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► **To cite this version:**

Yves Grandati. Multistep DBT and regular rational extensions of the isotonic oscillator. 2011. hal-00614879v1

**HAL Id: hal-00614879**

**<https://hal.science/hal-00614879v1>**

Preprint submitted on 17 Aug 2011 (v1), last revised 18 Nov 2011 (v2)

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# Multistep DBT and regular rational extensions of the isotonic oscillator

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In some recent articles we developed a new systematic approach to generate solvable rational extensions of primary translationally shape invariant potentials. In this generalized SUSY QM partnership, the DBT are built on the excited states Riccati-Schrödinger (RS) functions regularized via specific discrete symmetries of the considered potential. In the present paper, we prove that this scheme can be extended in a multistep formulation. Applying this scheme to the isotonic oscillator, we obtain new towers of regular rational extensions of this potential which are strictly isospectral to it. We give explicit expressions for their eigenstates which are associated to the recently discovered exceptional Laguerre polynomials and show explicitly that these extensions inherit of the shape invariance properties of the original potential.

PACS numbers:

## I. INTRODUCTION

Since the pionnering work of Gomez-Ullate et al [4], the exceptional orthogonal polynomials (EOP) and their connexion with rational extensions of solvable quantum potentials have been an active research subject [5–19, 21–23]. The EOP appear to be the constitutive elements of the eigenstates of solvable rational extensions of the second category primary translationally shape-invariant potentials (TSIP) [1–3, 24]. In a series of recent papers [21–23] we have developed a new approach which allows to generate infinite sets of such regular solvable extensions starting from every TSIP in a very direct and systematic way without taking recourse to any ansatz or ad hoc deforming functions. This approach is based on a generalization of the usual SUSY partnership built from excited states. The corresponding Darboux-Bäcklund Transformations (DBT), which are covariance transformations for the class of Riccati-Schrödinger (RS) equations [24], are based on regularized RS functions corresponding to unphysical (negative energy) eigenfunctions of the initial hamiltonian. They are obtained by using discrete symmetries acting on the parameters of the considered family of potentials. If the use of negative energy states has been already proposed in early years of SUSY MQ development [25], such a systematic scheme has never been envisaged. In the case of the isotonic oscillator, this gives the three infinite sets  $L1$ ,  $L2$  and  $L3$  of regular rationally solvable extensions of this potential, the  $L1$  and  $L2$  series potentials being strictly isospectral to the isotonic one and inherits of its shape-invariance property. Very recently Gomez-Ullate, Kamran and Milson [43] introduce a new family of EOP built from the classical Laguerre polynomials via two step Darboux transformations. In the present article we show that the scheme developed in [21–23] admits a very natural  $n$ -step extension which allows to recover and generalize the result of [43]. We obtain new infinite towers of rational extensions of the isotonic potential strictly isospectral to this last. Their eigenfunctions are associated to the new EOP families, the one introduced by Gomez-Ullate et al corresponding to the  $n = 2$  particular case.

The paper is organized as follows. First we recall the basic elements of the method in its one step form and the results obtained for the isotonic system. The regularity of the RS functions is here obtained in a self consistent way, using a classical result of Bôcher [46] without recourse of the Kienast-Lawton-Hahn theorem [48, 49]. Then we present on a formal level the general  $m$ -step scheme and in the specific case of the isotonic oscillator, we give a criterion of concrete application in terms of asymptotic behaviour near the origin. Next we consider in a detailed manner the particular  $m = 2$  case, initially envisaged by Gomez-Ullate et al [43]. We then prove by induction a sufficient condition to be able to build a  $m$ -step chain of regular and strictly isospectral extended potentials. We finally show that the shape invariance properties of the isotonic potential are hereditary and reached by all these extended potentials.

During the writing of this article, two papers on the same subject appeared. The first one [44] contains results parallels to those developed here. The authors adopt a point of view similar to the one initiated in [21–23] and extended in the present paper. The so-called "virtual states deletion method" corresponding exactly to the use of DBT based on unphysical eigenfunctions associated to regularized RS functions. The second one [45] discusses quite extensively the 2 and 3 steps cases under a slightly different angle and proposes an interesting conjecture on possible "degeneracies" in the set of the new generated ELP. The content of the present article has been established independently of these two works.

## II. DARBOUX-BÄCKLUND TRANSFORMATIONS (DBT) AND REGULAR EXTENSIONS

### A. General scheme

Consider a family of one-dimensional hamiltonians indexed by a multiparameter  $a$

$$\widehat{H}(a) = -d^2/dx^2 + V(x; a), \quad a \in \mathbb{R}^m, \quad x \in I \subset \mathbb{R}, \quad (1)$$

with zero ground level  $E_0(a) = 0$ .

If  $\psi_\lambda(x; a)$  is an eigenstate of  $\widehat{H}(a)$  associated to the eigenvalue  $E_\lambda(a)$ , then its logarithmic derivative  $w_\lambda(x; a) = -\psi'_\lambda(x; a)/\psi_\lambda(x; a)$ , that we will call a Riccati-Schrödinger (RS) function, satisfies a particular Riccati equation of the following form

$$-w'_\lambda(x; a) + w_\lambda^2(x; a) = V(x; a) - E_\lambda(a). \quad (2)$$

Eq(2) is called the Riccati-Schrödinger (RS) equation [24] for the level  $E_\lambda(a)$ . The RS function  $w_\lambda(x; a)$  presents a simple pole at each node of the eigenstates  $\psi_\lambda(x; a)$ .

It is a well-known fact that the set of general Riccati equations is invariant under the group  $\mathcal{G}$  of smooth  $SL(2, \mathbb{R})$ -valued curves  $Map(\mathbb{R}, SL(2, \mathbb{R}))$  [52, 53]. The particular of Riccati-Schrödinger equations is, as for it, preserved by a specific subset of  $\mathcal{G}$ . These transformations, called Darboux-Bäcklund Transformations (DBT), are build from any solution  $w_\nu(x; a)$  of the initial RS equation Eq(2) as [24, 52, 53]

$$w_\lambda(x; a) \xrightarrow{A(w_\nu)} w_\lambda^{(\nu)}(x; a) = -w_\nu(x; a) + \frac{E_\lambda(a) - E_\nu(a)}{w_\nu(x; a) - w_\lambda(x; a)}, \quad (3)$$

where  $E_\lambda(a) > E_\nu(a)$ . Then  $w_\lambda^{(\nu)}$  is a solution of the RS equation:

$$-w_\lambda^{(\nu)'}(x; a) + \left(w_\lambda^{(\nu)}(x; a)\right)^2 = V^{(\nu)}(x; a) - E_\lambda(a), \quad (4)$$

with the same energy  $E_\lambda(a)$  as in Eq(2) but with a modified potential

$$V^{(\nu)}(x; a) = V(x; a) + 2w'_\nu(x; a). \quad (5)$$

This can be schematically resumed as

$$\left\{ \begin{array}{l} w_\lambda \xrightarrow{A(w_\nu)} w_\lambda^{(\nu)} \\ V \xrightarrow{A(w_\nu)} V^{(\nu)} \end{array} \right. \quad (6)$$

The corresponding eigenstate of  $\widehat{H}^{(\nu)}(a) = -d^2/dx^2 + V^{(\nu)}(x; a)$  can be written

$$\psi_\lambda^{(\nu)}(x; a) = \exp\left(-\int dx w_\lambda^{(\nu)}(x; a)\right) \sim \frac{1}{\sqrt{E_\lambda(a) - E_\nu(a)}} \widehat{A}(w_\nu) \psi_\lambda(x; a), \quad (7)$$

where  $\widehat{A}(a)$  is a first order operator given by

$$\widehat{A}(w_\nu) = d/dx + w_\nu(x; a). \quad (8)$$

From  $V$ , the DBT generates a new potential  $V^{(\nu)}$  (quasi) isospectral to the original one and its eigenfunctions are directly obtained from those of  $V$  via Eq(7). If the initial system is exactly solvable, which is the case of the translationally shape invariant potentials (TSIP), this scheme allows to build new exactly solvable potentials.

Nevertheless, in general,  $w_\nu(x; a)$  and then the transformed potential  $V^{(\nu)}(x; a)$  are singular at the nodes of  $\psi_\nu(x; a)$ . For instance, if  $\psi_n(x; a)$  ( $\nu = n$ ) is a bound state of  $\widehat{H}(a)$ ,  $V^{(n)}$  is regular only when  $n = 0$ , that is when  $\psi_{n=0}$  is the ground state of  $\widehat{H}$ , and we recover the usual SUSY partnership in quantum mechanics. Starting from an excited state, that is for  $n \geq 1$ , the transformed potential presents exactly  $n$  second order poles and a priori we cannot use  $A(w_n)$  to build a regular potential. We can however envisage to use any other regular solution of Eq(2) as long as it has no zero on the considered real interval  $I$ , even if it does not correspond to a physical state. As shown in the case of the isotonic oscillator, we can obtain such solutions by using specific discrete symmetries  $\Gamma_i$  which are covariance transformations for the considered family of potentials

$$\left\{ \begin{array}{l} a \xrightarrow{\Gamma_i} a_i \\ V(x; a) \xrightarrow{\Gamma_i} V(x; a_i) = V(x; a) + U(a). \end{array} \right. \quad (9)$$

$\Gamma_i$  acts on the parameters of the potential and transforms the RS function of a physical eigenstate  $w_n$  into a unphysical RS function  $v_{n,i}(x; a) = \Gamma_i(w_n(x; a)) = w_n(x; a_i)$  associated to the eigenvalue  $\mathcal{E}_{n,i}(a) = \Gamma_i(E_n(a)) = U(a) - E_n(a_i) < 0$ .

$$-v'_{n,i}(x; a) + v_{n,i}^2(x; a) = V(x; a) - \mathcal{E}_{n,i}(a). \quad (10)$$

We cannot ensure that  $V(x; a) > \mathcal{E}_{n,i}(a)$  on the whole interval  $I$  but we can make use of the following theorem of Bôcher [46]:

*Bôcher's theorem*

If there exists a continuously derivable function  $W(x)$  such that  $\forall x \in [A, B] \subset \mathbb{R}$

$$W'(x) + W^2(x) < G(x), \quad (11)$$

then every non zero solution of

$$(K(x)\Phi'(x))' - G(x)\Phi(x) = 0, \quad K(x) > 0 \quad (12)$$

admits at most one zero on  $[A, B]$ .

Since  $\forall x \in I$

$$-w'_0(x; a) + w_0^2(x; a) < V(x; a) - \mathcal{E}_{n,i}(a), \quad (13)$$

we satisfy the condition Eq(11) with  $W(x) = -w_0(x; a)$  and  $G(x) = V(x; a) - \mathcal{E}_{n,i}(a)$ . Consequently

$$\phi_{n,i}(x; a) = \exp\left(-\int dx v_{n,i}(x; a)\right) \quad (14)$$

being a solution of

$$\phi''_{n,i}(x; a) - (V(x; a) - \mathcal{E}_{n,i}(a))\phi_{n,i}(x; a) = 0 \quad (15)$$

on  $I$ , it admits at most one simple zero on every compact subset of  $I$ . In the case where it has no zero,  $v_n(x; a) = -\phi'_n(x; a)/\phi_n(x; a)$  is regular on  $I$ . Then the extended potential (see Eq(5) and Eq(7))

$$V^{(n,i)}(x; a) = V(x; a) + 2v'_{n,i}(x; a) \quad (16)$$

is a regular extension of  $V(x; a)$  isospectral to this last and the eigenstates of which being given by (see Eq(3))

$$\left\{ \begin{array}{l} w_k^{(n,i)}(x; a) = -v_{n,i}(x; a) + \frac{E_k(a) - \mathcal{E}_{n,i}(a)}{v_{n,i}(x; a) - w_k(x; a)} \\ \psi_k^{(n,i)}(x; a) = \exp\left(-\int dx w_k^{(n,i)}(x; a)\right) \sim \frac{1}{\sqrt{E_k(a) - \mathcal{E}_{n,i}(a)}} \widehat{A}(v_{n,i}) \psi_k(x; a) \end{array} \right. , \quad (17)$$

for the respective energies  $E_k(a)$ . If we suppose that this isospectrality is strict, by applying the same DBT to an unphysical eigenstate  $\phi_{k,i}$  we obtain an unphysical eigenstate  $\phi_{k,j}^{(n,i)}$  of  $V^{(n,i)}$  with the same negative energy  $\mathcal{E}_{k,j}$

$$\left\{ \begin{array}{l} v_{k,j}^{(n,i)}(x; a) = -v_{n,i}(x; a) + \frac{\mathcal{E}_{k,j}(a) - \mathcal{E}_{n,i}(a)}{v_{n,i}(x; a) - v_{k,j}(x; a)} \\ \phi_{k,j}^{(n,i)}(x; a) = \exp\left(-\int dx v_{k,j}^{(n,i)}(x; a)\right) \sim \frac{1}{\sqrt{\mathcal{E}_{k,j}(a) - \mathcal{E}_{n,i}(a)}} \widehat{A}(v_{n,i}) \phi_{k,j}(x; a). \end{array} \right. \quad (18)$$

