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# On Classical Ideal Gases

Jacques ARNAUD \*, Laurent CHUSSEAU †, Fabrice PHILIPPE ‡

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## Abstract

The air density on earth decays exponentially as a function of altitude. To derive this law one usually invokes the Boltzmann factor, itself derived from statistical considerations. We show that this (barometric) law may be derived solely from the democritian concept of corpuscles moving in vacuum. We employ a principle of simplicity, namely that this law is *independent* of the laws of physics, aside from the law of conservation of energy. This viewpoint puts aside restrictive assumptions that are source of confusion. Similar observations apply to the ideal-gas law. It is usually derived under the assumption that the temperature is proportional to the corpuscle average kinetic energy, or else, from a form of the quantum theory. In contradistinction, we show that it follows solely from the postulate that the ideal-gas law is independent of the law of corpuscle motion. On the physical side we employ only the concept of potential energy. Most of the end results are known, but the method appears to be novel. The mathematics being elementary (no integration is needed), the present paper should facilitate the understanding of the physical meaning of the barometric and ideal-gas laws, even though not-usually-taught concepts are being introduced. An appendix presents a model of gravity based on light pressure. A report is quoted in appendix.

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# 1 Introduction

The purpose of this paper is to show that the barometric and ideal-gas laws may be obtained on the sole basis of the democritian model according to which nature consists of corpuscles moving in a vacuum, plus a principle of simplicity: namely that these fundamental laws are *independent* of the law of corpuscle motion (non-relativistic, relativistic, or otherwise: see Appendix A). The temperature  $\theta$  enters solely for dimensional reasons. We show from the general expressions of the gas internal energy and of the force (or pressure) that the heat delivered by the gas is  $\theta dS$ , an expression of the entropy  $S$  being given. This result enables us to prove that the formally-introduced temperature  $\theta$  is a thermodynamic temperature. Indeed, we recover for ideal gases the general Carnot result asserting that the maximum efficiency of thermal engines is:  $1 - \theta_l/\theta_h$ , where  $\theta_l$  denotes the cold bath temperature and  $\theta_h$  the hot bath temperature. An alternative derivation of the Carnot result, also solely based on the concept of potential energy, is in [1, 2].

In the title of this paper (“Classical Ideal Gases”), the word “classical” means: non-quantum, but the non-relativistic approximation is not made, except in examples. According to *general relativity*, thermal energy has weight. But this (so-called Tolman) effect entails a temperature variation on the order of  $10^{-12}$  on earth. It is entirely negligible. “Ideal Gases” refer to non-interacting identical corpuscles. Accordingly, the corpuscles are independent, and we may consider a single corpuscle. As a result of the slight quivering of the container wall, there is an exchange of energy between the corpuscle and the heat bath (of course, for a static wall, even a rough one, the corpuscle energy would be a constant of motion). This is the so-called “canonical ensemble”. Under those conditions heat engines and heat pumps may very well be constructed with a single corpuscle, or atom [3]. In contradistinction, the paper [4]: “Microcanonical single-particle distributions for an ideal gas in a gravitational field” is valuable because it establishes the gas law in the micro-canonical ensemble with the effect of gravity taken into account as in the present paper, in the limit of a very large number of corpuscles. In that case, no energy exchange with the outside is needed. However, that paper is complicated, both from the mathematical and physical viewpoints. The reader may wish to compare that micro-canonical many-corpuscle paper with our canonical single-corpuscle paper in terms of simplicity and pedagogical value. Of course the final results are the same.

The reader may feel that our statement that the above invariance principle *implies* the barometric and ideal-gas laws, without anything else, is surprising. Yet, we hope that we can convince him/her that this is indeed the case. On the other hand, a quick reading of standard books on Thermodynamics may lead other readers to believe that this is a well-known fact. For example, Callen [5] states correctly that: “The essence of the ideal-gas law is that molecules of the gas do not interact. This simple fact *implies* that  $PV \propto NT$ ”. However, in order to reach this conclusion, the author needs postulate the Newtonian law of motion, quantum theory, and the Boltzmann factor. The claim that corpuscle independence entails the ideal-gas law (without anything else) therefore does not appear to have been justified before.

Let us emphasize that our goal is to derive the barometric [6] and ideal-gas laws from first principles, only conservation of potential energy being assumed. We do not use the concept of kinetic energy, nor do we postulate any particular law of corpuscle motion. Accordingly, a given potential  $\phi(z) = wz$  for a weight  $w$  does not imply any specific law of motion  $z(t)$ . Because the presence of a corpuscle affects negligibly the gravitational potential, one may consider the latter to be an *external* potential.

To our knowledge, the concept presented in this paper has not been considered before. However, the final expressions relating to the ideal-gas law in the presence of gravity are known, see for

example [4, 7].

**Pedagogical usefulness:** Admittedly, the present paper departs much from usually-taught concepts in Thermodynamics [8]. We indeed ignore the concepts of kinetic energy, of micro-states, and the laws of Physics that would provide the motion  $z = z(t)$  (to within arbitrary  $t$  and  $z$  translations) of corpuscles of given energy. The temperature  $\theta$  (to within a constant factor) follows from dimensional considerations and not as usual from the derivative of the average energy  $U(S, h)$  with respect to the entropy  $S$  at constant volume (here denoted  $h$ ). Quite to the contrary, an explicit expression of the (classical ideal-gas) entropy function  $S(\theta, h)$  is derived from the average force  $\langle F \rangle$  (or pressure) and average energy  $U$  introducing, for convenience only, an intermediate quantity: the free-energy  $A$ .

This change of perspective suggests that one should first emphasize, in teaching the subject to students, the Democritian concept of corpuscles moving in vacuum, and the concept of potential energy (e.g., from cords and pulleys experiments). Next one may draw on the blackboard the up and down motion of a corpuscle thrown upward from  $z = 0$  in the earth gravitational field (supposed to be static and independent of the vertical  $z$  coordinate), the particular function  $z(t)$  being left undefined. One may at that point recall the *experimental* result obtained by Galileo: for small speeds, the corpuscle round-trip time  $\tau$  (from  $z = 0$  and back) is proportional to the square root of  $z_m$ , where  $z_m$  denotes the maximum corpuscle altitude [9]. But one should emphasize that the barometric and ideal-gas laws, to be derived next, *do not depend* on the Galileo observation. For example, had Galileo incorrectly stated that  $\tau \propto z_m^2$ , the barometric and ideal-gas laws derived from this erroneous observation would be unaffected. At that point, it should be obvious to the students, looking at Fig. 1, that the fraction of time during which the corpuscle is located above some altitude  $z \leq z_m$  is:  $\tau(z_m - z)/\tau(z_m)$ , see (2).

Next, it should be pointed out that, as a result of weak thermal contact with a bath (that needs not be specified further), the maximum altitude  $z_m$  varies slowly in the course of time. This variation is described statistically by some probability law  $\omega(z_m)$ . The question at that point is: what should be that probability law in order for the average time spent by the corpuscle above some level to be *independent* of the  $\tau(z_m)$ -function introduced above. The answer is very simple, as shown in Section 2. The end result is the barometric law: The probability that the corpuscle be above  $z$  decays exponentially.

