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Quantum-electrodynamical approach to the Casimir force problem

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Abstract

We derive the Casimir force expression from Maxwell's stress tensor by means of original quantum-electro-dynamical cavity modes. In contrast with similar calculations, our method does not rely on formal mathematical extrapolation relations [5] with not necessarily rigorous properties [6].

Introduction

The effect of the quantum nature of the electro-magnetic field manifests itself if retarded potentials between atoms and molecules are considered, as shown by Casimir and Polder [1]. A similar effect appears in Lifshitz's theory [2,3] of forces between dielectric solid bodies. In the limiting case, where dielectrics are replaced by perfectly conducting plates, i.e. by letting the dielectric constants go to infinity, Lifshitz recovers the attractive force for which Casimir derived in 1948 expression [4]

$$(1) \quad F_z = -\frac{\pi^2}{240} \frac{\hbar c}{a^4} \quad z = a$$

with the plates located at $z = 0$ and $z = a$.

In his derivation Casimir considers the zero point energy inside the cavity, which is a purely quantum-electro-dynamical notion, and he obtains the force as represented by its derivative with respect to the distance a .

Here a difficulty appears given the fact that if the zero-point energy density

$$(2) \quad \varepsilon = \frac{1}{2} \hbar \omega_{\vec{k}}$$

is summed over all possible modes \vec{k} , an infinite meaningless result appears. Casimir has solved the problem by demonstrating how unphysical infinities disappear by introducing compensations from the surroundings outside the cavity. However, this delicate problem has given rise to

numerous subsequent studies [5], most of them based on purely formal mathematical extrapolation methods.

Another way, chosen by some authors for computing the Casimir force, consists in writing down an expression derived directly from Maxwell's stress tensor instead of differentiating the energy relation [5,6]. Unfortunately this method does not remove the divergencies brought about by summation over modes so that here again convergence generation procedures are required. **Some time ago Brown and Maclay [7] presented a very special Green's function method for dealing with divergencies. Numerous other methods have been published since for which we refer to the book of K.A.Milton [5].**

In this work we evaluate Maxwell's stress tensor by means of **specific** quantum-electrodynamical cavity modes which to our knowledge have not been considered so far in the literature **in this context**. The advantage of using these modes lies in the fact that they allow to perform a rigorous mathematical treatment, with at the end a clear identification of the term that has to be discarded as unphysical.

We thus try to show that it is possible to obtain the correct zero-point result by simply using adequate cavity mode functions and perform a straightforward summation over all modes. Naturally, Green's function methods, initiated e.g. by Lifshitz [2] and developed by many authors, will always remain attractive as they approach the problem from the general angle of quantum field fluctuation theory.

The cavity modes

We consider a cavity made of two plane parallel perfectly conducting metallic plates located at distances $z = 0$ and $z = a$ respectively. We write the electric field vector inside the cavity in the form

$$(3) \quad \hat{E} = \vec{E} \left(\hat{a} e^{-i\omega t} + \hat{a}^{\dagger} e^{i\omega t} \right)$$

with, for a given frequency, \vec{E} a function of the space coordinates $\vec{r} = (x, y, z)$ and \hat{a} , \hat{a}^{\dagger} the usual annihilation and creation operators respectively. Introducing the expectation value of the quantity \hat{E}^2 in the vacuum state $|0\rangle$ we then have

$$(4) \quad \langle 0 | \hat{E}^2 | 0 \rangle = \vec{E}^2 \langle 0 | \hat{a} \hat{a}^{\dagger} | 0 \rangle = \vec{E}^2$$

Considering now the zero point energy density in the cavity as given by the expression

$$(5) \frac{1}{2} \hbar \omega / V = \frac{1}{2} \frac{\hbar \omega}{L^2 a}$$

where L^2 is the surface of the square shaped plates, the quantity given by eq.(4) can be linked to the electro-magnetic energy density, i.e.twice its electric part, by means of the following relation:

$$(6) 2\epsilon_0 \bar{E}^2 = \frac{1}{2} \frac{\hbar \omega_k}{L^2 a}$$

Here we have added the index k as the corresponding wave number.