If  $1/\phi_n(x; a)$  satisfies the appropriate boundary conditions, then it is a physical eigenstate of  $\widehat{H}^{(n,i)}(a) = -d^2/dx^2 + V^{(n,i)}(x; a)$  for the eigenvalue since the corresponding RS function  $-v_{n,i}(x; a)$  is a solution of

$$-w'(x; a) + (w(x; a))^2 = V^{(n,i)}(x; a) - \mathcal{E}_{n,i}(a). \quad (19)$$

In this case we only have quasi-isospectrality between  $V(x; a)$  and  $V^{(n,i)}(x; a)$  and the unphysical eigenstates can a priori be transformed into physical ones. This construction can be summarized by the following diagram

$$\begin{array}{rcccl} E_k > 0 : & w_k & \xrightarrow{A(v_{n,i})} & w_k^{(n,i)} & \} \text{Physical RS functions} \\ & \Gamma_j \downarrow & & & \\ \mathcal{E}_{k,j} < 0 : & v_{k,j} & \xrightarrow{A(v_{n,i})} & v_{k,j}^{(n,i)} & \} \text{Regularized (unphysical) RS functions} \\ & \underbrace{\hspace{1cm}}_V & \xrightarrow{A(v_{n,i})} & \underbrace{\hspace{1cm}}_{V^{(n,i)}} & \text{Potentials} \end{array} \quad (20)$$

with

$$\left\{ \begin{array}{l} \psi_k^{(n,i)} = \widehat{A}(v_{n,i}) \psi_k \\ \phi_{k,j}^{(n,i)} = \widehat{A}(v_{n,i}) \phi_{k,j}. \end{array} \right. \quad (21)$$

The above procedure can be viewed as a "generalized SUSY QM partnership" where the DBT can be based on excited states RS functions properly regularized by the symmetry  $\Gamma_j$ .

## B. Isotonic oscillator

Consider the isotonic oscillator potential (ie the radial effective potential for a three dimensional isotropic harmonic oscillator with zero ground-state energy)

$$V(x; \omega, a) = \frac{\omega^2}{4} x^2 + \frac{a(a-1)}{x^2} + V_0(\omega, a), \quad x > 0, \quad (22)$$

with  $a = l + 1 \geq 1$  and  $V_0(\omega, a) = -\omega(a + \frac{1}{2})$ . Its physical spectrum is characterized by the (unnormalized) eigenfunctions ( $z = \omega x^2/2$ ,  $\alpha = a - 1/2$ )

$$E_n(a) = 2n\omega, \quad \psi_n(x; \omega, a) = x^\alpha e^{-z/2} L_n^\alpha(z), \quad (23)$$

or, equivalently by the associated RS functions [24]

$$w_n(x; \omega, a) = w_0(x; \omega, a) + R_n(x; \omega, a), \quad (24)$$

with

$$w_0(x; \omega, a) = \frac{\omega}{2}x - \frac{a}{x} \quad (25)$$

and

$$\begin{aligned} R_n(x; \omega, a) &= \frac{-2n\omega}{\omega x - (2a+1)/x} \uparrow \dots \uparrow \frac{2(n-j+1)\omega}{\omega x - (2(a+j)-1)/x} \uparrow \dots \uparrow \frac{2\omega}{\omega x - (2(a+n)-1)/x} \\ &= -\omega x (\log(L_n^\alpha(z)))'. \end{aligned} \quad (26)$$

We have three possible discrete symmetries (see Eq(9) ) given by

1)

$$\omega \xrightarrow{\Gamma_+} (-\omega), \begin{cases} V(x; \omega, a) \xrightarrow{\Gamma_+} V(x; \omega, a) + \omega(2a+1) \\ w_n(x; \omega, a) \xrightarrow{\Gamma_+} v_{n,1}(x; \omega, a) = w_n(x; -\omega, a), \end{cases} \quad (27)$$

2)

$$a \xrightarrow{\Gamma_-} 1-a, \begin{cases} V(x; \omega, a) \xrightarrow{\Gamma_-} V(x; \omega, a) + \omega(2a-1) \\ w_n(x; \omega, a) \xrightarrow{\Gamma_-} v_{n,2}(x; \omega, a) = w_n(x; \omega, 1-a), \end{cases} \quad (28)$$

3)

$$(\omega, a) \xrightarrow{\Gamma_3 = \Gamma_+ \circ \Gamma_-} (-\omega, 1-a) \begin{cases} V(x; \omega, a) \xrightarrow{\Gamma_3} V(x; \omega, a) + 2\omega \\ w_n(x; \omega, a) \xrightarrow{\Gamma_3} v_{n,3}(x; \omega, a) = w_n(x; -\omega, 1-a). \end{cases} \quad (29)$$

In the  $(\omega, \alpha)$  parameters plane,  $\Gamma_+$  and  $\Gamma_-$  correspond respectively to the reflections with respect to the axes  $\omega = 0$  and  $\alpha = 0$ .

To each of these symmetries is associated family of DBT  $A(v_{n,i})$  which generate the three infinite series  $L1$ ,  $L2$  and  $L3$  of regular rationally solvable extensions of the isotonic potential, the physical eigenstates of the  $L1$  and  $L2$  series being expressible in terms of exceptional Laguerre polynomials of the  $L1$  and  $L2$  types respectively.

### C. L1 series

For the  $L1$  series we have, writing  $\phi_{k,+}(x; \omega, a) = \exp(-\int dx v_{k,+}(x; \omega, a))$

$$\begin{cases} \widehat{H}(\omega, a)\phi_{n,+}(x; \omega, a) = \mathcal{E}_{n,+}(\omega, a)\phi_{n,+}(x; \omega, a) \\ -v'_{n,+}(x; \omega, a) + v_{n,+}^2(x; \omega, a) = V(x; \omega, a) - \mathcal{E}_{n,+}(\omega, a) \end{cases} ; \mathcal{E}_{n,+}(\omega, a) = E_{-(n+a+1/2)}(\omega) < 0, \quad (30)$$

where ( $z = \omega x^2/2$ ,  $\alpha = a - 1/2$ )

$$\begin{cases} \phi_{n,+}(x; \omega, a) = x^a \exp(z/2) L_n^\alpha(-z) \\ v_{n,+}(x; \omega, a) = v_{0,+}(x; \omega, a) + Q_{n,+}(x; \omega, a) \end{cases}, \quad (31)$$

with

$$v_{0,+}(x; \omega, a) = -\frac{\omega}{2}x - \frac{a}{x} \quad (32)$$

and

$$\begin{aligned} Q_{n,+}(x; \omega, a) &= -\frac{2n\omega}{\omega x + (2a+1)/x+} \uparrow \dots \uparrow \frac{2(n-j+1)\omega}{\omega x + (2(a+j)-1)/x+} \uparrow \dots \uparrow \frac{2\omega}{\omega x + (2(a+n)-1)/x} \\ &= -\omega x (\log(L_n^\alpha(-z)))'. \end{aligned} \quad (33)$$

Since the eigenvalue  $\mathcal{E}_{n,+}(\omega, a)$  associated to  $\phi_{n,+}(x; \omega, a)$  is negative, this last not a physical state and,  $\phi_{n,+}$  being non oscillatory, it has at most one node on the positive half line. Since we have

$$\begin{cases} L_n^\alpha(z) \underset{x \rightarrow 0^+}{\rightarrow} \frac{(\alpha+1)\dots(\alpha+n)}{n!} \\ L_n^\alpha(z) \underset{x \rightarrow +\infty}{\sim} \frac{(-1)^n}{n!} z^n \end{cases}, \quad (34)$$

then

$$\begin{cases} \phi_{n,+}(x; \omega, a) \underset{x \rightarrow +\infty}{\rightarrow} \pm\infty \\ \phi_{n,+}(x; \omega, a) \underset{x \rightarrow 0^+}{\rightarrow} 0, \end{cases} \quad (35)$$

and  $\phi_{n,+}$  keeps a constant sign on  $]0, +\infty[$  which means that  $v_{n,+}(x; \omega, a)$  is regular on this interval. Note that this result implies in particular that  $L_n^\alpha(-z)$  has no zero on the positive half line which agrees with the Kienast-Lawton-Hahn theorem [48, 49].

$w_k^{(n,+)}(x; \omega, a)$  and  $\psi_k^{(n,+)}(x; \omega, a) = \exp\left(-\int dx w_k^{(n,+)}(x; \omega, a)\right)$  satisfy then

$$\begin{cases} \widehat{H}^{(n,+)}(\omega, a) \psi_k^{(n,+)}(x; \omega, a) = E_k(\omega) \psi_k^{(n,+)}(x; \omega, a) \\ -w_k^{(n,+)\prime}(x; \omega, a) + \left(w_k^{(n,+)}(x; \omega, a)\right)^2 = V^{(n,+)}(x; \omega, a) - E_k(\omega) \end{cases}, \quad (36)$$

with

$$V^{(n,+)}(x; \omega, a) = V(x; \omega, a) + 2v_{n,+}'(x; \omega, a). \quad (37)$$

For every  $n \geq 0$ ,  $V^{(n,+)}(x; \omega, a)$  is regular on the positive half line and strictly isospectral to  $V(x; \omega, a)$  ( $1/\phi_{n,+}(x; \omega, a) = \exp\left(\int dx v_{k,+}(x; \omega, a)\right)$  is not a physical eigenstate although it is an eigenfunction of  $\widehat{H}^{(n,+)}(\omega, a) = -d^2/dx^2 + V^{(n,1)}(x; \omega, a)$  associated to the eigenvalue  $\mathcal{E}_{n,+}(\omega, a) < 0$ )

$$V^{(n,+)}(x; \omega, a) = V(x; \omega, a_{-1}) + 2Q_{n,+}' \underset{iso}{(x; \omega, a)} \equiv V(x; \omega, a). \quad (38)$$

All the (unnormalized) physical eigenfunctions of  $\widehat{H}^{(n,+)}$  are then of the form

$$\psi_k^{(n,+)}(x; \omega, a) = \widehat{A}(v_{n,+}) \psi_k(x; \omega, a), \quad k \geq 0. \quad (39)$$

More precisely, using the identity  $L_n^\alpha(z) + L_{n-1}^{\alpha+1}(z) = L_n^{\alpha+1}(z)$

$$\begin{aligned} \psi_k^{(n,+)}(x; \omega, a) &= (v_{n,+}(x; \omega, a) - w_k(x; \omega, a)) \psi_k(x; \omega, a) \\ &= -\omega L_{n,k,\alpha}^+(z) \frac{x^{\alpha+1} \exp(-z/2)}{L_n^\alpha(-z)}. \end{aligned} \quad (40)$$

where

$$L_{n,k,\alpha}^+(z) = L_n^\alpha(-z) L_k^{\alpha+1}(z) + L_{n-1}^{\alpha+1}(-z) L_k^\alpha(z), \quad (41)$$

which is a polynomial of degree  $n + k$ , namely an exceptional Laguerre polynomial (ELP) of the  $L1$  series. From the orthogonality conditions on the eigenstates for fixed values of  $n$  and  $a$ , the  $L1$  ELP  $L_{n,k,\alpha}^+(z)$  constitute an orthogonal family with respect to the weight

$$W_n^+(z) = \frac{z^{\alpha+1} \exp(-z)}{(L_n^\alpha(-z))^2}, \quad (42)$$

which is complete in  $L_{W_n^+}^2[0, +\infty[$ .

