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SAMUEL AMSTUTZ AND NICOLAS VAN GOETHEM

ABSTRACT. In this paper we introduce a family of smooth perimeter approximating functionals designed to be incorporated within topology optimization algorithms. The required mathematical properties, namely the Γ-convergence and the compactness of sequences of minimizers, are first established. Then we propose several methods for the solution of topology optimization problems with perimeter penalization showing different features. We conclude by some numerical illustrations in the contexts of least square problems and compliance minimization.

1. INTRODUCTION

In several areas of applied sciences, models where the perimeter of an unknown set plays a crucial role may be considered. Such problems include multiphase problems where the interface between two liquid phases is assumed to minimize a free energy while keeping its area bounded [18, 23], or image segmentation models with Mumford-Shah type functionals [6]. Another important field where the perimeter comes into play is the optimal design of shapes [5], such as load bearing structures or electromagnetic devices, where it aims at rendering the problem well-posed in the sense of the existence of optimal domains. However it is known that the major difficulty of standard perimeter penalization is that the sensitivity of the perimeter to topology changes is of lower order compared to usual cost functionals, like the volume (see e.g. [17, 22, 8] for the topological sensitivity of other functionals) and thus prohibits successful numerical solution. In this paper we propose a regularization of the perimeter that overcomes this drawback and show simple applications in topology optimization and source identification. Since we believe that applications of our method could show useful in other areas of applied sciences, a brief overview of the physical motivation of our approach is first proposed.

The Ericksen-Timoshenko bar [21] was designed as an alternative to strain-gradient models to simulate microstructures of finite scale ξ, where an energy functional

$$G_\xi(u,v) = \int_0^1 \left(\frac{\alpha\xi^2}{2}(v')^2 + \frac{\alpha}{2}(u-v)^2 \right) dx$$

depending on two variables u, the longitudinal strain, and v, an internal variable assumed to measure all deviations from $1D$ deformations, is minimized in (u,v). Seeking a minumum in the second variable amounts to finding v_ξ solution of the Euler-Lagrange equation $-\xi^2 v_\xi'' + v_\xi = u$ with $v_\xi'(0) = v_\xi'(1) = 0$. Hence the problem can be restated as

$$u_\xi \in \text{argmin } F_\xi(u) := \frac{1}{\xi} G_\xi(u, v_\xi) = \frac{\alpha}{2\xi}\langle u - v_\xi, u\rangle, \tag{1.1}$$

where the brackets denote the L^2 scalar product. Moreover, it is observed that v_ξ also minimizes $\tilde{G}_\xi(u,v) := \frac{1}{\xi}G_\xi(u,v) + \frac{\alpha}{2\xi}\langle u, 1-u\rangle$ which, in two papers of Gurtin and Fried [15, 16], is identified with the free energy of (a particular choice of[1]) some thermally induced phase transition models where u stands for the scaled temperature variation and v represents a scalar "order parameter". In [16], the authors consider a dimensional analysis where $\xi = \varepsilon$ is allowed to tend to 0.

In this paper we show that for any space dimension N,

$$\tilde{F}_\varepsilon(u) := \frac{\alpha}{2\varepsilon}\langle 1 - v_\varepsilon, u\rangle \tag{1.2}$$

for $u \in [0,1]$ is the relaxation of $F_\varepsilon(u)$ for $u \in \{0,1\}$ in the weak-$*$ topology, and converges as $\varepsilon \to 0$ in a suitable sense and for a particular value of α independent of N to the perimeter $Per(A)$ of $A \subset \Omega$ as soon as u is the characteristic function of A. As a consequence we can address topology optimization

[1] In particular a model with dissipationless kinetics and vanishing specific heat.

problems where the perimeter is approximated by $\tilde{F}_\varepsilon(\chi_A)$ with A in some admissible class of shapes, which is thereby continuous for the weak-$*$ topology of characteristic functions and obviously free of any gradient term.

Let us emphasize that while the addition of the perimeter in several shape and topology optimization problems is by now quite standard, it is usually done in an ad-hoc manner to penalize an optimization algorithm, see [11] and the references therein. To our knowledge a proper mathematical justification is still missing and we believe that the contribution of this paper is also to propose a theoretical response to this important issue.

Here, by observing that $\tilde{F}_\varepsilon(u) = F_\varepsilon(u)$ as soon as u takes binary values (usually 1 or 0 in topology optimization), homogenization (i.e., intermediate values of u) can be considered in the converging optimization process. Moreover, since we intend to analyze the convergence of minimizers as $\varepsilon \to 0$, a more general notion of convergence of functionals, namely the Γ-convergence [14, 13], must be considered. In this setting, the Modica-Mortola approach to approximate the perimeter is well-known and widely used. In image segmentation [6] or fracture mechanics [12], the length of the jump set of the unknown u is added to quadratic terms integrated over the smooth regions, whose joint regularization is provided by the Ambrosio-Tortorelli functional [7]. Let us emphasize that, as they involve a gradient term $\|\nabla u\|^2_{L^2}$, none of these two functionals are well-suited to approximate optimal solutions in topology optimization. Indeed, they are defined for H^1 functions and not for characteristic functions, hence they would require to extend the cost function to the intermediate values, typically by a relaxation which is not always doable. In addition, they are not compatible with a discretization of u by piecewise constant finite elements, which are yet the most frequently used in topology optimization.

In a previous paper [9] the pointwise convergence of a variant of $F_\varepsilon(\chi_A)$ to $Per(A)$ for A with suitable regularity has been studied. Moreover, the topological sensitivity (or derivative) of the approximating functionals has been explicitly computed. With our approximating functionals $\tilde{F}_\varepsilon$ or F_ε we are able to nucleate holes, in particular, we can compute the corresponding topological derivatives at the only additional cost of computing an adjoint state solution to a well-posed elliptic PDE similar to the aforementioned Euler-Lagrange equation with appropriate right hand-side. Moreover, if topology optimization is intended without using the concept of topological derivative, our formulation allows one to relax the cost function, while the perimeter term might be approximated and relaxed by $\tilde{F}_\varepsilon(u)$, allowing for minimizing sequences showing intermediate "homogenized" values, but nevertheless converging to a characteristic function. From a numerical point of view another direct benefit of our approach, besides the absence of lower order terms in the topological sensitivity which has been mentioned already, is that the solutions of topology optimization problems is explicitely written as a multiple infima problem, which is easy to handle (e.g., by gradient or alternated directions algorithms), as shown in three basic examples at the end of our paper.

Moreover, it is rather remarkable that $\tilde{F}_\varepsilon(u)$ seems not only to be arbitrarily proposed to get better numerical algorithms, but also has an intrinsic meaning in terms of physical modelling, i.e., as a free-energy type functional depending on a small parameter and where v is interpreted as a slow *internal variable* which tracks the fast variable u. Our approach can therefore be a tool to study limit models as $\varepsilon \to 0$. In mechanics one may think for instance of fracture models approximated by damage models, where the damage variable is the scalar v, ε is the "thickness" of the crack, and u the displacement field, while the cost function is a Griffith-type energy [4, 12]. Let us remark that it is a general limitation of our method that the numerical solutions to optimal structures have at least a thickness of the order of ε, itself limited by the mesh stepsize. Coming back to our first motivation example, the Eriksen-Timoshenko bar, there is an interest to replace strain-gradient models by models with internal variables and free energy functionals reading as our F_ε. We believe that several other problems in physics where the perimeter enters the model could also find appropriate intepretations and/or extension in the light of our functional.

2. Description of the approximating functionals

Let Ω be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary. For all $u \in L^2(\Omega)$ we define

$$F_\varepsilon(u) = \inf_{v \in H^1(\Omega)} \left\{ \frac{\varepsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|v - u\|^2_{L^2(\Omega)} \right\}, \tag{2.1}$$

i.e., F_ε is equal to $\varepsilon/2$ times the Moreau-Yosida regularization with constant $1/\varepsilon^2$ of the function $v \in H^1(\Omega) \mapsto \|\nabla v\|^2_{L^2(\Omega)}$. For later purposes we also introduce the relaxation of F_ε for the weak-$*$ topology of $L^\infty(\Omega)$ (see Proposition 2.3), given by

$$\tilde{F}_\varepsilon(u) = F_\varepsilon(u) + \frac{1}{2\varepsilon}\langle u, 1-u\rangle, \tag{2.2}$$

or equivalently,

$$\tilde{F}_\varepsilon(u) = \inf_{v \in H^1(\Omega)} \left\{ \frac{\varepsilon}{2}\|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\left(\|v\|^2_{L^2(\Omega)} + \langle u, 1-2v\rangle\right)\right\}. \tag{2.3}$$

Throughout we use the notation $\langle u, v\rangle := \int_\Omega uv dx$ for every pair of functions u, v having suitable regularity. Practical expressions of $F_\varepsilon(u)$ and $\tilde{F}_\varepsilon(u)$ are provided below.

