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Entanglement and Squeezing with Identical Particles: Ultracold Atoms Quantum Metrology

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Abstract. In quantum metrological applications based on ultracold atom systems, entangled initial states are thought necessary to achieve sub-shot-noise accuracies. This conclusion, although strictly true for systems of distinguishable particles, does no longer hold for systems of identical particles. Indeed, while quantum non-locality is necessary, it can be encoded into the interferometric apparatus and not into the initial states. In particular, no preliminary spin-squeezing is necessary to reach quantum performances in metrological applications of ultracold atom physics.

Ultracold gases trapped in optical lattices provide among the most successful physical setups for studying quantum many-body effects in condensed matter physics [1] and quantum information [2, 3]; more recently, these systems have been proposed as experimental realizations of quantum metrological protocols [4, 5]

In all these applications, the property of entanglement [6] plays a crucial role; no longer an epistemological curiosity, entanglement is indeed becoming more and more an experimentally accessible resource, with applications in many areas of many-body physics. In particular, entangled N -qubit states have been proposed as means to beat the so-called shot-noise limit accuracy in parameter estimation [7, 8].

A first step in this direction using many-body systems has been recently realized: entangled states in systems of ultra-cold atoms have been generated through spin-squeezing techniques [9, 10], with the aim of using them as input states in interferometric apparatuses, specifically constructed for quantum enhanced metrological applications. In such devices, the initial N -qubit states are rotated by means of collective spin operators. In the case of distinguishable qubits, all these rotations are local; therefore, preliminary spin squeezing of separable states is necessary in order to introduce (non-local) quantum correlations, thus opening the possibility of sub-shot-noise accuracies in parameter measurements.

For systems of distinguishable particles, the notion of quantum entanglement is well-established [6]: a generic state of N qubits is defined to be entangled if it is not fully separable, namely if it cannot be written as

$$\rho_{sep} = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \cdots \otimes \rho_k^{(N)}, \quad (1)$$

where the $p_k \geq 0$ are weights, $\sum_k p_k = 1$, while each $\rho_k^{(j)}$ is a density matrix for the j -th qubit acting on the corresponding Hilbert space \mathbb{H}_j .

However, in the case of trapped ultra-cold atoms, the qubits involved are identical, a fact that has often not been fully appreciated in the recent literature dealing on quantum metrology. Indeed, the indistinguishability of the particles asks for a rethinking of definitions and concepts originally introduced in the case of distinguishable qubits. More specifically, the notion of separability based on Eq.(1) is strictly associated with the tensor product structure of the Hilbert space, $\mathbb{H}_N = \bigotimes_{j=1}^N \mathbb{H}_j$, which is natural in the case of N distinguishable particles. On the other hand, pure bosonic states must be symmetric under exchange of particles and mixed states must be convex combinations of projections onto such states. This fact demands a different approach to the notions of non-locality and entanglement based not on a structure related to the particle aspect of first quantization, as in (1), rather on the behavior of correlation functions of commuting observables more generally related to the mode description typical of second quantization.‡

In this spirit, a generalized notion of entanglement has been introduced in [14] which reduces to the standard one for distinguishable qubits. Although the notion of entanglement in many-body systems has already been addressed and discussed in the literature (for instance, see [2], [11]–[13], [15]–[27] and references therein), only limited results actually apply to the case of identical particles; the new definition of separability in [14] is instead very general, suitable to all physical situations and notably to systems of trapped ultracold atoms.

In the following, we show that, according to such new definition, neither entangled states, nor spin-squeezing are necessary in order to achieve sub-shot noise accuracies in quantum metrological applications based on systems of identical particles. Though necessary, quantum non-locality comes not from the states, rather from the rotations that are implemented in the interferometric apparatus. Indeed, while all these rotations are local for distinguishable qubits, in the case of identical qubits some of those operations, implemented through the action of collective spin operators, happen to be non-local and might allow sub-shot noise accuracies without the need of a preliminary squeezing.

In general, density matrices as in (1) are not allowed bosonic identical particle states, not even when the single-particle states are all equal in each term of the sum, *i.e.* $\rho_k^{(i)} = \rho_k$ for all i . Indeed, consider two qubits and fix an orthonormal basis $\mathbb{C}^4 \ni |ij\rangle$, $i, j = \uparrow, \downarrow$. If they are bosons, their states should have vanishing components on the anti-symmetric state $|\Psi_-\rangle = (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)/\sqrt{2}$. Since $\langle\Psi_-|\rho \otimes \rho|\Psi_-\rangle = \text{Det}(\rho)$, the density matrix $\rho \otimes \rho$ cannot correspond to a state of two bosonic qubits unless ρ is a projection (*i.e.* $\text{Det}(\rho) = 0$).