#### D. L2 series

For the  $L2$  series we have, writing  $\phi_{k,-}(x; \omega, a) = \exp(-\int dx v_{k,-}(x; \omega, a))$

$$\begin{cases} \widehat{H}(\omega, a)\phi_{n,-}(x; \omega, a) = \mathcal{E}_{n,-}(\omega, a)\phi_{n,-}(x; \omega, a) \\ -v'_{n,-}(x; \omega, a) + v_{n,-}^2(x; \omega, a) = V(x; \omega, a) - \mathcal{E}_{n,-}(\omega, a) \end{cases} ; \mathcal{E}_{n,-}(\omega, a) = E_{n+1/2-a}(\omega), \quad (43)$$

where

$$\begin{cases} \phi_{n,-}(x; \omega, a) = x^{1-a} \exp(-z/2) L_n^{-\alpha}(z) \\ v_{n,-}(x; \omega, a) = v_{0,-}(x; \omega, a) + Q_{n,-}(x; \omega, a) \end{cases} \quad (44)$$

with

$$v_{0,-}(x; \omega, l) = \frac{\omega}{2}x + \frac{a-1}{x} \quad (45)$$

and

$$\begin{aligned} Q_{n,-}(x; \omega, a) &= \frac{-2n\omega}{\omega x + (2a-3)/x-} \uparrow \dots \uparrow \frac{2(n-j+1)\omega}{\omega x + (2(a-j)-1)/x-} \uparrow \dots \uparrow \frac{2\omega}{\omega x + (2(a-n)-1)/x} \\ &= -\omega x (\log(L_n^{-\alpha}(z)))'. \end{aligned} \quad (46)$$

Only for  $\alpha = a - 1/2 > n$ , the eigenvalue  $\mathcal{E}_{n,-}(\omega, a)$  associated to  $\phi_{n,-}(x; \omega, a)$  is negative and this last is not a physical state of  $\widehat{H}(\omega, a)$ . Moreover,  $\phi_{n,-}$  is non oscillatory and has at most one node on the positive half line. Since we have (see Eq(34))

$$\begin{cases} \phi_{n,-}(x; \omega, a) \xrightarrow{x \rightarrow +\infty} 0^\mp \\ \phi_{n,-}(x; \omega, a) \xrightarrow{x \rightarrow 0} \pm\infty, \end{cases} \quad (47)$$

where  $\pm = (-1)^n$ ,  $\phi_{n,-}$  keeps a constant sign on  $]0, +\infty[$  which means that  $v_{n,-}(x; \omega, a)$  is regular on this interval. Note that this result implies in particular that  $L_n^{-\alpha}(z)$  has no zero on the positive half line when  $\alpha > n$  and again we recover the content of the Kienast-Lawton-Hahn theorem [48, 49].

$w_k^{(n,-)}(x; \omega, a)$  and  $\psi_k^{(n,-)}(x; \omega, a) = \exp(-\int dx w_k^{(n,-)}(x; \omega, a))$  satisfy

$$\begin{cases} \widehat{H}^{(n,-)}(\omega, a)\psi_k^{(n,-)}(x; \omega, a) = E_k(\omega)\psi_k^{(n,-)}(x; \omega, a) \\ -w_k^{(n,-)'}(x; \omega, a) + (w_k^{(n,-)}(x; \omega, a))^2 = V^{(n,-)}(x; \omega, a) - E_k(\omega) \end{cases}, \quad (48)$$

with

$$V^{(n,-)}(x; \omega, a) = V(x; \omega, a) + 2v'_{n,-}(x; \omega, a). \quad (49)$$

$V^{(n,-)}(x; \omega, a)$  is regular on the positive half line and strictly isospectral to  $V(x; \omega, a)$  ( $1/\phi_{n,-}(x; \omega, a) = \exp(\int dx v_{k,-}(x; \omega, a))$ ) is not a physical eigenstate although it is an eigenfunction of  $\widehat{H}^{(n,-)}(\omega, a) = -d^2/dx^2 + V^{(n,-)}(x; \omega, a)$  associated to the eigenvalue  $\mathcal{E}_{n,-}(\omega, a) < 0$ )

$$V^{(n,-)}(x; \omega, a) = V(x; \omega, a_{-1}) + 2Q'_{n,2}(x; \omega, a) \stackrel{iso}{=} V(x; \omega, a). \quad (50)$$

All the (unnormalized) physical eigenfunctions of  $\widehat{H}^{(n,-)}$  are then of the form

$$\psi_k^{(n,-)}(x; \omega, a) = \widehat{A}(v_{n,-}) \psi_k(x; \omega, a), \quad k \geq 0. \quad (51)$$

More precisely, using the identity  $zL_{n-1}^{\alpha+1}(z) = (n+\alpha)L_n^\alpha(z) - nL_n^\alpha(z)$

$$\begin{aligned} \psi_k^{(n,-)}(x; \omega, a) &= (v_{n,-}(x; \omega, a) - w_k(x; \omega, a)) \psi_k(x; \omega, a) \\ &= \frac{1}{2} L_{n,k,\alpha}^-(z) \frac{x^{\alpha-1} \exp(-z/2)}{L_n^{-\alpha}(z)}, \end{aligned} \quad (52)$$

where

$$L_{n,k,\alpha}^-(z) = (k+n+\alpha)L_k^\alpha(z)L_n^{-\alpha}(z) - (-n+\alpha)L_k^\alpha(z)L_{n-1}^{-\alpha}(z) - (k+\alpha)L_{k-1}^\alpha(z)L_n^{-\alpha}(z) \quad (53)$$

is a polynomial of degree  $n+k$ , namely an exceptional Laguerre polynomial (ELP) of the  $L2$  series. From the orthogonality conditions on the eigenstates for fixed values of  $n$  and  $a$  (with  $\alpha > n$ ), the  $L2$  ELP  $L_{n,k,\alpha}^-(z)$  constitute an orthogonal family with respect to the weight

$$W_n^-(z) = \frac{z^\alpha \exp(-z)}{(L_n^{-\alpha}(z))^2}. \quad (54)$$

### E. L3 series

For the  $L3$  series we have, writing  $\phi_{k,3}(x; \omega, a) = \exp(-\int dx v_{k,3}(x; \omega, a))$

$$\begin{cases} \widehat{H}(\omega, a)\phi_{n,3}(x; \omega, a) = \mathcal{E}_{n,3}(\omega, a)\phi_{n,3}(x; \omega, a) \\ -v'_{n,3}(x; \omega, a) + v_{n,3}^2(x; \omega, a) = V(x; \omega, a) - \widetilde{E}_{n,3}(\omega) \end{cases}; \quad \mathcal{E}_{n,3}(\omega, a) = E_{-(n+1)}(\omega) < 0. \quad (55)$$

where

$$\begin{cases} \phi_{n,3}(x; \omega, a) = x^{1-a} \exp(z/2) L_n^{-\alpha}(-z) \\ v_{n,3}(x; \omega, a) = v_{0,3}(x; \omega, a) + Q_{n,3}(x; \omega, a) \end{cases}, \quad (56)$$

with

$$v_{0,3}(x; \omega, l) = -\frac{\omega}{2}x + \frac{a-1}{x} \quad (57)$$

and

$$\begin{aligned} Q_{n,3}(x; \omega, a) &= \frac{2n\omega}{\omega x - (2a-3)/x+} \uparrow \dots \uparrow \frac{2(n-j+1)\omega}{\omega x - (2(a-j)-1)/x+} \uparrow \dots \uparrow \frac{2\omega}{\omega x - (2(a-n)-1)/x} \\ &= -(\log(L_n^{-\alpha}(-\omega x^2/2)))'. \end{aligned} \quad (58)$$

The eigenvalue  $\mathcal{E}_{n,3}(\omega)$  associated to  $\phi_{n,3}(x; \omega, a)$  is always negative and this last is not a physical state of  $\widehat{H}(\omega, a)$ . Moreover,  $\phi_{n,3}$  is non oscillatory and has at most one node on the positive half line. Since we have (see Eq(34))

$$\begin{cases} \phi_{n,3}(x; \omega, a) \xrightarrow{x \rightarrow +\infty} +\infty \\ \phi_{n,3}(x; \omega, a) \xrightarrow{x \rightarrow 0} \pm\infty, \end{cases} \quad (59)$$

where  $\pm = (-1)^n$ ,  $\phi_{n,3}$  keeps a constant sign on  $]0, +\infty[$  when  $n$  is even (in which case  $v_{n,3}(x; \omega, a)$  is regular on this interval) and presents one node on  $]0, +\infty[$  when  $n$  is odd (in which case  $v_{n,3}(x; \omega, a)$  has a singularity). Note that this result implies in particular that  $L_n^{(-\alpha)}(-z)$  has 0 or 1 zero on the positive half line in relation with the parity of  $n$ . This result is again coherent with the Kienast-Lawton-Hahn theorem [48, 49]

$w_k^{(n,3)}(x; \omega, a)$  and  $\psi_k^{(n,3)}(x; \omega, a) = \exp\left(-\int dx w_k^{(n,3)}(x; \omega, a)\right)$  satisfy

$$\begin{cases} \widehat{H}^{(n,3)}(\omega, a)\psi_k^{(n,3)}(x; \omega, a) = E_k(\omega)\psi_k^{(n,3)}(x; \omega, a) \\ -w_k^{(n,3)'}(x; \omega, a) + \left(w_k^{(n,3)}(x; \omega, a)\right)^2 = V^{(n,3)}(x; \omega, a) - E_k(\omega) \end{cases}, \quad (60)$$

with

$$V^{(n,3)}(x; \omega, a) = V(x; \omega, a) + 2v_{n,3}'(x; \omega, a). \quad (61)$$

For  $n$  even  $V^{(n,3)}(x; \omega, a)$  is regular on the positive half line and quasi isospectral to  $V(x; \omega, a)$

$$V^{(n,3)}(x; \omega, a) = V(x; \omega, a_{-1}) + 2Q_{n,3}'(x; \omega, a) \underset{iso}{=} V(x; \omega, a). \quad (62)$$

Indeed, in this case  $1/\phi_{n,3}(x; \omega, a) = \exp\left(\int dx v_{k,3}(x; \omega, a)\right)$  is normalizable and then admissible as physical eigenstate of lowest eigenvalue for  $\widehat{H}^{(n,3)}(\omega, a) = -d^2/dx^2 + V^{(n,3)}(x; \omega, a)$ . All the (unnormalized) physical eigenfunctions of  $\widehat{H}^{(n,3)}$  are then of the form

$$\begin{cases} \psi_-^{(n,3)}(x; \omega, a) = 1/\phi_{n,3}(x; \omega, a) = \exp\left(\int dx v_{k,3}(x; \omega, a)\right) \\ \psi_k^{(n,3)}(x; \omega, a) = \widehat{A}(v_{n,3})\psi_k(x; \omega, a), \quad k \geq 0, \end{cases} \quad (63)$$

for the corresponding energies

$$\begin{cases} E_-^{(n,3)}(\omega) = E_{-(n+1)}(\omega) < 0 \\ E_k^{(n,3)}(\omega) = E_k(\omega), \quad k \geq 0. \end{cases} \quad (64)$$

More precisely, using the identities  $zL_{n-1}^{\alpha+1}(z) = (n+\alpha)L_{n-1}^\alpha(z) - nL_n^\alpha(z)$

$$\begin{aligned} \psi_k^{(n,3)}(x; \omega, a) &= (v_{n,3}(x; \omega, a) - w_k(x; \omega, a))\psi_k(x; \omega, a) \\ &= (-2)L_{n,k,\alpha}^3(z) \frac{x^{\alpha-1} \exp(-z/2)}{L_n^{-\alpha}(-z)}, \end{aligned} \quad (65)$$

where

$$L_{n,k,\alpha}^3(z) = zL_k^\alpha(z)L_n^{-\alpha+1}(-z) + zL_{k-1}^{\alpha+1}(z)L_n^{-\alpha}(-z) - \alpha L_k^\alpha(z)L_n^{-\alpha}(-z) \quad (66)$$

is a polynomial of degree  $n+k+1$ . From the orthogonality conditions on the eigenstates for fixed values of  $n$  and  $a$ , the  $L_{n,k,\alpha}^3(z)$  constitute an orthogonal family with respect to the weight

$$W_n^3(z) = \frac{z^\alpha \exp(-z)}{(L_n^{-\alpha}(-z))^2}. \quad (67)$$

### III. MULTI-STEP DBT AND NEW ELP

#### A. General formulation

The question of successive iterations of DBT is very natural and is at the center of the construction of the hierarchy of hamiltonians in the usual SUSY QM scheme [30]. Staying at the formal level, it can be simply described by the following straightforward generalization of Eq(6)

$$\begin{cases} w_\lambda \xrightarrow{A(w_{\nu_1})} w_\lambda^{(\nu_1)} \xrightarrow{A(w_{\nu_2}^{(\nu_1)})} w_\lambda^{(\nu_1, \nu_2)} \dots \xrightarrow{A(w_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})})} w_\lambda^{(\nu_1, \dots, \nu_m)} \\ V \xrightarrow{A(w_{\nu_1})} V^{(\nu_1)} \xrightarrow{A(w_{\nu_2}^{(\nu_1)})} V^{(\nu_1, \nu_2)} \dots \xrightarrow{A(w_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})})} V^{(\nu_1, \dots, \nu_m)}, \end{cases} \quad (68)$$

where  $w_\lambda^{(\nu_1, \dots, \nu_m)}$  is a RS function associated to the eigenvalue  $E_\lambda$  of the potential

$$V^{(\nu_1, \dots, \nu_m)}(x; a) = V(x; a) + 2 \sum_{j=1}^{m-1} \left( w_{\nu_j}^{(\nu_1, \dots, \nu_{j-1})}(x; a) \right)' . \quad (69)$$

