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VOEVODSKY'S MOTIVES AND WEIL RECIPROCITY

BRUNO KAHN AND TAKAO YAMAZAKI

Abstract. We describe Somekawa's K-group associated to a finite collection of semi-abelian varieties (or more general sheaves) in terms of the tensor product in Voevodsky's category of motives. While Somekawa's definition is based on Weil reciprocity, Voevodsky's category is based on homotopy invariance. We apply this to explicit descriptions of certain algebraic cycles.

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1. Introduction

1.1. In this article, we construct an isomorphism

$$K(k;\mathcal{F}_1,\dots,\mathcal{F}_n)\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z},\mathcal{F}_1[0]\otimes\cdots\otimes\mathcal{F}_n[0]). \tag{1.1}$$

Here k is a perfect field, and $\mathcal{F}_1,\dots,\mathcal{F}_n$ are homotopy invariant Nisnevich sheaves with transfers in the sense of [30]. On the right hand side, the tensor product $\mathcal{F}_1[0]\otimes\cdots\otimes\mathcal{F}_n[0]$ is computed in Voevodsky's triangulated category $\mathbf{DM}_-^{\mathrm{eff}}$ of effective motivic complexes. The group $K(k;\mathcal{F}_1,\dots,\mathcal{F}_n)$ will be defined in Definition 5.1 by an explicit set of generators and relations: it is a generalization of the group which was defined by K. Kato and studied by M. Somekawa in [22] when $\mathcal{F}_1,\dots,\mathcal{F}_n$ are semi-abelian varieties.

1.2. In the introduction of [22], Somekawa wrote that he expected an isomorphism of the form

$$K(k;G_1,\dots,G_n)\simeq\mathrm{Ext}^n_{MM}(\mathbf{Z},G_1[-1]\otimes\cdots\otimes G_n[-1])$$

where MM is a conjectural abelian category of mixed motives over k, $G_1,\dots,G_n$ are semi-abelian varieties over k, and $G_1[-1],\dots,G_n[-1]$ are the corresponding 1-motives. Since we do not have such a category MM at hand, (1.1) provides the closest approximation to Somekawa's expectation.

1.3. The most basic case of (1.1) is $\mathcal{F}_1=\cdots=\mathcal{F}_n=\mathbf{G}_m$. By [22, Theorem 1.4], the left hand side is isomorphic to the usual Milnor K-group $K_n^M(k)$. The right hand side is almost by definition the motivic cohomology group $H^n(k,\mathbf{Z}(n))$. Thus, when k is perfect, we get a new and less combinatorial proof of the Suslin-Voevodsky isomorphism [26, Thm. 3.4], [16, Thm. 5.1]

$$K_n^M(k)\simeq H^n(k,\mathbf{Z}(n)). \tag{1.2}$$

1.4. The isomorphism (1.1) also has the following application to algebraic cycles. Let X be a k-scheme of finite type. Write $\underline{CH}_0(X)$ for the homotopy invariant Nisnevich sheaf with transfers

$$U\mapsto CH_0(X\times_k k(U))\quad (U\text{ smooth connected})$$

see [7, Th. 2.2]. Let $i,j\in\mathbf{Z}$. We write $CH_i(X,j)$ for Bloch's homological higher Chow group [14, 1.1]: if X is equidimensional of dimension d, it agrees with the group $CH^{d-i}(X,j)$ of [4].

1.5. **Theorem.** *Suppose* $\mathrm{char}\,k=0$. *Let* $X_1,\dots,X_n$ *be quasi-projective* k*-schemes. Put* $X=X_1\times\cdots\times X_n$. *For any* $r\geq 0$, *we have an*

isomorphism

$$K(k; \underline{CH}_0(X_1), \ldots, \underline{CH}_0(X_n), \mathbf{G}_m, \ldots, \mathbf{G}_m) \xrightarrow{\sim} CH_{-r}(X, r), \tag{1.3}$$

where we put r copies of $\mathbf{G}_m$ on the left hand side.[1]

1.6. When $X_1, \ldots, X_n$ are smooth projective,[2] special cases of (1.3) were previously known. The case $r = 0$ was proved by Raskind and Spieß [19, Corollary 4.2.6], and the case $n = 1$ was proved by Akhtar [1, Theorem 6.1] (without assuming k to be perfect). The extension to non smooth projective varieties is new and nontrivial.

1.7. Theorem 1.5 is proven using the Borel-Moore motivic homology introduced in [6, §9]. We also have a variant which involves motivic homology, see Theorem 12.3. Here is an application. Let C_1, C_2 be two smooth connected curves over our perfect field k, and put $S = C_1 \times C_2$. Assume that C_1 and C_2 both have a 0-cycle of degree 1. Then the special case $n = 2, r = 0$ of Theorem 12.3 gives an isomorphism

$$\mathbf{Z} \oplus \mathrm{Alb}_S(k) \oplus K(k; A_1, A_2) \xrightarrow{\sim} H_0(S, \mathbf{Z})$$

where A_i is the Albanese variety of C_i (compare [30, Th. 3.4.2]), $\mathrm{Alb}_S = A_1 \times A_2$ is the Albanese variety of S and the right hand side in this case is Suslin homology [27], see §12.2.

