations. The first 10000 observations are not displayed for a better clarity.

5.3 Stochastic volatility model

Consider the following stochastic volatility model:

$$X_{t+1} = \phi X_t + \sigma U_t, \quad Y_t = \beta e^{\frac{X_t}{2}} V_t,$$

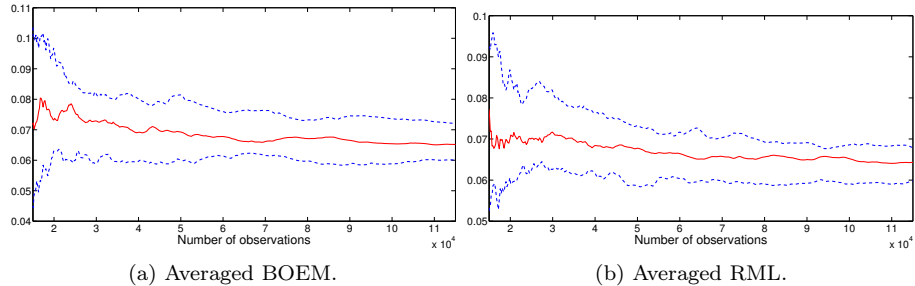
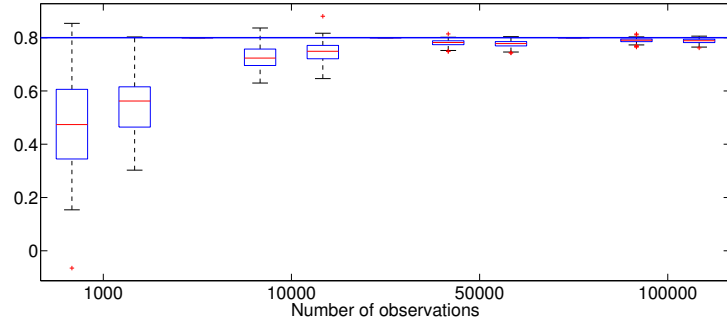


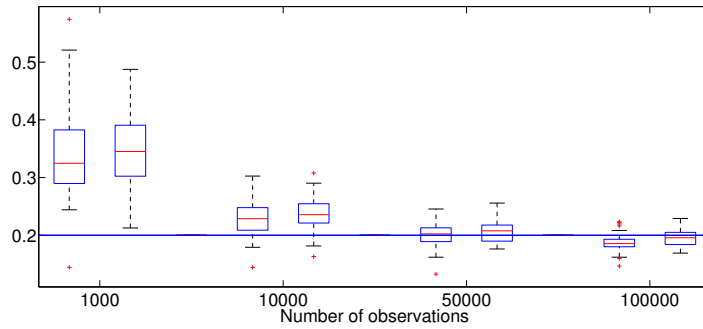
Figure 5: Empirical median (bold line) and first and last quartiles (dotted line) for the estimation of $m(1, 2)$ using the averaged RML algorithm (right) and the averaged BOEM algorithm (left). The true values is $m(1, 2) = 0.05$ and the averaging procedure is starter after 10000 observations. The first 10000 observations are not displayed for a better clarity.

where $X_0 \sim \mathcal{N}(0, (1 - \phi^2)^{-1} \sigma^2)$ and $(U_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ are two sequences of i.i.d. standard Gaussian r.v., independent from X_0 . Data are sampled using $\phi = 0.8$, $\sigma^2 = 0.2$ and $\beta^2 = 1$. All runs are started with $\phi = 0.1$, $\sigma^2 = 0.6$ and $\beta^2 = 2$.

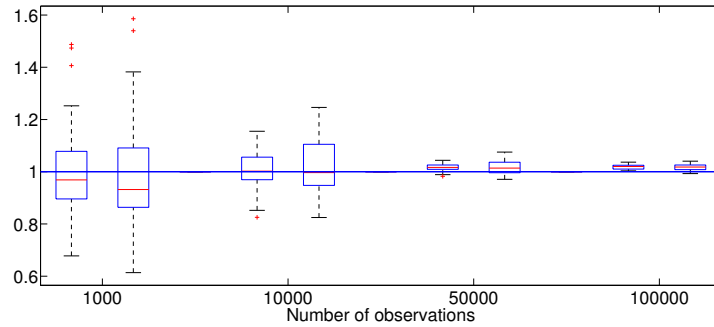
In this model, the smoothed sufficient statistics $\{\bar{S}_{\tau_n}^{\chi, T_{n-1}}(\theta_{n-1}, \mathbf{Y})\}_{n \geq 1}$ can not be computed explicitly. We thus propose to replace the exact computation by a Monte Carlo approximation based on particle filtering. The performance of the Stochastic BOEM is compared to the online EM algorithm given in [2] (see also [5]). To our best knowledge, there do not exist results on the asymptotic behavior of the algorithms by [2, 5]; these algorithms rely on many approximations that make the proof quite difficult (some insights on the asymptotic behavior are given in [2]). Despite there are no results in the literature on the rate of convergence of the Online EM algorithm by [2] we choose the step size γ_n in [2] and the block size τ_n s.t. $\gamma_n = n^{-0.6}$ and $\tau_n \propto n^{3/2}$ (see [8, Section 3.2.1] for a discussion on this choice). 50 particles are used for the approximation of the filtering distribution by Particle filtering. We report in Figure 6, the boxplots for the estimation of the three parameters (β, ϕ, σ^2) for the Polyak-Ruppert [10] averaged Online EM and the averaged BOEM. Both average versions are started after 20000 observations. Figure 6 displays the estimation of ϕ , σ^2 and β^2 . This figure shows that both algorithms have the same behavior. Similar conclusions are obtained by considering other true values for ϕ (such as $\phi = 0.95$). Therefore, the intuition is that online EM and Stochastic BOEM have the same asymptotic behavior. The main advantage of the second approach is that it relies on approximations which can be controlled in such a way that we are able to show that the limiting points of the particle version of the Stochastic BOEM algorithms are the stationary points of the limiting normalized log-likelihood of the observations.



(a) Estimation of ϕ .



(b) Estimation of σ^2 .



(c) Estimation of β^2 .

Figure 6: Estimation of ϕ , σ^2 and β^2 using the averaged online EM algorithm (left) and the averaged BOEM (right), after $n = \{1000, 10k, 50k, 100k\}$ observations. The true value of ϕ is 0.8.

We now compare the two algorithms when the true value of ϕ is (in absolute value) closer to 1: we choose $\phi = 0.95$, β^2 and σ^2 being the same as in the previous experiment. As illustrated on Figure 7, the same conclusions are drawn for greater values of ϕ .

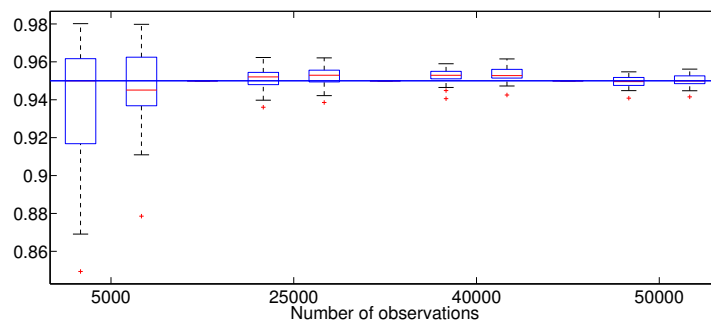


Figure 7: Estimation of ϕ using the averaged online EM algorithm (left) and the averaged BOEM algorithm (right), after $n = \{5k, 25k, 40k, 50k\}$ observations. The true value of ϕ is 0.95.

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