Proposition 2.1. *Let $u \in L^2(\Omega)$ be given and $v_\varepsilon \in H^1(\Omega)$ be the (weak) solution of*

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } \Omega \\ \partial_n v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then we have

$$F_\varepsilon(u) = \frac{1}{2\varepsilon}\langle u - v_\varepsilon, u\rangle, \tag{2.5}$$

$$\tilde{F}_\varepsilon(u) = \frac{1}{2\varepsilon}\langle 1 - v_\varepsilon, u\rangle. \tag{2.6}$$

Moreover, $\tilde{F}_\varepsilon(u)$ is differentiable with respect to ε with derivative

$$\frac{d}{d\varepsilon}\tilde{F}_\varepsilon(u) = \frac{1}{2\varepsilon^2}\left[3\langle u, v_\varepsilon\rangle - 2\|v_\varepsilon\|^2_{L^2(\Omega)} - \langle 1, u\rangle\right]. \tag{2.7}$$

Proof. The Euler-Lagrange equations of the minimization problems (2.1) and (2.3) are identical and read for the solution v_ε

$$\varepsilon^2\langle \nabla v_\varepsilon, \nabla\varphi\rangle + \langle v_\varepsilon - u, \varphi\rangle = 0 \qquad \forall \varphi \in H^1(\Omega), \tag{2.8}$$

which is the weak formulation of (2.4). It holds in particular

$$\varepsilon^2\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \|v_\varepsilon\|^2_{L^2(\Omega)} = \langle v_\varepsilon, u\rangle. \tag{2.9}$$

Plugging (2.9) into (2.1) and (2.3) entails (2.5) and (2.6). Let $\dot{v}_\varepsilon$ denote the derivative of v_ε with respect to ε, whose existence is easily deduced from the implicit function theorem. Differentiating (2.6) by the chain rule yields

$$\frac{d}{d\varepsilon}\tilde{F}_\varepsilon(u) = -\frac{1}{2\varepsilon^2}\langle 1 - v_\varepsilon, u\rangle - \frac{1}{2\varepsilon}\langle \dot{v}_\varepsilon, u\rangle.$$

Using (2.8) we obtain

$$\frac{d}{d\varepsilon}\tilde{F}_\varepsilon(u) = -\frac{1}{2\varepsilon^2}\langle 1 - v_\varepsilon, u\rangle - \frac{1}{2\varepsilon}\left[\varepsilon^2\langle \nabla v_\varepsilon, \nabla \dot{v}_\varepsilon\rangle + \langle v_\varepsilon, \dot{v}_\varepsilon\rangle\right]. \tag{2.10}$$

Now differentiating (2.8) provides

$$2\varepsilon\langle \nabla v_\varepsilon, \nabla\varphi\rangle + \varepsilon^2\langle \nabla \dot{v}_\varepsilon, \nabla\varphi\rangle + \langle \dot{v}_\varepsilon, \varphi\rangle = 0 \qquad \forall \varphi \in H^1(\Omega).$$

Choosing $\varphi = v_\varepsilon$ yields

$$\varepsilon^2\langle \nabla \dot{v}_\varepsilon, \nabla v_\varepsilon\rangle + \langle \dot{v}_\varepsilon, v_\varepsilon\rangle = -2\varepsilon\|\nabla v_\varepsilon\|^2_{L^2(\Omega)}.$$

It follows from (2.10) that

$$\frac{d}{d\varepsilon}\tilde{F}_\varepsilon(u) = -\frac{1}{2\varepsilon^2}\langle 1 - v_\varepsilon, u\rangle + \|\nabla v_\varepsilon\|^2_{L^2(\Omega)}.$$

Using (2.9) and rearranging yields (2.7). □

We define the sets

$$\mathcal{E} = L^\infty(\Omega, \{0, 1\}), \qquad \tilde{\mathcal{E}} = L^\infty(\Omega, [0, 1]),$$

remarking that $\tilde{\mathcal{E}}$ is obviously the convex hull of $\mathcal{E}$. Let us now prove that $\tilde{F}_\varepsilon$ is the relaxation, i.e. the lower semicontinuous envelope, of F_ε:

$$\tilde{F}_\varepsilon(u) = \inf\{\liminf_{n\to\infty} F_\varepsilon(u_n) : \ (u_n) \ s.t. \ u = \lim_{n\to\infty} u_n\}.$$

Lemma 2.2. *The functional $\tilde{F}_\varepsilon$ is continuous on $\tilde{\mathcal{E}}$ for the weak-$*$ topology of $L^\infty(\Omega)$.*

Proof. We first note that $\tilde{\mathcal{E}}$, endowed with the weak-$*$ topology of $L^\infty(\Omega)$, is metrizable. Thus continuity is equivalent to sequential continuity. Assume that $u_n, u \in \tilde{\mathcal{E}}$ satisfy $u_n \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$. Set $v_n = (-\varepsilon^2\Delta + I)^{-1}u_n$ and $v = (-\varepsilon^2\Delta + I)^{-1}u$, so that by Proposition 2.1

$$\tilde{F}_\varepsilon(u_n) = \frac{1}{2\varepsilon}\langle 1 - v_n, u_n\rangle, \qquad \tilde{F}_\varepsilon(u) = \frac{1}{2\varepsilon}\langle 1 - v, u\rangle.$$

For all test function $\varphi \in L^2(\Omega)$ we have

$$\langle v_n, \varphi\rangle = \langle u_n, (-\varepsilon^2\Delta + I)^{-1}\varphi\rangle \to \langle u, (-\varepsilon^2\Delta + I)^{-1}\varphi\rangle = \langle v, \varphi\rangle,$$

hence $v_n \rightharpoonup v$ weakly-$*$ in $L^2(\Omega)$. By standard elliptic operator theory, $\|v_n\|_{H^1(\Omega)}$ is uniformly bounded. By the Rellich theorem, one can extract a non-relabeled subsequence such that $v_n \to w$ strongly in $L^2(\Omega)$, for some $w \in L^2(\Omega)$. By uniqueness of the weak limit, we have $w = v$ and convergence of the whole sequence (v_n). Finally, as product of strongly and weakly convergent sequences, we get $\tilde{F}_\varepsilon(u_n) \to \tilde{F}_\varepsilon(u)$. □

Proposition 2.3. *The function $\tilde{F}_\varepsilon : \tilde{\mathcal{E}} \to \mathbb{R}$ is the relaxation of the function*

$$u \in \tilde{\mathcal{E}} \mapsto \begin{cases} F_\varepsilon(u) & \text{if } u \in \mathcal{E} \\ +\infty & \text{if } u \notin \mathcal{E} \end{cases} \tag{2.11}$$

for the weak-$*$ topology of $L^\infty(\Omega)$.

Proof. Denote by G_ε the function defined by (2.11). According to Proposition 11.1.1 of [10], the problem amounts to establishing the two following assertions:

$$\forall (u_n) \in \tilde{\mathcal{E}},\ u_n \rightharpoonup u \Rightarrow \tilde{F}_\varepsilon(u) \le \liminf_{n\to\infty} G_\varepsilon(u_n),$$

$$\forall u \in \tilde{\mathcal{E}}\ \exists (u_n) \in \tilde{\mathcal{E}} \text{ s.t. } u_n \rightharpoonup u, \tilde{F}_\varepsilon(u) = \lim_{n\to\infty} G_\varepsilon(u_n).$$

Using that $G_\varepsilon(u) \ge \tilde{F}_\varepsilon(u)$ for all $u \in \tilde{\mathcal{E}}$, the first assertion is a straightforward consequence of Lemma 2.2. Let now $u \in \tilde{\mathcal{E}}$ be arbitrary. A standard construction (see e.g. [19] proposition 7.2.14) enables to define a sequence $(u_n) \in \mathcal{E}$ such that $u_n \rightharpoonup u$. By Lemma 2.2 there holds

$$\tilde{F}_\varepsilon(u) = \lim_{n\to\infty} \tilde{F}_\varepsilon(u_n) = \lim_{n\to\infty} G_\varepsilon(u_n).$$

□

3. Γ-convergence of the approximating functionals

This section addresses the Γ-convergence of the sequence of functionals $(\tilde{F}_\varepsilon)$ when $\varepsilon \to 0$. Note that, when a sequence is indexed by the letter ε, we actually mean any sequence of indices (ε_k) of positive numbers going to zero.

3.1. **Definition and basic properties of Γ-convergence.** The notion of Γ-convergence (see, e.g., [14, 13, 10]) is a powerful tool of calculus of variations in function spaces. Given a metrizable space (X, d) (in our case $X = \tilde{\mathcal{E}}$ endowed with distance induced by the L^1-norm) one would like the maps

$$F \mapsto \inf_X F \quad \text{and} \quad F \mapsto \operatorname*{argmin}_X F$$

to be sequentially continuous on the space of extended real-valued functions $F : X \to \mathbb{R} \cup \{+\infty\}$.

Definition 3.1. Let $(\tilde{F}_\varepsilon)$ be a sequence of functions $\tilde{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ and $\tilde{F} : X \to \mathbb{R} \cup \{+\infty\}$. We say that $\tilde{F}_\varepsilon$ Γ-converges to $\tilde{F}$ iff for all $u \in X$ the two following conditions hold:

(1) for all sequences $(u_\varepsilon) \in X$ such that $d(u_\varepsilon, u) \to 0$ it holds $\tilde{F}(u) \le \liminf_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_\varepsilon)$,
(2) there exists a sequence $(\bar{u}_\varepsilon) \in X$ such that $d(\bar{u}_\varepsilon, u) \to 0$ and $\tilde{F}(u) \ge \limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon(\bar{u}_\varepsilon)$.

The key theorem we shall use in this paper reads ([10] Theorem 12.1.1):

Theorem 3.2. *Let $\tilde{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ Γ-converge to $\tilde{F} : X \to \mathbb{R} \cup \{+\infty\}$.*

(1) *If u_ε is an approximate minimizer of $\tilde{F}_\varepsilon$, i.e.*

$$\tilde{F}_\varepsilon(u_\varepsilon) \le \inf_{u\in X} \tilde{F}_\varepsilon(u) + \lambda_\varepsilon,$$

with $\lambda_\varepsilon \to 0$, then $\inf_{u\in X} \tilde{F}_\varepsilon(u) \to \inf_{u\in X} \tilde{F}(u)$ and every cluster point of (u_ε) is a minimizer of $\tilde{F}$.

(2) *If $\tilde{J} : X \to \mathbb{R}$ is continuous, then $\tilde{J} + \tilde{F}_\varepsilon$ Γ-converge to $\tilde{J} + \tilde{F}$.*

Let us emphasize that the consideration of approximate minimizers is of major importance as soon as numerical approximations are made.

3.2. **Preliminary results.** It turns out that the Γ-convergence can be straightforwardly deduced from the pointwise convergence if the sequence of functionals under consideration is nondecreasing and lower semicontinuous (see, e.g., [14] Proposition 5.4). The subsequent Lemma 3.6 as well as several numerical tests based on the expression (2.7) of the derivative lead us to conjecture that $\tilde{F}_\varepsilon$ is indeed nondecreasing when ε decreases. In addition, the pointwise convergence can be established, at least under some regularity assumptions, by harmonic analysis techniques, similarly to [9]. However, the motonicity being unproven, we will proceed more directly.

