# Transverse Momentum Transfer in Atom-Light Scattering

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B.A. van Tiggelen,[1] A. Nussle,[1] and G.L.J.A. Rikken[2]

[1]*Université Grenoble 1/CNRS, LPMMC UMR 5493, B.P. 166, 38042 Grenoble, France*
[2]*LNCMI, UPR 3228 CNRS/INSA/UJF Grenoble 1/UPS, Toulouse & Grenoble, France*
(Dated: January 2, 2013)

We predict a photon Hall effect in the optical cross-section of atomic hydrogen, which is caused by the interference between an electric quadrupole transition and an electric dipole transition from the ground state to $3D_{3/2}$ and $3P_{3/2}$. This induces a magneto-transverse acceleration comparable to a fraction of $g$. In atoms with a two level electric dipole transition, a much smaller transverse force is generated only when the atom is moving.

Light scattering exchanges momentum between matter and radiation, and thus induces a force on the matter. Classical light scattering is well known to be affected by a magnetic field. A specific feature, the photon Hall effect (PHE), was first predicted in multiple light scattering [1], and observed shortly afterwards [2] with typical changes in the magneto-transverse photon flux of order $10^{-5}$ per Tesla of applied magnetic field. A Mie theory for the PHE [4] agreed quantitatively with the experiments. Given the wave number $\mathbf{k}$ of the incident photon flux and the magnetic field $\mathbf{B}$ , the PHE induces an exchange of momentum between scatterer and radiation in the magneto-transverse ("upward") direction along $\mathbf{B} \times \mathbf{k}$. A light flux of $10^4$ W/m$^2$ incident on a micron-sized particle with a relative PHE of $10^{-5}$ per Tesla experiences a transverse force of $10^{-19}$ N/T, roughly equivalent to the Lorentz force on a charge $e$ moving with a velocity of 1 m/s. The magneto-transverse acceleration for a $10\,\mu$m TiO$_2$ particle would be as small as $10^{-11}$ m/s$^2$ in a field of 10 Tesla.

Atoms are strong light scatterers that can achieve elastic optical cross-sections as large as the maximum unitary limit $\lambda^2$ near optical transitions, and with promising applications in mesoscopic physics [3]. When the typical Zeeman splitting $\frac{1}{2}\omega_c$ ($\omega_c = eB/m_e = 17.5$ MHz/Gauss is the cyclotron circular frequency) equals the atomic line width (typically $\gamma \approx 100$ MHz), the optical cross-section is significantly altered by the magnetic field, typically true for a few Gauss. Since atoms have small mass, the magneto-transverse recoil would be much larger than for Mie particles. The magneto-cross-section of an atomic resonance with width $\gamma$ and Zeeman splitting $\omega_c$ can be estimated as $\frac{1}{2}(\omega_c/\gamma)\lambda^2/\pi^2$. If we would assume the Hall cross-section to be of this order the magneto-transverse acceleration would be as large as 4 km/s$^2$ per Gauss when tuned the 1S-2P transition in Strontium exposed to a small flux of 100 W/m$^2$. Unfortunately, no PHE can occur for pure electric-dipole (ED) transitions, since the ED imposes a symmetry between forward and backward scattering, as well as between upward and downward directions in the magneto-cross-section [4]. The PHE induced by the scattering from pairs of atoms in a cold $^{88}$Sr gaz, is estimated to be a few percent [5].

Can the PHE of a single atom exist at all, and how large will the magneto-transverse momentum transfer to the atom be? Two striking differences exist between classical Mie scattering and light-atom scattering. First, given a monochromatic incident laser beam, the atom is usually subject to inelastic transitions to levels that are no longer excited by the same beam, thus preventing a stationary scattering process. Secondly, given the small mass of atoms, one must anticipate significant velocity recoils that change the resonant frequency via the Doppler effect, and finally reduce the light scattering.

The optical cross-section of an atom is expressed by the Kramers-Heisenberg formula [8],

$$\frac{d\sigma}{d\Omega}\left(\omega\mathbf{k}\varepsilon \to \omega_s\mathbf{k}_s\varepsilon_s\right) = \alpha^2 \frac{\omega_s^3}{\omega^3}\,\big|f_{ED}(\omega,\varepsilon,\varepsilon_s)$$
$$+ f_{EQ}(\omega,\varepsilon,\varepsilon_s,\mathbf{k},\mathbf{k}_s) + \cdots\big|^2$$