The corresponding eigenfunction is given by

$$\psi_\lambda^{(\nu_1, \dots, \nu_m)}(x; a) = \widehat{A} \left( w_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})} \right) \psi_\lambda^{(\nu_1, \dots, \nu_{m-1})}(x; a) = \widehat{A} \left( w_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})} \right) \dots \widehat{A} \left( w_{\nu_1} \right) \psi_\lambda(x; a), \quad (70)$$

that is,

$$\psi_\lambda^{(\nu_1, \dots, \nu_m)}(x; a) = \left( w_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})}(x; a) - w_\lambda^{(\nu_1, \dots, \nu_{m-1})}(x; a) \right) \psi_\lambda^{(\nu_1, \dots, \nu_{m-1})}(x; a) = \frac{W \left( \psi_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})}, \psi_\lambda^{(\nu_1, \dots, \nu_{m-1})} \mid x \right)}{\psi_{\nu_m}^{(\nu_1, \dots, \nu_{m-1})}(x; a)}, \quad (71)$$

where  $W(y_1, \dots, y_m \mid x)$  is the wronskian of the functions  $y_1, \dots, y_m$

$$W(y_1, \dots, y_m \mid x) = \begin{vmatrix} y_1(x) & \dots & y_m(x) \\ \vdots & & \vdots \\ y_1^{(m-1)}(x) & \dots & y_m^{(m-1)}(x) \end{vmatrix}. \quad (72)$$

Other useful representations for the extended potentials and of their eigenfunctions are possible. From Sylvester's theorem we can deduce [47] the following property for the Wronskians

$$W(y_1, \dots, y_m, y \mid x) = \frac{W(W(y_1, \dots, y_m), W(y_1, \dots, y_{m-1}, y) \mid x)}{W(y_1, \dots, y_{m-1} \mid x)}. \quad (73)$$

Using the identity [47]

$$W(uy_1, \dots, uy_m \mid x) = u^m W(y_1, \dots, y_m \mid x), \quad (74)$$

this gives

$$\frac{W(y_1, \dots, y_m, y \mid x)}{W(y_1, \dots, y_m \mid x)} = \frac{W \left( \frac{W(y_1, \dots, y_m)}{W(y_1, \dots, y_{m-1})}, \frac{W(y_1, \dots, y_{m-1}, y)}{W(y_1, \dots, y_{m-1})} \mid x \right)}{\frac{W(y_1, \dots, y_m \mid x)}{W(y_1, \dots, y_{m-1} \mid x)}} \quad (75)$$

and comparing to Eq(71), we obtain

$$\psi_\lambda^{(\nu_1, \dots, \nu_m)}(x; a) = \frac{W(\psi_{\nu_1}, \dots, \psi_{\nu_m}, \psi_\lambda | x)}{W(\psi_{\nu_1}, \dots, \psi_{\nu_m} | x)}, \quad (76)$$

which is the well known Crum formula [26, 29] for the eigenfunctions.

Inserting this result in Eq(69), we then deduce the Crum formula for the potential [26, 29]

$$V^{(\nu_1, \dots, \nu_m)}(x; a) = V(x; a) + 2(\log W(\psi_{\nu_1}, \dots, \psi_{\nu_m} | x))'' . \quad (77)$$

Since all the functions  $\psi_{\nu_j}$  implied in the wronskians in Eq(76) and Eq(77) are eigenfunctions of the same hamiltonian  $\widehat{H}(a)$ , the properties of the determinants allow us to replace in these wronskians the even derivatives  $\psi_{\nu_j}^{(2l)}$  by  $(-E_{\nu_j})^l \psi_{\nu_j}$  and the odd derivatives  $\psi_{\nu_j}^{(2l+1)}$  by  $(-E_{\nu_j})^l \psi'_{\nu_j}$ .

For instance

$$W(\psi_{\nu_1}, \dots, \psi_{\nu_{2l}} | x) = \begin{vmatrix} \psi_{\nu_1}(x; a) & \dots & \psi_{\nu_{2l}}(x; a) \\ \psi'_{\nu_1}(x; a) & \dots & \psi'_{\nu_{2l}}(x; a) \\ \dots & \dots & \dots \\ (-E_{\nu_1})^{l-1} \psi_{\nu_1}(x; a) & \dots & (-E_{\nu_{2l}})^{l-1} \psi_{\nu_{2l}}(x; a) \\ (-E_{\nu_1})^{l-1} \psi'_{\nu_1}(x; a) & \dots & (-E_{\nu_{2l}})^{l-1} \psi'_{\nu_{2l}}(x; a) \end{vmatrix} = \Delta^{(\nu_1, \dots, \nu_{2l})}(x; a) \prod_{j=1}^{2l} \psi_{\nu_j}(x; a), \quad (78)$$

where

$$\Delta^{(\nu_1, \dots, \nu_{2l})}(x; a) = (-1)^{l^2-1} \begin{vmatrix} 1 & \dots & 1 \\ w_{\nu_1}(x; a) & \dots & w_{\nu_{2l}}(x; a) \\ \dots & \dots & \dots \\ (E_{\nu_1})^{l-1} & \dots & (E_{\nu_{2l}})^{l-1} \\ (E_{\nu_1})^{l-1} w_{\nu_1}(x; a) & \dots & (E_{\nu_{2l}})^{l-1} w_{\nu_{2l}}(x; a) \end{vmatrix} \quad (79)$$

and

$$W(\psi_{\nu_1}, \dots, \psi_{\nu_{2l+1}} | x) = \begin{vmatrix} \psi_{\nu_1}(x; a) & \dots & \psi_{\nu_{2l+1}}(x; a) \\ \psi'_{\nu_1}(x; a) & \dots & \psi'_{\nu_{2l+1}}(x; a) \\ \dots & \dots & \dots \\ (-E_{\nu_1})^{l-1} \psi'_{\nu_1}(x; a) & \dots & (-E_{\nu_{2l+1}})^{l-1} \psi'_{\nu_{2l+1}}(x; a) \\ (-E_{\nu_1})^l \psi_{\nu_1}(x; a) & \dots & (-E_{\nu_{2l+1}})^l \psi_{\nu_{2l+1}}(x; a) \end{vmatrix} = \Delta^{(\nu_1, \dots, \nu_{2l+1})}(x; a) \prod_{j=1}^{2l+1} \psi_{\nu_j}(x; a), \quad (80)$$

where

$$\Delta^{(\nu_1, \dots, \nu_{2l+1})}(x; a) = (-1)^{l^2+l-1} \begin{vmatrix} 1 & \dots & 1 \\ w_{\nu_1}(x; a) & \dots & w_{\nu_{2l+1}}(x; a) \\ \dots & \dots & \dots \\ (E_{\nu_1})^{l-1} w_{\nu_1}(x; a) & \dots & (E_{\nu_{2l+1}})^{l-1} w_{\nu_{2l+1}}(x; a) \\ (E_{\nu_1})^l & \dots & (E_{\nu_{2l+1}})^l \end{vmatrix}. \quad (81)$$

Then we obtain the representation of the  $\psi_\lambda^{(\nu_1, \dots, \nu_m)}$  in terms the so-called Crum-Krein determinants  $\Delta^{(\nu_1, \dots, \nu_m)}$  [26, 27, 33]

$$\psi_\lambda^{(\nu_1, \dots, \nu_m)}(x; a) = \psi_\lambda(x; a) \frac{\Delta^{(\nu_1, \dots, \nu_m, \lambda)}(x; a)}{\Delta^{(\nu_1, \dots, \nu_m)}(x; a)}. \quad (82)$$

Until now, all these results are purely formal. The central problem is now to choose in an appropriate way the family of eigenfunctions  $(\psi_{\nu_1}, \dots, \psi_{\nu_m})$  from which the successive DBT are built (see Eq(76) and Eq(69)) in order to ensure the regularity of the successive extended potentials  $V^{(\nu_1, \dots, \nu_m)}$ .

A first answer has been given more than half a century ago by Crum [26]. The Crum proposal [26] is to take the successive eigenstates of the discrete bound states spectrum starting from the ground state

$$(\nu_1, \dots, \nu_m) = (0, \dots, m-1) \quad (83)$$

Krein [27, 33, 34] extended this result showing it is possible to choose sets of unnecessarily successive eigenstates if they satisfy a given condition, namely

$$(\nu_1, \dots, \nu_m) = (n_1, \dots, n_m) \in \mathbb{N}^m, \quad (84)$$

with

$$(n - n_1)(n - n_2) \dots (n - n_m) \geq 0, \quad \forall n \in \mathbb{N}. \quad (85)$$

This last is verified if the set  $(n_1, \dots, n_m)$  is constituted by "aggregates" of an even number of eigenstates. The possibility to employ sets of two juxtaposed eigenstates has been rediscovered later by Adler [28]. These results have been used extensively in the context of higher order SUSY (see for instance [25, 30–33, 35–41]). In [37], Samsonov has still extended the Krein-Adler result. Note finally that Fernandez et al [38] employed the specific "Wick rotated" eigenfunctions of negative energies introduced by Shnol' [21, 35, 42] to build successive extensions of the harmonic potential.

Our aim is to consider the possibility to generalize our construction in a multi-step version on the basis of the regularized RS functions (or unphysical eigenfunctions) obtained above. Staying at a formal level, it would correspond to the following generalization of the diagram Eq(20) (see Eq(68))

$$\begin{array}{ccccccc}
 E_k > 0 : & w_k & \xrightarrow{A(v_{n_1, i_1})} & w_k^{(n_1, i_1)} & \xrightarrow{A(v_{n_2, i_2}^{(n_1, i_1)})} & w_k^{(n_1, i_1, n_2, i_2)} & \dots & \xrightarrow{A(v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})})} & w_k^{(n_1, i_1, \dots, n_m, i_m)} \\
 & \Gamma_i \downarrow & & & & & & & \\
 \mathcal{E}_{k,i} < 0 : & v_{k,i} & \xrightarrow{A(v_{n_1, i_1})} & v_{k,i}^{(n_1, i_1)} & \xrightarrow{A(v_{n_2, i_2}^{(n_1, i_1)})} & v_{k,i}^{(n_1, i_1, n_2, i_2)} & \dots & \xrightarrow{A(v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})})} & v_{k,i}^{(n_1, i_1, \dots, n_m, i_m)} \\
 & \underbrace{\phantom{V}}_V & \xrightarrow{A(v_{n_1, i_1})} & \underbrace{\phantom{V}}_{V^{(n_1, i_1)}} & \xrightarrow{A(v_{n_2, i_2}^{(n_1, i_1)})} & \underbrace{\phantom{V}}_{V^{(n_1, i_1, n_2, i_2)}} & \dots & \xrightarrow{A(v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})})} & \underbrace{\phantom{V}}_{V^{(n_1, i_1, \dots, n_m, i_m)}}
 \end{array} \quad (86)$$

with  $v_k^{(n_0)} = v_k$ . Under this way, at, we generate chains of isospectral extensions  $V^{(n_1, i_1, \dots, n_m, i_m)}$  of the potential  $V$ , the  $w_k^{(n_1, i_1, \dots, n_m, i_m)}$  and the  $v_{k,i}^{(n_1, i_1, \dots, n_m, i_m)}$  being RS functions of the extended potential (see Eq(69) and Eq(77))

$$V^{(n_1, i_1, \dots, n_m, i_m)}(x; a) = V(x; a) + 2 \sum_{j=1}^{m-1} \left( v_{n_j, i_j}^{(n_1, i_1, \dots, n_{j-1}, i_{j-1})}(x; a) \right)' \quad (87)$$

$$= V(x; a) + 2 (\log W(\phi_{n_1, i_1}, \dots, \phi_{n_m, i_m} | x))'' \quad (88)$$

associated respectively to the eigenvalues  $E_k$  and  $\mathcal{E}_{k,i}$ .