It should be explained (or demonstrated) next that a weight measured by a high-inertia balance does not depend on possible up and down motions of the weight. Accordingly, for a unit weight, the weight impact is equal to the motion period. This observation enables us to express the force  $F$  in terms of round-trip times when a rigid plate is present at  $z = h$ . Again, it is the condition that the average force be *independent* of the  $\tau(z_m)$ -function introduced above that provides the ideal-gas law.

Students having followed the reasoning so far should be able to appreciate subsequent derivations of the ideal-gas law (generalized here to account for a uniform gravity), of the average energy  $U$ , and of the concept of entropy, since only straightforward mathematics is then being employed. We have considered above a single corpuscle; but the results obviously hold for any fixed number  $N$  of non-interacting identical classical corpuscles, the average force, energy and entropy being then multiplied by  $N$ .

To conclude, the approach advocated in this paper enables us to recover classical results in Thermodynamics in a considerably generalized form (but only for independent classical corpuscles, and laws of Physics that are independent of time and altitude translations). Admittedly, our

presentation goes against the order in which basic concepts are usually taught. But once the students are used to it they should be able to appreciate the greater simplicity.

**Concepts relating to heat since the antiquity:** From Boyle's time (*circa* 1660) to Carnot's time (*circa* 1824), the most popular theory, usually referred to as the "caloric theory of heat", considered heat as a kind of fluid consisting of weightless mutually repelling particles, attracted by matter. This fluid could flow from hotter to colder matter, see Levermore [10]. This hypothesis, however, did not explain Rumford's experiments showing that friction alone generates heat. Doubts about the validity of the caloric theory were expressed in Carnot's notes, published after his early death. As a matter of fact, the alternative model of heat as a form of *matter* motion (without any additional fluid) had already been enunciated in the antiquity.

Democritus, who lived about 300 years B.C., described nature as a collection of corpuscles that cannot be split, moving in vacuum. These corpuscles differ from one-another in form, position and weight. In the case of a gas, interaction between corpuscles may often be neglected, but they collide with the container's walls. Platon [11] ascribed heat to corpuscular motion: "Heat and fire are generated by impact and friction, but that's motion". Much later, Francis Bacon (1561-1626) wrote: "The very nature of heat is motion, and nothing else". This view-point is more explicit in Daniel Bernoulli writing (1738): "Gas atoms are moving randomly, and pressure is nothing else but the impact of the atoms on their container walls". Lastly, Carnot introduced energy considerations *circa* 1830: "Heat is nothing but motive power, or rather another form of motion. When motive power is destroyed, heat is generated precisely in proportion of the motive power destroyed. Likewise, when heat is destroyed, motive power is generated".

In the present paper, we employ the law of conservation of potential energy, well known since the antiquity from cords and pulleys experiments, rather than the concept of kinetic energy introduced by Leibniz. The Leibniz form is now recognized as being valid at small temperatures only ( $\theta \equiv k_B T \ll mc^2$ , where  $mc^2$  denotes the corpuscle rest energy). It should be recalled that the thermodynamic temperature is generally *not* proportional to the average kinetic energy.

**Experimental results relating to air:** The first accurate experiments relating to gases relied on the invention of the thermometer by Galileo and of the barometer by his assistant Torricelli. Then, to Pascal experiments on atmospheric pressure. Pascal ascribed the diminution of the height of a mercury column as a function of altitude to the reduction of the weight of the air above the barometer. It was later shown that the pressure decays exponentially (or more precisely as the sum of two exponentials, one relating to nitrogen, and one relating to oxygen).

Let us recall the crucial experiments performed in the seventeenth century concerning the properties of air. Air, consisting mostly of di-atomic nitrogen, may be viewed as an ideal gas. When a tight box contains some amount of air, the volume-pressure product is a constant at room temperature, a law enunciated for the first time by Boyle in 1660: "Pressure and expansion are in reciprocal proportions". Boyle employed a J-shaped glass tube, with the sealed small side full of air, and the other full of mercury. The J left-side height was a measure of volume and the right-side height a measure of pressure. Subsequent experiments have shown that this law is applicable at any constant temperature, for example at various liquid boiling temperatures, within some experimental range. We call "generalized Boyle law" the expression:  $V(P, \theta) = f(\theta)/P$ , where P represents pressure, V volume, and  $f(\theta)$  some temperature measure.

From an experimental standpoint we could define temperature as the pressure relating to some given amount of matter contained within some fixed volume. As the temperature gets higher

the pressure increases. This pressure may be used to define  $\theta$ . Of course, different temperature scales would be obtained for different substances, but such thermometers may be calibrated one against another because temperatures tend to equalize in equilibrium. Rarefied helium may be described with great accuracy as a collection of independent corpuscles, except perhaps at very low temperatures when quantum effects become significant and at very high temperatures when the helium atoms may get ionized. The theory presented in this paper shows on the basis of the corpuscular model that  $\theta$ , initially introduced formally from dimensional considerations, coincides with the thermodynamic temperature. This is the temperature that enters in the expression of thermal-engine efficiencies.

Gay-Lussac has shown in 1802 that, at atmospheric pressure, the volume increment of various gases from freezing to boiling water temperatures is 37.5 %. The empirical Gay-Lussac law makes sense only if one specifies which thermometer is being employed. One may employ a gas thermometer from a selected gas such as helium in two ways. One method consists of defining the temperature as the cylinder height (or volume) at a fixed pressure, for example at the standard atmospheric pressure. In the following, we assume that a second method is being employed instead: the temperature is defined as the force that must be exerted on the piston to maintain the height at a fixed value, for example one meter. If the Gay-Lussac experiment were applied to a gas identical to the gas employed in the thermometer (helium in our example) the fact that pressure is proportional to temperature would be obvious. The importance of the Gay-Lussac experiment is that the proportionality law is found to be approximately valid for any gas. It was subsequently discovered that the Gay-Lussac law is reasonably accurate only at very small pressures. The theoretical reason that explains this observation is that, at low pressures, the molecules of the tested gas and those of the thermometer gas may both be considered as independent non-interacting corpuscles.

In 1803, Dalton, on the basis of his studies of chemical compounds and gaseous mixtures suggested that matter consists of atoms of different masses that combine in simple ratios. He discovered the partial-pressure law according to which the total pressure exerted by a gas mixture is equal to the sum of the pressure that each one of the gases would exert if it occupied the full volume alone. Finally, in 1811, Avogadro concluded that equal volumes of gases at the same temperature and pressure contain the same number of molecules (or corpuscles). This entails that  $PV/\theta$  is proportional to  $N$ , now interpreted as the number of corpuscles. On empirical grounds, the ideal-gas law may therefore be written as:

$$PV = N\theta \tag{1}$$

where  $V$  denotes the volume,  $P$  the pressure,  $N$  the number of corpuscles, and  $\theta \equiv k_B T$  the temperature.

The ideal-gas law has been partly explained on the basis of a kinetic theory by Waterston [12] in 1843, the kinetic theory being based itself on non-relativistic mechanics. Let us now recall the basic assumptions on which rest the usual proofs of the barometric and ideal-gas laws, and then present our model.

**Usual kinetic and statistical theories** A recent reference by Miller [13] lists the assumptions on which the gas kinetic theory is supposed to rest. Some of them express the democritian hypothesis and are indeed essential. The usefulness of the others [14] may be questioned. As we shall show, none are needed. It suffices that the (perhaps unique) corpuscle be in thermal contact with the ground. Besides, the condition that the average kinetic energy *depends* only on the system temperature (see the footnote) is insufficient to establish the ideal-gas law because the temperature remains

undefined. In the non-relativistic approximation the appropriate assumption should read: “The average kinetic energy is *proportional* to the system temperature”. But at arbitrary temperatures this assumption is false.