Therefore, the tensor product structure which is natural for distinguishable particles is not appropriate for discussing the entanglement properties of systems of identical

‡ The necessity of this change of perspective in dealing with systems of identical particles, with emphasis on observables rather than states, has already been remarked in [11]–[13].

particles. These should rather be investigated within the second quantization formalism whereby one introduces creation a_i^\dagger and annihilation operators a_i of single-particle orthonormal basis states $|i\rangle$, obeying the canonical commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$.

Entanglement in such a context should correspond to whether, given a state ω of the system, there are non-classical correlations among commuting observables [14] as, for instance, those localized in non-overlapping spatial regions V_1 and V_2 [12]. Specifically, this can be inspected by considering the structure of two-point functions of the form $\omega(P_{V_1}P_{V_2})$, where P_{V_α} , $\alpha = 1, 2$, are polynomials in the creation and annihilation operators $a^\dagger(\psi_\alpha)$, $a(\psi_\alpha)$, of states $|\psi_\alpha\rangle$ spatially localized in the region V_α . The relevant fact is that the spatially local algebras generated by the polynomials P_{V_α} commute: $[a(\psi_1), a^\dagger(\psi_2)] = \langle\psi_1|\psi_2\rangle = 0$.

More in general, one can argue about the entanglement between observables belonging to two generic commuting sub-algebras $(\mathcal{A}, \mathcal{B})$ of the entire algebra generated by creation and annihilation operators, which we shall refer to as *algebraic bipartition* [14]. We shall call an observable (operator) \mathcal{O} “ $(\mathcal{A}, \mathcal{B})$ -local” if it is of the form $\mathcal{O} = AB$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and a state ω “ $(\mathcal{A}, \mathcal{B})$ -separable” if the expectation values $\omega(AB)$ of local operators AB can be decomposed into a convex linear combination of product of expectations:

$$\omega(AB) = \sum_i \lambda_i \omega_i^a(A) \omega_i^b(B), \quad \lambda_i > 0, \quad \sum_i \lambda_i = 1, \quad (2)$$

in terms of other states ω_i^a, ω_i^b ; otherwise, ω is said $(\mathcal{A}, \mathcal{B})$ -entangled.

Remark 1: Algebraic and standard entanglement. As already mentioned, the previous definition reduces to the standard notion of entanglement for distinguishable particles [14]. This becomes apparent in the case of two qubits, by choosing the algebraic bipartition $\mathcal{A} = \mathcal{B} = M_2$, where M_2 is the algebra of 2×2 matrices over \mathbb{C}^2 , and the expectation value $\omega(AB) = \text{Tr}(\rho A \otimes B)$ defined in the usual way through the trace with the two-qubit density matrix ρ . \square

Let us stress that in the case of identical particles, there is no a priori given bi-partition so that questions about entanglement and separability, non-locality and locality are meaningful only with reference to a specific class of (commuting) observables; the general definition given above precisely stems from this fundamental observation.

A concrete application of the previous considerations is the second quantization of a single-particle with Hilbert space \mathbb{C}^2 which, in the Bose-Hubbard approximation, effectively describes N ultracold atoms confined by a double-well potential. Then, the state $|\downarrow\rangle$ describes one atom located within the left well and the state $|\uparrow\rangle$ an atom localized within the right one. Let $|0\rangle$ be the vacuum state and a^\dagger, b^\dagger the creation operators of a particle in the states $|\downarrow\rangle$ and $|\uparrow\rangle$, that is $a^\dagger|0\rangle = |\downarrow\rangle$, $b^\dagger|0\rangle = |\uparrow\rangle$.

When the total number N is conserved, the symmetric Fock space of this two-mode system is generated by $N + 1$ orthonormal eigenvectors of the number operator

$\mathcal{N} = a^\dagger a + b^\dagger b$:

$$|k\rangle = \frac{(a^\dagger)^k (b^\dagger)^{N-k}}{\sqrt{k!(N-k)!}} |0\rangle, \quad 0 \leq k \leq N. \quad (3)$$

Because of the orthogonality of the spatial modes, by considering the norm-closures of all polynomials P_a in a, a^\dagger , respectively P_b in b, b^\dagger , one obtains two commuting subalgebras \mathcal{A} and \mathcal{B} .

According to the definition in 2), the states $|k\rangle$ are $(\mathcal{A}, \mathcal{B})$ -separable; indeed, they are created by the $(\mathcal{A}, \mathcal{B})$ -local operators $(a^\dagger)^k (b^\dagger)^{N-k}$. More in general, one can show that $(\mathcal{A}, \mathcal{B})$ -separable density matrices must be convex combinations of projections $|k\rangle\langle k|$ [14]:

$$\rho = \sum_{k=0}^N p_k |k\rangle\langle k|, \quad p_k > 0, \quad \sum_{k=0}^N p_k = 1. \quad (4)$$

Consider instead the following operators,

$$J_x = \frac{1}{2}(a^\dagger b + ab^\dagger), \quad J_y = \frac{1}{2i}(a^\dagger b - ab^\dagger), \quad J_z = \frac{1}{2}(a^\dagger a - b^\dagger b), \quad (5)$$

that satisfy the $SU(2)$ algebraic relations $[J_x, J_y] = i J_z$ and their cyclic permutations. They are all non-local with respect to the algebraic bipartition $(\mathcal{A}, \mathcal{B})$ and such are the exponentials $e^{i\theta J_x}$ and $e^{i\theta J_y}$, while $e^{i\theta J_z} = e^{i\theta a^\dagger a} e^{-i\theta b^\dagger b}$ is $(\mathcal{A}, \mathcal{B})$ -local.