We define the potential function $W : \mathbb{R} \to \mathbb{R}_+$ by

$$W(s) = \begin{cases} s(1-s) & \text{if } 0 \le s \le 1, \\ -s & \text{if } s \le 0, \\ s-1 & \text{if } s \ge 1. \end{cases}$$

We set for all $u, v \in L^1(\Omega) \times L^1(\Omega)$

$$\tilde{G}_\varepsilon(u,v) = \begin{cases} \dfrac{\varepsilon}{2}\|\nabla v\|^2_{L^2(\Omega)} + \dfrac{1}{2\varepsilon}\|v-u\|^2_{L^2(\Omega)} + \dfrac{1}{2\varepsilon}\displaystyle\int_\Omega W(u) & \text{if } (u,v) \in L^2(\Omega) \times H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that, if $(u, v) \in \tilde{\mathcal{E}} \times H^1(\Omega)$, then

$$\begin{aligned} \tilde{G}_\varepsilon(u,v) &= \frac{\varepsilon}{2}\|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|v-u\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\langle u, 1-u\rangle \\ &= \frac{\varepsilon}{2}\|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\left(\|v\|^2_{L^2(\Omega)} + \langle u, 1-2v\rangle\right). \end{aligned}$$

Therefore we have for all $u \in \tilde{\mathcal{E}}$

$$\tilde{F}_\varepsilon(u) = \inf_{v\in H^1(\Omega)} \tilde{G}_\varepsilon(u,v).$$

The following theorem, taken from [23], will play a central role in our proof. We recall (see e.g. [10]) that, when u belongs to the space $BV(\Omega)$ of functions of bounded variations on Ω, its distributional derivative Du is a Borel measure whose total mass is denoted by $|Du|(\Omega)$. If u is the characteristic function of some subset A of Ω with finite perimeter, then $|Du|(\Omega)$ corresponds to the relative perimeter of A in Ω, namely, the $N-1$ dimensional Hausdorff measure of $\partial A \setminus \partial\Omega$.

Theorem 3.3. *When $\varepsilon \to 0$, the functionals $\tilde{G}_\varepsilon$ Γ-converge in $L^1(\Omega) \times L^1(\Omega)$ to the functional*

$$\tilde{G}(u,v) = \begin{cases} \kappa|Du|(\Omega) & \text{if } u = v \in BV(\Omega, \{0,1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

The constant κ is given by

$$\kappa = \frac{1}{2}\inf\left\{\int_{\mathbb{R}} W(\varphi)dx + \frac{1}{4}\int_{\mathbb{R}^2} e^{-|x-y|}(\varphi(x)-\varphi(y))^2 dxdy,\ \varphi(-\infty) = 0, \varphi(+\infty) = 1\right\}.$$

Before stating our Γ-convergence result for $\tilde{F}_\varepsilon$, we shall prove three technical lemmas.

Lemma 3.4. *Let Φ_ε be the fundamental solution of the operator $-\varepsilon^2\Delta + I$ on $\mathbb{R}^N$. For all $u \in L^1(\mathbb{R}^N, [0,1])$ we have*

$$\lim_{\varepsilon\to 0} \|\Phi_\varepsilon * u - u\|_{L^1(\mathbb{R}^N)} = 0.$$

Proof. Let $\lambda > 0$ be arbitrary. A classical density result gives the existence of $v \in \mathcal{C}(\mathbb{R}^N, [0,1])$ with compact support such that $\|u - v\|_{L^1(\mathbb{R}^N)} \leq \lambda$. We have

$$\|\Phi_\varepsilon * u - u\|_{L^1(\mathbb{R}^N)} \leq \|\Phi_\varepsilon * v - v\|_{L^1(\mathbb{R}^N)} + \|\Phi_\varepsilon * (u - v)\|_{L^1(\mathbb{R}^N)} + \|u - v\|_{L^1(\mathbb{R}^N)}.$$

Using that $\Phi_\varepsilon \geq 0$ (from the maximum principle) and $\int_{\mathbb{R}^N} \Phi_\varepsilon = 1$ (from $-\varepsilon^2 \Delta \Phi_\varepsilon + \Phi_\varepsilon = \delta$), we obtain

$$\|\Phi_\varepsilon * u - u\|_{L^1(\mathbb{R}^N)} \leq \|\Phi_\varepsilon * v - v\|_{L^1(\mathbb{R}^N)} + 2\|u - v\|_{L^1(\mathbb{R}^N)}. \tag{3.1}$$

Now let $\mu > 0$. By uniform continuity of v (Heine's theorem) there exists $\eta > 0$ such that

$$|x - y| \leq \eta \Rightarrow |v(x) - v(y)| \leq \mu.$$

We have for any $x \in \mathbb{R}^N$

$$\begin{aligned}
|(\Phi_\varepsilon * v - v)(x)| &= \left| \int_{\mathbb{R}^N} \Phi_\varepsilon(x - y)(v(y) - v(x)) dy \right| \\
&\leq \int_{[|y-x| \leq \eta]} \Phi_\varepsilon(x - y)|v(y) - v(x)| dy + \int_{[|y-x| > \eta]} \Phi_\varepsilon(x - y)|v(y) - v(x)| dy \\
&\leq \mu + 2 \int_{[|y-x| > \eta]} \Phi_\varepsilon(x - y) dy.
\end{aligned}$$

By change of variable we have $\Phi_\varepsilon(z) = \frac{1}{\varepsilon^N} \Phi_1(\frac{z}{\varepsilon})$. Due to the exponential decay of Φ_1 at infinity, we get for ε small enough $\int_{[|y-x|>\eta]} \Phi_\varepsilon(x - y) dy \leq \mu$. This shows that $|(\Phi_\varepsilon * v - v)(x)| \to 0$ uniformly on $\mathbb{R}^N$. This entails $\|\Phi_\varepsilon * v - v\|_{L^1(\mathbb{R}^N)} \to 0$. Consequently, we have for ε small enough $\|\Phi_\varepsilon * v - v\|_{L^1(\mathbb{R}^N)} \leq \lambda$. Going back to (3.1) we arrive at

$$\|\Phi_\varepsilon * u - u\|_{L^1(\mathbb{R}^N)} \leq 3\lambda.$$

As λ is arbitrary this proves the desired convergence. □

We define the projection (or truncation) operator $P_{[0,1]} : \mathbb{R} \to [0, 1]$ by

$$P_{[0,1]}(s) = \max(0, \min(1, s)). \tag{3.2}$$

Lemma 3.5. *Let $(u, v) \in L^2(\Omega) \times H^1(\Omega)$ and set $\tilde{u} = P_{[0,1]}(u)$, $\tilde{v} = P_{[0,1]}(v)$. Then*

$$\tilde{G}_\varepsilon(\tilde{u}, \tilde{v}) \leq \tilde{G}_\varepsilon(u, v).$$

Proof. We shall show that each term in the definition of $\tilde{G}_\varepsilon$ is decreased by truncation. Suppose that $(u, v) \in L^2(\Omega) \times H^1(\Omega)$. For the first one we have $\nabla \tilde{v} = \chi_{[0<v<1]} \nabla v$. Hence $\|\nabla \tilde{v}\|^2_{L^2(\Omega)} \leq \|\nabla v\|^2_{L^2(\Omega)}$. For the second term we use that the pojection $P_{[0,1]}$ is 1-Lipschitz, which yields

$$|\tilde{v}(x) - \tilde{u}(x)| \leq |v(x) - u(x)| \qquad \forall x \in \Omega.$$

This obviously implies that $\|\tilde{v} - \tilde{u}\|^2_{L^2(\Omega)} \leq \|v - u\|^2_{L^2(\Omega)}$. As to the last term we notice that, by construction of W, we have

$$0 \leq W(P_{[0,1]}(s)) \leq W(s) \qquad \forall s \in \mathbb{R}.$$

□

Although the third lemma holds only in dimension $N = 1$, it will have useful consequences in arbitrary space dimension.

Lemma 3.6. *Let $a < 0 < b$, $\Omega =]a, b[$ and $u = \chi_{]0,b[}$. We have*

$$\lim_{\varepsilon \to 0} F_\varepsilon(u) = \frac{1}{4},$$

$$\frac{d}{d\varepsilon} F_\varepsilon(u) \leq 0 \qquad \forall \varepsilon > 0.$$

Proof. We make the splitting $u_\varepsilon = v_\varepsilon + w_\varepsilon$ with

$$\begin{cases} -\varepsilon^2 v_\varepsilon'' + v_\varepsilon = \chi_{\mathbb{R}_+} \text{ on } \mathbb{R}, \\ v_\varepsilon'(-\infty) = v_\varepsilon'(+\infty) = 0, \end{cases}$$

$$\begin{cases} -\varepsilon^2 w_\varepsilon'' + w_\varepsilon = 0 \text{ on } [a, b], \\ w_\varepsilon'(a) = -v_\varepsilon'(a), \; w_\varepsilon'(b) = -v_\varepsilon'(b), \end{cases}$$

and we find the solutions

$$v_\varepsilon(x) = \begin{cases} \frac{1}{2}e^{x/\varepsilon} & \text{if } x \le 0, \\ 1 - \frac{1}{2}e^{-x/\varepsilon} & \text{if } x \ge 0, \end{cases}$$

$$w_\varepsilon(x) = -\frac{1}{2}\frac{e^{-2a/\varepsilon} - 1}{e^{2(b-a)/\varepsilon} - 1}e^{x/\varepsilon} + \frac{1}{2}\frac{e^{2b/\varepsilon} - 1}{e^{2(b-a)/\varepsilon} - 1}e^{-x/\varepsilon} \qquad \forall x \in \mathbb{R}.$$

After some algebra we arrive at

$$F_\varepsilon(u) = \frac{1}{4}\frac{(e^{-2a/\varepsilon} - 1)(e^{2b/\varepsilon} - 1)}{e^{2(b-a)/\varepsilon} - 1}.$$

Setting $t = 2/\varepsilon$, we obtain

$$F_\varepsilon(u) = \frac{1}{4}\frac{(e^{-ta} - 1)(e^{tb} - 1)}{e^{t(b-a)} - 1} = \frac{1}{4}\frac{(1 - e^{ta})(1 - e^{-tb})}{1 - e^{-t(b-a)}}.$$

Clearly, $F_\varepsilon(u) \to 1/4$ as $t \to +\infty$. Set now $h = b - a > 0$, $r = -a/(b-a) \in [0,1]$, so that $a = -rh$, $b = (1-r)h$, and

$$F_\varepsilon(u) = \frac{1}{4}\frac{(1 - e^{-trh})(1 - e^{-t(1-r)h})}{1 - e^{-th}}.$$

The change of variable $s = e^{-th}$ leads to

$$F_\varepsilon(u) = \frac{1}{4}\frac{(1 - s^r)(1 - s^{1-r})}{1 - s}.$$

Now differentiating with respect to s yields

$$\frac{d}{ds}F_\varepsilon(u) = \frac{1}{4(1-s)^2}\left[2 - r(s^{r-1} + s^{1-r}) - (1-r)(s^{-r} + s^r)\right].$$

Set

$$f(s,r) = \frac{1}{2}\left[r(s^{r-1} + s^{1-r}) + (1-r)(s^{-r} + s^r)\right].$$

We have

$$f(e^\tau, r) = r\cosh((1-r)\tau) + (1-r)\cosh(r\tau) =: g_r(\tau).$$

For fixed $r \in [0,1]$, the function g_r is clearly even and nondecreasing on $\mathbb{R}_+$. Hence $g_r(\tau) \ge g_r(0) = 1$ for all $\tau \in \mathbb{R}$. This implies that $f(s,r) \ge 1$ for all $(s,r) \in \mathbb{R}_+^* \times [0,1]$, therefore

$$\frac{d}{ds}F_\varepsilon(u) \le 0 \qquad \forall (s,r) \in \mathbb{R}_+^* \times [0,1].$$

Recalling that $s = e^{-2h/\varepsilon}$, we derive

$$\frac{d}{d\varepsilon}F_\varepsilon(u) \le 0 \qquad \forall \varepsilon > 0.$$

□

3.3. **Main result.** With Theorem 3.3 and the three above lemmas at hand we are now able to state and prove our Γ-convergence result.