Let us sketch the most usual derivations of the barometric and ideal-gas laws to remind the readers of the underlying assumptions. Note that the barometric law may be obtained from the ideal-gas law, and conversely, if one postulates that weightless infinitely-thin plates may be introduced or removed at will in the gas at various altitudes. However, this postulate is at best plausible. Norton [15] derived the ideal-gas law from the barometric equation. But the latter involves the Boltzmann factor that requires other physical considerations (see below), while in the present paper this factor comes in naturally, that is, for purely mathematical reasons [16].

The traditional derivation of the ideal-gas law, on the other hand, is based on non-relativistic mechanics. For a one-dimensional model, one considers a corpuscle moving back and forth between two plates separated by a distance  $h$ , one of them playing the role of a piston. If  $v$  denotes the speed of a corpuscle, an impact on a plate delivers to it an impulse  $2mv$  where  $m$  denotes the corpuscle mass, and this occurs every  $2h/v$  time units. It is concluded that the force  $F$  exerted on the piston is:  $2mv/(2h/v) = mv^2/h$ , that is:  $Fh = 2K$ , where  $K = \frac{1}{2}mv^2$  denotes the kinetic energy. It is recognised that there may be a distribution of kinetic energies. *Postulating* that the temperature  $\theta$  is proportional to the average kinetic energy one obtains for the average force the ideal-gas law:  $\langle F \rangle h \propto \theta$ . Alternatively, one may quantize the corpuscle wave-function and employ the Boltzmann factor.

The procedure described above has been generalized to relativistic motion ( $k_B T \sim mc^2$ ), where  $mc^2$  denotes the corpuscle rest energy, but the relation of  $v$  to  $p$  is different, and averaging requires a generalized Boltzmann factor. The same ideal-gas law is valid at any temperature (within the corpuscular model). Our thesis is that the ideal-gas law has simply nothing to do with the law of corpuscle motion, and that it is therefore not surprising that it holds for the Galileo, Einstein, and many other laws of motion. These are the reasons why we feel that the traditional proofs are unsatisfactory. An alternative is offered below. We consider in succession the barometric law, the ideal-gas law, and the internal energy. We finally show that the formally introduced temperature  $\theta$  is the thermodynamic temperature. Expressions for the round-trip time of a corpuscle from and back to the ground  $z=0$  level, denoted by  $\tau$ , is derived in Appendix A. The case of potentials not proportional to  $z$  ( $z$ -dependent weights) was treated by the same method in [17].

## 2 The barometric law

We are considering an unit-area cylinder with vertical axis in uniform gravity, resting on the ground ( $z = 0$ ) at some temperature. We consider the motion along the vertical  $z$ -axis of a single corpuscle, denoted in general by  $z = z(t; E)$ , where  $t$  denotes time. The corpuscle energy is defined as:  $E \equiv w z_m$  where  $w$  is the corpuscle weight and  $z_m$  the maximum altitude reached. To make the following expressions more concise we set initially  $w = 1$ , so that the corpuscle energy  $E$  coincides with  $z_m$ . With this convention, the dimension of the temperature  $\theta$  that we shall introduce is that of an energy divided by a force, that is, a length. (The reader is advised to write the following expressions in full, that is, restoring  $w$ , to clarify the dimensions). In the following, some regularity of the  $z(t)$ -function is assumed, but no specific law is presumed, except in examples. We set for convenience  $t = 0$  at the top of the trajectory, that is:  $z(0) = z_m$ ,  $z'(0) = 0$ , where a prime denotes a derivative with respect to  $t$ .

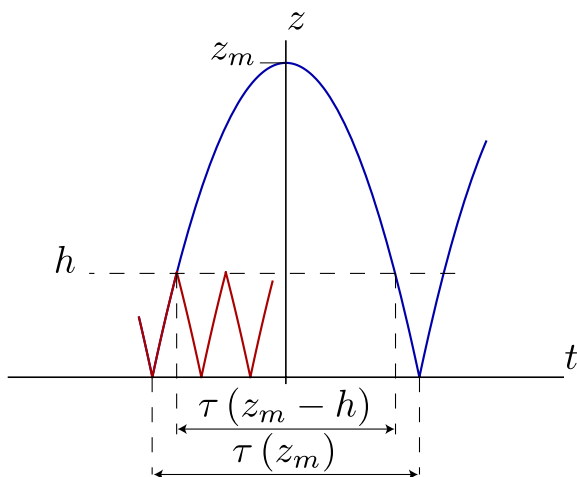


Figure 1: Space-time  $(z, t)$  trajectory of a corpuscle of unit weight ( $w = 1$ ) bouncing off the ground ( $z = 0$ ). The maximum altitude reached by the corpuscle is  $z_m = E$ , where  $E$  denotes the energy. The motion is periodic with period  $\tau(z_m)$ , where  $\tau(Z)$  denotes the corpuscle round-trip time at a distance  $Z$  from the top of the trajectory. When the altitude is restricted to  $h$  by a plate (dashed horizontal line) the motion remains periodic with a period evidently equal to:  $\tau(z_m) - \tau(z_m - h)$ . Note that this expression holds even if the motion is not symmetric in time.

Consider a single period of corpuscle motion as shown in Fig. 1. Let the corpuscle distance from the top of its trajectory be denoted by  $Z \equiv z_m - z \geq 0$ , at times  $t_1$  and  $t_2 \geq t_1$ . We call [18]: “round-trip time”  $\tau(Z) \equiv t_2 - t_1$ . Because gravity is static and uniform (that is, independent of time and altitude, respectively) this  $\tau$ -function depends only on  $Z$ . As an example, for non-relativistic motion:  $Z(t) = \frac{1}{2}gt^2$ . As said earlier the corpuscle weight  $w \equiv mg$ , where  $m$  denotes the corpuscle mass and  $g$  the gravitational acceleration, is set equal to unity for brevity. In that example:  $\tau(Z) = 2\sqrt{2Z/g} \propto \sqrt{Z}$ ,  $g$  being a constant.

The period of motion of a corpuscle bouncing off the ground ( $z = 0$ ) without any loss of energy (rigid walls and negligible gas friction), and having energy  $z_m$ , is according to the above definitions:  $\tau(z_m)$ . The time spent by the corpuscle above some  $z$ -level is obviously zero if  $z > z_m$ . The fraction of time during which the corpuscle is above  $z$  is:

$$\text{fraction of time above } z = \frac{\tau(z_m - z)}{\tau(z_m)}, \quad (2)$$

if  $z \leq z_m$ , as suggested on Fig. 1, and zero otherwise.

We now suppose that the ground on which the corpuscle is bouncing off is in contact with a heat bath [19]. This means that the ground level ( $z=0$ ) is not perfectly static as assumed above, but instead is quivering. Concretely, the ground level could be moving up and down according to some zero-mean fluctuation of small amplitude so that, upon impacting on the ground, the corpuscle may loose energy (when the ground level is moving downward), or gain energy (when the ground level is moving upward). We will not need the explicit form of this fluctuation. We only assume that the amplitude of that fluctuation is so small that the corpuscle energy does not vary significantly over many periods. Yet it may evolve slowly. The energy distribution  $\omega(z_m)$  refers to averages over



arbitrarily long times, and is presently unknown; it will be determined by enforcing the condition that the law of interest (presently the barometric law) does not depend on the laws of physics (except of the law of conservation of energy, which follows from the time-translation invariance of the system).