Since Somekawa's groups are defined in an explicit manner, one can sometimes determine the structure of $K(k; A_1, A_2)$ completely. For instance, when k is finite, we have $K(k; A_1, A_2) = 0$ by [9]. This immediately implies the bijectivity of the generalized Albanese map

$$a_S : H_0(S, \mathbf{Z})^{\deg=0} \to \mathrm{Alb}_S(k)$$

of Ramachandran and Spieß-Szamuely [23]. Note that a_S is not bijective for a smooth projective surface S in general, see [12, Prop. 9].

1.8. We conclude this introduction by pointing out the main difficulty and main ideas in the proof of (1.1).

The definitions of the two sides of (1.1) are quite different: the left hand side is based on *Weil reciprocity*, while the right hand side is based on *homotopy invariance*. Thus it is not even obvious how to define a map (1.1) to start with. Our solution is to write both sides as quotients of a common larger group, and to prove that one quotient factors through the other. This provides a map (1.1) which is automatically surjective (Theorem 5.3).

[1] Using recent results of Shane Kelly [13], one may remove the characteristic zero hypothesis if we invert the exponential characteristic of k.

[2] In this case, Theorem 1.5 is valid in any characteristic, see Remark 12.4.

The proof of its injectivity turns out to be much more difficult. We need to find many relations coming from Weil reciprocity. Our main idea, inspired by [22, Theorem 1.4] (recalled in §1.3), is to use the *Steinberg relation* to create Weil reciprocity relations. To show that this provides us with enough such relations, we need to carry out a heavy computation of symbols in §11.

Acknowledgements. Work in this direction had been done previously by Mochizuki [17]. The surjective map (1.1) was announced in [24, Remark 10 (b)]. This research was started by the first author, who wrote the first part of this paper [11]. The collaboration began when the second author visited the Institute of Mathematics of Jussieu in October 2010. Somehow, the research accelerated after the earthquake on March 11, 2011 in Japan. We wish to acknowledge the pleasure of such a fruitful collaboration, along these circumstances.

We acknowledge the depth of the ideas of Milnor, Kato, Somekawa, Suslin and Voevodsky. Especially we are impressed by the relevance of the Steinberg relation in this story.

2. Mackey functors and presheaves with transfers

2.1. A *Mackey functor* over k is a contravariant additive (i.e., commuting with coproducts) functor A from the category of étale k-schemes to the category of abelian groups, provided with a covariant structure verifying the following exchange condition: if

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

is a cartesian square of étale k-schemes, then the diagram

$$\begin{array}{ccc} A(Y') & \xrightarrow{f'^*} & A(Y) \\ {\scriptstyle g'_*}\downarrow & & {\scriptstyle g_*}\downarrow \\ A(X') & \xrightarrow{f^*} & A(X) \end{array}$$

commutes. Here, * denotes the contravariant structure while $_*$ denotes the covariant structure. The Mackey functor A is *cohomological* if we further have

$$f_* f^* = \deg(f)$$

for any $f : X' \to X$, with X connected. We denote by **Mack** the abelian category of Mackey functors, and by $\mathbf{Mack}_c$ its full subcategory of cohomological Mackey functors.

2.2. Classically [28, (1.4)], a Mackey functor may be viewed as a contravariant additive functor on the category **Span** of "spans" on étale k-schemes, defined as follows: objects are étale k-schemes. A morphism from X to Y is an equivalence class of diagram (span)

$$X \xleftarrow{g} Z \xrightarrow{f} Y. \tag{2.1}$$

Composition of spans is defined via fibre product in an obvious manner. If A is a Mackey functor, the corresponding functor on **Span** has the same value on objects, while its value on a span (2.1) is given by $g_* f^*$.

Note that **Span** is a preadditive category: one may add (but not subtract) two morphisms with same source and target. We may as well view a Mackey functor as an additive functor on the associated additive category $\mathbf{Z}\,\mathbf{Span}$.