Here, $\alpha$ is the fine structure constant, $\omega$ and $\omega_s < \omega$ are incident and scattered frequency, $\varepsilon$ and $\varepsilon_s$ are the polarization vectors of incident and scattered radiation; $f(\omega)$ is the complex scattering amplitude associated with transitions in the atom, that can be either elastic or inelastic, and driven by either electric dipole (ED) or quadrupole (EQ). The above expression does not take into account stimulated emission (SE). For this to be true we require
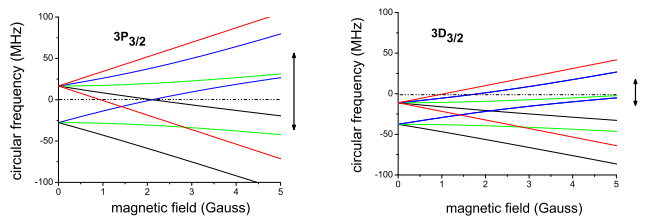


FIG. 1: Hyperfine structure of the $3P_{3/2}$ (left) and $3D_{3/2}$ (right) level of atomic hydrogen, as a function of magnetic field. Equal colors indicate equal values for the hyperfine magnetic quantum number $m$. The height of the vertical bar on the right indicates the line width $\gamma$. The zero in frequency is chosen at the fine structure level of $3P_{3/2}$. The one of $3D_{3/2}$ is 21.07 radMHz lower.

that $W(\omega_s, \mathbf{k}_s, \varepsilon_s) < W_0(\omega_s)$, with $W$ the radiation density per steradian, per bandwidth, per polarization, and $W_0 = \hbar\omega_s^3/(2\pi c_0)^3$ its value for the quantum vacuum.

We will first focus on the simplest atom, atomic hydrogen, whose physics in a magnetic field has been studied in great detail [6, 7]. This atom has the *unique* property that the fine-structure levels $3P_{3/2}$ and $3D_{3/2}$ strongly overlap, despite their hyperfine structure (HF), The anomalous Zeeman effect of the latter is shown in Fig. 1. For not too large magnetic fields all levels are energetically close and can thus interfere constructively. It is instructive to first simply ignore the spin of both electron and proton, and to adopt a simple $1S$ ground state and excited levels $3P$ and $3D$ separated by the (fine-structure energy) of 18 radHz ($= 2\pi s^{-1}$). The electronic transitions $1S \to 3P \to 1S$ and $1S \to 3D \to 1S$ are now both elastic. The $1S - 3D$ transition, however, is ED forbidden and requires an EQ transition. The ED transition between the ground state $1S$ and the $3P$ level reads

$$f_{ED}(\omega) = \frac{\omega^2}{c_0} \sum_{m=0,\pm 1} \frac{\{\langle 1S|\mathbf{r}|3P_m\rangle \cdot \varepsilon_s\}\{\langle 3P_m|\mathbf{r}|1S\rangle \cdot \varepsilon\}}{\omega - \omega_m^P(B) + i\gamma_P} \equiv \frac{\omega^2 r_{3P}^2}{c_0} \left[ P_0(\varepsilon_s \cdot \varepsilon) + [P_1 - P_0](\varepsilon_s \cdot \hat{\mathbf{z}})(\varepsilon \cdot \hat{\mathbf{z}}) + P_2(\omega) i\varepsilon_s \cdot (\varepsilon \times \hat{\mathbf{z}}) \right]$$

(1)

Here $\langle i|\mathbf{r}|j\rangle$ is the ED matrix element between states $i$ and $j$. It depends on orbital momentum but has constant radial part $r_{3P} = 0.517a_0$. The second expression is obtained by inserting the orbital eigenfunctions, and by putting $P_i = 1/(\omega - \omega_m(i, B) + i\gamma_P)$. In our simplified picture, the Zeeman effect behaves normally ($m\omega_c/2$). We choose $\hat{\mathbf{k}} = \hat{\mathbf{x}}$, $\hat{\mathbf{B}} = \hat{\mathbf{z}}$ and let $\hat{\mathbf{B}} \times \hat{\mathbf{k}} = \hat{\mathbf{y}}$ be the Hall direction. For the elastic EQ transition via the $3D$ level we find,

$$f_{EQ}(\omega) = \frac{\omega^2}{c_0} \sum_{m=0,\pm 1,\pm 2} \frac{\{\mathbf{k}_s \cdot \langle 1S|\frac{1}{2}\mathbf{rr}|3D_m\rangle \cdot \varepsilon_s\}\{\mathbf{k} \cdot \langle 3D_m|\frac{1}{2}\mathbf{rr}|1S\rangle \cdot \varepsilon\}}{\omega - \omega_m^D(B) + i\gamma_D}$$

with $\gamma_D = 32$ radMHz the natural line width of the $3D$ level. This expression can again be developed by inserting the orbital eigenfunctions associated with the $3D$ level, at fixed radial matrix element $q_{3D} = 0.867a_0^2$.