On the average, the fraction of time  $\mathcal{B}$  during which the corpuscle is above some  $z$ -level is, according to (2):

$$\mathcal{B}(z) = \frac{\int_z^\infty dz_m \omega(z_m) \tau(z_m - z) / \tau(z_m)}{\int_0^\infty dz_m \omega(z_m)}. \quad (3)$$

The lower limit of the integral in the numerator is  $z$  since the fractional time is equal to zero when  $z_m \leq z$ . Note that the dimension of  $\omega$  is arbitrary since it appears both in the numerator and the denominator of that expression. We shall give it the dimension of a time.

We now argue that  $\omega(z_m)$  *must be*:  $\exp(-w z_m / \theta) \tau(z_m)$ , where  $\theta$  is a constant having the dimension of an energy, as is required by the fact that the argument of an exponential is dimensionless [20]. First, let us introduce this distribution law in (3). We obtain (setting  $w = 1$ ):

$$\begin{aligned} \mathcal{B}(z) &= \frac{\int_z^\infty dz_m \exp(-z_m / \theta) \tau(z_m - z)}{\int_0^\infty dz_m \exp(-z_m / \theta) \tau(z_m)} \\ &= \frac{\exp(-z / \theta) \int_z^\infty dz_m \exp(-(z_m - z) / \theta) \tau(z_m - z)}{\int_0^\infty dz_m \exp(-z_m / \theta) \tau(z_m)} \\ &= \frac{\exp(-z / \theta) \int_0^\infty dz'_m \exp(-z'_m / \theta) \tau(z'_m)}{\int_0^\infty dz_m \exp(-z_m / \theta) \tau(z_m)} \\ &= \exp(-z / \theta). \end{aligned} \quad (4)$$

On the third line, we have used as an integration variable  $z'_m \equiv z_m - z$  instead of  $z_m$ . The last line follows from the fact that  $z_m, z'_m$  are dummy variables, so that we may replace  $z'_m$  by  $z_m$ . Note that, even though we have introduced integral signs, no integration has been performed.

The distribution  $\omega(z_m)$  introduced above is the only one that leads to a result (last line of (4)) independent of the particular form of the  $\tau(Z)$ -function, and therefore of the law of motion. For a purely mathematical reason, the term:  $\tau(z_m)$  must be there to cancel a similar term in the denominator of the expression of the fractional time. On the other hand, the only function of  $u \equiv z_m / \theta$  that may cause the integral in the numerator to go from 0 to infinity and cancel out with the integral in the denominator is:  $\exp(-u)$ , the argument being defined only to within a constant factor. Indeed, the exponential is the only function  $f(a)$  such that  $f(a + b) = f(a)f(b)$  for any  $a, b$ . In order to obtain the distribution  $\omega(z_m)$ , one would need to know the function  $\tau(Z)$ . But, remarkably, this distribution is not needed.

The gas density, defined as the probability that the corpuscle be located between  $z$  and  $z + dz$ , divided by  $dz$ , is, restoring  $w$ :  $\rho(z) = -d\mathcal{B}(z)/dz = (w/\theta) \exp(-w z / \theta)$ . This is the barometric law. Since  $w$  and  $\theta$  are constants the density decays exponentially as a function of altitude  $z$ . In the earth atmosphere the density of di-atomic oxygen decays faster than the density of di-atomic nitrogen because the weight of an oxygen atom exceeds that of a nitrogen atom in the ratio  $\approx 16/14$ .

### 3 Average force exerted by a corpuscle on a piston

We now treat the ideal-gas law by similar methods. We are considering again a unit-area cylinder with vertical axis in uniform gravity, resting on the ground ( $z = 0$ ) at some temperature. A tight piston can move in the vertical  $z$  direction. The cylinder height is denoted by  $h$  and contains a single corpuscle of weight  $w = 1$ . In our one-dimensional model, the pressure  $P$  corresponds to the average force  $\langle F \rangle$ , the volume  $V$  to the height  $h$ , and  $N = 1$ . Our result provides the ideal-gas law in a generalized form, taking into account gravity. In that case, the pressure varies as a function of altitude. More precisely, the force exerted by the corpuscle on the lower end of the cylinder exceeds the force exerted on the upper end (or piston) by the corpuscle weight. But in the absence of gravity, the forces exerted on both ends would be the same.

We are introducing (static and uniform as stated earlier) gravity mainly because this helps clarify the concept of corpuscle energy: the corpuscle energy  $E$  is the maximum altitude  $z_m$  that the corpuscle would reach in the absence of the piston when the corpuscle weight is unity. The corpuscle bounces elastically off the ground, that is, without any loss or gain of energy.

Consider first the case where  $h$  is infinite, that is, in the absence of a piston. The time period is denoted as before by  $\tau(Z)$  with  $Z = z_m$ . The average force exerted on the ground, equal to the corpuscle weight  $w = 1$ , is the product of the impulse  $i$  and the number of impulses per unit time. Thus  $1 = i/\tau(z_m)$  or  $i = \tau(z_m)$ . In other words, the impulse transmitted to a piston when the corpuscle impacts on it is equal to the motion period.

If the plate is located at  $z = h$  the impulse is:  $i_h = \tau(z_m - h)$ . When the corpuscle is moving back and forth between the planes at  $z = 0$  and  $z = h$  (instead of being located above  $h$ ) the impulse is just opposite to  $i_h$ . It is therefore in absolute value equal to  $\tau(z_m - h)$ .

Next, we introduce a rigid thin plate at  $z = h$ , viewed as a piston, and consider a corpuscle bouncing on the  $z = 0$  and  $z = h$  plates alternately. The time period becomes:  $\tau(z_m) - \tau(z_m - h)$ , as one can see from inspection of the figure. We call  $F$  the force exerted on the  $z = h$  plate, averaged over a time period. It follows from the previous expressions that:

$$\begin{cases} F = 0 & z_m \leq h, \\ F = \frac{i_h}{\tau(z_m) - \tau(z_m - h)} = \frac{\tau(z_m - h)}{\tau(z_m) - \tau(z_m - h)} & z_m > h. \end{cases} \quad (5)$$

Because the cylinder lower end is supposed to be in contact with a bath, there is a slight quivering (thermal motion), and the corpuscle energy  $z_m$  slowly varies in the course of time. The force  $F$  just defined must be weighed by some energy distribution  $\omega(z_m)$  in such a way that the average force  $\langle F \rangle$  be *independent* of the corpuscle equation of motion, and thus of the  $\tau(\cdot)$ -function. In the limit where  $h \ll \theta$ , a condition that amounts to ignoring gravity, we obtain the ideal-gas law in the form:  $\langle F \rangle = \theta/h$ . We later on prove that  $\theta$  is the thermodynamic temperature.