2.3. Let **Cor** be Voevodsky's category of finite correspondences on smooth k-schemes, denoted by $SmCor(k)$ in [30, §2.1]. The category $\mathbf{Z}\,\mathbf{Span}$ is isomorphic to its full subcategory consisting of smooth k-schemes of dimension 0 (= étale k-schemes). In particular, any presheaf with transfers in the sense of Voevodsky [30, Def. 3.1.1] restricts to a Mackey functor over k. By [29, Cor. 3.15], the restriction of a *homotopy invariant* presheaf with transfers yields a cohomological Mackey functor. In other words, we have exact functors

$$\rho : \mathbf{PST} \to \mathbf{Mack} \tag{2.2}$$

$$\rho : \mathbf{HI} \to \mathbf{Mack}_c \tag{2.3}$$

where **PST** denotes the category of presheaves with transfers (contravariant additive functors from **Cor** to abelian groups) and **HI** is its full subcategory consisting of homotopy invariant presheaves with transfers.

2.4. There is a tensor product of Mackey functors $\overset{M}{\otimes}$, originally defined by L. G. Lewis (unpublished): it extends naturally the symmetric monoidal structure $(X, Y) \mapsto X \times_K Y$ on $\mathbf{Z}\,\mathbf{Span}$ via the additive Yoneda embedding (see §A.7). If either A or B is cohomological, $A \overset{M}{\otimes} B$ is cohomological. This tensor product is the same as the one defined in [8, §5] and [9]: this follows from (A.2) and the fact that $\mathbf{Z}\,\mathbf{Span}$ is rigid, all objects being self-dual (indeed, $\mathbf{Z}\,\mathbf{Span}$ is canonically isomorphic to the category of Artin Chow motives with integral coefficients).

For the reader's convenience, we recall the definition of $\overset{M}{\otimes}$. Let $A_1, \dots, A_n$ be Mackey functors. For any étale k-scheme X, we define

$$(A_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} A_n)(X) := [\bigoplus_{Y \to X} A_1(Y) \otimes \dots \otimes A_n(Y)]/R,$$

where $Y \to X$ runs through all finite étale morphisms, and R is the subgroup generated by all elements of the form

$$a_1 \otimes \dots \otimes f_*(a_i) \otimes \dots a_n - f^*(a_1) \otimes \dots \otimes a_i \otimes \dots f^*(a_n),$$

where $Y_1 \xrightarrow{f} Y_2 \to Y$ is a tower of étale morphisms, $1 \le i \le n$, $a_i \in A_i(Y_1)$ and $a_j \in A_j(Y_2)$ $(j = 1, \dots, i-1, i+1, \dots, n)$.

2.5. There is a tensor product on presheaves with transfers defined exactly in the same way [30, p. 206].

2.6. By definition, the functor (2.2) equals i^*, where i is the inclusion $\mathbf{Z\,Span} \to \mathbf{Cor}$. This inclusion has a left adjoint π_0 (scheme of constants). Both functors i and π_0 are symmetric monoidal: for π_0, reduce to the case where k is separably closed.

2.7. By §§A.2 and A.8, this implies that (2.2) is symmetric monoidal. In other words, if $\mathcal{F}$ and $\mathcal{G}$ are presheaves with transfers, then

$$\rho\mathcal{F} \overset{M}{\otimes} \rho\mathcal{G} \simeq \rho(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.4}$$

The left hand side is sometimes abbreviated to $\mathcal{F} \overset{M}{\otimes} \mathcal{G}$.

2.8. The inclusion functor $\mathbf{HI} \to \mathbf{PST}$ has a left adjoint h_0, and the symmetric monoidal structure of $\mathbf{PST}$ induces one on $\mathbf{HI}$ via h_0. In other words, if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}$, we define

$$\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G} = h_0(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.5}$$

Note that (2.3) is *not* symmetric monoidal (since it is the restriction of (2.2)).

2.9. For any $\mathcal{F} \in \mathbf{PST}$, the unit morphism $\mathcal{F} \to h_0(\mathcal{F})$ induces a surjection

$$\mathcal{F}(k) \twoheadrightarrow h_0(\mathcal{F})(k). \tag{2.6}$$

This is obvious from the formula $h_0(\mathcal{F}) = \mathrm{Coker}(C_1(\mathcal{F}) \to \mathcal{F})$.

2.10. We shall also need to work with Nisnevich sheaves with transfers. We denote by $\mathbf{NST}$ the category of Nisnevich sheaves with transfers (objects of $\mathbf{PST}$ which are sheaves in the Nisnevich topology). By [30, Theorem 3.1.4], the inclusion functor $\mathbf{NST} \to \mathbf{PST}$ has an exact left

adjoint $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}$ (sheafification). The category **NST** then inherits a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{NST}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{\mathrm{Nis}}.$$

Similarly, we define $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST}$. The sheafification functor restricts to an exact functor $\mathbf{HI} \to \mathbf{HI}_{\mathrm{Nis}}$ [30, Theorem 3.1.11], and $\mathbf{HI}_{\mathrm{Nis}}$ gets a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G})_{\mathrm{Nis}}.$$

To summarize, all functors in the following naturally commutative diagram are symmetric monoidal:

$$\begin{tikzcd} \mathbf{PST} \arrow[r, "\mathrm{Nis}"] \arrow[d, "h_0"'] & \mathbf{NST} \arrow[d, "h_0^{\mathrm{Nis}}"'] \\ \mathbf{HI} \arrow[r, "\mathrm{Nis}"] & \mathbf{HI}_{\mathrm{Nis}}\,. \end{tikzcd} \tag{2.7}$$

where each functor is left adjoint to the corresponding inclusion.