By means of a Bogolubov transformation to other creation and annihilation operators (c^\dagger, c) and (d^\dagger, d) , such that $a = (c+d)/\sqrt{2}$ and $b = (c-d)/\sqrt{2}$, together with their hermitian conjugate, one obtains another bipartition $(\mathcal{C}, \mathcal{D})$ and rewrites

$$J_x = \frac{1}{2}(c^\dagger c - d^\dagger d), \quad J_y = \frac{1}{2i}(d^\dagger c - dc^\dagger), \quad J_z = \frac{1}{2}(c^\dagger d + cd^\dagger).$$

Relatively to $(\mathcal{C}, \mathcal{D})$, it is now $e^{i\theta J_x} = e^{i\theta c^\dagger c} e^{-i\theta d^\dagger d}$ which acts locally.

Remark 2: Mode entanglement vs. particle entanglement. In first quantization, an $(\mathcal{A}, \mathcal{B})$ -separable state for a $N = 2$ bosonic qubits like $a^\dagger b^\dagger |0\rangle$ corresponds to $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$. Such state is surely entangled for distinguishable qubits, while, according to our definition, it is no longer so for identical bosonic qubits; the reason is that its entanglement is only formal as it comes from the necessary symmetrization of the separable state $|\uparrow\downarrow\rangle$ [15]. A Bogolubov transformation as the one above corresponds to a change of basis in the single particle Hilbert space, from the one of spatially localized states, to the one of $c^\dagger |0\rangle = (|\downarrow\rangle + |\uparrow\rangle)/2$, $d^\dagger |0\rangle = (|\downarrow\rangle - |\uparrow\rangle)/2$. Physically speaking, such states are eigenstates of the single particle Hamiltonian in the Bose-Hubbard approximation with a highly penetrable barrier. The change to the *energy bipartition* $(\mathcal{C}, \mathcal{D})$ is non-local with respect to the *spatial bipartition* $(\mathcal{A}, \mathcal{B})$, though it corresponds to a local unitary rotation in first quantization. \square

In the case of a system of N distinguishable qubits, the collective angular momentum operators J_α , $\alpha = x, y, z$ and the corresponding rotations are sums of single

particle spin operators, $J_\alpha^{(i)}$, $\alpha = x, y, z$, $i = 1, \dots, N$. $J_\alpha = \sum_{i=1}^N J_\alpha^{(i)}$. These operators are local with respect to the tensor product structure in (1).

Based on this, the variance $\Delta^2 J_{\vec{n}}$ of the collective spin $J_{\vec{n}} = n_x J_x + n_y J_y + n_z J_z$ along the unit spatial direction $\vec{n} = (n_x, n_y, n_z)$, with respect to separable vector states $|\Psi\rangle = \bigotimes_{i=1}^N |\psi_i\rangle$ results [28, 29]

$$\Delta^2 J_{\vec{n}} = \langle \Psi | J_{\vec{n}}^2 | \Psi \rangle - \langle \Psi | J_{\vec{n}} | \Psi \rangle^2 = \frac{N}{4} - \sum_{j=1}^N \left(\langle \psi_j | J_n^{(j)} | \psi_j \rangle \right)^2 \leq \frac{N}{4}. \quad (6)$$

Therefore, $\Delta^2 J_{\vec{n}}$ is an entanglement witness for pure states, in the sense that if $\Delta^2 J_{\vec{n}} > N/4$ then the pure state $|\Psi\rangle$ cannot be fully separable.

This is no longer the case for N identical bosonic qubits. Indeed, consider the number states $|k\rangle$ in (3); using (5), one gets

$$\langle k | J_{\vec{n}} | k \rangle = \frac{n_z}{2} (2k - N) \quad (7)$$

$$\langle k | J_{\vec{n}}^2 | k \rangle = \frac{N + 2k(N - k)}{4} + n_z^2 \frac{N(N - 1) - 6k(N - k)}{4} \quad (8)$$

$$\Delta^2 J_{\vec{n}} = \frac{1 - n_z^2}{4} (N + 2k(N - k)). \quad (9)$$

Therefore, if $k \neq 0, N$, for all \vec{n} that satisfy

$$n_z^2 < \frac{2k(N - k)}{N + 2k(N - k)} \leq 1,$$

the states $|k\rangle$, though $(\mathcal{A}, \mathcal{B})$ -separable, nevertheless yield $\Delta^2 J_{\vec{n}} > N/4$; therefore, $\Delta^2 J_{\vec{n}}$ is not an entanglement witness for pure states of bosonic qubits.