The above condition obtains from (5) if one selects the following energy distribution:

$$\begin{cases} \omega(z_m) = \exp(-z_m/\theta)\tau(z_m) & z_m \leq h, \\ \omega(z_m) = \exp(-z_m/\theta)(\tau(z_m) - \tau(z_m - h)) & z_m > h, \end{cases} \quad (6)$$

where  $\theta$  has the dimension of an energy when non-unity weights  $w$  are considered. When  $z_m \leq h$ , the  $\omega(z_m)$ -distribution is of course the same as in Section 2. The average force becomes, using (5)

and (6):

$$\begin{aligned} \langle F \rangle &= \frac{\int_h^\infty dz_m \exp(-z_m/\theta) \tau(z_m - h)}{\int_0^h dz_m \exp(-z_m/\theta) \tau(z_m) + \int_h^\infty dz_m \exp(-z_m/\theta) (\tau(z_m) - \tau(z_m - h))} \\ &= \frac{w}{\exp(w h/\theta) - 1} \end{aligned} \quad (7)$$

if we restore  $w$ . In the above integrals going from  $h$  to  $\infty$  we have replaced  $\exp(-z_m/\theta)$  by  $\exp(-h/\theta) \exp(-(z_m - h)/\theta)$  and introduced the variable  $z'_m \equiv z_m - h$ , so that all the integrals go from zero to infinity and cancel out. Note that no integration has been performed.

For a collection of  $N$  independent corpuscles having weights  $w_i$ ,  $i = 1, \dots, N$  respectively, the force is a sum of  $N$  terms of the form given in (7). In the case of zero gravity ( $w=0$  or more precisely:  $w h \ll \theta$ ), the above expression gives:  $\langle F \rangle h = \theta$ . Thus we have obtained the ideal-gas law:  $\langle F \rangle h = N \theta$ .

The average force may be obtained alternatively from the barometric law established in Section 2:  $\rho(z) = C \exp(-w z/\theta)$ , where  $C$  is a constant, if one uses the concept that plates may be added or removed at some altitude  $z = h$ .

**Average force for a three-dimensional space:** We suppose that the cylinder radius is very large compared with  $h$ , and we do not consider the force exerted by the corpuscle on the cylinder wall. Motion of the corpuscle along directions perpendicular to  $z$  (say,  $x$  and  $y$ ) *does* affect the round-trip time function  $\tau(Z)$ . However, since the average force does not depend on this function, the ideal-gas law is unaffected. This is so for any physical system involving a single corpuscle provided the physical laws are invariant under a  $z$ -translation (besides being static).

The internal energy, to be discussed in the following section, though, is incremented. One can prove that in the non-relativistic approximation and in the absence of gravity the internal energy is multiplied by 3. It would be incremented further by corpuscle rotation or vibration, not considered here. Using conventional methods, Landsberg [7] in Eq. (2.6), and Louis-Martinez [21] in Eq. (68), obtain exactly the same result as given above (except for the factor 3 in the expression of the internal energy, relating to the number of space dimensions considered).

## 4 Internal energy

The gas internal energy  $U$  is the average value of  $E$ , the gravitational energy being accounted for. Note that only corpuscle motion along the  $z$ -axis is being considered. In the present section a non-unity (but constant) weight  $w$  is considered to clarify the dimensions of the quantities introduced. The expression of  $U$  is, using the energy distribution given in (6):

$$\begin{aligned} U &= \frac{\int_0^{w h} dE E \exp(-E/\theta) \tau(E/w) + \int_{w h}^\infty dE E \exp(-E/\theta) (\tau(E/w) - \tau(E/w - h))}{\int_0^{w h} dE \exp(-E/\theta) \tau(E/w) + \int_{w h}^\infty dE \exp(-E/\theta) (\tau(E/w) - \tau(E/w - h))} \\ &= \frac{\int_0^\infty dE E \exp(-E/\theta) \tau(E/w)}{\int_0^\infty dE \exp(-E/\theta) \tau(E/w)} - \frac{w h}{\exp(w h/\theta) - 1}. \end{aligned} \quad (8)$$

The internal energy divided by the temperature:  $U(\theta, h)/\theta$ , thus is the sum of two terms. The second one is an explicit function of the dimensionless quantity  $w h/\theta$ , which tends to -1 when

$h \rightarrow 0$ . The first term is a function of  $\theta$  but not of  $h$ . To evaluate this first term we need to know the round-trip time  $\tau(Z)$  to within an arbitrary proportionality factor, and an integration must be performed in that case.

In order to write the above expression in terms of dimensionless quantities, we must introduce a universal constant,  $c$ , having the dimension of speed. Defining:  $x \equiv E/\theta$ ,  $\theta' \equiv \theta/mc^2$ ,  $h' \equiv h/(c^2/g)$ , (8) becomes, remembering that  $w = mg$ :

$$\frac{U}{\theta} = \frac{\int_0^\infty dx x \exp(-x) f(\theta' x)}{\int_0^\infty dx \exp(-x) f(\theta' x)} - \frac{h'/\theta'}{\exp(h'/\theta') - 1}, \quad (9)$$

where the function  $f(y)$  depends on the round-trip time.

In the special case of non-relativistic motion  $y \ll 1$ , we have  $f(y) \propto \sqrt{y}$ , see the appendix. The first term in (9) is then equal to  $3/2$  (note that:  $2 \int_0^\infty dx x \exp(-x) \sqrt{x} = 3 \int_0^\infty dx \exp(-x) \sqrt{x}$ ), so that in the absence of gravity the internal energy  $U$  is equal to  $3\theta/2 - \theta = \theta/2$ , a well-known result [22].

## 5 The energy $\theta$ is the thermodynamic temperature

We prove in this section that  $\theta$ , introduced in previous sections on dimensional grounds only, is the thermodynamic temperature (to within an arbitrary constant). We do this by showing that the maximum efficiency of a thermal cycle employing ideal gases is:  $1 - \theta_l/\theta_h$ , where  $\theta_l$  is the cold-bath temperature and  $\theta_h$  the hot bath temperature: this is the accepted Kelvin definition of absolute temperatures.

The expressions given earlier for the average force  $\langle F \rangle$  in (7) and the internal energy  $U$  in (8) may be written, setting  $\beta \equiv 1/\theta$ , as:

$$\langle F \rangle = \frac{\partial \ln(Z)}{\beta \partial h} \quad U = -\frac{\partial \ln(Z)}{\partial \beta}$$

$$Z(\beta, h) = (\exp(-\beta w h) - 1) \int_0^\infty dE \exp(-\beta E) \tau(E/w). \quad (10)$$

$Z$  is essentially the quantity called in statistical mechanics the partition function. It becomes dimensionless if it is divided by the reduced Planck constant  $\hbar$ , which however plays here no physical role. The continuous energy  $E$  in the integral may be replaced by closely-spaced discrete energies  $\varepsilon_i$ ,  $i = 1, 2, \dots$ , the spacing between adjacent energies accounting for the  $\tau$ -function. This procedure is the one employed in the numerical evaluation of integrals; it does not in itself imply quantization. The factor preceding the integral in (10) entails that the energies  $\varepsilon_i$ ,  $i = 1, 2, \dots$  are multiplied by some function of  $h$ .