2.11. Let $\mathcal{F}$ be a presheaf on Sm/k, and let $\mathcal{F}_{\mathrm{Nis}}$ be the associated Nisnevich sheaf. Then we have an isomorphism

$$\mathcal{F}(k) \xrightarrow{\sim} \mathcal{F}_{\mathrm{Nis}}(k). \tag{2.8}$$

Indeed, any covering of $\operatorname{Spec} k$ for the Nisnevich topology refines to a trivial covering. In particular, the functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}(k)$ is exact.

This applies in particular to a presheaf with transfers and the associated Nisnevich sheaf with transfers.

2.12. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Then (2.4) yields a canonical isomorphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k) \simeq (\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k). \tag{2.9}$$

Composing (2.9) with the unit morphism $Id \Rightarrow h_0^{\mathrm{Nis}}$ from (2.7) and taking (2.5) into account, we get a canonical morphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k) \to (\mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n)(k). \tag{2.10}$$

which is surjective by §§2.9 and 2.11.

2.13. If G is a commutative k-group scheme whose identity component is a quasi-projective variety, then G has a canonical structure of Nisnevich sheaf with transfers ([23, proof of Lemma 3.2] completed by [2, Lemma 1.3.2]). This applies in particular to semi-abelian varieties and also to the "full" Albanese scheme [18] of a smooth variety (which

is an extension of a lattice by a a semi-abelian variety). In particular, if $G_1, \dots, G_n$ are such k-group schemes, (2.10) yields a canonical surjection

$$(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(k) \to (G_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G_n)(k), \tag{2.11}$$

where the G_i are considered on the left as Mackey functors, and on the right as homotopy invariant Nisnevich sheaves with transfers.

3. Presheaves with transfers and motives

3.1. The left adjoint h_0^{Nis} in (2.7) "extends" to a left adjoint C_* of the inclusion

$$\mathbf{DM}_-^{\mathrm{eff}} \to D^-(\mathbf{NST})$$

where the left hand side is Voevodsky's triangulated category of effective motivic complexes [30, §3, esp. Prop. 3.2.3].

More precisely, $\mathbf{DM}_-^{\mathrm{eff}}$ is defined as the full subcategory of objects of $D^-(\mathbf{NST})$ whose cohomology sheaves are homotopy invariant. The canonical t-structure of $D^-(\mathbf{NST})$ induces a t-structure on $\mathbf{DM}_-^{\mathrm{eff}}$, with heart $\mathbf{HI}_{\mathrm{Nis}}$. The functor C_* is right exact with respect to these t-structures, and if $\mathcal{F} \in \mathbf{NST}$, then $H_0(C_*(\mathcal{F})) = h_0^{\mathrm{Nis}}(\mathcal{F})$.

3.2. The tensor structure of §2.10 on $\mathbf{NST}$ extends to one on $D^-(\mathbf{NST})$ [30, p. 206]. Via C_*, this tensor structure descends to a tensor structure on $\mathbf{DM}_-^{\mathrm{eff}}$ [30, p. 210], which will simply be denoted by $\otimes$. The relationship between this tensor structure and the one of §2.10 is as follows: if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, then

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G} = H^0(\mathcal{F}[0] \otimes \mathcal{G}[0]) \tag{3.1}$$

where $\mathcal{F}[0], \mathcal{G}[0]$ are viewed as complexes of Nisnevich sheaves with transfers concentrated in degree 0.

We shall need the following lemma, which is not explicit in [30]:

3.3. **Lemma.** *The tensor product $\otimes$ of $\mathbf{DM}_-^{\mathrm{eff}}$ is right exact with respect to the homotopy t-structure.*

Proof. By definition,

$$C \otimes D = C_*(C \overset{L}{\otimes} D)$$

for $C, D \in \mathbf{DM}_-^{\mathrm{eff}}$, where $\overset{L}{\otimes}$ is the tensor product of $D^-(\mathbf{NST})$ defined in [30, p. 206]. We want to show that, if C and D are concentrated in degrees ≤ 0, then so is $C \otimes D$. Using the canonical left resolutions of loc. cit., it is enough to do it for C and D of the form $C_*(L(X))$

and $C_*(L(Y))$ for two smooth schemes X, Y. Since C_* is symmetric monoidal, we have

$$C_*(L(X)) \otimes C_*(L(Y)) \xleftarrow{\sim} C_*(L(X) \overset{L}{\otimes} L(Y)) = C_*(L(X \times Y))$$

and the claim is obvious in view of the formula for C_* [30, p. 207]. $\square$

3.4. Let $C \in \mathbf{DM}_-^{\mathrm{eff}}$. For any $X \in Sm/k$ and any $i \in \mathbf{Z}$, we have

$$\mathbb{H}^i_{\mathrm{Nis}}(X, C) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(M(X), C[i])$$

where $M(X) = C_*(L(X))$ is the motive of X computed in $\mathbf{DM}_-^{\mathrm{eff}}$ (cf. [30, Prop. 3.2.7]).