In greater generality, inequalities for mean values and variances of collective spin operators with respect to any (mixed) separable state of distinguishable qubits as in (1) have been derived in [30]; these are called *spin squeezing inequalities* and read:§

$$\langle J_{\vec{n}_1}^2 \rangle + \langle J_{\vec{n}_2}^2 \rangle + \langle J_{\vec{n}_3}^2 \rangle - \frac{N(N + 2)}{4} \leq 0, \quad (10)$$

$$\Delta^2 J_{\vec{n}_1} + \Delta^2 J_{\vec{n}_2} + \Delta^2 J_{\vec{n}_3} - \frac{N}{2} \geq 0, \quad (11)$$

$$\langle J_{\vec{n}_1}^2 \rangle + \langle J_{\vec{n}_2}^2 \rangle - \frac{N}{2} - (N - 1)\Delta^2 J_{\vec{n}_3} \leq 0, \quad (12)$$

$$(N - 1)(\Delta^2 J_{\vec{n}_1} + \Delta^2 J_{\vec{n}_2}) - \langle J_{\vec{n}_3}^2 \rangle - \frac{N(N - 2)}{4} \geq 0, \quad (13)$$

where $\vec{n}_1, \vec{n}_2, \vec{n}_3$ denotes any triplet of unit vectors corresponding to orthogonal spatial directions and $\langle X \rangle$ the mean value of an operator X .

It is thus interesting to study whether these inequalities are also satisfied by $(\mathcal{A}, \mathcal{B})$ -separable states (4) of N identical qubits. Denote by $\langle k \rangle = \sum_{k=0}^N p_k k$ and $\langle k^2 \rangle = \sum_{k=0}^N p_k k^2$ the first and second moments of the $N + 1$ -valued stochastic variable

§ In [30], these inequalities are derived with respect to the standard triplet $\vec{n}_1 = \hat{x}$, $\vec{n}_2 = \hat{y}$, $\vec{n}_3 = \hat{z}$. The result easily extends to more general triplets.

k with respect to the probability distribution $\pi = \{p_k\}_{k=0}^N$. Using (7) and (9), mean-values and variances of the collective spin operators $J_{\vec{n}}$ with respect to the states in (4) read

$$\langle J_{\vec{n}} \rangle = \frac{n_z}{2} (2\langle k \rangle - N) \quad (14)$$

$$\langle J_{\vec{n}}^2 \rangle = \frac{N(1 + 2\langle k \rangle) - 2\langle k^2 \rangle}{4} + \frac{n_z^2}{4} (N(N-1) - 6N\langle k \rangle + 6\langle k^2 \rangle) \quad (15)$$

$$\Delta^2 J_{\vec{n}} = \frac{N(1 + 2\langle k \rangle) - 2\langle k^2 \rangle}{4} + \frac{n_z^2}{4} (6\langle k^2 \rangle - 2\langle k \rangle(N + 2\langle k \rangle) - N) . \quad (16)$$

From the orthogonality of the triplet $\vec{n}_1, \vec{n}_2, \vec{n}_3$, it follows that $n_{1z}^2 + n_{2z}^2 + n_{3z}^2 = 1$; one can thus check that all the previous inequalities but (12) are satisfied by $(\mathcal{A}, \mathcal{B})$ -separable states. Concerning (12), its left hand side reads

$$\delta = \frac{N}{2} (\Delta^2 k - \langle k \rangle(N - \langle k \rangle)) + \frac{n_{3z}^2}{2} ((N + 2)\langle k \rangle(N - \langle k \rangle) - 3N\Delta_k^2) , \quad (17)$$

where $\Delta^2 k := \langle k^2 \rangle - \langle k \rangle^2$ is the variance of k with respect to $\pi = \{p_k\}_{k=0}^N$. If π is chosen such that $a := \langle k \rangle(N - \langle k \rangle) > N\Delta^2 k$, then δ becomes positive and thus (12) results violated by the corresponding $(\mathcal{A}, \mathcal{B})$ -separable states (4) for all orthogonal triplets with

$$\frac{N(a - \Delta^2 k)}{(N + 2)a - 3N\Delta^2 k} < n_{3z}^2 \leq 1 . \quad (18)$$

Consider the pure states $|\ell\rangle$ in (3) with $\ell \neq 0, N$; in such a case, $p_k = \delta_{k\ell}$, $\langle k \rangle = \ell$ and $\Delta^2 k = 0$, so that (12) is violated for $N/(N + 2) < n_{3z}^2 \leq 1$.