The differential cross-section for incident unpolarized, broadband light is obtained from the interference between the two transitions, averaged over incident polarization, and summed over outgoing polarization. The Hall terms are defined by the difference in flux up and down along the vector $\hat{\mathbf{y}}$, and are all characterized by a factor $i(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}})$ that emerges in the cross-product of the two scattering amplitudes. We shall write this as

$$\frac{d\sigma}{d\Omega} = \alpha^2 \frac{1}{\Delta} \int_\Delta d\omega \sum_{\varepsilon, \varepsilon_s} \mathrm{Re}\, f_{ED}^* f_{EQ} = \frac{d\sigma^0}{d\Omega}$$
$$+ \alpha^2 \frac{\omega^6}{\Delta c_0^4} r_{3P}^2 q_{3D}^2 \mathrm{Re}\, i(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}}) \sum_{m,m'} \bar{F}_n(\omega, B) A_{mm'}(\mathbf{k}, \hat{\mathbf{k}}_s, \hat{\mathbf{B}})$$

(2)

The Hall cross-section is a sum over $3 \times 5 = 15$ cross-products among the magnetic sublevels. The factor $\bar{F}_{mm'}$ is the frequency-averaged cross-product of the complex line profiles. If the bandwidth $\Delta$ largely exceeds the line widths then

$$\bar{F}_{mm'}(\omega) = \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_P(m) - i\gamma_P} \frac{1}{\omega - \omega_D(m') + i\gamma_D}$$
$$= \frac{2\pi i}{\omega_P(m) - \omega_D(m') + i(\gamma_D + \gamma_P)}$$

(3)

It can be seen that only 6 functions $A_{mm'}$ actually generate a PHE, with $A_{0,m'=\pm 1} = m'[1 - 2(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{z}})^2]/60$ and $A_{m=\pm 1,m'=\pm 2} = [m + \frac{1}{2}m' + (-2m + \frac{1}{2}m')(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{x}})^2 - \frac{1}{2}m'\frac{1}{2}(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}})^2]/60$. Note that this simplified picture highlights the PHE as a "which-way" event inside the hydrogen atom. It is straightforward to calculate from Eq. (2) the total magneto-transverse recoil force (black line in Fig. 2).

The present picture poses three problems. First we know that excited $3P$ atoms will have a significant probability to decay inelastically to the meta-stable state $2S$ so
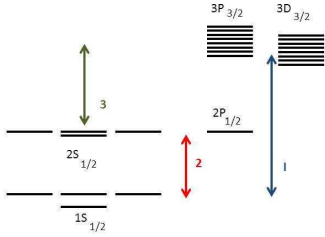
FIG. 2: Three different broadband laser beams are necessary to generate a stationary process with magneto-transverse recoil. Laser 1 induces the transition to $3P_{3/2}$ and $3D_{3/2}$ levels that generate the PHE. Laser 2 reassures that the inelastic decay to $2S_{1/2}$ is pumped back to $3P_{3/2}$. Finally, beam 3, propagating opposite to the beam 1 , compensates the longitudinal photon recoil produced by laser 1.



FIG. 3: Magneto transverse recoil acceleration of the hydrogen atom in the presence of a broadband laser beam with flux $I = 10$ kW/m$^2$ and bandwidth $\Delta = 10$ radGHz (thus resolving the HF ground state). The black curve follows from the model with neglect of (hyper)fine structure, the red curve takes into account the full hyperfine structure of the $1S_{1/2}$ ground state and the $3D_{3/2}$ and $3P_{3/2}$ levels, which results in a smaller PHE recoil though with same sign.

that the PHE process would rapidly come to an end. Secondly, absorbed photons will transfer momentum to the atom that will rapidly become Doppler detuned from the incident laser. Finally, the inclusion of hyperfine structure (HF) considerably complicates the above picture.

The longitudinal photon recoil to the atom of the first laser can be compensated by a *second* laser beam (intensity $I_2$) opposite to the first, and exciting the atom to the $2P$ transition. If $I_2 \approx 3I_1$ the average recoil rate is equal to zero (see Figure 2). The occurrence of inelastic decay to $2S$ must be compensated by a *third* laser beam (intensity $I_3$) that pumps $2S$ atoms back to $3P$. A straightforward analysis shows that detailed balance results in $N_{2S}/N_{1S} = (\omega_{23}/\omega_{13})^3 I_3/I_1$. If the two laser intensities are roughly equal we infer that $N_{2S} \ll N_{1S}$, so that the PHE with $1S$ as initial state is maintained. Note that the $2S - (3P, 3D)$ transitions also induce a PHE which we will not discuss in view of its much smaller transverse recoil.