Theorem 3.7. *When $\varepsilon \to 0$, the functionals $\tilde{F}_\varepsilon$ Γ-converge in $\tilde{\mathcal{E}}$ endowed with the strong topology of $L^1(\Omega)$ to the functional*

$$\tilde{F}(u) = \begin{cases} \frac{1}{4}|Du|(\Omega) & \text{if } u \in BV(\Omega, \{0,1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. (1) Let $(u_\varepsilon), u \in \tilde{\mathcal{E}}$ be such that $u_\varepsilon \to u$ in $L^1(\Omega)$. For each $\varepsilon > 0$ there exists a (unique) function $v_\varepsilon \in H^1(\Omega)$ such that $\tilde{F}_\varepsilon(u_\varepsilon) = \tilde{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$. This is the solution of

$$\begin{cases} -\varepsilon^2\Delta v_\varepsilon + v_\varepsilon = u_\varepsilon & \text{in } \Omega, \\ \partial_n v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Set $w_\varepsilon = \Phi_\varepsilon * u_\varepsilon$, where u_ε is here extended by zero outside Ω. By the Lax-Milgram theorem we have

$$\frac{1}{2}\left(\varepsilon^2\|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + \|v_\varepsilon\|_{L^2(\Omega)}^2\right) - \langle u_\varepsilon, v_\varepsilon\rangle \le \frac{1}{2}\left(\varepsilon^2\|\nabla w_\varepsilon\|_{L^2(\Omega)}^2 + \|w_\varepsilon\|_{L^2(\Omega)}^2\right) - \langle u_\varepsilon, w_\varepsilon\rangle.$$

Adding to both sides $\frac{1}{2}\|u_\varepsilon\|^2_{L^2(\Omega)}$ results in

$$\varepsilon^2\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \|v_\varepsilon - u_\varepsilon\|^2_{L^2(\Omega)} \leq \varepsilon^2\|\nabla w_\varepsilon\|^2_{L^2(\Omega)} + \|w_\varepsilon - u_\varepsilon\|^2_{L^2(\Omega)}.$$

Yet the right hand side is bounded from above by

$$\begin{aligned}\varepsilon^2\|\nabla w_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \|w_\varepsilon - u_\varepsilon\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left(-\varepsilon^2\Delta w_\varepsilon + w_\varepsilon\right) w_\varepsilon - 2u_\varepsilon w_\varepsilon + u_\varepsilon^2 \\ &= \int_{\mathbb{R}^N} u_\varepsilon w_\varepsilon - 2u_\varepsilon w_\varepsilon + u_\varepsilon^2 \\ &= \int_{\mathbb{R}^N} (u_\varepsilon - w_\varepsilon)u_\varepsilon.\end{aligned}$$

We obtain

$$\begin{aligned}\|v_\varepsilon - u_\varepsilon\|^2_{L^2(\Omega)} &\leq \|u_\varepsilon - w_\varepsilon\|_{L^1(\Omega)} \\ &\leq \|u - \Phi_\varepsilon * u\|_{L^1(\Omega)} + \|u - u_\varepsilon\|_{L^1(\Omega)} + \|\Phi_\varepsilon * (u_\varepsilon - u)\|_{L^1(\Omega)}.\end{aligned}$$

By virtue of Lemma 3.4 and the Young inequality for convolutions the right hand side goes to zero, hence $\|v_\varepsilon - u_\varepsilon\|_{L^2(\Omega)} \to 0$. Next we have

$$\|v_\varepsilon - u\|_{L^1(\Omega)} \leq \|v_\varepsilon - u_\varepsilon\|_{L^1(\Omega)} + \|u_\varepsilon - u\|_{L^1(\Omega)} \leq |\Omega|^{1/2}\|v_\varepsilon - u_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon - u\|_{L^1(\Omega)}.$$

It follows that $\|v_\varepsilon - u\|_{L^1(\Omega)} \to 0$. We infer using Theorem 3.3:

$$\begin{aligned}\liminf_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \tilde{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \\ &\geq \tilde{G}(u,u) = 4\kappa\tilde{F}(u).\end{aligned}$$

(2) Suppose that $u \in \tilde{\mathcal{E}}$. By Theorem 3.3 there exists $(u_\varepsilon, v_\varepsilon) \in L^2(\Omega) \times H^1(\Omega)$ such that $u_\varepsilon \to u$, $v_\varepsilon \to u$ in $L^1(\Omega)$, and

$$\limsup_{\varepsilon\to 0} \tilde{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \tilde{G}(u,u).$$

By truncation (see Lemma 3.5), one may assume that $u_\varepsilon, v_\varepsilon \in \tilde{\mathcal{E}}$. Yet $\tilde{F}_\varepsilon(u_\varepsilon) \leq \tilde{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$, which entails

$$\limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_\varepsilon) \leq 4\kappa\tilde{F}(u).$$

(3) Finally, observing that κ is independent of Ω and the dimension N, we deduce by identification from Lemma 3.6 that $\kappa = 1/4$.

□

4. Solution of topology optimization problems with perimeter penalization

In this section we propose solution methods for the optimization of shape functionals involving the perimeter. The functionals under consideration will be of the form $j_\alpha(A) = J_\alpha(\chi_A)$, with $J_\alpha(u) = J(u) + 4\alpha|Du|(\Omega)$. Through a continuation procedure, J_α will be approximated by a sequence of auxiliary functionals of the form $\tilde{J}_{\alpha,\varepsilon}(u) = \tilde{J}(u) + 4\alpha\tilde{F}_\varepsilon(u)$. The issue is then to study the convergence (up to a subsequence) of sequences of minimizers of $\tilde{J}_{\alpha,\varepsilon}(u)$. As is well-known, the Γ-convergence of the functionals is not sufficient to guarantee this property, which is yet essential.

4.1. Preliminary results.

Lemma 4.1. *Let (u_ε) be a sequence of $\tilde{\mathcal{E}}$ such that $(\tilde{F}_\varepsilon(u_\varepsilon))$ is bounded. For each $\varepsilon > 0$ let $v_\varepsilon \in H^1(\Omega)$ be the solution of (2.4) with right hand side u_ε. Then (v_ε) admits a subsequence which converges strongly in $L^1(\Omega)$.*

Proof. We have by definition

$$\tilde{F}_\varepsilon(u_\varepsilon) = \frac{\varepsilon}{2}\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\left(\|v_\varepsilon\|^2_{L^2(\Omega)} + \langle u_\varepsilon, 1 - 2v_\varepsilon\rangle\right),$$

and, as $0 \leq u_\varepsilon \leq 1$,

$$\langle u_\varepsilon, 1 - 2v_\varepsilon\rangle \geq \int_\Omega \min(0, 1 - 2v_\varepsilon)dx.$$

Setting

$$\mathcal{W}(s) = s^2 + \min(0, 1 - 2s)$$

we obtain

$$\tilde{F}_\varepsilon(u_\varepsilon) \geq \int_\Omega \left(\frac{\varepsilon}{2} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon} \mathcal{W}(v_\varepsilon) \right) dx. \tag{4.1}$$

Straightforward calculations show that the function $\mathcal{W}$ is nonnegative, symmetric with respect to 1/2, and vanishes only in 0 and 1 (see Figure 1). We now use a classical argument due to Modica [20], which consists in applying successively to the right hand side of (4.1) the elementary Young inequality and the chain rule. This entails

$$\tilde{F}_\varepsilon(u_\varepsilon) \geq \int_\Omega |\nabla v_\varepsilon| \sqrt{\mathcal{W}(v_\varepsilon)} dx = \int_\Omega |\nabla w_\varepsilon| dx,$$

where ψ is an arbitrary primitive of $\sqrt{\mathcal{W}}$ and $w_\varepsilon = \psi \circ v_\varepsilon$. The weak maximum principle implies that $0 \leq v_\varepsilon \leq 1$, hence $\psi(0) \leq w_\varepsilon \leq \psi(1)$. It follows that (w_ε) is bounded in $L^1(\Omega)$. By the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$, (w_ε) admits a subsequence which converges strongly in $L^1(\Omega)$ to some function w. By construction, ψ is an increasing homeomorphism of $\mathbb{R}$ into itself. Denoting by ψ^{-1} the inverse function, we have $v_\varepsilon = \psi^{-1} \circ w_\varepsilon$. Up to a subsequence, we have $w_\varepsilon \to w$ almost everywhere, thus $v_\varepsilon \to \psi^{-1} \circ w =: v$ almost everywhere. The Lebesgue dominated convergence theorem yields that $v_\varepsilon \to v$ in $L^1(\Omega)$. □

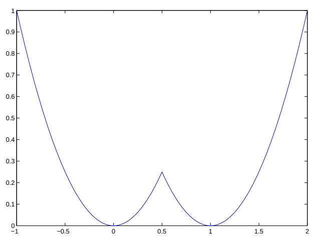

FIGURE 1. Plot of the function $\mathcal{W}$.