Remark 3: Additional squeezing inequalities failing for identical qubits.

Different spin-squeezing inequalities have also been presented in [31]. Although derived for distinguishable qubits, they have been directly applied to signal entanglement also in the case of systems of identical particles. A simple instance of their failure for $(\mathcal{A}, \mathcal{B})$ -separable states is provided by inequality (10) in [31]: $N\Delta^2 J_{\vec{n}} + \langle J_{\vec{n}} \rangle^2 < N/4$; indeed, it is always satisfied by any Fock state (3) with $0 < k < N$ when \vec{n} is chosen along the z direction, thus erroneously signaling entanglement in such states. \square

The preceding results indicate that spin-squeezing inequalities that are derived for distinguishable qubits can not directly be used as entanglement witnesses in the context of identical qubits. Since the use of spin-squeezed states for metrological purposes have recently become the focus of much theoretical investigations [7, 8, 29, 32], we now discuss the impact of particle indistinguishability on such an issue.

For any orthogonal triplet of space-directions $\vec{n}_1, \vec{n}_2, \vec{n}_3$, the Heisenberg uncertainty relations for the $SU(2)$ operators $J_{\vec{n}}$ read

$$\Delta^2 J_{\vec{n}_1} \Delta^2 J_{\vec{n}_2} \geq \frac{1}{4} \langle J_{\vec{n}_3} \rangle^2 . \quad (19)$$

One speaks of spin-squeezing when one of the variances can be made smaller than $\frac{1}{2} |\langle J_{\vec{n}_3} \rangle|$. The relevance of states satisfying this condition for achieving otherwise unavailable accuracies has been studied in relation to the measure of an angle θ by

interferometric techniques. These are based on a rotation of an input state ρ into the final state

$$\rho_\theta = e^{-i\theta J_{\vec{n}_1}} \rho e^{i\theta J_{\vec{n}_1}} , \quad (20)$$

and upon measuring on ρ_θ the collective spin $J_{\vec{n}_2}$, where $\vec{n}_2 \perp \vec{n}_1$. By choosing the remaining orthogonal unit vector \vec{n}_3 such that $\langle J_{\vec{n}_3} \rangle = \text{Tr}(\rho J_{\vec{n}_3}) \neq 0$, by error propagation, the uncertainty $\delta\theta$ in the determination of θ can be estimated by [8]:

$$\delta^2\theta = \frac{\Delta^2 J_{\vec{n}_1}}{\left(\partial_\theta \langle J_{\vec{n}_2} \rangle_\theta \Big|_{\theta=0}\right)^2} = \frac{\Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_3} \rangle^2} = \frac{\xi_W^2}{N} , \quad (21)$$

in terms of the spin-squeezing parameter

$$\xi_W^2 := \frac{N \Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_3} \rangle^2} . \quad (22)$$

The value $\delta^2\theta = 1/N$ is called shot-noise limit; in the case of distinguishable qubits, it gives the lower bound to the attainable accuracies when the input state ρ is separable. Indeed, in such a case one finds $\xi_W^2 \geq 1$. This result follows from the inequality $\xi_W^2 \geq \xi_S^2$ where the new spin-squeezing parameter

$$\xi_S^2 := \frac{N \Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2} \rangle^2 + \langle J_{\vec{n}_3} \rangle^2} , \quad (23)$$

has been introduced in [32]; by means of the local structure of the collective spin operators J_α , $\alpha = x, y, z$, one can prove that ξ_S^2 is always ≥ 1 for separable states of distinguishable qubits. Therefore, using distinguishable qubits, the shot-noise limit can be beaten, namely accuracies better than $1/N$ can be achieved only if $\xi_S^2 < 1$, that is only by means of entangled states.

Let us instead consider N identical bosonic qubits in the $(\mathcal{A}, \mathcal{B})$ -separable pure states $|k\rangle$ and any triplet of orthogonal spatial directions $\vec{n}_1, \vec{n}_2, \vec{n}_3$, with $\vec{n}_1 \neq \hat{z}$. Using (7)-(9) and $n_{1z}^2 + n_{2z}^2 + n_{3z}^2 = 1$, one computes

$$\begin{aligned} \xi_S^2 &= \frac{N \Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2} \rangle^2 + \langle J_{\vec{n}_3} \rangle^2} = N \frac{1 - n_{1z}^2}{n_{2z}^2 + n_{3z}^2} \frac{N + 2k(N - k)}{(2k - N)^2} \\ &= \frac{N(N + 2k(N - k))}{(2k - N)^2} \geq 1 , \quad 0 \leq k \leq N . \end{aligned} \quad (24)$$