The inclusion of HF structure is a straightforward process that we shall not discuss in detail. The HF eigenfunctions $\left| 3P(D)_{j=\frac{3}{2}}, f = 1, 2, m = -f, .., f \right\rangle$ can be constructed from the product states of orbital momentum, electron and proton spin with appropriate Clebsch-Gordan coefficients. In the presence of a magnetic field the magnetic sublevels $m = 0, \pm 1$ of the HF levels $f = 1, 2$ mix, thus giving the 8 sublevels whose Zeeman effect is shown in Fig. 1. The $1S_{1/2}$ ground state splits into one singlet and a triplet at 8.9 radGHz higher in energy. The PHE can be determined by collecting all cross products among the transitions from the 4 $1S_{1/2}$ to the 8 $3D_{3/2}$ and $3P_{3/2}$ levels. The result of this cumbersome task is shown as the red line in Figure 3. The HF splitting decreases the recoil because the overlap between the
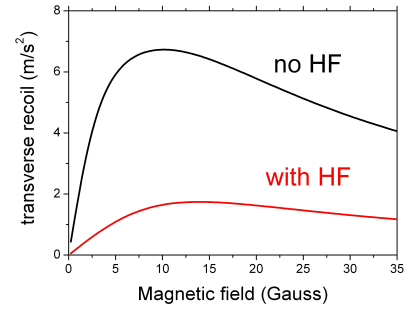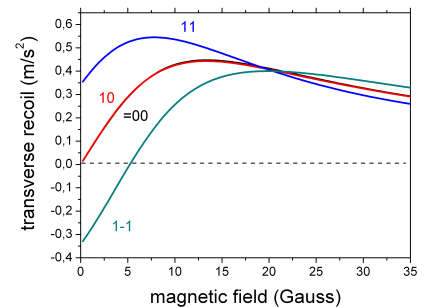


FIG. 4: Magneto transverse recoil of the hydrogen atom, as in previous figure, here specified for transitions from the 4 HF ground states $|f = 0, 1, m = -f, +f\rangle$. The red line in Fig 3 is the sum of the 4. The curves for the two levels $f = 0, 1$, $m = 0$ overlap.

levels decreases, roughly by a factor 4. In this calculation it has been assumed that all HF levels are equally populated, as a result of the presence of the two additional laser beams and the broad band incident beam. In Figure 3 we show the individual contributions of the 4 HF $1S_{1/2}$ states to the total recoil. The spin-polarized states $|f = 1, m = \pm 1\rangle$ have each a nonzero, though opposite PHE at zero field that vanishes for equal level population (a "spin Hall effect", unobservable in the present configuration with inelastic transitions that mix all up). In addition, the PHE recoil from the unpolarized singlet state $|f = 0, m = 0\rangle$ and the unpolarized triplet state $|f = 1, m = 0\rangle$ are equal.

We finally show that a PHE can be induced by an ED transition once the atom is moving. Let $K'$ be the frame that moves with the atom, and $K$ the one in which the

atom moves with velocity $\mathbf{v}$. The ED magneto-transverse scattering cross-section for a two-level atom in frame $K'$ follows from the interference between first and third term in Eq. (1),

$$\frac{d\sigma'}{d\Omega'}\left(\omega'\mathbf{k}' \to \omega'_s\mathbf{k}'_s\right) = \frac{1}{9}\omega'^3_s\omega'\frac{\alpha^2 r^4_{01}}{c^2_0}\mathrm{Im}(P_0 P^*_2)$$
$$\times [\hat{\mathbf{k}}'_s \cdot \hat{\mathbf{k}}'][\hat{\mathbf{k}}'_s \cdot (\hat{\mathbf{B}} \times \hat{\mathbf{k}}')] \quad (4)$$

This cross-section exhibits no net PHE since forward and backward contribution cancel the flux along $\mathbf{k} \times \mathbf{B}$. However, this cancelation is perturbed by the Doppler effect. The cross-section $d\sigma$ relates an incident flux $\rho(\omega, \mathbf{k})/c_0$ to an outgoing current $\rho_s(\omega_s, \mathbf{k}_s)d\Omega r^2/c_0$ at a distance $r$ in the far field. Since $r$ is unaffected by a Lorentz transformation to order $v/c_0$, and since the radiation density $\rho$ transforms as $\rho' = (\omega'^3/\omega^3)\rho$ [9], the cross-section in frame $K$ is

$$\frac{d\sigma}{d\Omega}\left(\omega\mathbf{k} \to \omega_s\mathbf{k}_s\right) \approx \left(\frac{\omega'\omega_s}{\omega\omega'_s}\right)^3 \frac{d\sigma'}{d\Omega'}\left(\omega'\mathbf{k}' \to \omega'_s\mathbf{k}'_s\right)$$