If we introduce the Helmholtz free-energy (the letter  $A$  is from the German ‘‘Arbeit’’ or work):  $A(\theta, h) \equiv -\theta \ln(Z(\theta, h))$  the expressions in (10) are conveniently written [23]:

$$\langle F \rangle = -\frac{\partial A}{\partial h} \quad U = A - \theta \frac{\partial A}{\partial \theta}. \quad (11)$$

From (11) we obtain:

$$-\delta Q \equiv dU + \langle F \rangle dh = dA - \frac{\partial A}{\partial \theta} d\theta - \frac{\partial A}{\partial h} dh - \theta d\left(\frac{\partial A}{\partial \theta}\right) = \theta dS \quad S \equiv -\frac{\partial A}{\partial \theta}, \quad (12)$$

where  $\delta Q$  represents the heat released by the gas, from the law of conservation of energy. For any function  $f(\theta, h)$  such as  $U, A, S$ :  $df \equiv \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial h} dh$ . Note that we employ only two independent variables, namely  $\theta$  and  $h$ , so-that partial derivatives are un-ambiguous. If the gas is in contact with a thermal bath ( $\theta$ =constant),  $\delta Q$  is the heat gained by the bath. The quantity  $S$  defined above is called the “entropy”. In particular, if heat cannot go through the gas container wall (adiabatic transformation) we have  $\delta Q = 0$  that is, according to the above result:  $dS = 0$ . Thus adiabatic transformations are isentropic.

**The Carnot cycle:** A Carnot cycle consists of two isothermal transformations at temperatures  $\theta_l$  and  $\theta_h$ , and two intermediate adiabatic transformations ( $dS = 0$ ). After a complete cycle, the entropy recovers its original value and therefore  $dS_l + dS_h = 0$ . According to (12):  $-\delta Q_l = \theta_l dS_l$ ,  $-\delta Q_h = \theta_h dS_h$  and therefore  $\delta Q_l/\theta_l + \delta Q_h/\theta_h = 0$ . Energy conservation gives the work  $\delta W$  performed over a cycle from:  $\delta W + \delta Q_l + \delta Q_h = 0$ . The cycle efficiency is defined as the ratio of  $\delta W$  and the heating  $-\delta Q_h$  supplied by the hot bath. We have therefore:  $\eta \equiv \frac{\delta W}{-\delta Q_h} = \frac{\delta Q_h + \delta Q_l}{\delta Q_h} = 1 - \frac{\theta_l}{\theta_h}$ , from which we conclude that  $\theta$  is the “thermodynamic temperature”. Since Kelvin time, thermodynamics temperatures are strictly defined from Carnot (or other reversible) cycles efficiency. In practice, temperatures are measured by other means as discussed in the introduction, and employed in other circumstances.

We have implicitly assumed in the above discussion that the working medium (presently an ideal gas) has reached the bath temperature before being contacted with it. Otherwise, there would be at that time a jump in entropy, and the cycle would no longer be reversible. Given initial  $\theta, h$  values, the temperature change  $d\theta$  for an increment  $dh$  in the isentropic regime ( $dS = 0$ ) follows from the relation:  $d\theta = -\left(\frac{\partial S/\partial h}{\partial S/\partial \theta}\right)dh$ , where  $S(\theta, h)$  may be expressed in terms of  $Z(\theta, h)$ , (10), from the above expressions. The details will be omitted. It suffices to know that  $\theta$  may be varied by varying  $h$ , in a calculable manner, in an adiabatic transformation.

**Practical units:** The energy  $\theta = \langle F \rangle h$  has been defined so far only to within a multiplicative factor, from dimensional considerations. This factor is fixed by agreeing that  $\theta = 273.16 k_B$  exactly when the cylinder is in thermal equilibrium with water at its triple point. Here  $k_B = 1.38066... \cdot 10^{-23}$  joules, is considered as an energy unit (akin to the calorie = 4.182... joules). This manner of defining  $\theta$  is equivalent to the usual one, though expressed differently. The dimensionless quantity  $T \equiv \theta/k_B$  is the usual unit of thermodynamic temperature, expressed in kelvin.

Next, measurements have shown that the number of atoms in 0.012 kg of carbon 12 is:  $N_A \approx 6.0221367 \cdot 10^{23}$ . For this quantity of matter (called a mole), the ideal-gas law therefore reads:  $\langle F \rangle h = N_A \theta$ , or:  $PV = RT$ , with the ideal-gas constant:  $R \equiv N_A k_B \approx 8.31451$  joules per kelvin per mole.

## 6 Conclusion

Let us briefly recall the concepts introduced in the present paper. One can imagine that after having introduced the corpuscular concept, Democritus observed the elastic bounces of a unit weight on a balance and defined the weight “impulse” from the motion period. Not knowing the nature of the motion (parabolic? hyperbolic?), he may have thought of introducing a weight factor such that the average force  $\langle F \rangle$  *does not depend* on the law of motion. This, as we have seen, may be done. This weight factor involves for dimensional reasons a quantity  $\theta$  having the dimension of energy.

Considering a thermal engine operating between two baths at temperatures  $\theta_l, \theta_h$  one finds on the basis of the principles just stated that the maximum efficiency is:  $1 - \theta_l/\theta_h$ . This allows us to call  $\theta$  the thermodynamic temperature.

The present paper provides explicit expressions of gases internal energy provided the round-trip time function  $\tau(Z)$  is known. It does provide a first-principle proof of the ideal-gas law, including a possible effect of uniform gravity, and of the barometric equation with no knowledge of the  $\tau(Z)$  function required. We have recovered the usual thermodynamical and semi-classical statistical-mechanical expressions for the special case of ideal gases submitted to gravity. The theory presented is strictly classical. As such, it does not depend on the numerical values of the universal constants  $\hbar$  and  $c$ . However, the introduction of these quantities is needed to make the results dimensionless. Generalisation to two (or more) colliding corpuscles seems to be possible, perhaps with the help of numerical methods.

William of Ockham (1287-1347) set as a matter of principle that one should not employ more concepts than those that are strictly necessary to explain the observed phenomenas. (Some authors consider that the Ockham philosophy predates the advent of modern science by insisting on facts and the kind of reasoning employed rather than on speculations about essences). Accordingly, it seems important to elucidate the assumptions on which rest, in particular, the barometric and ideal-gas laws that play an essential role in theoretical physics and many applications. Our thesis is that these laws may be obtain on the sole basis of the Democritus model of corpuscles and vacuum. It is indeed unnecessary to specify the laws of motion. One can further show that the ideal-gas internal energy depends only on temperature (in the absence of gravity). To evaluate explicitly this energy it is, however, necessary to know the round-trip corpuscule time. From a pedagogical standpoint and in application of Ockham's concept one should not postulate principles which, without being erroneous, are unnecessary. In a text-book presentation of Physics, it seems to us that it would be appropriate to give first considerations similar to those presented in this paper, since the explicit laws of motion are not needed.

## A General equations of motion

Laws of motion are conveniently written in the Hamiltonian form. If  $H(z, p)$  denotes the Hamilton function of coordinate  $z$  and momentum  $p$ , we have:

$$v(t) \equiv \frac{dz(t)}{dt} = \frac{\partial H(z, p)}{\partial p} \quad \frac{dp(t)}{dt} = -\frac{\partial H(z, p)}{\partial z}. \quad (13)$$

Because the Hamilton function does not depend explicitly on  $t$ , the energy of a corpuscle is the constant:  $E = H(z, p)$  for some initial  $z, p$  values.