Specializing to the case $X = \operatorname{Spec} k$ ($M(X) = \mathbf{Z}$) and taking §2.11 into account, we get

$$\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, C[i]) \simeq H^i(C)(k). \tag{3.2}$$

Combining (3.1), (2.8) and (3.2), we get:

3.5. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then we have a canonical isomorphism*

$$(\mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n)(k) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]). \tag{3.3}$$

3.6. Summarizing, for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ we get a surjective homomorphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k) \to \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]). \tag{3.4}$$

by composing (2.9), (2.6), (2.5), (2.8) and (3.3).

4. Presheaves with transfers and local symbols

4.1. Given a presheaf with transfers $\mathcal{G}$, recall from [29, p. 96] the presheaf with transfers $\mathcal{G}_{-1}$ defined by the formula

$$\mathcal{G}_{-1}(U) = \mathrm{Coker}\left(\mathcal{G}(U \times \mathbf{A}^1) \to \mathcal{G}(U \times (\mathbf{A}^1 - \{0\}))\right). \tag{4.1}$$

Suppose that $\mathcal{G}$ is homotopy invariant. Let $X \in Sm/k$ (connected), $K = k(X)$ and $x \in X$ be a point of codimension 1. By [29, Lemma 4.36], there is a canonical isomorphism

$$\mathcal{G}_{-1}(k(x)) \simeq H^1_x(X, \mathcal{G}_{\mathrm{Zar}}) \tag{4.2}$$

yielding a canonical map

$$\partial_x : \mathcal{G}(K) \to \mathcal{G}_{-1}(k(x)). \tag{4.3}$$

The following lemma follows from the construction of the isomorphisms (4.2). It is part of the general fact that $\mathcal{G}$ defines a cycle module in the sense of Rost (cf. [5, Prop. 5.4.64]).

4.2. **Lemma.** *a) Let $f : Y \to X$ be a dominant morphism, with Y smooth and connected. Let $L = k(Y)$, and let $y \in Y^{(1)}$ be such that $f(y) = x$. Then the diagram*

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \mathcal{G}_{-1}(k(y)) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle ef^*}\uparrow \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

commutes, where e is the ramification index of v_y relative to v_x.
b) If f is finite surjective, the diagram

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \bigoplus_{y\in f^{-1}(x)} \mathcal{G}_{-1}(k(y)) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

commutes. □

4.3. **Proposition.** *Let $\mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$. There is a canonical isomorphism*

$$\mathcal{G}_{-1} = \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{G}).$$

Proof. This may not be the most economic proof, but it is quite short. The statement means that $\mathcal{G}_{-1}$ represents the functor

$$\mathcal{H} \mapsto \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathcal{G}).$$

By [29, Lemma 4.35], we have

$$\mathcal{G}_{-1} = \mathrm{Coker}(\mathcal{G} \to p_*p^*\mathcal{G})$$

where $p : \mathbf{A}^1 - \{0\} \to \operatorname{Spec} k$ is the structural morphism and p_*, p^* are computed with respect to the Zariski topology. By [29, Theorem 5.7], we may replace the Zariski topology by the Nisnevich topology. Moreover, by [29, Prop. 5.4 and Prop. 4.20], we have $R^ip_*p^*\mathcal{G} = 0$ for $i > 0$, hence $p_*p^*\mathcal{G}[0] \xrightarrow{\sim} Rp_*p^*\mathcal{G}[0]$.

By [30, Prop. 3.2.8], we have

$$Rp_*p^*\mathcal{G}[0] = \underline{\mathrm{Hom}}(M(\mathbf{A}^1 - \{0\}), \mathcal{G}[0])$$

where $\underline{\mathrm{Hom}}$ is the (partially defined) internal Hom of $\mathbf{DM}_-^{\mathrm{eff}}$. By [30, Prop. 3.5.4] (Gysin triangle) and homotopy invariance, we have an exact triangle, split by any rational point of $\mathbf{A}^1 - \{0\}$:

$$\mathbf{Z}(1)[1] \to M(\mathbf{A}^1 - \{0\}) \to \mathbf{Z} \xrightarrow{+1}$$

To get a canonical splitting, we may choose the rational point $1 \in \mathbf{A}^1 - \{0\}$.