In the case of the $(\mathcal{A}, \mathcal{B})$ -separable density matrices (4), first observe that, thanks to the Cauchy-Schwartz inequality, one has $\sum_{k=0}^N p_k \langle J_{\vec{n}} \rangle_k^2 \geq \langle J_{\vec{n}} \rangle^2$, where $\langle X \rangle_k^2$ denotes the mean-value of X with respect to the number state $|k\rangle$. Then,

$$\begin{aligned} \xi_S^2 &\geq N \frac{\sum_{k=0}^N p_k \Delta_k^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2} \rangle^2 + \langle J_{\vec{n}_3} \rangle^2} = \frac{1 - n_{1z}^2}{n_{2z}^2 + n_{3z}^2} \frac{\sum_{k=0}^N p_k N (N + 2k(N - k))}{\left(\sum_{k=0}^N p_k (N - 2k)\right)^2} \\ &\geq \frac{\sum_{k=0}^N p_k \left(k - \frac{N}{2}\right)^2}{\left(\sum_{k=0}^N p_k \left(k - \frac{N}{2}\right)\right)^2} , \end{aligned}$$

where the last inequality follow from the second line of (24). A further application of the Cauchy-Schwartz inequality to the right hand side of the last inequality yields $\xi_S^2 \geq 1$ for all $(\mathcal{A}, \mathcal{B})$ -separable states when $\vec{n}_1 \neq \hat{z}$.

If one chooses $\vec{n}_1 = \hat{z}$, in the case of $(\mathcal{A}, \mathcal{B})$ -separable mixed states, one finds $\Delta^2 J_z \neq 0$ and $\langle J_{\vec{n}_2} \rangle = \langle J_{\vec{n}_3} \rangle = 0$; therefore, ξ_S^2 (and ξ_W^2) diverges. Instead, for $(\mathcal{A}, \mathcal{B})$ -separable pure states $|k\rangle$ also $\Delta_k J_z = 0$ whence ξ_S^2 (and ξ_W^2) are not defined and must thus be computed by means of suitable limiting procedures.

Let us consider the $(\mathcal{A}, \mathcal{B})$ -entangled vector state $|\Psi\rangle = \sum_{k=0}^N \sqrt{p_k} |k\rangle$, with real coefficients from a probability distribution $\pi = \{p_k\}_{k=0}^N$ over the stochastic variable k . Then, from (7)–(9) it follows that $\langle J_y \rangle = 0$ and $\Delta^2 J_z = \Delta^2 k$; therefore $\xi_S^2 = \xi_W^2$ and

$$\xi_W^2 = \frac{N \Delta^2 J_z}{\langle J_x \rangle^2} = \frac{N \Delta^2 k}{\left(\sum_{k=1}^N \sqrt{k(N-k+1)} \sqrt{p_k p_{k-1}} \right)^2}. \quad (25)$$

In the case of a Gaussian distribution peaked around $k = \ell \neq 0, N$,

$$p_k = \frac{1}{Z} \exp \left(-\frac{(k-\ell)^2}{\sigma^2} \right), \quad Z = \sum_{k=0}^N \exp \left(-\frac{(k-\ell)^2}{\sigma^2} \right), \quad (26)$$

one finds

$$\xi_W^2 = \frac{2N + \mathcal{O} \left(e^{-\frac{1}{2\sigma^2}} \right)}{\left(\sqrt{(\ell+1)(N-\ell)} + \sqrt{\ell(N-\ell+1)} \right)^2}. \quad (27)$$

Thus, for sufficiently small σ , $\xi_W^2 < 1$ for all $\ell \neq 0, N$.

On the other hand, by choosing

$$|\Psi\rangle = \frac{p}{N} \sum_{k=0, k \neq N/2}^N |k\rangle + (1-p) |N/2\rangle, \quad 0 < p < 1, \quad (28)$$

it turns out that $\Delta^2 k = p(N+2)(N+1)/12$ so that (25) yields

$$\xi_W^2 = \frac{N(N+1)}{12(\sqrt{1-p} + q)^2},$$

where

$$q = \sqrt{\frac{p}{N^2(N+2)}} \sum_{k \neq N/2, N/2+1} \sqrt{k(N-k+1)}.$$

Letting $p \rightarrow 0$ one gets $|\psi\rangle \rightarrow |N/2\rangle$ and, if $N > 3$,

$$\xi_W^2 \rightarrow N(N+1)/12 > 1.$$

Remark 4: Drawbacks of spin-squeezing for identical qubits. The above two examples show that, when $\vec{n}_1 = \hat{z}$, the spin-squeezing parameters are not well-defined: different values for ξ_W^2 and ξ_S^2 can be obtained by approaching a state $|k\rangle$ via different limit procedures. This fact is also of practical importance: indeed, in [5] approximations to Fock states $|k\rangle$ have been experimentally constructed that are characterized by

spin-squeezing parameters $\xi_W^2 < 1$. This property arises from the fact that those approximations are $(\mathcal{A}, \mathcal{B})$ -entangled states. The previous discussion shows that some care has to be taken in constructing the perturbations of $|k\rangle$; indeed, not all $(\mathcal{A}, \mathcal{B})$ -entangled states arbitrarily close to it automatically have $\xi_{W,S}^2 \geq 1$. Therefore, the spin-squeezing parameters $\xi_{W,S}^2$ are not always useful for metrological applications, a better quantity is the so-called quantum Fisher information [33], which, as we shall show below, is continuous and well defined for all bosonic qubits. \square