Lemma 4.2. *Let (u_ε) be a sequence of $\tilde{\mathcal{E}}$ which converges weakly-$*$ in $L^\infty(\Omega)$ to $u \in \tilde{\mathcal{E}}$. For each $\varepsilon > 0$ let $v_\varepsilon \in H^1(\Omega)$ be the solution of (2.4) with right hand side u_ε. Then $v_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$.*

Proof. The variational formulation for v_ε reads

$$\int_\Omega \left(\varepsilon^2 \nabla v_\varepsilon . \nabla \varphi + v_\varepsilon \varphi \right) dx = \int_\Omega u_\varepsilon \varphi dx \qquad \forall \varphi \in H^1(\Omega). \tag{4.2}$$

Choosing $\varphi = v_\varepsilon$ and using the Cauchy-Schwarz inequality yields

$$\begin{aligned} \varepsilon^2 \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \|v_\varepsilon\|^2_{L^2(\Omega)} &\leq \|u_\varepsilon\|_{L^2(\Omega)} \|v_\varepsilon\|_{L^2(\Omega)} \\ &\leq \|u_\varepsilon\|_{L^2(\Omega)} \sqrt{\varepsilon^2 \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \|v_\varepsilon\|^2_{L^2(\Omega)}}, \end{aligned}$$

which results in

$$\varepsilon^2 \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \|v_\varepsilon\|^2_{L^2(\Omega)} \leq \|u_\varepsilon\|^2_{L^2(\Omega)} \leq |\Omega|.$$

In particular we infer that

$$\|v_\varepsilon\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}, \qquad \|\nabla v_\varepsilon\|_{L^2(\Omega)} \leq \frac{\sqrt{|\Omega|}}{\varepsilon}. \tag{4.3}$$

Coming back to (4.2) we derive that, for every $\varphi \in H^1(\Omega)$,

$$\int_\Omega v_\varepsilon \varphi dx = \int_\Omega u_\varepsilon \varphi dx - \varepsilon^2 \int_\Omega \nabla v_\varepsilon . \nabla \varphi dx.$$

Passing to the limit, we get with the help of (4.3)

$$\int_\Omega v_\varepsilon \varphi dx \to \int_\Omega u \varphi dx. \tag{4.4}$$

Choose now an arbitrary test function $\psi \in L^2(\Omega)$, and fix $\rho > 0$. By density of $H^1(\Omega)$ in $L^2(\Omega)$, there exists $\varphi \in H^1(\Omega)$ such that $\|\varphi - \psi\|_{L^2(\Omega)} \leq \rho$. From (4.4), there exists $\eta > 0$ such that

$$\left| \int_\Omega (v_\varepsilon - u)\varphi dx \right| \leq \rho \qquad \forall \varepsilon < \eta.$$

We obtain for any $\varepsilon < \eta$

$$\begin{aligned} \left| \int_\Omega (v_\varepsilon - u)\psi dx \right| &\leq \left| \int_\Omega (v_\varepsilon - u)\varphi dx \right| + \left| \int_\Omega v_\varepsilon(\psi - \varphi) dx \right| + \left| \int_\Omega u(\psi - \varphi) dx \right| \\ &\leq \rho(1 + 2\sqrt{|\Omega|}). \end{aligned}$$

Hence $v_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$. □

Lemma 4.3. *Let $(u_\varepsilon) \in \tilde{\mathcal{E}}$ be a sequence such that $u_\varepsilon \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$. If $u \in \mathcal{E}$, then $u_\varepsilon \to u$ strongly in $L^1(\Omega)$.*

Proof. We have by definition

$$\int_\Omega (u_\varepsilon - u)\varphi dx \to 0 \qquad \forall \varphi \in L^1(\Omega). \tag{4.5}$$

Since $u \in \mathcal{E}$ and $u_\varepsilon \in \tilde{\mathcal{E}}$, we have

$$\int_\Omega |u_\varepsilon - u| dx = \int_{[u=0]} u_\varepsilon dx + \int_{[u=1]} (1 - u_\varepsilon) dx.$$

From (4.5) with $\varphi = \chi_{[u=0]}$ we get

$$\int_{[u=0]} u_\varepsilon dx \to 0.$$

Choosing now $\varphi = \chi_{[u=1]}$ results in

$$\int_{[u=1]} (1 - u_\varepsilon) dx \to 0,$$

which completes the proof. □

The three above lemmas can be summarized in the following Proposition.

Proposition 4.4. *Let (u_ε) be a sequence of $\tilde{\mathcal{E}}$ such that $(\tilde{F}_\varepsilon(u_\varepsilon))$ is bounded. For each $\varepsilon > 0$ let $v_\varepsilon \in H^1(\Omega)$ be the solution of (2.4) with right hand side u_ε. If $u_\varepsilon \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$ then, for some subsequence, there holds:*

1. *$v_\varepsilon \to u$ strongly in $L^1(\Omega)$,*
2. *$u \in \mathcal{E}$,*
3. *$u_\varepsilon \to u$ strongly in $L^1(\Omega)$.*

Proof. By Lemma 4.2, we have $v_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$, thus also weakly in $L^1(\Omega)$ since Ω is bounded. By Lemma 4.1, we have for a subsequence $v_\varepsilon \to v \in \tilde{\mathcal{E}}$ strongly in $L^1(\Omega)$, and subsequently by uniqueness of the weak limit we have $v = u$.

Next, we have in view of (2.6)

$$\tilde{F}_\varepsilon(u_\varepsilon) = \frac{1}{2\varepsilon} \int_\Omega (1 - v_\varepsilon) u_\varepsilon dx \geq 0.$$

Therefore, the boundedness of $(\tilde{F}_\varepsilon(u_\varepsilon))$ entails

$$\int_\Omega (1 - v_\varepsilon) u_\varepsilon dx \to 0.$$

Yet, there holds

$$\int_\Omega (1 - v_\varepsilon) u_\varepsilon dx - \int_\Omega (1 - u) u dx = \int_\Omega (u_\varepsilon - u)(1 - u) dx - \int_\Omega u_\varepsilon (v_\varepsilon - u) dx.$$

Since, on one hand, $u_\varepsilon \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$ and, on the other hand, $v_\varepsilon \to u$ strongly in $L^1(\Omega)$ and $u_\varepsilon \in \tilde{\mathcal{E}}$, both integrals at the right hand side of the above equality tend to zero. We arrive at

$$\int_\Omega (1 - u) u dx = 0.$$

In addition, due to the closedness of $\tilde{\mathcal{E}}$ in the weak-$*$ topology of $L^\infty(\Omega)$, we have $u \in \tilde{\mathcal{E}}$. We infer that $u(x) \in \{0,1\}$ for almost every $x \in \Omega$.

Finally, Lemma 4.3 implies that $u_\varepsilon \to u$ strongly in $L^1(\Omega)$. □

4.2. **Existence and convergence of minimizers.** Let a functional $J : \mathcal{E} \to \mathbb{R}_+$ and a parameter $\alpha > 0$ be given. We want to solve the minimization problem

$$I := \inf_{u \in BV(\Omega,\{0,1\})} \{J(u) + 4\alpha |Du|(\Omega)\}. \tag{4.6}$$

Proposition 4.5. *Assume that J is lower semi-continuous on $\mathcal{E}$ for the strong topology of $L^1(\Omega)$. Then the infimum in (4.6) is attained.*

Proof. Let $(u_n) \in BV(\Omega,\{0,1\})$ be a minimizing sequence. By boundedness of Ω and definition of the objective functional, $\|u_n\|_{L^1(\Omega)} + |Du_n|(\Omega)$ is uniformly bounded. Therefore, due to the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$, one can extract a subsequence (not relabeled) such that $u_n \to u$ in $L^1(\Omega)$, for some $u \in L^1(\Omega)$. In addition, for a further subsequence, $u_n \to u$ almost everywhere in Ω, thus $u \in \mathcal{E}$. Using the sequential lower semi-continuity of J and $u \mapsto |Du|(\Omega)$, we obtain

$$J(u) + 4\alpha |Du|(\Omega) \le \liminf_{n\to\infty} J(u_n) + 4\alpha |Du_n|(\Omega) = I.$$

It follows that u is a minimizer. □

Let $\tilde{J} : \tilde{\mathcal{E}} \to \mathbb{R}_+$ be an extension of J, i.e., a function such that $\tilde{J}(u) = J(u)$ for all $u \in \mathcal{E}$. By Theorem 3.7 we have

$$I = \inf_{u\in\tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha \tilde{F}(u) \right\}. \tag{4.7}$$

Given $\varepsilon > 0$ we introduce the approximate problem:

$$I_\varepsilon := \inf_{u\in\tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u) \right\}. \tag{4.8}$$

It turns out (cf. Proposition 4.8), that the approximate subproblem (4.8) needs to be solved only approximately. However, the existence of exact minimizers is an information of interest regarding the design and analysis of a solution method.

Proposition 4.6. *Assume that $\tilde{J} : \tilde{\mathcal{E}} \to \mathbb{R}_+$ is lower semi-continuous for the weak-$*$ topology of $L^\infty(\Omega)$. Then the infimum in (4.8) is attained.*

Proof. By Lemma 2.2, the functional $u \in \tilde{\mathcal{E}} \to \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u)$ is lower semi-continuous for the weak-$*$ topology of $L^\infty(\Omega)$. In addition, the set $\tilde{\mathcal{E}}$ is compact for the same topology. The claim results from standard arguments. □

Thanks to Proposition 4.4 the so-called equicoercivity property might be formulated as follows.

Proposition 4.7. *Let $\tilde{J} : \tilde{\mathcal{E}} \to B$ be a given cost function, where B is a bounded subset of $\mathbb{R}$. Consider a sequence $(u_\varepsilon) \in \tilde{\mathcal{E}}$ such that*

$$\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \le I_\varepsilon + \lambda_\varepsilon,$$

with (λ_ε) bounded. There exists $u \in \mathcal{E}$ and a subsequence of indices such that $u_\varepsilon \to u$ strongly in $L^1(\Omega)$.

Proof. By the *limsup* inequality of the Γ-convergence, there exists a sequence $(z_\varepsilon) \in \tilde{\mathcal{E}}$ such that $z_\varepsilon \to 0$ in $L^1(\Omega)$ and $\tilde{F}_\varepsilon(z_\varepsilon) \to \tilde{F}(0) = 0$. For this particular sequence we have

$$\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \le \tilde{J}(z_\varepsilon) + \alpha \tilde{F}_\varepsilon(z_\varepsilon) + \lambda_\varepsilon,$$

which entails that $(\tilde{F}_\varepsilon(u_\varepsilon))$ is bounded.