By [30, Cor. 3.4.3], we have an isomorphism $\mathbf{Z}(1)[1] \simeq \mathbf{G}_m[0]$. Hence, in $\mathbf{DM}_-^{\mathrm{eff}}$, we have an isomorphism

$$\mathcal{G}_{-1}[0] \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m[0], \mathcal{G}[0]).$$

Let $\mathcal{H} \in \mathbf{HI}_{\mathrm{Nis}}$. We get:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathcal{H}[0], \mathcal{G}_{-1}[0]) &\simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathcal{H}[0] \otimes \mathbf{G}_m[0], \mathcal{G}[0]) \\ &\simeq \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(H^0(\mathcal{H}[0] \otimes \mathbf{G}_m[0]), \mathcal{G}) =: \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathcal{G})\end{aligned}$$

as desired (see (3.1)). For the second isomorphism, we have used the right exactness of $\otimes$ (Lemma 3.3). □

4.4. *Remark.* The proof of Proposition 4.3 also shows that, in $\mathbf{DM}_-^{\mathrm{eff}}$, we have an isomorphism

$$\underline{\mathrm{Hom}}(\mathbf{G}_m[0], \mathcal{G}[0]) \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{G})[0]$$

where the left $\underline{\mathrm{Hom}}$ is computed in $\mathbf{DM}_-^{\mathrm{eff}}$ and the right $\underline{\mathrm{Hom}}$ is computed in $\mathbf{HI}_{\mathrm{Nis}}$. In particular, $\underline{\mathrm{Hom}}(\mathbf{G}_m[0], -) : \mathbf{DM}_-^{\mathrm{eff}} \to \mathbf{DM}_-^{\mathrm{eff}}$ is t-exact.

4.5. **Proposition.** *Let C be a smooth, proper, connected curve over k, with function field K. There exists a canonical homomorphism*

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(C, \mathcal{G}) \to \mathcal{G}_{-1}(k)$$

such that, for any $x \in C$, the composition

$$\mathcal{G}_{-1}(k(x)) \simeq H^1_x(C, \mathcal{G}) \to H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\mathrm{Tr}_C} \mathcal{G}_{-1}(k)$$

equals the transfer map $\mathrm{Tr}_{k(x)/k}$ associated to the finite surjective morphism $\mathrm{Spec}\, k(x) \to \mathrm{Spec}\, k$.

Proof. By [30, Prop. 3.2.7], we have

$$H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(C, \mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(M(C), \mathcal{G}[1]).$$

The structural morphism $C \to \mathrm{Spec}\, k$ yields a morphism of motives $M(C) \to \mathbf{Z}$ which, by Poincaré duality, yields a canonical morphism

$$\mathbf{G}_m[1] \simeq \mathbf{Z}(1)[2] \to M(C).$$

(One may view this morphism as the image of the canonical morphism $\mathbb{L} \to h(C)$ in the category of Chow motives.)

Therefore, by Proposition 4.3 and Remark 4.4, we get a map

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(X, \mathcal{G}) \to \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{G}_m[1], \mathcal{G}[1]) = \mathcal{G}_{-1}(k).$$

It remains to prove the claimed compatibility. Let $M^x(C)$ be the motive of C with supports in x, defined as $C_*(\mathrm{Coker}(L(C - \{x\}) \to L(C))$. Let $\mathbf{Z}_{k(x)} = M(\mathrm{Spec}\, k(x))$. By [30, proof of Prop. 3.5.4], we

have an isomorphism $M^x(C) \simeq \mathbf{Z}_{k(x)}(1)[2]$, and we have to show that the composition

$$\mathbf{Z}(1)[2] \to M(C) \xrightarrow{g_x} \mathbf{Z}_{k(x)}(1)[2]$$

is $\mathrm{Tr}_{k(x)/k}$, up to twisting and shifting. To see this, we observe that g_x is the image of the morphism of Chow motives

$$h(C) \to h(\operatorname{Spec} k(x))(1)$$

dual to the morphism $h(\operatorname{Spec} k(x)) \to h(C)$ induced by the inclusion $\operatorname{Spec} k(x) \to C$: this is easy to check from the definition of g_x in [30] (observe that in this special case, $Bl_x(C) = C$ and that we may use a variant of the said construction replacing $C \times \mathbf{A}^1$ by $C \times \mathbf{P}^1$ to stay within smooth projective varieties). The conclusion now follows from the fact that the composition

$$\operatorname{Spec} k(x) \to C \to \operatorname{Spec} k$$

is the structural morphism of $\operatorname{Spec} k(x)$. □

4.6. **Proposition** (Reciprocity)**.** *Let C be a smooth, proper, connected curve over k, with function field K. Then the sequence*

$$\mathcal{G}(K) \xrightarrow{(\partial_x)} \bigoplus_{x\in C} \mathcal{G}_{-1}(k(x)) \xrightarrow{\sum_x \mathrm{Tr}_{k(x)/k}} \mathcal{G}_{-1}(k)$$

is a complex.