The corresponding eigenfunctions are given by (see Eq(71) and Eq(76))

$$\left\{ \begin{array}{l}
 \psi_k^{(n_1, i_1, \dots, n_m, i_m)}(x; a) = \frac{W\left(\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}, \psi_k^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})} | x\right)}{\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; a)} = \frac{W(\phi_{n_1, i_1}, \dots, \phi_{n_m, i_m}, \psi_k | x)}{W(\phi_{n_1, i_1}, \dots, \phi_{n_m, i_m} | x)} \\
 \phi_{k,i}^{(n_1, i_1, \dots, n_m, i_m)}(x; a) = \frac{W\left(\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}, \phi_{k,i}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})} | x\right)}{\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; a)} = \frac{W(\phi_{n_1, i_1}, \dots, \phi_{n_m, i_m}, \phi_{k,i} | x)}{W(\phi_{n_1, i_1}, \dots, \phi_{n_m, i_m} | x)}.
 \end{array} \right. \quad (89)$$

Suppose that at the step  $m-1$  all the extended potentials in the chain are regular and strictly isospectral. The problem is now to control the regularity of the potential  $V^{(n_1, i_1, \dots, n_m, i_m)}$  and of its physical eigenstates  $\psi_k^{(n_1, i_1, \dots, n_m, i_m)}$  as well as its strict isospectrality with  $V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$ . This is achieved if the unphysical eigenfunction  $\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  is nodeless and if  $1/\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  is not in the set of physical eigenstates.

The Bôcher's theorem guarantees that the  $v_{n_j, i_j}^{(n_1, i_1, \dots, n_{j-1}, i_{j-1})}$  is at most singular in one point. The regularity of the extended potential  $V^{(n_1, i_1, \dots, n_m, i_m)}$  and its strict isospectrality with the preceding one in the chain are then satisfied as soon as the unphysical eigenfunction associated to the DBT  $A(v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})})$  satisfies the appropriate boundary conditions, which was already the argument used above in the one step case. Namely, the strict isospectrality and the regularity are satisfied when  $\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  tends to infinity at a limit of the interval and to zero at the other limit of the definition interval.

### B. Application to the isotonic oscillator

In the case of the isotonic oscillator, there are three type of regularization transformations  $\Gamma_i$ ,  $i = +, -, 3$  associated respectively to the three series of EOP  $L1$ ,  $L2$  and  $L3$ . At each step we can a priori use any of the three type if regularized RS function. However, since the use of DBT associated to RS functions of the  $L3$  series leads only to quasi isospectrality, the argument above is no more valid. Consequently, in the following we exclude this case and consider only DBT built from the RS functions of the  $L1$  or  $L2$  series. Note that for  $j = +, -$ , we can write synthetically (see Eq(31) and Eq(44))

$$\begin{cases} \mathcal{E}_{n,j}(\omega, a) = -2\omega(a + j(n + 1/2)) \\ \phi_{n,j}(x; \omega, a) = \phi_{0,j}(x; \omega, a)L_n^{\alpha}(-jz) \end{cases}, \quad (90)$$

with

$$\phi_{0,j}(x; \omega, a) \sim x^{j\alpha+1/2} \exp(jz/2), \quad (91)$$

where  $\alpha$  is supposed to satisfy the constraint  $\alpha > -jn$ .

As for the corresponding RS functions, they are

$$v_{n,j}(x; \omega, a) = v_{0,j}(x; \omega, a) - j \frac{L_{n-1}^{j\alpha+1}(-jz)}{L_n^{j\alpha}(-jz)}, \quad j = \pm, \quad (92)$$

with

$$v_{0,j}(x; \omega, a) = -\frac{1}{x}(j(z + \alpha) + 1/2). \quad (93)$$

We have then to ensure the regularity and the strict isospectrality of the potentials  $V^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a)$  on the positive half line. As mentioned above, this is achieved if each  $\phi_{n_j, i_j}^{(n_1, i_1, \dots, n_{j-1}, i_{j-1})}(x; \omega, a)$  satisfies the appropriate boundary conditions on  $]0, +\infty[$ , that is, if it tends to zero at one end and  $\pm\infty$  at the other end of this interval. Indeed, in this case its inverse cannot be a physical eigenstate and, from Bôcher's theorem it cannot have any zero on  $]0, +\infty[$ . From Eq(82), we see that in the present case

$$\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; a) = \phi_{0, i_m}(x; a) R^{(n_1, i_1, \dots, n_m, i_m)}(x; a), \quad (94)$$

where

$$R^{(n_1, i_1, \dots, n_m, i_m)}(x; a) = L_{n_m}^{i_m \alpha}(-i_m z) \frac{\Delta^{(n_1, i_1, \dots, n_m, i_m)}(x; a)}{\Delta^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; a)} \quad (95)$$

is a rational function. Due to the presence of the exponential in  $\phi_{0, i_m}$ , the behaviour at infinity of  $\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  is the same as for  $\phi_{0, i_m}$ , namely

$$\begin{cases} \phi_{n_m,+}^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}(x;\omega,a) \xrightarrow{x \rightarrow +\infty} \pm\infty \\ \phi_{n_m,-}^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}(x;\omega,a) \xrightarrow{x \rightarrow +\infty} 0. \end{cases} \quad (96)$$

The absence of node for  $\phi_{n_m,i_m}^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}$  and the strict isospectrality between  $V^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}$  and  $V^{(n_1,i_1,\dots,n_m,i_m)}$ , are simultaneously ensured if and only if we have

$$\begin{cases} \phi_{n_m,+}^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}(x;\omega,a) \xrightarrow{x \rightarrow 0} 0 \\ \phi_{n_m,-}^{(n_1,i_1,\dots,n_{m-1},i_{m-1})}(x;\omega,a) \xrightarrow{x \rightarrow 0} \pm\infty. \end{cases} \quad (97)$$

### C. Two-step DBT

Before to consider the general case, we first consider the two-step case. We then have to check that the unphysical eigenstates  $\phi_{n_2,i_2}^{(n_1,i_1)}$  satisfy the appropriate boundary conditions. Using Eq(78), we have

$$W(\phi_{n_1,i_1}, \phi_{n_2,i_2} | x) = \phi_{n_1,i_1}(x;a) \phi_{n_2,i_2}(x;a) \Delta^{(n_1,i_1,n_2,i_2)}(x;a) \quad (98)$$

with  $\alpha > -i_j n_j$ ,  $j = 1, 2$  and where

$$\begin{aligned} \Delta^{(n_1,i_1,n_2,i_2)}(x;a) &= (v_{n_2,i_2}(x;a) - v_{n_1,i_1}(x;a)) \\ &= \frac{2}{x L_{n_1}^{i_1 \alpha}(-i_1 z) L_{n_2}^{i_2 \alpha}(-i_2 z)} \left( \varepsilon_{i_1,i_2}(z+\alpha) L_{n_1}^{i_1 \alpha}(-i_1 z) L_{n_2}^{i_2 \alpha}(-i_2 z) + z \begin{vmatrix} L_{n_1}^{i_1 \alpha}(-i_1 z) & L_{n_2}^{i_2 \alpha}(-i_2 z) \\ i_1 L_{n_1-1}^{i_1 \alpha+1}(-i_1 z) & i_2 L_{n_2-1}^{i_2 \alpha+1}(-i_2 z) \end{vmatrix} \right) \\ &\underset{x \rightarrow 0}{\sim} \frac{2}{x \binom{n_1+i_1 \alpha}{n_1} \binom{n_2+i_2 \alpha}{n_2}} \left( \alpha \varepsilon_{i_1,i_2} \binom{n_1+i_1 \alpha}{n_1} \binom{n_2+i_2 \alpha}{n_2} + z \begin{vmatrix} \binom{n_1+i_1 \alpha}{n_1+i_1 \alpha} & \binom{n_2+i_2 \alpha}{n_2} \\ i_1 \binom{n_1+i_1 \alpha}{n_1-1} & i_2 \binom{n_2+i_2 \alpha}{n_2-1} \end{vmatrix} + O(z^2) \right), \end{aligned} \quad (99)$$

since

$$L_k^\alpha(0) = \binom{k+\alpha}{k} = \frac{(k+\alpha) \dots (1+\alpha)}{k!}. \quad (100)$$

If  $i_1 = i_2 = i$ , then

$$\Delta^{(n_1,i,n_2,i)}(x;a) \underset{x \rightarrow 0}{\sim} i x \begin{vmatrix} \binom{n_1+i \alpha}{n_1} & \binom{n_2+i \alpha}{n_2} \\ \binom{n_1+i \alpha}{n_1-1} & \binom{n_2+i \alpha}{n_2-1} \end{vmatrix} \underset{x \rightarrow 0}{\sim} x \quad (101)$$

and (see Eq(82))

$$\phi_{n_2,i}^{(n_1,i)}(x;\omega,a) = \phi_{n_2,i}(x;a) \Delta^{(n_1,i,n_2,i)}(x;a) \underset{x \rightarrow 0}{\sim} x^{i \alpha + 3/2}, \quad (102)$$

that is,

$$\begin{cases} \phi_{n_2,+}^{(n_1,+)}(x;\omega,a) \underset{x \rightarrow 0}{\sim} x^{a+1} \rightarrow 0, \text{ if } a > 1 \\ \phi_{n_2,-}^{(n_1,-)}(x;\omega,a) \underset{x \rightarrow 0}{\sim} x^{2-a} \rightarrow \infty, \text{ if } a > 2 \end{cases}. \quad (103)$$

If  $i_1 = -i_2 = -i$ , then

$$\Delta^{(n_1,-i,n_2,i)}(x;a) \underset{x \rightarrow 0}{\sim} \frac{2}{x} \alpha \quad (104)$$

and

$$\phi_{n_2, i}^{(n_1, -i)}(x; \omega, a) = \phi_{n_2, i}(x; \omega, a) \Delta^{(n_1, -i, n_2, i)}(x; a) \underset{x \rightarrow 0}{\sim} x^{i\alpha-1/2}, \quad (105)$$

that is,

$$\begin{cases} \phi_{n_2, +}^{(n_1, -)}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{\alpha-1/2} = x^a \rightarrow 0 \\ \phi_{n_2, -}^{(n_1, +)}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{-\alpha-1/2} = x^{-a} \rightarrow \infty \end{cases}, \text{ if } a > 1. \quad (106)$$

When  $a > \sup(2, -i_1 n_1)$ , we have the correct asymptotic behaviour which ensures the regularity of  $V^{(n_1, i_1, n_2, i_2)}(x; \omega, a)$  and  $V^{(n_1, i_1)}(x; \omega, a)$  as their strict isospectrality with  $V(x; \omega, a)$  for every choice of  $(n_1, i_1, n_2, i_2)$  with  $n_1 \neq n_2$ .