**Non-relativistic approximation for a constant weight:** In that approximation,  $H(z, p) = \frac{p^2}{2m} + wz$  for a weight (or force)  $w \equiv mg$ . It follows that  $v = p/m$  and  $dp/dt = -w$ . With the initial conditions:  $z(0) = 0$  and  $p(0) = 0$ , the corpuscle motion is described by:  $z(t) = -\frac{1}{2}gt^2$ . Hence,  $\tau(Z) \equiv 2t(-z) = 2\sqrt{2Z/g} \propto \sqrt{Z}$ . For consistency with subsequent expressions one may add to  $H(z, p)$  the constant  $mc^2$ .

**Relativistic expression for constant electrical forces:** We presently have for a weightless corpuscle with electrical charge  $e$  and submitted to a constant electrical field  $\mathcal{E}$ , setting  $w \equiv e\mathcal{E}$ ,

(see Landau [24], p.115, for the relativistic form without gravity in three dimensions):

$$\begin{aligned}
H(z, p) &= mc^2 \sqrt{1 + (p/mc)^2} + wz \rightarrow p = -wt, \\
\frac{dz(t)}{dt} &= \frac{\partial H(z, p)}{\partial p} = c \frac{\tilde{p}}{\sqrt{1 + \tilde{p}^2}} \quad \tilde{p} \equiv \frac{p}{mc}.
\end{aligned} \tag{14}$$

From these expressions we may obtain the equation of motion with  $z(0) = z_m$ , and the round-trip time as a function of  $z_m - z$ .

The traditional methods of calculating  $\langle F \rangle$  and  $U$  in the absence of gravity consist instead in calculating the average values of  $pv$  and  $H(p)$ , respectively, using as a weight the Boltzmann factor:  $\exp(-H(p)/\theta)$ .

**Gravity modeled by radiation pressure:** In this paragraph we replace the uniform gravity postulated in the main text by radiation pressure. Specifically, we consider photons incident at speed  $-c$  on the corpuscle of interest, colliding with it elastically. We no longer need to add to the free-space hamiltonian a potential term  $wz$ , and only free-space motion is considered. Remarkably, the final exact result is very simple. For generality we first consider “drops” with speed  $V$ , and later specialize the results to the photon case:  $V = c$ .

Let the corpuscle motion be described by  $H(p)$  and a drop motion by  $D(P)$ , where  $H, D$  are energies and  $p, P$  momenta. When an elastic corpuscle-drop collision occurs, the total momentum and total energy are conserved. Thus, setting:  $\tilde{p} \equiv p + \Delta p$ , we have:

$$\begin{aligned}
p + P &= \tilde{p} + \tilde{P} \rightarrow \tilde{P} = P - \Delta p \\
H(p) + D(P) &= H(\tilde{p}) + D(\tilde{P}) = H(p + \Delta p) + h(P - \Delta p),
\end{aligned} \tag{15}$$

where the tildes refer to the momenta after a collision.

Expanding the above relation up to second order in  $\Delta p$  we obtain:

$$H(p) + D(P) = H(p) + H'(p)\Delta p + \frac{1}{2}H''(p)\Delta p^2 + D(P) - D'(P)\Delta p + \frac{1}{2}D''(P)\Delta p^2, \tag{16}$$

where primes (double primes) refer to first (second) derivatives with respect to the arguments. After simplification we have:

$$\Delta p = -2 \frac{H'(p) - D'(P)}{H''(p) + D''(P)}. \tag{17}$$

Let the spacing between two successive drops at a given instant be denoted  $\epsilon$ . The corpuscle and drop speeds are  $v = H'(p)$  and  $V = D'(P)$ , respectively. We suppose that  $V < 0$ , thereby simulating a downward gravity force. If a collision occurred at  $z = 0$ ,  $t = 0$  the next collision occurs when  $\epsilon + V \Delta t = v \Delta t$ , from which it follows that:

$$\Delta t = \frac{\epsilon}{v - V} = \frac{\epsilon}{H'(p) - D'(P)}, \tag{18}$$

which is positive provided that  $v > V$ . We suppose that  $\epsilon$  and individual drop momenta are so small that the corpuscle momentum may be considered a continuous function of time. Remember that  $P$  and  $\epsilon$  are fixed quantities while  $p$  varies in the course of time. We assume that  $p = p_o > 0$  at  $t = 0$ . Ultimately for  $t \rightarrow \infty$ , the corpuscle speed reaches the drop speed, that is:  $v(\infty) \approx V < 0$ .

Equations (17) and (18) give the force  $F(v) = dp/dt$  experienced by the corpuscle. We have:

$$\frac{dt}{dp} = -\frac{\varepsilon}{2} \frac{H''(p) + D''(P)}{(H'(p) - D'(P))^2}. \quad (19)$$

from which we may obtain by integration  $t(p)$ , with an integration constant determined by the relation  $t(p_o) = 0$  for some initial momentum  $p_o$  taken as a parameter. Given that  $v = H'(p)$ , we may obtain  $v(t; p_o)$ . Finally, since  $v = dz/dt$ , we obtain by integration  $z(t)$ , to within an additional integration constant.

We have for a corpuscle of mass  $m$ :  $H(p) = \sqrt{(mc^2)^2 + c^2 p^2}$ . It follows that:  $H'(p) = c^2 p/H(p)$ ,  $H''(p) = m^2 c^4/H(p)^3$ . As a special case with  $m = 0$  (photons) we have:  $D(P) = cP, D'(P) = c, D'' = 0$ . Then (19) becomes:

$$\frac{dt}{dp} = -\frac{\varepsilon}{2} \frac{H''(p) + D''(P)}{(H'(p) - D'(P))^2} = -\frac{\varepsilon m^2 c^2}{2H(p)(H(p) - cp)^2}, \quad (20)$$

Setting:  $c = 1, m = 1$  with an appropriate choice of units, and:  $p \equiv \sinh(x)$  we obtain:  $H(p) = \cosh(x)$ ,  $dp/H(p) = dx$ ,  $H(p) - p = \cosh(x) - \sinh(x) = \exp(-x)$  and  $v = p/H(p) = \tanh(x) = \frac{\exp(2x)-1}{\exp(2x)+1}$ ,  $v_o = \frac{\exp(2x_o)-1}{\exp(2x_o)+1}$ . Thus:  $dt = -(\varepsilon/2) \exp(2x)dx$  and, to within a constant factor:

$$t = \exp(2x) - \exp(2x_o), \quad (21)$$

or conversely:

$$v = 1 - 2 \frac{\frac{1-v_o}{2}}{1 + \frac{1-v_o}{2}t}. \quad (22)$$

Since the speed  $v \equiv dz/dt$ , integrating with respect to  $t$  we obtain the corpuscle motion in the (exact) form:

$$z(t) = t - 2 \ln\left(1 + \frac{1-v_o}{2}t\right). \quad (23)$$

Expanding the  $\ln(\cdot)$ -function to second order:

$$z(t) \approx v_o t - \frac{(1-v_o)^2}{2} t^2. \quad (24)$$

which is the expected result for Galilean motion for a corpuscle submitted to a constant force if  $v_o \ll 1$ . The term:  $(1-v_o)^2$  gives a first-order relativistic correction.