We insert $\hat{\mathbf{k}}'_{(s)} \approx \hat{\mathbf{k}}_{(s)}(1 + \hat{\mathbf{k}}_{(s)} \cdot \mathbf{v}/c_0) - \mathbf{v}/c_0$ and $\omega'_{(s)} \approx \omega_{(s)}(1 - \mathbf{v} \cdot \hat{\mathbf{k}}_{(s)}/c_0)$ and assume for simplicity that the atom moves either parallel or opposite to the incident wave vector $\mathbf{k}$. Note that transformation factors involving $\omega'_s$ in the formula above cancel and do not contribute to the PHE. The only contribution is the Lorentz transform of the angle-dependent factor in Eq. (4) that generates $-[1 - 2(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{k}})^2](\mathbf{k}_s \cdot (\mathbf{v} \times \hat{\mathbf{B}}))/c_0$. The momentum transfer to the atom exhibits a magneto-transverse force,

$$\mathbf{F} = -\frac{1}{\hbar\omega}I(\mathbf{k})\int d\Omega_s \,\hbar\mathbf{k}_s \frac{d\sigma}{d\Omega}\left(\omega\mathbf{k} \to \omega_s\mathbf{k}_s\right)$$
$$= I(\mathbf{k})\frac{4\pi}{45}\frac{\alpha^2\omega^4 r^4_{01}}{c^3_0}\mathrm{Im}(P_0 P^*_2)(\omega \mp \frac{v\omega}{c_0}, B)\frac{\mathbf{v}}{c_0} \times \hat{\mathbf{B}}$$
$$(5)$$

with $I$ the incident flux in W/m$^2$. Apart from the Doppler shift $\mp v/c_0$ of the incident frequency, this force is equal for motion parallel $(-)$ or opposite to the incident beam. This is useful since the normal scattering of the light beam also induces a longitudinal acceleration. The presence of two opposite beams with properly chosen intensities allows to induce a magneto-transverse recoil and *at the same time* select a specific constant velocity. For typical parameters in a total flux of 100 W/m$^2$ (small enough for stimulated emission to be small) applied to $^{88}$Sr we find the stable velocity $v = 4.4$ m/s and a magneto-transverse acceleration $a = 43$ $\mu$m /s$^2$ (Fig. 5). This is much smaller than what we found for hydrogen, since $^{88}$Sr is heavier and $v/c_0 \ll \alpha^2$, but is still measurable.

In conclusion, we have quantified the magneto-transverse scattering of light from unpolarized atomic
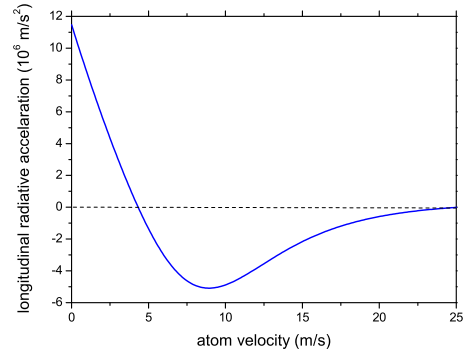


FIG. 5: Longitudinal acceleration of $^{88}$Sr by two opposite beams with total flux ($I = 100$ W/m$^2$), as a function of atom velocity. The atomic transition is taken red-tuned by one line width ($\gamma = 101$ MHz) in a transverse magnetic field of $B = 1$ Gauss. The beam parallel to the speed is 50% more intense so that a stable velocity $v = 4.4$ m/s is selected (acceleration $a = 0$).

hydrogen. It is caused by the interference of en electric dipole transition and a electric quadrupole transition. A transverse recoil of several m/s$^2$ is predicted, i.e. a fraction of $g$. The generalization to other atoms seems difficult since one needs overlapping transitions with different (orbital) symmetry. these are often excluded by (hyper)fine splitting. Maybe the application of high magnetic fields may induce level-crossing of remote transitions, this causing a PHE. An electric dipole *alone* is found to induce a magneto-transverse scattering only when the atom is moving, though with much smaller accelerations of order $\mu$m/s$^2$ It could be interesting to study the atomic spin-Hall effect in the spin-polarized $S$-state of atomic hydrogen [7], and to make a link with previous predictions [10]

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