We now turn to the function \bar{E} which for a given mode with wave vector \vec{k} has to fulfill two conditions:

i) the zero charge condition inside the cavity expressed by the relation

$$(7) \nabla \cdot \bar{E} = 0$$

ii) the continuity condition of the parallel components at the boundaries, which for conducting plates amounts to setting

$$(8) E_x = E_y = 0 \text{ for } z = 0, z = a.$$

A candidate obeying these conditions is given by the expression [8]

$$(9) \begin{aligned} E_x &= A_x \cos(k_x x) \sin(k_y y) \sin(k_z z) \\ E_y &= A_y \sin(k_x x) \cos(k_y y) \sin(k_z z) \\ E_z &= A_z \sin(k_x x) \sin(k_y y) \cos(k_z z) \end{aligned} \quad \vec{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \frac{\pi n_x}{L} \\ \frac{\pi n_y}{L} \\ \frac{\pi n_z}{a} \end{pmatrix}$$

Moreover, by averaging over space coordinates eq.(6) then takes the form

$$(10a) \epsilon_0 2 \times \frac{1}{8} A^2 = \frac{1}{2} \frac{\hbar \omega_k}{L^2 a}$$

$$(10b) A^2 = \frac{2}{\epsilon_0} \frac{\hbar \omega_k}{L^2 a}$$

with

$$(11) A^2 = A_x^2 + A_y^2 + A_z^2$$

The Casimir force

The force exerted by the cavity field on the conducting plates can be deduced from Maxwell's stress tensor

$$(12) \quad \sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \left(\epsilon_0 \bar{E}^2 + \frac{1}{\mu_0} \bar{B}^2 \right) \delta_{ij} \quad \text{with } i,j = x,y,z$$

where the electric and magnetic field quantities refer to their values at the boundaries. For the electric field these are given by eq.(8) and (9). For the magnetic field they can be deduced from the relation

$$(13) \quad \vec{B} = -\frac{1}{i\omega} \nabla \times \vec{E}$$

yielding the following results

$$(14) \quad B_z = 0$$

$$(15) \quad \nabla \times \vec{E} = A_z \left\{ k_y \sin(k_x x) \cos(k_y y) - k_x \cos(k_x x) \sin(k_y y) \right\}$$

Taking the square of the quantity given by eq.(13) and averaging over coordinates x,y, we then obtain

$$(16) \quad \langle \bar{B}^2 \rangle_{xy} = \frac{1}{\omega^2} \frac{1}{4} A_z^2 \kappa^2 = \frac{1}{4} A_z^2 \frac{1}{c^2} \frac{\kappa^2}{k^2}$$

with

$$(17) \quad \kappa^2 = k_x^2 + k_y^2 = k^2 - k_z^2$$

Replacing the quantities of eq.(12) by these averages together with, as a result of eq.(9),

$$(18) \quad \langle E_z^2 \rangle_{xy} = \langle \bar{E}^2 \rangle_{xy} = \frac{1}{4} A_z^2$$

we arrive at the relation

$$(19) \quad \sigma_z = \epsilon_0 \frac{1}{4} A_z^2 - \frac{1}{2} \left\{ \frac{1}{4} \epsilon_0 A_z^2 + \frac{1}{4} \epsilon_0 A_z^2 \frac{\kappa^2}{k^2} \right\} = \frac{1}{8} \epsilon_0 A_z^2 \frac{k_z^2}{k^2}$$

recalling that $\frac{1}{c^2} = \mu_0 \epsilon_0$.

In order to determine the value of the amplitude factor A_z we notice that the average energy density of the fluctuating field inside the cavity must be homogeneous everywhere including the border region. This fact can be expressed by the relation

$$(20) \quad \langle \bar{E}^2 \rangle_{xyz} \text{ (volume)} = \langle \bar{E}^2 \rangle_{xy} \text{ (border)}$$

From the definitions of eq.(9) we thus obtain

$$(21) \quad \frac{1}{8} A^2 = \frac{1}{4} A_z^2 \rightarrow A_z^2 = \frac{1}{2} A^2$$

With A^2 given by eq.(10b) we have

$$(22) \quad \sigma_z = \frac{\hbar c k k_z^2}{L^2 a k^2} = \frac{1}{4} \frac{\hbar c}{L^2 a} \frac{\left(\frac{n_z \pi}{a}\right)^2}{\sqrt{k^2 + \left(\frac{n_z \pi}{a}\right)^2}}$$

where the explicit expression of eq.(7) for k_z together with

$$(23) \quad k = \sqrt{k^2 + k_z^2} = \sqrt{k^2 + \left(\frac{n_z \pi}{a}\right)^2}$$

has been introduced.

We now have to sum over mode numbers n_x, n_y, n_z . Letting the auxiliary quantity L go to

infinity we make the usual replacement $\sum_{n_x, n_y} \rightarrow \frac{L^2}{\pi^2} dk_x dk_y = \frac{L^2}{\pi^2} d^2 \kappa$ This then yields for the

force the expression:

$$(24) \quad F_z = \frac{1}{a} \frac{\hbar c}{4\pi^2} \sum_n \int d^2 \kappa \frac{\frac{n^2 \pi^2}{a^2}}{\sqrt{k^2 + \left(\frac{n^2 \pi^2}{a^2}\right)}} \quad (\text{summing over all positive integers})$$

This expression is highly divergent as it should, given the facts mentioned in the introduction.