**Uniform acceleration  $g$ :** The Hamilton equations of motion in one space dimension ( $z$ ) derive from a hamiltonian function of  $z$  and  $p$  (see Louis-Martinez [21], Eq(5) with  $N = 1$ ):

$$H(z, p) = (mc^2 + mgz) \sqrt{1 + (p/mc)^2} \quad (25)$$

$$v(t) \equiv \frac{dz(t)}{dt} = \frac{\partial H(z, p)}{\partial p} = c \left(1 + \frac{z}{z_g}\right) \frac{\tilde{p}}{\sqrt{1 + \tilde{p}^2}} \quad \tilde{p} \equiv \frac{p}{mc} \quad z_g \equiv \frac{c^2}{g} \quad (26)$$

$$\frac{dp(t)}{dt} = -\frac{\partial H(z, p)}{\partial z} = -mg \sqrt{1 + \tilde{p}^2} \quad \rightarrow \quad \frac{d\tilde{p}(t)}{dt} = -\frac{g}{c} \sqrt{1 + \tilde{p}^2}. \quad (27)$$



We take the time origin such that:  $p(0) = 0$ . Note that the term  $mc^2 + mgz_m \equiv mc^2(1 + z_m/z_g)$ , where the maximum altitude  $z_m$  represents the corpuscle energy. The round-trip time is now a function of  $\zeta_m - \zeta$ , where  $\zeta = \ln(1 + z/z_g)$ . The non-relativistic expressions are recovered when  $z, ct \ll z_g$ . An Hamiltonian function that would account for Quantum Gravity has been proposed by Chandra [25].

## B A report to the American Journal of Physics. Further comments on this paper desired

The authors propose to establish the barometric formula in particular, or the ideal gas law in general, straight from atomistic principles (a la Democrit), independent of any particular laws of motion. True, in most text-books this follows as an afterthought, after the derivation of Boltzmann, which may be overkill. (A minor exception to this, which I always found cute, is the derivation of the Maxwell distribution in O'Hanian's book on modern physics, which is based only on the isotropy of space but relies on a purely classical notion of velocity and I don't know whether it can be generalized.)

Assuming only that the time spent in a region is related to the potential against which a particle would have to climb, which follows from the energy-work theorem, an exponential distribution is obtained as a function of height, independent of whether the motion is relativistic or not.

The presentation is clear and insightful, which will greatly benefit the reader, and inspire similar thought on other taken-for-granted assumptions. I can not judge, though, how original these ideas are. I am almost certain that there are many insiders for whom this would be obvious, but I am not sure whether anybody would have written it down in such a pedagogical and thoughtful manner.

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- [14] We quote: Gases consist in corpuscles having non-zero mass. The corpuscles are quickly moving. They are perfectly spherical and elastic. The average kinetic energy depends only on the system temperature. Relativistic effects are negligible. Motion laws are time reversible. The number of molecules is so large that a statistical treatment is appropriate.
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- [16] The derivation of the Boltzmann factor is usually based on a quantization of the energy and the postulated equi-probability of the micro-states. Let the discrete (non-degenerate) energy levels be denoted by  $\varepsilon_1, \varepsilon_2, \dots$ . If distinguishable corpuscles are distributed among the energy levels, with  $n_1$  corpuscles in level 1,  $n_2$  corpuscles in level 2, and so on, the number of ways of doing that is inversely proportional to:  $n_1!n_2!\dots$ . It is postulated that this number reaches its maximum value at equilibrium under the constraint that  $n_1 + n_2 + \dots = N$ , the total number of corpuscles, and  $n_1\varepsilon_1 + n_2\varepsilon_2 + \dots = E$ , the total energy. In the limit of large  $N$  values, one finds that:  $n_i \propto \exp(-\varepsilon_i/\theta)$ , for some  $\theta$ -value that depends on  $N$  and  $E$ . Even though physicists are now-a-day very familiar with that procedure, it is not so easy to explain it to students. Besides, it rests on a number of assumptions.
- [17] J. Arnaud, L. Chusseau, and F. Philippe, “On the ideal gas law,” arXiv 1105.2643 v2, May 2011. [Online]. Available: <http://arxiv.org/abs/1105.2643>
- [18] We do not assume that  $z(t) = z(-t)$  or  $t_2 + t_1 = 0$ . Asymmetric laws of motion occur if one employs clock synchronisation rules different from the one proposed by Einstein. For example, if a light pulse emitted from  $z = 0$  at  $t = 0$  propagating upward is employed to synchronise

clocks located at different altitudes, the apparent upward speed of light is, by this convention, infinite. The downward speed of light is then  $c/2$  if  $c$  denotes the Einstein speed of light, so that the two-way speed of light remains equal to  $c$ , in agreement with very precise experiments. The anisotropy mentioned in the present note refers to a change of formalism, not of physics. It is of some importance that the laws discussed in this paper do not depend on such conventional changes. On the historical side note that in his crucial experiment Galileo could have chosen to throw corpuscles upward, measuring the round-trip time  $\tau$  as a function of the altitude  $z_m$  reached, in which case a single clock at  $z = 0$  is needed. The problem of clock synchronization at two different altitudes was not important at the time of Galileo but it proved to be crucial later on.

- [19] We are implicitly considering the so-called “canonical ensemble”: the ground plate does not allow corpuscles to go through, but it moves slightly so that a transfer of energy between the bath and the corpuscle is allowed.
- [20] The exponential term in the expression of  $\omega(z_m)$  corresponds to the so-called “Boltzmann factor”, while the second term accounts for the fact that the quantum-theory level energies are not evenly spaced. Let us emphasize that the main-text theory is strictly classical. The present note purpose is only aimed at establishing a connection with the perhaps better-known quantum theory. Put another way, the energy distribution may be written as  $\exp(-E(f)/\theta)$ , where  $f$  denotes the action, equal to the  $z(t)$  motion area for one period, and  $df/dE = \tau(E/w)$ . In quantum mechanics the action  $f$  is set equal to an integer (1,2,..., ignoring a small constant) times the reduced Planck constant  $\hbar$ . The term  $\exp(-E/\theta)$ , usually referred to as the “Boltzmann factor”, enters here solely by mathematical necessity [2].
- [21] D. J. Louis-Martinez, “Classical relativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated reference frame,” arXiv 1012.3063, Dec. 2010. [Online]. Available: <http://arxiv.org/abs/1012.3063>
- [22] A plot of  $U$  from low to high temperatures in the case of an external force much larger than gravity (e.g., an electrical force) can be seen in [2]. A three-dimensional expression of  $U$  in terms of Bessel functions can be found, for example, in Greiner, p. 234. [26].
- [23] Note incidentally that:  $\frac{\partial U}{\partial \hbar} = \theta \frac{\partial \langle F \rangle}{\partial \theta} - \langle F \rangle$ . It follows that, given  $\langle F \rangle$ , we may obtain  $U(\theta, \hbar)$  by an integration over  $\hbar$ , provided an unknown function of  $\theta$  be added. This relation is in fact applicable to any working medium, as one can show by considering a Carnot cycle with small temperature differences (see for example Kubo [27], p. 81).
- [24] L. D. Landau and E. M. Lifshitz, *Statistical Physics*. Oxford: Pergamon Press, 1980, vol. 5, Part I.
- [25] N. Chandra and S. Chatterjee, “Thermodynamics of ideal gas in doubly special relativity,” arXiv 1108.0896, Aug. 2011. [Online]. Available: <http://arxiv.org/abs/1108.0896>
- [26] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics*. New-York: Springer-Verlag, 1997.
- [27] R. Kubo, *Thermodynamics*. Amsterdam: North-Holland, 1968.