Now, since $\tilde{\mathcal{E}}$ is weakly-$*$ compact in $L^\infty(\Omega)$, there exists $u \in \tilde{\mathcal{E}}$ and a non-relabeled subsequence such that $u_\varepsilon \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$. Using Proposition 4.4, we infer that $u \in \mathcal{E}$ as well as $u_\varepsilon \to u$ strongly in $L^1(\Omega)$. □

Combining Theorem 3.2, Theorem 3.7 and Proposition 4.7 leads to the following result.

Theorem 4.8. *Let u_ε be an approximate minimizer of* (4.8)*, i.e.*

$$\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \leq I_\varepsilon + \lambda_\varepsilon,$$

with $\lim_{\varepsilon\to 0} \lambda_\varepsilon = 0$. Assume that $\tilde{J}$ is continuous on $\tilde{\mathcal{E}}$ for the strong topology of $L^1(\Omega)$. Then we have $\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \to I$. Moreover, (u_ε) admits cluster points, and each of these cluster points is a minimizer of (4.7)*.*

Theorem 4.8 shows in particular that, when (4.7) admits a unique minimizer u^*, then the whole sequence (u_ε) converges in $L^1(\Omega)$ to u^*. We have now a solid background to address the algorithmic issue.

4.3. **Algorithms for topology optimization with perimeter penalization.** As already said, we propose to use a continuation method with respect to ε. Namely, we construct a sequence (ε_k) going to zero and solve at each iteration k the minimization problem (4.8) using the previous solution as initial guess.

Several methods may be used to solve (4.8). The specific features of the functional $\tilde{J}$ may guide the choice.

(1) The most direct approach consists in using methods dedicated to the solution of optimization problem with box constraints, for instance the projected gradient method.
(2) When $\tilde{J}$ is continuous for the weak-$*$ topology of $L^\infty(\Omega)$ one can restrict the feasible set to $\mathcal{E}$ and use topology optimization methods to find an approximate minimizer.
(3) Another alternative is to come back to the definition of $\tilde{F}_\varepsilon$ by (2.3), and write

$$I_\varepsilon = \inf_{u\in\tilde{\mathcal{E}}} \inf_{v\in H^1(\Omega)} \left\{ \tilde{J}(u) + \alpha \left[\frac{\varepsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \left(\|v\|^2_{L^2(\Omega)} + \langle u, 1-2v\rangle \right) \right] \right\}.$$

Then one can use an alternating minimization algorithm with respect to the pair of variables (u, v).

In the subsequent sections we present three examples of application. The first one illustrates the method (1) in the context of least square problems. The last two ones deal with self-adjoint problems for which, as we shall see, the method (3) is particularly relevant. We refer to [9] for some examples of application of the approach (2).

For the discretization of all the PDEs involved, we use piecewise linear finite elements on a structured triangular mesh. For each example different values of the penalization parameter α are considered. Note that choosing α too small requires, in order to eventually obtain a *binary* solution (i.e. $u \in \mathcal{E}$), to drive ε towards very small values, which in turn necessitates the use of a very fine mesh to solve (2.4) with a good accuracy. This is why, to enable comparisons of solutions obtained with identical meshes and a wide range of values of α, we always use relatively fine meshes.

5. First application: source identification for the Poisson equation

5.1. **Problem formulation.** For all $u \in L^2(\Omega)$ we denote by $y_u \in H^1_0(\Omega)$ the solution of

$$\begin{cases} -\Delta y_u = u & \text{in } \Omega, \\ y_u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we set

$$\tilde{J}(u) = \frac{1}{2} \|y_u - y^\dagger\|^2_{L^2(\Omega)}.$$

Proposition 5.1. *The functional $\tilde{J}$ is continuous on $\tilde{\mathcal{E}}$ strongly in $L^1(\Omega)$ and also weakly-$*$ in $L^\infty(\Omega)$.*

Proof. First we remark that if (u_n) is a sequence of $\tilde{\mathcal{E}}$ such that $u_n \to u$ strongly in $L^1(\Omega)$, then $u_n \to u$ almost everywhere (for a subsequence), which implies that $u_n \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$ by dominated convergence.

Thus, let us assume that $u_n \rightharpoonup u$ weakly-$*$ in $L^\infty(\Omega)$. As $(\|y_{u_n}\|_{H^1(\Omega)})$ is bounded, we can extract a subsequence such that $y_{u_n} \rightharpoonup y \in H^1_0(\Omega)$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. This implies, on one hand, that $y = y_u$, thus by uniqueness of this cluster point the whole sequence (y_{u_n}) converges to y for the aforementioned topologies, and, on the other hand, that $\tilde{J}(u_n) \to \tilde{J}(u)$. $\square$

5.2. **Algorithm and examples.** In our simulations $y^\dagger$ is defined by

$$y^\dagger = y^\sharp + n,$$

where $y^\sharp$ solves

$$\begin{cases} -\Delta y^\sharp = u^\sharp & \text{in } \Omega, \\ y^\sharp = 0 & \text{on } \partial\Omega, \end{cases}$$

for some given $u^\sharp \in L^2(D)$ and $n \in L^2(D)$. More precisely, n is of the form $\beta\bar{n}$, with $\beta > 0$ and $\bar{n}$ a random Gaussian noise with zero mean. The function $u^\sharp$ is chosen as the characteristic function of a subdomain $\Omega^\sharp \subset\subset \Omega$.

The domain Ω is the unit square $]0,1[\times]0,1[$. We initialize ε to 1 and divide it by 2 until it becomes less that 10^{-6}. The initial guess is $u \equiv 1$. In order to solve the approximate problems we use a projected gradient method with line search. Here the mesh contains 80401 nodes. The results of computations performed with different values of the coefficients α and β are depicted on Figure 2. Rather than β, we indicate the noise to signal ratio, viz.,

$$R = \frac{\|n\|_{L^2(D)}}{\|y^\dagger\|_{L^2(D)}}.$$

We observe, as expected, that the higher the noise level is, the larger the penalization parameter α must be chosen in order to achieve a proper reconstruction. The price to pay, of course, is that the reconstructed shapes are smoothed.

6. Second application: conductivity optimization

6.1. **Problem formulation.** We consider a two-phase conductor Ω with source term $f \in L^2(\Omega)$. For all $u \in \tilde{\mathcal{E}}$ we define the conductivity

$$\gamma_u := \gamma_0(1-u) + \gamma_1 u,$$

where $\gamma_1 > \gamma_0 > 0$ are given constants. Our objective functional is the power dissipated by the conductor augmented by a volume term, i.e.,

$$\tilde{J}(u) = \int_\Omega f y dx + \ell \int_\Omega u dx, \tag{6.1}$$

where ℓ is a fixed positive multiplier and y solves

$$\begin{cases} -\operatorname{div}(\gamma_u \nabla y) = f & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Note that the Dirichlet boundary condition has been chosen merely for simplicity of the presentation. Alternatively, this functional can be expressed in terms of the complementary energy (see, e.g., [2])

$$\tilde{J}(u) = \inf_{\tau\in\Sigma} \left\{ \int_\Omega \gamma_u^{-1} |\tau|^2 dx \right\} + \ell \int_\Omega u dx, \tag{6.3}$$

with

$$\Sigma = \{\tau \in L^2(\Omega)^N, -\operatorname{div}\tau = f \text{ in } \Omega\}.$$

When it occurs that $u \in \mathcal{E}$ we set $J(u) := \tilde{J}(u)$. Given $\alpha > 0$, we want to solve

$$\inf_{u\in\mathcal{E}} \{J(u) + 4\alpha|Du|(\Omega)\}, \tag{6.4}$$

which amounts to solving

$$\inf_{u\in\tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha\tilde{F}(u) \right\}.$$

Proposition 6.1. *The functional $\tilde{J}$ defined by* (6.1) *is continuous on $\tilde{\mathcal{E}}$ strongly in $L^1(\Omega)$.*

Proof. Assume that $u_n \to u$ strongly in $L^1(\Omega)$, and denote by y_n, y the corresponding states. Obviously, $\gamma_{u_n} \to \gamma_u$ strongly in $L^1(\Omega)$. Then $y_n \rightharpoonup y$ weakly in $H_0^1(\Omega)$, see [10] Theorem 16.4.1 or [1] Lemma 1.2.22. It follows straightforwardly that $\tilde{J}(u_n) \to \tilde{J}(u)$. ☐

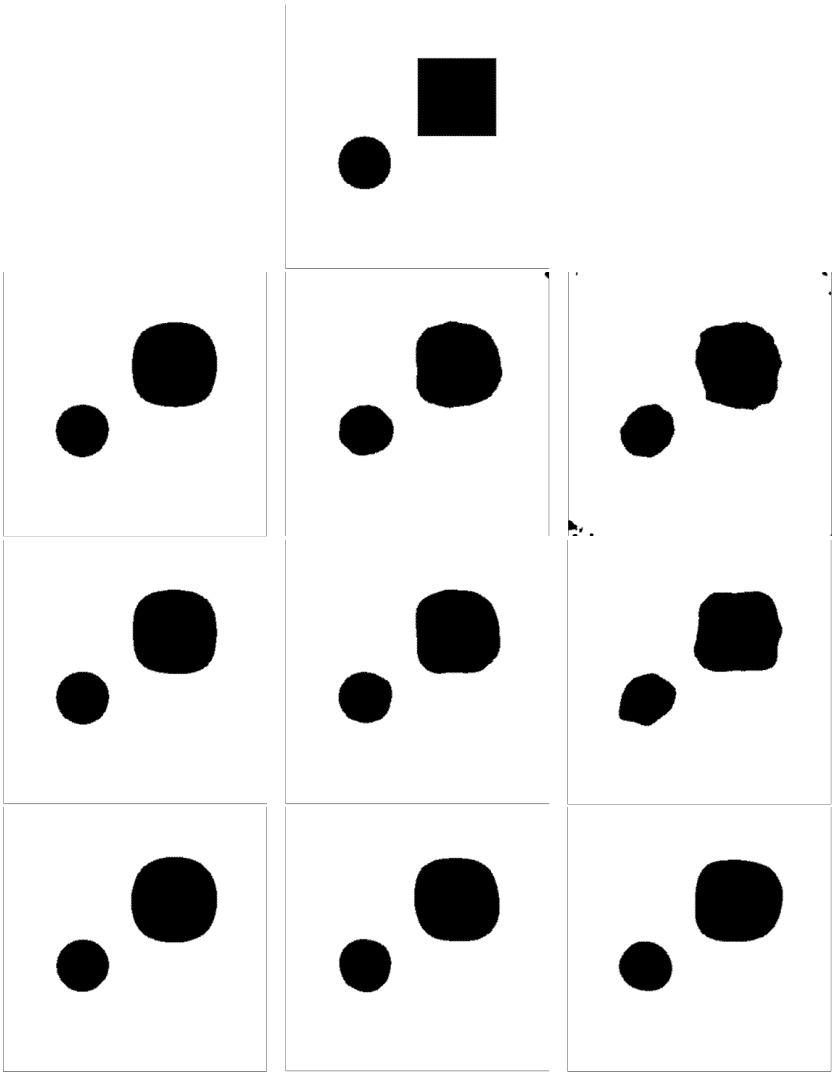

FIGURE 2. Source identification. Top: true sources. Then reconstructed sources for $\alpha = 10^{-8}$ (first line), $\alpha = 10^{-7}$ (second line) and $\alpha = 10^{-6}$ (third line), with $R = 0$ (first column), $R = 33\%$ (second column) and $R = 58\%$ (third column).