We then obtain a chain with two strictly isospectral regular successive extensions of  $V$

$$V \xrightarrow{A(v_{n_1, i_1})} V^{(n_1, i_1)} \xrightarrow{A(v_{n_2, i_2}^{(n_1, i_1)})} V^{(n_1, i_1, n_2, i_2)}. \quad (107)$$

Under the above condition, the eigenstates of  $V^{(n_1, i_1, n_2, i_2)}$  are given by

$$\psi_k^{(n_1, i_1, n_2, i_2)}(x; a) = \frac{W(\phi_{n_1, i_1}, \phi_{n_2, i_2}, \psi_k | x)}{W(\phi_{n_1, i_1}, \phi_{n_2, i_2} | x)}, \quad (108)$$

where, using Eq(74)

$$\begin{aligned} W(\phi_{n_1, i_1}, \phi_{n_2, i_2}, \psi_k | x) &= W(\phi_{0, i_1}(x; \omega, a) L_{n_1}^{i_1 \alpha}(-i_1 z), \phi_{0, i_2}(x; \omega, a) L_{n_2}^{i_2 \alpha}(-i_2 z), \psi_0(x; \omega, a) L_k^\alpha(z) | x) \\ &= (\psi_0(x; \omega, a))^3 W(\delta_{0, i_1}(x) L_{n_1}^{i_1 \alpha}(-i_1 z), \delta_{0, i_2}(x) L_{n_2}^{i_2 \alpha}(-i_2 z), L_k^\alpha(z) | x), \end{aligned} \quad (109)$$

with

$$\delta_{0, j}(z) = \frac{\phi_{0, j}(x; \omega, a)}{\psi_0(x; \omega, a)} = \begin{cases} e^z, & \text{if } j = + \\ \left(\frac{z}{\omega}\right)^{-\alpha} z^{-\alpha}, & \text{if } j = -. \end{cases} \quad (110)$$

Using [47]

$$W(y_1, \dots, y_m | x) = \left(\frac{dz}{dx}\right)^{m(m-1)/2} W(y_1, \dots, y_m | z), \quad (111)$$

this gives

$$W(\phi_{n_1, i_1}, \phi_{n_2, i_2}, \psi_k | x) = (\omega x \psi_0(x; \omega, a))^3 W(\delta_{0, i_1}(x) L_{n_1}^{i_1 \alpha}(-i_1 z), \delta_{0, i_2}(x) L_{n_2}^{i_2 \alpha}(-i_2 z), L_k^\alpha(z) | z). \quad (112)$$

In the same manner

$$\begin{aligned} W(\phi_{n_1, i_1}, \phi_{n_2, i_2} | x) &= W(\phi_{0, i_1}(x; \omega, a) L_{n_1}^{i_1 \alpha}(-i_1 z), \phi_{0, i_2}(x; \omega, a) L_{n_2}^{i_2 \alpha}(-i_2 z) | x) \\ &= \omega x (\phi_{0, i}(x; \omega, a))^2 W(\delta_{i, i_1}(x) L_{n_1}^{i_1 \alpha}(-i_1 z), \delta_{i, i_2}(x) L_{n_2}^{i_2 \alpha}(-i_2 z) | z), \end{aligned} \quad (113)$$

$i = \pm$ , with

$$\delta_{i,j}(z) = \frac{\phi_{0,j}(x; \omega, a)}{\phi_{0,i}(x; \omega, a)} = \begin{cases} 1, & \text{if } j = i \\ \left(\frac{2}{\omega}\right)^{-i\alpha} z^{-i\alpha} e^{-iz}, & \text{if } j = -i. \end{cases} \quad (114)$$

We can make use of the following identities [50]

$$\begin{cases} (L_n^\alpha(z))' = -L_{n-1}^{\alpha+1}(z) \\ (z^\alpha L_n^\alpha(z))' = (n+\alpha) z^{\alpha-1} L_n^{\alpha-1}(z) \\ (e^{-z} L_n^\alpha(z))' = -e^{-z} L_n^{\alpha+1}(z) \\ (z^\alpha e^{-z} L_n^\alpha(z))' = (n+1) z^{\alpha-1} e^{-z} L_{n+1}^{\alpha-1}(z) \end{cases} \quad (115)$$

to obtain

$$\begin{aligned} W(\phi_{n_1,+}, \phi_{n_2,+}, \psi_k | x) &= \omega^3 x^{3(a+1)} e^{-3z/2} W(e^z L_{n_1}^\alpha(-z), e^z L_{n_2}^\alpha(-z), L_k^\alpha(z) | z) \\ &= \omega^3 x^{3(a+1)} e^{z/2} \begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & L_{n_2}^{\alpha+1}(-z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & L_{n_2}^{\alpha+2}(-z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix} \end{aligned} \quad (116)$$

and

$$W(\phi_{n_1,+}, \phi_{n_2,+} | x) = \omega x^{2a+1} e^z \begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) \\ L_{n_1-1}^{\alpha+1}(-z) & L_{n_2-1}^{\alpha+1}(-z) \end{vmatrix}, \quad (117)$$

that is,

$$\psi_k^{(n_1,+,n_2,+)}(x; a) \sim x^{a+2} e^{-z/2} \frac{\begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & L_{n_2}^{\alpha+1}(-z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & L_{n_2}^{\alpha+2}(-z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix}}{\begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) \\ L_{n_1-1}^{\alpha+1}(-z) & L_{n_2-1}^{\alpha+1}(-z) \end{vmatrix}}. \quad (118)$$

Proceeding in a similar way we find

$$\begin{aligned} &W(\phi_{n_1,-}, \phi_{n_2,-}, \psi_k | x) \\ &= \omega^3 \left(\frac{2}{\omega}\right)^{-2\alpha} x^{3(a+1)} e^{-3z/2} z^{-3\alpha} W(L_{n_1}^{-\alpha}(z), L_{n_2}^{-\alpha}(z), z^\alpha L_k^\alpha(z) | z) \\ &\sim x^{3(a+1)} e^{-3z/2} z^{-3\alpha} \begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(-z) & z^\alpha L_k^\alpha(z) \\ -L_{n_1-1}^{-\alpha+1}(z) & -L_{n_2-1}^{-\alpha+1}(-z) & (k+\alpha) z^{\alpha-1} L_k^{\alpha-1}(z) \\ L_{n_1-2}^{-\alpha+2}(z) & L_{n_2-2}^{-\alpha+2}(z) & (k+\alpha)(k+\alpha-1) z^{\alpha-2} L_k^{\alpha-2}(z) \end{vmatrix} \\ &\sim x^{-a+1} e^{-3z/2} \begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(-z) & z^2 L_k^\alpha(z) \\ -L_{n_1-1}^{-\alpha+1}(z) & -L_{n_2-1}^{-\alpha+1}(-z) & (k+\alpha) z L_k^{\alpha-1}(z) \\ L_{n_1-2}^{-\alpha+2}(z) & L_{n_2-2}^{-\alpha+2}(z) & (k+\alpha)(k+\alpha-1) L_k^{\alpha-2}(z) \end{vmatrix} \end{aligned} \quad (119)$$

and

$$W(\phi_{n_1,-}, \phi_{n_2,-} | x) \sim x^{3-2a} e^{-z} \begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{-\alpha+1}(z) & L_{n_2-1}^{-\alpha+1}(z) \end{vmatrix}, \quad (120)$$

that is,

$$\psi_k^{(n_1,-,n_2,-)}(x; a) \sim x^{a-2} e^{-z/2} \frac{\begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(-z) & z^2 L_k^\alpha(z) \\ -L_{n_1-1}^{-\alpha+1}(z) & -L_{n_2-1}^{-\alpha+1}(-z) & (k+\alpha) z L_k^{\alpha-1}(z) \\ L_{n_1-2}^{-\alpha+2}(z) & L_{n_2-2}^{-\alpha+2}(z) & (k+\alpha)(k+\alpha-1) L_k^{\alpha-2}(z) \end{vmatrix}}{\begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{-\alpha+1}(z) & L_{n_2-1}^{-\alpha+1}(z) \end{vmatrix}}. \quad (121)$$

Also

$$\begin{aligned}
& W(\phi_{n_1,+}, \phi_{n_2,-}, \psi_k | x) \\
&= \omega^3 \left(\frac{2}{\omega}\right)^{-\alpha} x^{3(a+1)} e^{-3z/2} W(e^z L_{n_1}^\alpha(-z), z^{-\alpha} L_{n_2}^{-\alpha}(z), L_k^\alpha(z) | z) \\
&\sim x^a e^{-z/2} \begin{vmatrix} L_{n_1}^{-\alpha}(-z) & z^2 L_{n_2}^{-\alpha}(z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & (n_2 - \alpha) z L_{n_2}^{-\alpha+1}(z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & (n_2 - \alpha)(n_2 - \alpha - 1) L_{n_2}^{-\alpha+2}(z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix}
\end{aligned} \tag{122}$$

and

$$\begin{aligned}
W(\phi_{n_1,+}, \phi_{n_2,-} | x) &= \omega \left(\frac{2}{\omega}\right)^{-\alpha} x^{2a+1} e^z W(L_{n_1}^\alpha(-z), z^{-\alpha} e^{-z} L_{n_2}^{-\alpha}(z) | z) \\
&\sim \begin{vmatrix} L_{n_1}^\alpha(-z) & z L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{\alpha+1}(-z) & (n_2 + 1) L_{n_2-1}^{\alpha+1}(z) \end{vmatrix},
\end{aligned} \tag{123}$$

that is,

$$\psi_k^{(n_1,+,n_2,-)}(x; a) \sim x^a e^{-z/2} \frac{\begin{vmatrix} L_{n_1}^{-\alpha}(-z) & z^2 L_{n_2}^{-\alpha}(z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & (n_2 - \alpha) z L_{n_2}^{-\alpha+1}(z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & (n_2 - \alpha)(n_2 - \alpha - 1) L_{n_2}^{-\alpha+2}(z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix}}{\begin{vmatrix} L_{n_1}^\alpha(-z) & z L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{\alpha+1}(-z) & (n_2 + 1) L_{n_2-1}^{\alpha+1}(z) \end{vmatrix}}. \tag{124}$$

If  $a > 2$ , all these states satisfy the physical boundary conditions  $\psi_k^{(n_1, i_1, n_2, i_2)}(0; a) = \psi_k^{(n_1, i_1, n_2, i_2)}(+\infty; a) = 0$ . They are of the following form

$$\psi_k^{(n_1, i_1, n_2, i_2)}(x; a) = \psi_0(x; a + i_1 + i_2) \frac{P_k^{(n_1, i_1, n_2, i_2)}(x; a)}{Q^{(n_1, i_1, n_2, i_2)}(x; a)}, \tag{125}$$

the  $Q^{(n_1, i_1, n_2, i_2)}$  and  $P_k^{(n_1, i_1, n_2, i_2)}$  being polynomials given by

$$\begin{cases} Q^{(n_1,+,n_2,+)}(x; a) = \begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) \\ L_{n_1-1}^{\alpha+1}(-z) & L_{n_2-1}^{\alpha+1}(-z) \end{vmatrix} \\ Q^{(n_1,-,n_2,-)}(x; a) = \begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{-\alpha+1}(z) & L_{n_2-1}^{-\alpha+1}(z) \end{vmatrix} \\ Q^{(n_1,+,n_2,-)}(x; a) = \begin{vmatrix} L_{n_1}^\alpha(-z) & z L_{n_2}^{-\alpha}(z) \\ L_{n_1-1}^{\alpha+1}(-z) & (n_2 + 1) L_{n_2-1}^{\alpha+1}(z) \end{vmatrix} \end{cases} \tag{126}$$

and

$$\begin{cases} P_k^{(n_1,+,n_2,+)}(x; a) = \begin{vmatrix} L_{n_1}^\alpha(-z) & L_{n_2}^\alpha(-z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & L_{n_2}^{\alpha+1}(-z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & L_{n_2}^{\alpha+2}(-z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix} \\ P_k^{(n_1,-,n_2,-)}(x; a) = \begin{vmatrix} L_{n_1}^{-\alpha}(z) & L_{n_2}^{-\alpha}(z) & z^2 L_k^\alpha(z) \\ -L_{n_1-1}^{-\alpha+1}(z) & -L_{n_2-1}^{-\alpha+1}(z) & (k + \alpha) z L_k^{\alpha-1}(z) \\ L_{n_1-2}^{-\alpha+2}(z) & L_{n_2-2}^{-\alpha+2}(z) & (k + \alpha)(k + \alpha - 1) L_k^{\alpha-2}(z) \end{vmatrix} \\ P_k^{(n_1,+,n_2,-)}(x; a) = \begin{vmatrix} L_{n_1}^{-\alpha}(-z) & z^2 L_{n_2}^{-\alpha}(z) & L_k^\alpha(z) \\ L_{n_1}^{\alpha+1}(-z) & (n_2 - \alpha) z L_{n_2}^{-\alpha+1}(z) & -L_{k-1}^{\alpha+1}(z) \\ L_{n_1}^{\alpha+2}(-z) & (n_2 - \alpha)(n_2 - \alpha - 1) L_{n_2}^{-\alpha+2}(z) & L_{k-2}^{\alpha+2}(z) \end{vmatrix}. \end{cases} \tag{127}$$

#### IV. GENERAL CASE

Consider now the general case of a  $m$ -step DBT. We suppose that we have built a chain of  $m - 1$  regular strictly isospectral potentials

$$V \xrightarrow{A(v_{n_1, i_1})} V^{(n_1, i_1)} \xrightarrow{A(v_{n_2, i_2})} \dots \xrightarrow{A(v_{n_{m-1}, i_{m-1}})} V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}. \quad (128)$$

We have to determine what is the constraint to implement for the unphysical eigenfunction  $\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  in order that it satisfies the good boundary conditions Eq(97) necessary to build a regular and strictly isospectral extension  $V^{(n_1, i_1, \dots, n_m, i_m)}$  at the next step.