Therefore a convergence factor has to be inserted which, following Fierz [9], we choose to be

$e^{-\lambda k}$.

Naturally the result will depend on the frequency cut-off involving the value of λ . The sum on the r.h.s. of eq.(24) can be done exactly as shown in the appendix, yielding the result

$$(25) \quad F_z = \frac{\hbar c}{\pi^2} \lambda^{-4} + \frac{\hbar c}{a^4} \pi^2 \frac{6}{2 \times 4!} B_4 \quad \text{with } B_4 = -\frac{1}{30} \text{ a Bernoulli number}$$

Hence the second term on the r.h.s. is equal to the Casimir force as given by eq.(1)

Discussion

A general rule in quantum physics stipulates that the result of any actual measurement involving the zero-point energy should be proportional to $(\Delta l)^{-4}$ with Δl a small characteristic length [10]. This is because the measurement reflects the average over a small but finite region in space. Applied to the Casimir effect with $\Delta l = a$, this means that in eq.(25) only the second term on the r.h.s. is physically significant, and that therefore the question of compensation of the a - independent infinity is irrelevant. This argument makes our method different from the numerous approaches which consider in detail compensation of singularities by various procedures.

Note finally that the expression (A1) represents twice (2 polarization states) the derivative with respect to a of the corresponding energy density expression, provided however that the derivative of the convergence factor can be ignored. This latter important condition has been investigated in detail in ref. [11] .

Conclusion

We present a derivation of the Casimir formula by evaluating Maxwell's stress tensor in terms of quantum electrodynamical cavity modes. The calculations do not involve any non rigorous mathematical manipulations and convergence is achieved on purely physical grounds.

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Appendix

With k given by eq.(23) we obtain for F_z the expression

$$(A1) F_z = \frac{1}{4\pi^2} \hbar c \frac{1}{a} \sum_n \int d^2\kappa \frac{\frac{n^2 \pi^2}{a^2}}{\sqrt{\kappa^2 + \frac{n^2 \pi^2}{a^2}}} e^{-\lambda \sqrt{\kappa^2 + \frac{n^2 \pi^2}{a^2}}}$$

We evaluate this quantity by making the following substitutions [9] :

$$d^2\kappa = 2\pi\kappa d\kappa \quad ; \quad \kappa^2 = \frac{n^2 \pi^2}{a^2} z$$

We thus obtain

$$(A2) F_z = \frac{\hbar c}{4\pi a} \sum_n \left(\frac{n^3 \pi^3}{a^3} \right) \int_0^\infty \frac{1}{\sqrt{z+1}} e^{-\lambda \frac{n\pi}{a} \sqrt{z+1}} dz$$

The integral can be calculated exactly with the result

$$(A3) F_z = \frac{\hbar c}{2\pi a} \sum_n \frac{n^2 \pi^2}{a^2} \frac{1}{\lambda} e^{-\lambda \frac{n\pi}{a}}$$

After generating the factor n^2 by deriving the exponential twice with respect to λ , the sum over n reduces to a geometric series. This yields the expression

$$(A4) F_z = \frac{\hbar c}{2\pi \lambda} \frac{1}{\partial \lambda^2} \frac{1}{1 - e^{-\lambda \frac{\pi}{a}}}$$

Introducing Bernoulli numbers B_h defined by

$$(A5) \frac{e^{-\lambda \frac{\pi}{a}}}{e^{-\lambda \frac{\pi}{a}} - 1} = \frac{\lambda \frac{\pi}{a}}{1 - e^{-\lambda \frac{\pi}{a}}} = \sum_0^\infty \frac{B_h}{h!} (-1)^h \left(\frac{\pi}{a} \right)^h \lambda^h$$

we expand the expression (A4) as follows:

$$(A6) F_z = \frac{\hbar c}{2\pi a} \sum_0^\infty \frac{B_h}{h!} (-1)^h \left(\frac{\pi}{a} \right)^{h-1} (h-1)(h-2)\lambda^{h-4}$$

Letting λ be an infinitesimal quantity, eq.(A6) reduces to

$$(A7) \quad F_z = \frac{\hbar c}{\pi^2} \lambda^{-4} + \frac{\hbar c}{a^4} \pi^2 \frac{6}{2 \times 4!} B_4$$

$$\text{with } B_0 = 1, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}$$

We thus recover the result of eq.(25)..