For $\varepsilon > 0$ fixed we solve the approximate problem

$$\inf_{u \in \tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u) \right\}. \tag{6.5}$$

Using (2.3) and (6.3), this can be rewritten as

$$\inf_{(u,v,\tau) \in \tilde{\mathcal{E}} \times H^1(\Omega) \times \Sigma} \left\{ \int_\Omega \gamma_u^{-1} |\tau|^2 dx + \ell \int_\Omega u dx + \alpha \left[\frac{\varepsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \left(\|v\|^2_{L^2(\Omega)} + \langle u, 1 - 2v \rangle \right) \right] \right\}. \tag{6.6}$$

Proposition 6.2. *The infima* (6.5) *and* (6.6) *are attained.*

Proof. Since the infima (2.3) and (6.3) are both attained, it suffices to consider (6.5). Let therefore (u_n) be a minimizing sequence for (6.5), whose corresponding solutions of (6.2) are denoted by (y_n). We extract a subsequence, still denoted (u_n), such that $u_n \rightharpoonup u \in \tilde{\mathcal{E}}$ weakly-$*$ in $L^\infty(\Omega)$. By the so-called compactness property of the G-convergence (see, e.g., [1] Theorem 1.2.16), we can extract a further subsequence such that the matrix-valued conductivity $\gamma_{u_n} I$, where I is the identity matrix of order N, G-converges to some A, where, at each $x \in \Omega$, $A(x)$ is a symmetric positive definite $N \times N$

matrix. This means that $y_n \rightharpoonup y$ weakly in $H_0^1(\Omega)$, where y solves

$$\begin{cases} -\operatorname{div}(A\nabla y) = f & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

By virtue of [1] Theorem 3.2.6, we have at each point $x \in \Omega$ $A\nabla y = \gamma_u \nabla y$. Therefore, by uniqueness, y is the state associated to u. By (6.1), $\tilde{J}(u_n) \to \tilde{J}(u)$ while we know by Lemma 2.2 that $\tilde{F}_\varepsilon(u_n) \to \tilde{F}_\varepsilon(u)$, which completes the proof. □

6.2. **Description of the algorithm.** In the spirit of [3], we use an alternating minimization algorithm, by performing successively a full minimization of (6.6) with respect to each of the variables u, v, τ. The minimization with respect to τ is equivalent to solving (6.2) and setting $\tau = \gamma_u \nabla y$. The minimization with respect to v is done by solving (2.4). Let us focus on the minimization with respect to u. We have to solve

$$\inf_{u \in \tilde{\mathcal{E}}} \left\{ \int_\Omega \Phi_{\varepsilon,v,\tau}(u(x))dx \right\}, \qquad \text{with } \Phi_{\varepsilon,v,\tau}(u) = \gamma_u^{-1}|\tau|^2 + \ell u + \frac{\alpha}{2\varepsilon}u(1-2v).$$

This means that, at every point $x \in \Omega$, we have to minimize the function $s \in [0,1] \mapsto \Phi_{\varepsilon,v,\tau}(s)$. From

$$\Phi_{\varepsilon,v,\tau}(s) = \frac{|\tau|^2}{\gamma_0 + (\gamma_1 - \gamma_0)s} + \left[\ell + \frac{\alpha}{2\varepsilon}(1-2v)\right] s$$

we readily find the minimizer

$$u = \begin{cases} 1 & \text{if } \ell + \dfrac{\alpha}{2\varepsilon}(1-2v) \leq 0, \\ P_{[0,1]}\left(\sqrt{\dfrac{|\tau|^2}{(\gamma_1-\gamma_0)\left(\ell + \frac{\alpha}{2\varepsilon}(1-2v)\right)}} - \dfrac{\gamma_0}{\gamma_1 - \gamma_0} \right) & \text{if } \ell + \dfrac{\alpha}{2\varepsilon}(1-2v) > 0. \end{cases}$$

where we recall that $P_{[0,1]}$ is the projection operator defined by (3.2).

6.3. **Numerical examples.** Our example is a conductor with one inlet and two outlets, see Figure 3. The domain Ω is the square $]0,1.5[\times]0,1.5[$. The conductivities of the two phases are $\gamma_0 = 10^{-3}$ and $\gamma_1 = 1$. The Lagrange multiplier is $\ell = 2$. We initialize ε to 1 and divide it by two each time a (local) minimizer of (6.6) has been found. The procedure is stopped when ε becomes less that $h/10$, with h the mesh stepsize. We use a mesh with 65161 nodes. The results of computations performed with different values of α are shown on Figure 4.

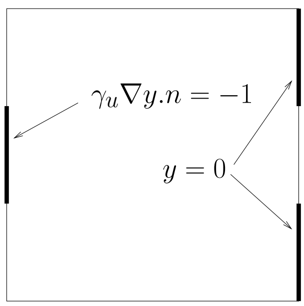

Figure 3. Boundary conditions for the V-shaped conductor. An homogeneous Neumann condition is prescribed on the non-specified boundaries.

7. Third application: compliance minimization in linear elasticity

7.1. **Problem formulation.** We assume now that Ω is occupied by a linear elastic material subject to a volume force $f \in L^2(\Omega)^N$. We denote by $A(x)$ the Hooke tensor at point x. We assume for simplicity, bu without loss of generality, that the medium is clamped on $\partial\Omega$. The compliance can be defined either by

$$C(A) = \int_\Omega f.y dx,$$

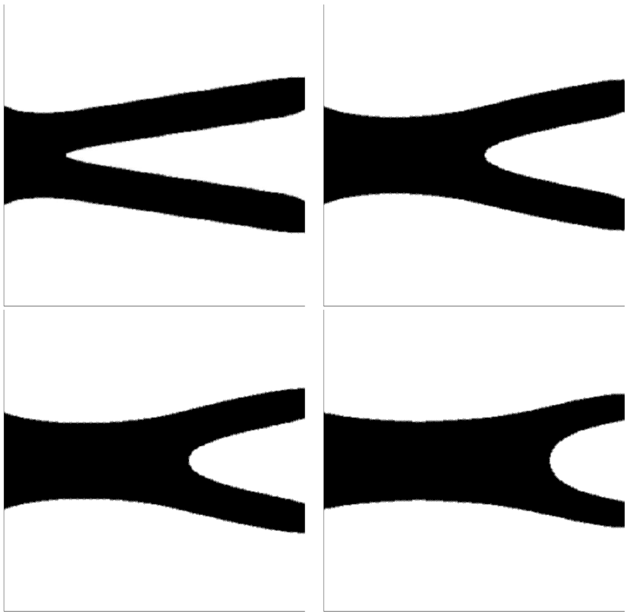

FIGURE 4. Optimized V-shaped conductor for $\alpha = 0.1, 0.5, 1, 3$, respectively.

where y solves

$$\begin{cases} -\operatorname{div}(A\nabla^s y) = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

with ∇^s the symmetrized gradient, or with the help of the complementary energy [2],

$$C(A) = \inf_{\sigma\in\Sigma} \left\{ \int_\Omega A^{-1}\sigma : \sigma dx \right\}, \tag{7.2}$$

with

$$\Sigma = \{\sigma \in L^2(\Omega)^{N\times N}, -\operatorname{div}\sigma = f \text{ in } \Omega\}.$$

Given $\ell, \alpha > 0$, we want to solve

$$\inf_{u\in\mathcal{E}} \{J(u) + 4\alpha|Du|(\Omega)\}, \tag{7.3}$$

with

$$J(u) = C(A(u)) + \ell\int_\Omega u dx, \qquad A(u)(x) = \begin{cases} A_0 & \text{if } u(x) = 0, \\ A_1 & \text{if } u(x) = 1. \end{cases}$$

Here, A_0, A_1 are given Hooke tensors. Typically, A_1 corresponds to a physical material, while A_0 represents a weak phase of small Young modulus meant to mimick void. The problem can be reformulated as

$$\inf_{u\in\tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha\tilde{F}(u) \right\},$$

where

$$\tilde{J}(u) = \inf_{A\in G_u} C(A) + \ell\int_\Omega u dx, \tag{7.4}$$

the convention $A \in G_u \iff A(x) \in G_{u(x)}$ for almost every $x \in \Omega$ is used, and, for each $x \in \Omega$, $G_{u(x)}$ is a set of fourth order tensors such that

$$G_{u(x)} = \begin{cases} \{A_0\} & \text{if } u(x) = 0, \\ \{A_1\} & \text{if } u(x) = 1. \end{cases}$$

Henceforth we choose, for all $x \in \Omega$, $G_{u(x)}$ as the set of all Hooke tensors obtained by homogenization of tensors A_0 and A_1 in proportion $1 - u(x)$ and $u(x)$, respectively (see e.g. [1] for details on homogenization).