We prove by induction the following result for the unphysical eigenfunctions of the potential obtained at the  $(m - 1)^{th}$  step

$$\phi_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{i_m(\alpha + q(i_1, \dots, i_m)) - 1/2} \quad (129)$$

or

$$\begin{cases} \phi_{n_m, +}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_m)) - 1/2} \left( \tilde{\phi}_{n_m, +}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(0) + O(z) \right), & i_m = + \\ \phi_{n_m, -}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{-(\alpha + q(i_1, \dots, i_m)) - 1/2} \left( \tilde{\phi}_{n_m, -}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(0) + O(z) \right), & i_m = - \end{cases}, \quad (130)$$

with  $\tilde{\phi}_{n_m, \pm}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(0) \neq 0$  and where  $q(i_1, \dots, i_m) = q_+(i_1, \dots, i_m) - q_-(i_1, \dots, i_m)$ ,  $q_{\pm}(i_1, \dots, i_m)$  being the number of  $i_j$  equal to  $\pm$  in the set  $(i_1, \dots, i_m)$ .  $q_{\pm}(i_1, \dots, i_m)$  can be viewed as the number of state of " $\pm$  charge" used in the chain of DBT at the  $m^{th}$  step and the difference  $q(i_1, \dots, i_m)$  as the total charge in this chain. We then have

$$\begin{cases} \phi_{n_m, +}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \underset{x \rightarrow 0}{\rightarrow} 0, & \text{if } a > -q(i_1, \dots, i_{m-1}, +) + 1 = -q(i_1, \dots, i_{m-1}) \\ \phi_{n_m, -}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \underset{x \rightarrow 0}{\rightarrow} \infty, & \text{if } a > -q(i_1, \dots, i_{m-1}, -) = 1 - q(i_1, \dots, i_{m-1}), \end{cases} \quad (131)$$

which are necessary conditions in order that the potentials  $V^{(n_1, i_1, \dots, n_m, +)}$  and  $V^{(n_1, i_1, \dots, n_m, -)}$  respectively are regular and strictly isospectral to  $V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$ . Note that these constraints are independent of the  $n_j$  but depend only on the type of the states used to build the chain.

Referring to the results obtained in the 2-step case, we can verify this property is verified for  $m = 2$  since

$$\begin{cases} q(+, +) = -q(-, -) = 2 \\ q(+, -) = q(-, +) = 0 \end{cases}. \quad (132)$$

Suppose that Eq(129) is verified till the  $(m - 1)^{th}$  step, that is,

$$\phi_{n, j}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{j(\alpha + q(i_1, \dots, i_{m-2}, j)) - 1/2}. \quad (133)$$

Since  $V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  is supposed regular,  $\phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$  must satisfy the good boundary conditions Eq(97)

$$\phi_{n_{m-1}, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_{m-2}, +)) - 1/2} \underset{x \rightarrow 0}{\rightarrow} 0, \quad (134)$$

if  $i_{m-1} = +$ , which implies  $a > -q(i_1, \dots, i_{m-2}, +)$  and

$$\phi_{n_{m-1}, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_{m-2}, +)) - 1/2} \underset{x \rightarrow 0}{\rightarrow} \infty, \quad (135)$$

if  $i_{m-1} = -$ , which implies  $a > 1 - q(i_1, \dots, i_{m-2}, -)$ .

Since  $\phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$  and  $\phi_{k, i}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$  are both eigenfunctions of  $\widehat{H}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$ , using the wronskian theorem [51] in Eq(89) we can write the unphysical eigenfunctions at the  $(m-1)^{th}$  step as

$$\begin{aligned} \phi_{k, i}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) &= \frac{(\mathcal{E}_{n_{m-1}, i_{m-1}} - \mathcal{E}_{k, i}) \int_{x_0}^x d\xi \phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a) \phi_{k, i}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a)}{\phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ &+ \frac{W\left(\phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}, \phi_{k, i}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})} \mid x_0\right)}{\phi_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a)}. \end{aligned} \quad (136)$$

We have to consider four cases corresponding to the four possible couples  $(i_{m-1}, i)$ .

Consider first the case  $i = i_{m-1} = +$  in which we take  $x_0 = 0$ . Using Eq(134), Eq(136) gives

$$\begin{aligned} \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2}, n_{m-1}, +)}(x; \omega, a) &= (\mathcal{E}_{n_{m-1}, +} - \mathcal{E}_{k, +}) \frac{\int_0^x d\xi \phi_{n_{m-1}, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a) \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a)}{\phi_{n_{m-1}, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ &= (\mathcal{E}_{n_{m-1}, +} - \mathcal{E}_{k, +}) x^{(\alpha + q(i_1, \dots, i_{m-2}) + 1/2) + 1} (const + O(z)) \\ &\underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_{m-2}, +, +)) - 1/2}, \end{aligned} \quad (137)$$

In the case  $i = i_{m-1} = -$  in which we take  $x_0 = +\infty$ , we have similarly with Eq(135) and Eq(136)

$$\begin{aligned} \phi_{k, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2}, n_{m-1}, -)}(x; \omega, a) &= (\mathcal{E}_{n_{m-1}, -} - \mathcal{E}_{k, -}) \frac{-\int_x^{+\infty} d\xi \phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a) \phi_{k, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(\xi; \omega, a)}{\phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ &= (\mathcal{E}_{n_{m-1}, -} - \mathcal{E}_{k, -}) x^{-(\alpha + q(i_1, \dots, i_{m-2}) - 1) - 1/2 + 1} (const + O(z)) \\ &\underset{x \rightarrow 0}{\sim} x^{-(\alpha + q(i_1, \dots, i_{m-2}, -, -)) - 1/2}, \end{aligned} \quad (138)$$

It remains to consider the cases  $i = + = -i_{m-1}$  and  $i = - = -i_{m-1}$ . Since  $\phi_{n, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$  and  $\phi_{n, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}$  have not the same behavior nor at 0 nor at  $\infty$ , formula Eq(136) is not adapted and we proceed by a direct calculation of the wronskian  $W\left(\phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}, \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})} \mid x\right)$ . Using (see Eq(130))

$$\left\{ \begin{array}{l} \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_{m-2}) + 1/2)} \left( \tilde{\phi}_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) + O(z) \right) \\ \phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) \underset{x \rightarrow 0}{\sim} x^{-(\alpha + q(i_1, \dots, i_{m-2}) + 1/2)} \left( \tilde{\phi}_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) + O(z) \right), \end{array} \right. \quad (139)$$

we obtain

$$\begin{aligned} &W\left(\phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}, \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})} \mid x\right) \\ &\underset{x \rightarrow 0}{\sim} ((\alpha + q(i_1, \dots, i_{m-2})) + 1/2) \tilde{\phi}_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) \tilde{\phi}_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) + O(z) \\ &- (-(\alpha + q(i_1, \dots, i_{m-2})) + 1/2) \tilde{\phi}_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) \tilde{\phi}_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) + O(z), \end{aligned} \quad (140)$$

that is,

$$\begin{aligned} &W\left(\phi_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}, \phi_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})} \mid x\right) \\ &\underset{x \rightarrow 0}{\sim} 2(\alpha + q(i_1, \dots, i_{m-2})) \tilde{\phi}_{k, +}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) \tilde{\phi}_{n_{m-1}, -}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) + O(z). \end{aligned} \quad (141)$$

Consequently

$$\begin{aligned}
& \phi_{k,+}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2}, n_{m-1}, -)}(x; \omega, a) \\
& \underset{x \rightarrow 0}{\sim} 2(\alpha + q(i_1, \dots, i_{m-2})) \tilde{\phi}_{k,+}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) x^{(\alpha + q(i_1, \dots, i_{m-2})) - 1/2} (1 + O(z)) \\
& \underset{x \rightarrow 0}{\sim} x^{(\alpha + q(i_1, \dots, i_{m-2}, -, +)) - 1/2}
\end{aligned} \tag{142}$$

and

$$\begin{aligned}
\phi_{k,-}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2}, n_{m-1}, +)}(x; \omega, a) &= \frac{W\left(\phi_{n_{m-1},+}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}, \phi_{k,-}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})} \mid x\right)}{\phi_{n_m,+}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\
&\underset{x \rightarrow 0}{\sim} -2(\alpha + q(i_1, \dots, i_{m-2})) \tilde{\phi}_{k,-}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(0) x^{-(\alpha + q(i_1, \dots, i_{m-2})) - 1/2} (1 + O(z)) \\
&\underset{x \rightarrow 0}{\sim} x^{-(\alpha + q(i_1, \dots, i_{m-2}, +, -)) - 1/2}.
\end{aligned} \tag{143}$$

The result Eq(129) is then still valid at the  $m^{\text{th}}$  step.

We still have to verify that this condition is compatible with the regularity of the preceding potentials in the chain. For that, note that at the  $m^{\text{th}}$  step the explicit expression of the resulting potential and its associated eigenfunctions are a priori independent of the ordering chosen for the sequence. This means that we can consider several chains leading to the same extended potential. Consider the case where in the set  $(n_1, i_1, \dots, n_{m-2}, i_{m-2}, n_{m-1}, i_{m-1})$  we group first all the  $n_j$  associated to a positive  $i_j$ , namely we arrange the chain as  $(n'_1, i'_1, \dots, n'_{q_+}, i'_{q_+}, n'_{q_++1}, i'_{q_++1}, \dots, n'_{m-1}, i'_{m-1}) = (n'_1, +, \dots, n'_{q_+}, +, n'_{q_++1}, -, \dots, n'_{m-1}, -)$ , where  $(n'_1, \dots, n'_{m-1})$  is a permutation of  $(n_1, \dots, n_{m-1})$  and where  $q_+ = q_+(i_1, \dots, i_{m-1})$ .

Suppose first that  $q_+ = m-1$ , that is,  $i'_j = +, \forall j \in \{1, \dots, m-1\}$ .  $\forall j \in \{1, \dots, m-1\}$  we have  $1 - q_+(i'_1, \dots, i'_j) \leq 0$  and since  $a > 1$ , the  $\phi_{n'_{j+1}, i'_{j+1}}^{(n'_1, i'_1, \dots, n'_j, i'_j)}$  are all nodeless implying the regularity of all the potentials  $V^{(n'_1, i'_1, \dots, n'_j, i'_j)}$  in the chain.

Suppose now that  $q_+ < m-1$ , that is,  $i'_{m-1} = -$ . The condition  $a > 1 - q(i'_1, \dots, i'_{m-1}) = 2 - q(i'_1, \dots, i'_{m-2}) > 1 - q(i'_1, \dots, i'_{m-2})$  implies the absence of node for  $\phi_{n'_{m-1}, -}^{(n'_1, i'_1, \dots, n'_{m-2}, i'_{m-2})}$  and consequently the regularity of  $V^{(n'_1, i'_1, \dots, n'_{m-1}, i'_{m-1})}$ .

We can iterate this reasoning to go backward along the chain and we see clearly that at each step, the opposite of the "total charge" decreases, ensuring the regularity of the associated potential. In this procedure, once we attain the level  $q_+$ , we have eliminated all the negative  $i'_j$  and retrieve then the first case.

Consequently, if we satisfy the condition  $a > 1 - q(i_1, \dots, i_{m-1})$  at the step  $m$ , we are not only ensured of the regularity of the corresponding  $m^{\text{th}}$  extension  $V^{(n_1, i_1, \dots, n_m, i_m)}$  but also of the existence of a chain of regular extensions which leads to  $V^{(n_1, i_1, \dots, n_m, i_m)}$ .