Proof. This follows from Proposition 4.5, since the composition

$$\mathcal{G}(K) \to \bigoplus_{x\in C} H^1_x(C, \mathcal{G}) \xrightarrow{(g_x)} H^1(C, \mathcal{G})$$

is 0. □

4.7**.** If $\mathcal{F}, \mathcal{G}$ are presheaves with transfers, there is a bilinear morphism of presheaves with transfers (i.e. a natural transformation over $\mathbf{PST} \times \mathbf{PST}$):

$$\begin{aligned}
&\mathcal{F}(U) \otimes \mathcal{G}_{-1}(V) = \\
&\quad \operatorname{Coker}\big(\mathcal{F}(U) \otimes \mathcal{G}(V \times \mathbf{A}^1) \to \mathcal{F}(U) \otimes \mathcal{G}(V \times (\mathbf{A}^1 - \{0\}))\big) \to \\
&\operatorname{Coker}\big((\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})(U \times V \times \mathbf{A}^1) \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})(U \times V \times (\mathbf{A}^1 - \{0\}))\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}(U \times V)
\end{aligned}$$

which induces a morphism

$$\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}_{-1} \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}. \tag{4.4}$$

In particular, for $\mathcal{G} = \mathbf{G}_m$, we get a morphism $\mathcal{F} \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathbf{G}_m)_{-1}$.

4.8. **Theorem.** *Suppose* $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. *Then*
a) The composition

$$\mathcal{F} \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathbf{G}_m)_{-1} \to (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)_{-1}$$

is the unit map of the adjunction between $-\otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m$ *and* $(-)_{-1}$ *stemming from Proposition 4.3.*
b) This composition is an isomorphism.

Proof. a) is an easy bookkeeping. For b), we compute again in $\mathbf{DM}_-^{\mathrm{eff}}$. By Proposition 4.3, we are considering the morphism in $\mathbf{HI}_{\mathrm{Nis}}$

$$\mathcal{F} \to \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m). \tag{4.5}$$

Consider the corresponding morphism in $\mathbf{DM}_-^{\mathrm{eff}}$

$$\mathcal{F}[0] \to \underline{\mathrm{Hom}}(\mathbf{G}_m[0], \mathcal{F}[0] \otimes \mathbf{G}_m[0]).$$

As recalled in the proof of Proposition 4.3, we have $\mathbf{G}_m[0] = \mathbf{Z}(1)[1]$, hence the above morphism amounts to

$$\mathcal{F}[0] \to \underline{\mathrm{Hom}}(\mathbf{Z}(1), \mathcal{F}[0](1))$$

which is an isomorphism by the cancellation theorem [32]. A fortiori, (4.5), which is (by Remark 4.4) the H^0 of this isomorphism, is an isomorphism. □

4.9. **Notation.** Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$ and $\mathcal{H} = \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}$. Let X, K, x be as in §4.1. For $(a, b) \in \mathcal{F}(K) \times \mathcal{G}(K)$, we denote by $a \cdot b$ the image of $a \otimes b$ in $\mathcal{H}(K)$ by the map

$$\mathcal{F}(K) \otimes \mathcal{G}(K) \to \mathcal{H}(K).$$

We define the *local symbol* on $\mathcal{F}$

$$\mathcal{F}(K) \times K^* \to \mathcal{F}(k(x))$$

to be the composition

$$\mathcal{F}(K) \times K^* \xrightarrow{\cdot} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)(K) \xrightarrow{\partial_x} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)_{-1}(k(x)) \cong \mathcal{F}(k(x))$$

where the first map is given by the above construction with $\mathcal{G} = \mathbf{G}_m$, and the last isomorphism is given by Theorem 4.8. The image of $(a, b) \in \mathcal{F}(K) \times K^*$ by the local symbol is denoted by $\partial_x(a, b) \in \mathcal{F}(k(x))$.

4.10. **Proposition** (cf. [5, Prop. 5.5.27])**.** *Let* $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$*, and consider the morphism induced by* (4.4)

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1} \xrightarrow{\Phi} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}.$$

Let X, K, x be as in §4.1. Then the diagram

$$\begin{tikzcd}
\mathcal{F}(\mathcal{O}_{X,x}) \otimes \mathcal{G}(K) \arrow[r] \arrow[d, "i_x^* \otimes \partial_x"] & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})(K) \arrow[dd, "\partial_x"] \\
\mathcal{F}(k(x)) \otimes \mathcal{G}_{-1}(k(x)) \arrow[d] & \\
(\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1})(k(x)) \arrow[r, "\Phi"] & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}(k(x))
\end{tikzcd}$$

commutes, where i_x^* is induced by the reduction map $\mathcal{O}_{X,x} \to k(x)$. In other words, with Notation 4.9 we have the identity

$$\partial_x(a \cdot b) = \Phi(i_x^* a \cdot \partial_x b) \tag{4.6}$$

for $(a, b) \in \mathcal{F}(\mathcal{O}_{X,x}) \times \mathcal{G}(K)$.