Proposition 7.1. *The functional $\tilde{J}$ is continuous on $\tilde{\mathcal{E}}$ strongly in $L^1(\Omega)$.*

Proof. Suppose that $(u_n) \in \tilde{\mathcal{E}}$ converges to $u \in \tilde{\mathcal{E}}$ strongly in $L^1(\Omega)$. Thus $u_n \to u$ almost everywhere for a non-relabeled subsequence. Thanks to the density of $\mathcal{E}$ in $\tilde{\mathcal{E}}$ for the weak-$*$ topology of $L^\infty(\Omega)$, we may assume that $(u_n) \in \mathcal{E}$. By compactness of the G-convergence and stability of G_{u_n} with respect to this convergence (see [1] Lemma 2.1.5), there exists $A_n^* \in G_{u_n}$ such that

$$C(A_n^*) = \inf_{A \in G_{u_n}} C(A).$$

Using again the compactness of the G-convergence, there exists a subsequence such that A_n^* G-converges to some A^*, thus $C(A_n^*) \to C(A^*)$. By [1] Lemma 2.1.7, there exists $c, \delta > 0$ such that

$$d(G_{u_n(x)}, G_{u(x)}) \leq c|u_n(x) - u(x)|^\delta \tag{7.5}$$

for every $x \in \Omega$, where d denotes the Hausdorff distance between sets. Hence there exists $A_n^\sharp \in G_u$ such that $|A_n^* - A_n^\sharp| \leq c|u_n - u|^\delta$ almost everywhere. By the dominated convergence theorem we get $\|A_n^* - A_n^\sharp\|_{L^1(\Omega)} \to 0$. Once more by compactness of the G-convergence, $A_n^\sharp$ G-converges to some $A^\sharp \in G_u$, up to a subsequence. It follows from [1] Proposition 1.3.44 that $A^* = A^\sharp \in G_u$.

Let now $A \in G_u$ be arbitrary, and denote by $A_n(x)$ the projection of $A(x)$ onto $G_{u_n(x)}$. Using again (7.5), we get $A_n(x) \to A(x)$ almost everywhere, therefore, by [1] Lemma 1.2.22, $C(A_n) \to C(A)$. By definition we have $C(A_n) \geq C(A_n^*)$ for all n. Passing to the limit yields $C(A) \geq C(A^*)$. This means that

$$C(A^*) = \inf_{A \in G_u} C(A).$$

Eventually we have obtained

$$\tilde{J}(u_n) = C(A_n^*) + \ell \int_\Omega u_n dx \to C(A^*) + \ell \int_\Omega u dx = \tilde{J}(u).$$

□

For $\varepsilon > 0$ fixed we solve the approximate problem

$$\inf_{u \in \tilde{\mathcal{E}}} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u) \right\}. \tag{7.6}$$

Using (2.3), (7.4) and (7.2), this can be rewritten as

$$\inf_{\substack{u \in \tilde{\mathcal{E}}, A \in G_u, \\ (v,\sigma) \in H^1(\Omega) \times \Sigma}} \left\{ \int_\Omega A^{-1}\sigma : \sigma dx + \ell \int_\Omega u dx + \alpha \left[\frac{\varepsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \left(\|v\|^2_{L^2(\Omega)} + \langle u, 1 - 2v \rangle \right) \right] \right\}. \tag{7.7}$$

Proposition 7.2. *The infima* (7.6) *and* (7.7) *are attained.*

Proof. First, we remark that both problems (7.6) and (7.7) amount to solving

$$\inf_{u \in \tilde{\mathcal{E}}, A \in G_u} \left\{ E_\varepsilon(u, A) := C(A) + \ell \int_\Omega u dx + \alpha \tilde{F}_\varepsilon(u) \right\}.$$

Let (u_n, A_n) be a minimizing sequence. Thanks to the density of $\mathcal{E}$ in $\tilde{\mathcal{E}}$ for the weak-$*$ topology of $L^\infty(\Omega)$ and the continuity of $E_\varepsilon(., A)$ for the same topology (see Lemma 2.2), we may assume that $(u_n) \in \mathcal{E}$. We extract a subsequence, still denoted (u_n), such that $u_n \rightharpoonup u \in \tilde{\mathcal{E}}$ weakly-$*$ in $L^\infty(\Omega)$. Further, by compactness of the G convergence, we can extract a subsequence such that (A_n) G-converges to some tensor field A. By construction, we have $A \in G_u$. By definition of the G-convergence, the sequence of the states (y_n), solutions of (7.1) with Hooke's tensor (A_n), converges weakly in $H_0^1(\Omega)$ to the state y associated to A. This implies that $C(A_n) \to C(A)$, and subsequently, using again Lemma 2.2, that $E_\varepsilon(u_n, A_n) \to E_\varepsilon(u, A)$. □

7.2. **Description of the algorithm.** We use again an alternating minimization algorithm, by performing successively a full minimization with respect to each of the variables $(u, A), v, \sigma$. The minimization with respect to σ is equivalent to solving the linear elasticity problem (7.1) and setting $\sigma = A\nabla^s y$. The minimization with respect to v is again done by solving (2.4). The minimization with respect to A for a given u reduces to the standard problem

$$\inf_{A\in G_u}\left\{\int_\Omega A^{-1}\sigma : \sigma dx\right\} =: f(u,\sigma).$$

When A_1 and A_0 are isotropic and $A_0 \to 0$, the minimization is achieved by using well-known lamination formulas, see [1]. We have

$$f(u,\sigma) = A_1^{-1}\sigma : \sigma + \frac{1-u}{u} f^*(\sigma),$$

with, in dimension $N = 2$,

$$f^*(\sigma) = \frac{2\mu+\lambda}{4\mu(\mu+\lambda)}(|\sigma_1| + |\sigma_2|)^2.$$

Above, λ, μ are the Lamé coefficients of the phase A_1, and σ_1, σ_2 are the principal stresses. Let us finally focus on the minimization with respect to u. We have to solve

$$\inf_{u\in\tilde{\mathcal{E}}}\left\{\int_\Omega \Phi_{\varepsilon,v,\sigma}(u(x))dx\right\}, \qquad \text{with } \Phi_{\varepsilon,v,\sigma}(u) = f(u,\sigma) + \ell u + \frac{\alpha}{2\varepsilon}u(1-2v).$$

This means that, at every point $x \in \Omega$, we have to minimize the function $s \in [0,1] \mapsto \Phi_{\varepsilon,v,\sigma}(s)$. From

$$\Phi_{\varepsilon,v,\sigma}(s) = A_1^{-1}\sigma : \sigma + \frac{1-s}{s} f^*(\sigma) + \left[\ell + \frac{\alpha}{2\varepsilon}(1-2v)\right] s$$

we obtain the minimizer

$$u = \begin{cases} 1 & \text{if } \ell + \frac{\alpha}{2\varepsilon}(1-2v) \le 0, \\ \min\left(1, \sqrt{\frac{f^*(\sigma)}{\ell + \frac{\alpha}{2\varepsilon}(1-2v)}}\right) & \text{if } \ell + \frac{\alpha}{2\varepsilon}(1-2v) > 0. \end{cases}$$

7.3. **Numerical examples.** We first consider the classical cantilever problem, where Ω is a rectangle of size 2×1. The left edge is clamped, and a unitary pointwise vertical force is applied at the middle of the right edge. We choose the Lagrange multiplier $\ell = 100$, and use a mesh containing 160601 nodes. Our findings are displayed on Figure 5.

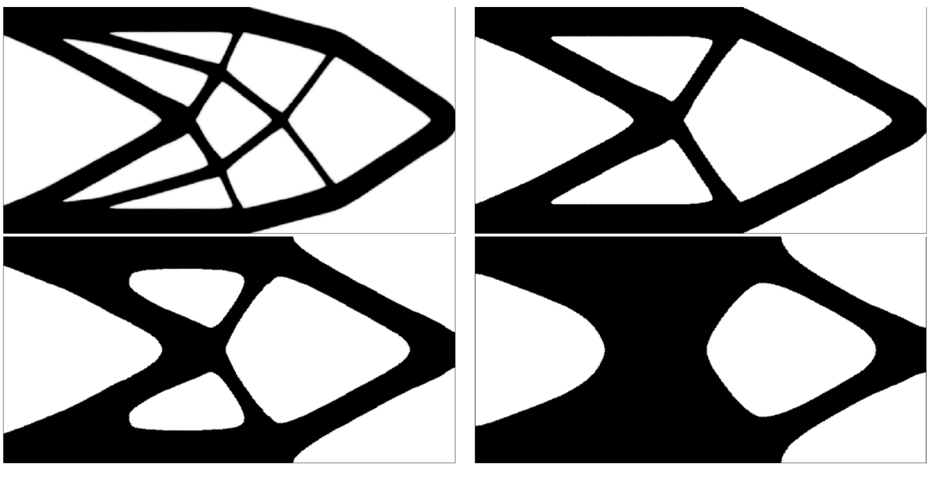

FIGURE 5. Cantilever for $\alpha = 0.1, 2, 20, 50$, respectively.

Next we address the bridge problem, where Ω is a rectangle of size 2×1.2. The structure is clamped on two segments of lengths 0.1 located at the tips of the bottom edge, and submitted to a unitary pointwise vertical force exerted at the middle of the bottom edge. The chosen Lagrange multiplier is $\ell = 30$, and the mesh contains 123393 nodes. Our results are depicted on Figure 6.

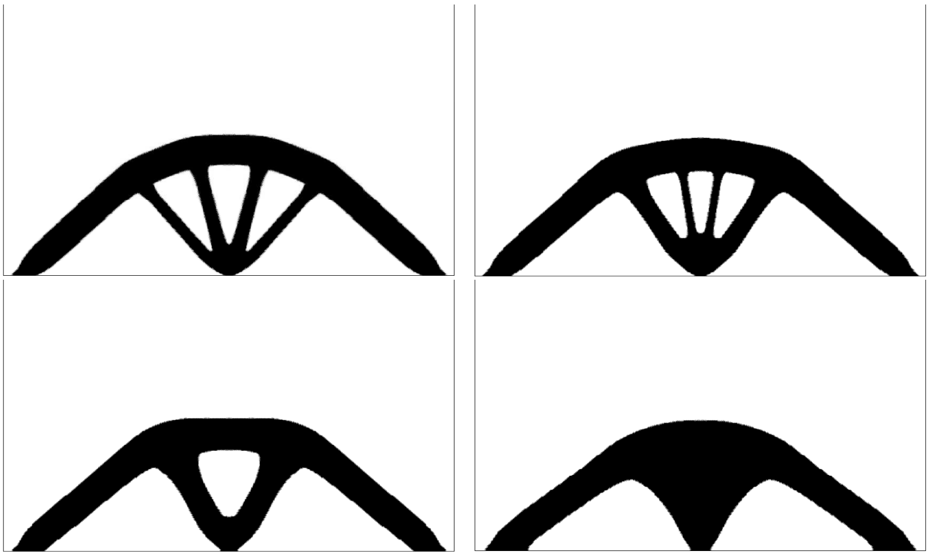

FIGURE 6. Bridge for $\alpha = 0.2, 1, 3, 10$, respectively.

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