## V. EIGENSTATES

We can generalize the calculations made in the two-step case to determine explicit expressions for the eigenstates of the successive extensions. For that we adopt the ordering mentioned above and consider the spectrum of the potential  $V^{(n_1, +, \dots, n_{q_+}, +, n_{q_++1}, -, \dots, n_m, -)}$  where  $q_+ = q_+(i_1, \dots, i_m)$ . Using Eq(74) and Eq(111), we have

$$\begin{aligned}
& W\left(\phi_{n_1,+}, \dots, \phi_{n_{q_+},+}, \dots, \phi_{n_{q_++1},-}, \dots, \phi_{n_m,-}, \psi_k \mid x\right) \\
&= (\omega x)^{m(m+1)/2} (\psi_0(x; \omega, a))^{m+1} W\left(\delta_{0,+} L_{n_1}^\alpha(-z), \dots, \delta_{0,+} L_{n_{q_+}}^\alpha(-z), \delta_{0,-} L_{n_{q_++1}}^{-\alpha}(z), \dots, \delta_{0,-} L_{n_m}^{-\alpha}(z), L_k^\alpha(z) \mid x\right) \\
&\sim x^{(m+1)(m/2+a)} e^{-(m+1)z/2} W\left(e^z L_{n_1}^\alpha(-z), \dots, e^z L_{n_{q_+}}^\alpha(-z), z^{-\alpha} L_{n_{q_++1}}^{-\alpha}(z), \dots, z^{-\alpha} L_{n_m}^{-\alpha}(z), L_k^\alpha(z) \mid x\right),
\end{aligned} \tag{144}$$

that is, with Eq(115)

$$\begin{aligned}
& W\left(\phi_{n_1,+}, \dots, \phi_{n_{q_+},+}, \dots, \phi_{n_{q_++1},-}, \dots, \phi_{n_m,-}, \psi_k \mid x\right) \\
& \sim x^{(m+1)(m/2+a)} e^{(q_+-(m+1)/2)z} z^{-(m-q_+)(\alpha+m)} \\
& \quad \cdot \begin{vmatrix} L_{n_1}^\alpha(-z) & \dots & L_{n_{q_+}}^\alpha(-z) & z^m L_{n_{q_++1}}^{-\alpha}(z) & \dots & z^m L_{n_m}^{-\alpha}(z) & L_k^\alpha(z) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n_1}^{\alpha+m}(-z) & \dots & L_{n_{q_+}}^{\alpha+m}(-z) & (n_{q_++1} - \alpha - m + 1)_{m-1} L_{n_{q_++1}}^{-\alpha-m}(z) & \dots & (n_m - \alpha - m + 1)_{m-1} L_{n_m}^{-\alpha-m}(z) & (-1)^m L_{k-m}^{\alpha+m}(z) \end{vmatrix}
\end{aligned}$$

where  $(x)_n = x(x+1)\dots(x+n-1)$  is the usual Pochhammer symbol.

In the same manner

$$\begin{aligned}
& W\left(\phi_{n_1,+}, \dots, \phi_{n_{q_+},+}, \dots, \phi_{n_{q_++1},-}, \dots, \phi_{n_m,-} \mid x\right) \\
& = (\omega x)^{m(m-1)/2} (\phi_{0,+}(x; \omega, a))^m W\left(L_{n_1}^\alpha(-z), \dots, L_{n_{q_+}}^\alpha(-z), \delta_{+,-} L_{n_{q_++1}}^{-\alpha}(z), \dots, \delta_{+,-} L_{n_m}^{-\alpha}(z) \mid x\right) \\
& \sim x^{m((m-1)/2+a)} e^{mz/2} W\left(L_{n_1}^\alpha(-z), \dots, L_{n_{q_+}}^\alpha(-z), z^{-\alpha} e^{-z} L_{n_{q_++1}}^{-\alpha}(z), \dots, z^{-\alpha} e^{-z} L_{n_m}^{-\alpha}(z) \mid x\right),
\end{aligned} \tag{146}$$

that is,

$$\begin{aligned}
& W\left(\phi_{n_1,+}, \dots, \phi_{n_{q_+},+}, \dots, \phi_{n_{q_++1},-}, \dots, \phi_{n_m,-} \mid x\right) \\
& \sim x^{m((m-1)/2+a)} z^{(m-q_+)(-\alpha-m+1)} e^{-(m/2-q_+)z} \\
& \quad \cdot \begin{vmatrix} L_{n_1}^\alpha(-z) & \dots & L_{n_{q_+}}^\alpha(-z) & z^{m-1} L_{n_{q_++1}}^{-\alpha}(z) & \dots & z^{m-1} L_{n_m}^{-\alpha}(z) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n_1}^{\alpha+m-1}(-z) & \dots & L_{n_{q_+}}^{\alpha+m-1}(-z) & (n_{q_++1} + 1)_{m-1} L_{n_{q_++1}+m-1}^{-\alpha-m+1}(z) & \dots & (n_m + 1)_{m-1} L_{n_m+m-1}^{-\alpha-m+1}(z) \end{vmatrix}.
\end{aligned} \tag{147}$$

Eq(89) gives then

$$\psi_k^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}(x; \omega, a) \sim \psi_0(x; \omega, a + q(i_1, \dots, i_m)) \frac{P_k^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}(z)}{Q^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}(z)}, \tag{148}$$

where the two polynomials  $Q^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}$  and  $P_k^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}$  are given by

$$\begin{aligned}
& P_k^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}(z) \\
& = \begin{vmatrix} L_{n_1}^\alpha(-z) & \dots & L_{n_{q_+}}^\alpha(-z) & z^m L_{n_{q_++1}}^{-\alpha}(z) & \dots & z^m L_{n_m}^{-\alpha}(z) & L_k^\alpha(z) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n_1}^{\alpha+m}(-z) & \dots & L_{n_{q_+}}^{\alpha+m}(-z) & (n_{q_++1} - \alpha - m + 1)_{m-1} L_{n_{q_++1}}^{-\alpha-m}(z) & \dots & (n_m - \alpha - m + 1)_{m-1} L_{n_m}^{-\alpha-m}(z) & (-1)^m L_{k-m}^{\alpha+m}(z) \end{vmatrix}
\end{aligned} \tag{149}$$

and

$$\begin{aligned}
& Q^{(n_1,+,\dots,n_{q_+},+,n_{q_++1},-,\dots,n_m,-)}(z) \\
& = \begin{vmatrix} L_{n_1}^\alpha(-z) & \dots & L_{n_{q_+}}^\alpha(-z) & z^{m-1} L_{n_{q_++1}}^{-\alpha}(z) & \dots & z^{m-1} L_{n_m}^{-\alpha}(z) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n_1}^{\alpha+m-1}(-z) & \dots & L_{n_{q_+}}^{\alpha+m-1}(-z) & (n_{q_++1} + 1)_{m-1} L_{n_{q_++1}+m-1}^{-\alpha-m+1}(z) & \dots & (n_m + 1)_{m-1} L_{n_m+m-1}^{-\alpha-m+1}(z) \end{vmatrix}.
\end{aligned} \tag{150}$$

The  $\psi_k^{(n_1,i_1,\dots,n_m,i_m)}$  are then obtained as the product of the modified "gauge factor" (ie fundamental state of the initial potential)  $\psi_0(x; \omega, a + q(i_1, \dots, i_m))$ , multiplied by the ratio of two polynomials, the polynomial denominator  $Q^{(n_1,i_1,\dots,n_m,i_m)}$  being common to all the eigenstates of the considered extension. The orthogonality condition on the  $\psi_k^{(n_1,i_1,\dots,n_m,i_m)}$  ensure that the  $P_k^{(n_1,i_1,\dots,n_m,i_m)}(z)$  constitute an orthogonal family with respect to the weight

$$W^{(n_1,i_1,\dots,n_m,i_m)}(z) = \frac{z^{\alpha+q(i_1,\dots,i_m)} e^{-z}}{(Q^{(n_1,i_1,\dots,n_m,i_m)}(z))^2}.$$

## VI. SHAPE INVARIANCE

Consider a chain of regular strictly isospectral extensions

$$V \xrightarrow{A(v_{n_1, i_1})} V^{(n_1, i_1)} \xrightarrow{A(v_{n_2, i_2}^{(n_1, i_1)})} \dots \xrightarrow{A(v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})})} V^{(n_1, i_1, \dots, n_m, i_m)}, \quad (151)$$

The superpartner of a potential  $V^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) + 2 \left( v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \right)'$  is defined as

$$\tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = V^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) + 2 \left( w_0^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) \right)', \quad (152)$$

$w_0^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a)$  being the RS function associated to the ground level of  $V^{(n_1, i_1, \dots, n_m, i_m)}$  ( $E_0(\omega) = 0$ ).

Since (see Eq(3))

$$w_0^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = -v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) - \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a)}, \quad (153)$$

we have

$$\tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \quad (154)$$

$$- 2 \left( \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a)} \right)'. \quad (155)$$

Suppose that the potential  $V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}$  has the same shape invariance properties than  $V$

$$\begin{aligned} \tilde{V}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) &= V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) + 2 \left( w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \right)' \\ &= V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a + 1) + 2\omega. \end{aligned} \quad (156)$$

As shown in [22], this is effectively the case for  $m = 2$ . It results

$$\begin{aligned} \tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) &= V^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a + 1) \\ &\quad - 2 \left( \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a)} + w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) \right)' \\ &= \tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a + 1) + 2\omega - 2 \left( \Delta^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) \right)', \end{aligned}$$

where

$$\begin{aligned} \Delta^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) &= \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a)} \\ &\quad + w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) + v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a + 1). \end{aligned} \quad (158)$$

Suppose also that

$$\Delta^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) = 0, \quad (159)$$

which again is verified for  $m = 2$  [22]. We can then write

$$v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a+1) = -w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)}. \quad (160)$$

Moreover

$$\begin{cases} w_0^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) = -v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-1}, i_{m-1})}(x; \omega, a) = -v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) + \frac{\mathcal{E}_{n_m, i_m}(\omega, a) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \end{cases} \quad (161)$$

and inserting Eq(161) and Eq(160) into Eq(158), we obtain

$$\begin{aligned} & \Delta^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) \quad (162) \\ &= \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{\frac{\mathcal{E}_{n_m, i_m}(\omega, a) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} + \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)}} \\ & - v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ & - v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a+1) - \frac{\mathcal{E}_{n_m, i_m}(\omega, a+1) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a+1)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a+1) - v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a+1)} \\ &= \frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{\frac{\mathcal{E}_{n_m, i_m}(\omega, a) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} + \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)}} \\ & - v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ & + w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) + \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} \\ & - \frac{\mathcal{E}_{n_m, i_m}(\omega, a+1) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a+1)}{\frac{\mathcal{E}_{n_m, i_m}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - v_{n_m, i_m}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)} - \frac{\mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a)}{v_{n_{m-1}, i_{m-1}}^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a) - w_0^{(n_1, i_1, \dots, n_{m-2}, i_{m-2})}(x; \omega, a)}} \end{aligned}$$

But using Eq(90) we deduce

$$\mathcal{E}_{n_m, i_m}(\omega, a+1) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a+1) = \mathcal{E}_{n_m, i_m}(\omega, a) - \mathcal{E}_{n_{m-1}, i_{m-1}}(\omega, a), \quad (163)$$

which after insertion in Eq(162) and a little elementary algebra gives

$$\Delta^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = 0. \quad (164)$$

By induction, this property is verified for every  $m$ . Then Eq(157) becomes

$$\tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a) = \tilde{V}^{(n_1, i_1, \dots, n_m, i_m)}(x; \omega, a+1) + 2\omega, \quad (165)$$

that is, the potential has also the same shape invariance properties than the isotonic potential. The translational shape invariance of the isotonic potential is then hereditary in such chain of extensions.

## VII. CONCLUSION

We have shown that the "generalized SUSY QM partnership" that we have previously elaborated in a one-step scheme can be extended in a multi-step formulation. We have proven the necessary conditions to obtain chains of regular extensions of the isotonic potential of arbitrary length and have given explicit expressions for their eigenstates. We also established explicitly the hereditary character of the shape invariance properties of the isotonic potential which are common to all the potentials in a given chain.

The case of the generic potentials of the second category of primary TSIP [24], namely Pöschl-Teller or Scarf potentials, can be considered in the same way. This work is in progress and a forthcoming paper is in preparation. For the first category exceptional TSIP, due to the strict isospectrality constraint, the only interesting case in which we can envisage such chain of extensions is the ERKC potential [23]. This is the object of further investigations.

## VIII. ACKNOWLEDGMENTS

I would like to thank A. Bérard, R. Milson and C. Quesne for stimulating exchanges and very interesting suggestions.

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