In a measurement of the angle θ based on the state rotation (20), the error $\Delta\theta$ given by a locally unbiased estimator of the angle θ is bounded by (see the Appendix)

$$\Delta^2\theta \geq \frac{1}{F[\rho, J_{\vec{n}_1}]} , \quad (29)$$

where $F[\rho, J_{\vec{n}_1}]$ is the so-called quantum Fisher information associated with the rotation of ρ around \vec{n}_1 .

In order to overcome the shot-noise limit $\Delta^2\theta = 1/N$, the quantum Fisher information must then be strictly larger than N . In the appendix it is also showed that, in full generality,

$$F[\rho, J_{\vec{n}_1}] \Delta^2 J_{\vec{n}_2} \geq \langle J_{\vec{n}_3} \rangle^2 , \quad (30)$$

where $\vec{n}_1, \vec{n}_2, \vec{n}_3$ is a triplet of orthogonal spatial directions. Thus, if $\langle J_{\vec{n}_3} \rangle \neq 0$, one gets the following relation between the quantum Fisher information and the squeezing parameter ξ_W^2 in (22):

$$F^{-1}[\rho, J_{\vec{n}_1}] \leq \frac{\Delta^2 J_{\vec{n}_2}}{\langle J_{\vec{n}_3} \rangle^2} = \frac{\xi_W^2}{N} . \quad (31)$$

In the case of distinguishable qubits, from (29) and (31) it follows that spin-squeezing, namely $\xi_W^2 < 1$, opens the possibility of achieving $\Delta^2\theta < 1/N$, thus of beating the shot-noise limit.

In the case of identical qubits and of $(\mathcal{A}, \mathcal{B})$ -separable states, the right hand side of the above inequality diverges when $\vec{n}_1 = \hat{z}$, since $\langle J_{\vec{n}_3} \rangle = 0$, while it does not make sense if $\vec{n}_2 = \hat{z}$ for then also $\Delta^2 J_{\vec{n}_2} = 0$; whence, as already observed, ξ_W^2 is not defined. However, the quantum Fisher information is always well-defined. Indeed, using (37) in the Appendix and (9), one finds that, if $k \neq 0, N$,

$$F[|k\rangle\langle k|, J_{\vec{n}}] = 4 \Delta^2 J_z = (1 - n_z^2) (N + 2k(N - k)) > N \quad (32)$$

for

$$0 \leq n_z^2 < \frac{2k(N - k)}{N + 2k(N - k)} < 1 . \quad (33)$$

In particular,

$$F[|k\rangle\langle k|, J_y] = N + 2k(N - k) > N ,$$

for $k \neq 0, N$, so that, according to (29), the $(\mathcal{A}, \mathcal{B})$ -separable pure states $|k\rangle$ might overcome the shot-noise limit; the largest value is obtained with the twin Fock state $|N/2\rangle$ yielding $F[\rho, J_y] = \mathcal{O}(N^2)$, thus allowing to approach the so-called Heisenberg limit $\Delta^2\theta = 1/N^2$.

Remark 5: Approximations of Fock states. Notice that even approximating a number state $|\ell\rangle$ by an experimentally more realistic superposition $|\ell, \sigma\rangle$ of states $|k\rangle$ with coefficients as in (26), may beat the shot noise limit. Indeed, one computes

$$F[|\ell, \sigma\rangle\langle\ell, \sigma|, J_y] = 4 \Delta^2 J_y = N + 2\ell(N - \ell) + \mathcal{O}\left(e^{-\frac{1}{\sigma^2}}\right),$$

which can be kept $> N$ by choosing a suitably small σ . \square

Instead, making the quantum Fisher information larger than N is impossible without $(\mathcal{A}, \mathcal{B})$ -non-locality; indeed, if both the state and the rotation operation are $(\mathcal{A}, \mathcal{B})$ -separable, one finds: $F[|\ell\rangle\langle\ell|, J_z] = 4 \Delta^2 J_z = 0$. Even considering the $(\mathcal{A}, \mathcal{B})$ -entangled perturbation $|\ell, \sigma\rangle$ does not help, since

$$F[|\ell, \sigma\rangle\langle\ell, \sigma|, J_z] = 4 \Delta^2 J_z = 8e^{-\frac{1}{\sigma^2}} + \mathcal{O}\left(e^{-\frac{2}{\sigma^2}}\right),$$

making the lower bound to the error in (29) arbitrarily large as $\sigma \rightarrow 0$.

When dealing with $(\mathcal{A}, \mathcal{B})$ -separable mixed states (4), by means of the equation (36) of the Appendix, one computes [14]:

$$F[\rho, J_{\vec{n}}] = (1 - n_z^2) \left(N + 2N\langle k \rangle - \langle k^2 \rangle - 4 \sum_{k=0}^N \frac{p_k p_{k+1}}{p_k + p_{k+1}} (k+1)(N-k) \right). \quad (34)$$

Thus, if (32) holds for a certain $|\ell\rangle$, then, by continuity, $F[\rho, J_{\vec{n}}] > N$ for any probability distribution $\pi = \{p_k\}_{k=0}^N$ suitably peaked around $k = \ell$. Hence, by feeding the interferometer apparatus even with a mixed separable state one might still be able to overcome the shot-noise limit. On the other hand, using the squeezing parameter criteria, we would wrongly discard such states as not useful for metrological applications, since, as shown before, we know that for all such mixed states one has: $\xi_W^2 \geq \xi_S^2 \geq 1$.

Conclusions. In the case of metrological applications based on ultracold atom systems, entangled states are not necessary to reach sub-shot noise accuracies in parameter estimation; indeed, as shown above, one can feed quantum interferometers with separable states of N identical qubits and yet obtain a value greater than N for the associated quantum Fisher information. This phenomenon is surely due to the non-local character of the system; however, the non-locality is not in the input states, which in fact are $(\mathcal{A}, \mathcal{B})$ -separable, rather in the $(\mathcal{A}, \mathcal{B})$ -non-local character of the state-rotation (20) generated by $J_{\vec{n}} \neq J_z$ in the experimental apparatus. Were the qubits distinguishable, neither the state nor the rotation would carry elements of non-locality so that in order to beat the shot-noise limit, the state should be turned into an entangled one before sending it to input port of the interferometer. This is exactly what is done via a spin-squeezing technique in the experiments reported in [9, 10].

Instead, the main point we make here is that in experiments involving identical qubits, no state entanglement, as for instance that obtained through a preliminary squeezing operation, is needed in order to beat the classical shot-noise accuracies. Even separable Fock states as in (3) could be used, the necessary quantum non-locality

being provided by the measuring procedure itself via the mode rotation operated by the interferometric device. In line of principle, a twin-Fock state $|N/2\rangle$ can approach the Heisenberg limit in phase estimation. Furthermore, although the experimental preparation of Fock states may be problematic, we have also shown that sub-shot-noise accuracies can be reached by states sufficiently close to Fock states as those obtained in [5]. Therefore, the results presented here open new perspectives in the actual realization of cold atom based interferometers able to outperform any apparatus so far constructed for measuring ultra-weak effects.

Appendix

A most used definition of quantum Fisher information $F[\rho, J_{\vec{n}}]$ is given by [34]

$$F[\rho, J_{\vec{n}}] := \text{tr}(\rho L^2), \quad (35)$$

where L , known as symmetric logarithmic derivative, $\partial_{\theta}\rho_{\theta}\big|_{\theta=0} = (\rho L + L\rho)/2 = -i[J_{\vec{n}}, \rho]$. Given a spectral decomposition $\rho = \sum_j r_j |r_j\rangle\langle r_j|$, one computes

$$F[\rho, J_{\vec{n}}] = 2 \sum_{i,j:r_i \neq r_j} \frac{(r_i - r_j)^2}{r_i + r_j} \left| \langle r_i | J_{\vec{n}} | r_j \rangle \right|^2. \quad (36)$$

From such an expression one sees that the quantum Fisher information is a continuous function of the state ρ and that, for pure states,

$$F[|\psi\rangle\langle\psi|, J_{\vec{n}}] = 4 \Delta_{\psi}^2 J_{\vec{n}}. \quad (37)$$

An estimator E is locally unbiased if $\partial_{\theta}\text{Tr}(\rho_{\theta} E)\big|_{\theta=0} = 1$; then, inequality (29) follows from applying to this relation the Cauchy-Schwartz inequality for matrices

$$\left| \text{Tr}(AB) \right|^2 \leq \text{Tr}(A^{\dagger} A) \text{Tr}(B^{\dagger} B).$$

Analogously, inequality (30) follows from the fact that $\text{Tr}(\rho_{\theta}(J_{\vec{n}_2} - \langle J_{\vec{n}_2} \rangle_{\theta})) = 0$ implies

$$\left| \text{Tr}(\partial_{\theta}\rho_{\theta}(J_{\vec{n}_2} - \langle J_{\vec{n}_2} \rangle_{\theta})) \right| = \left| \partial_{\theta}\langle J_{\vec{n}_2} \rangle_{\theta} \right| = \left| \langle J_{\vec{n}_3} \rangle_{\theta} \right|.$$

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