4.11. **Corollary.** *Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$; let X, K, x be as in §4.1 and let $(a, f) \in \mathcal{F}(K) \times K^*$.*
a) Suppose that there is $\tilde{a} \in \mathcal{F}(\mathcal{O}_{X,x})$ whose image in $\mathcal{F}(K)$ is a. Then we have

$$\partial_x(a, f) = v_x(f) a(x)$$

where $a(x)$ is the image of $\tilde{a}$ in $\mathcal{F}(k(x))$ (which is independent of the choice of $\tilde{a}$).
b) Suppose that $v_x(f - 1) > 0$. Then $\partial_x(a, f) = 0$.

Proof. a) This follows from Proposition 4.10 (applied with $\mathcal{G} = \mathbf{G}_m$) and Theorem 4.8. b) This follows again from Proposition 4.10, after switching the rôles of $\mathcal{F}$ and $\mathcal{G}$. □

4.12. **Proposition.** *Let G be a semi-abelian variety. The local symbol on G defined in Notation 4.9 agrees with Somekawa's local symbol* [22, (1.1)] *(generalising the Rosenlicht-Serre local symbol) on G.*

Proof. Up to base-changing from k to $\bar{k}$ (see Lemma 4.2 a)), we may assume k algebraically closed. By [21, Ch. III, Prop. 1], it suffices to show that the local symbol in Notation 4.9 satisfies the properties in [21, Ch. III, Def. 2] which characterize the Rosenlicht-Serre local symbol. In this definition, Condition i) is obvious, Condition ii) is Corollary 4.11 b), Condition iii) is Corollary 4.11 a) and Condition iv) is Proposition 4.6. □

5. K-groups of Somekawa type

5.1. **Definition.** Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.
a) A *relation datum of Somekawa type* for $\mathcal{F}_1, \dots, \mathcal{F}_n$ is a collection

$(C, h, (g_i)_{i=1,\dots,n})$ of the following objects: (i) a smooth proper connected curve C over k, (ii) $h \in k(C)^*$, and (iii) $g_i \in \mathcal{F}_i(k(C))$ for each $i \in \{1, \dots, n\}$; which satisfies the condition

(5.1) for any $c \in C$, there is $i(c)$ such that $c \in R_i$ for all $i \neq i(c)$,

where $R_i := \{c \in C \mid g_i \in \mathrm{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \to \mathcal{F}_i(k(C)))\}$.
b) We define the *K-group of Somekawa type* $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ to be the quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$\sum_{c \in C} \mathrm{Tr}_{k(c)/k}(g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)}, h) \otimes \dots \otimes g_n(c)) \tag{5.2}$$

where $(C, h, (g_i)_{i=1,\dots,n})$ runs through all relation data of Somekawa type.

5.2. *Remark.* In view of Proposition 4.12, our group $K(k; \mathcal{F}_1, \dots \mathcal{F}_n)$ coincides with the Milnor K-group defined in [22] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semi-abelian varieties over k.[3]

5.3. **Theorem.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The homomorphism* (2.10) *factors through $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we get a surjective homomorphism* (1.1).

Proof. Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \dots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n$. Let $(C, h, (g_i)_{i=1,\dots,n})$ be a relation datum of Somekawa type. We must show that the element (5.2) goes to 0 in $\mathcal{F}(k)$ via (2.10). Consider the element $g = g_1 \otimes \dots \otimes g_n \in \mathcal{F}(K)$. It follows from (4.6) that, for any $c \in C$, we have

$$\begin{aligned} &g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)}, h) \otimes \dots \otimes g_n(c) \\ &\qquad = g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)} \otimes \{h\}) \otimes \dots \otimes g_n(c) = \partial_c(g \otimes \{h\}). \end{aligned}$$

The claim now follows from Proposition 4.6. □

6. K-groups of geometric type

6.1. **Definition.** Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$.
a) A *relation datum of geometric type* for $\mathcal{F}_1, \dots, \mathcal{F}_n$ is a collection $(C, f, (g_i)_{i=1,\dots,n})$ of the following objects: (i) a smooth projective connected curve C over k, (ii) a surjective morphism $f : C \to \mathbf{P}^1$, (iii) $g_i \in \mathcal{F}_i(C')$ for each $i \in \{1, \dots, n\}$, where $C' = f^{-1}(\mathbf{P}^1 \setminus \{1\})$.
b) We define the *K-group of geometric type* $K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ to be the

[3] As was observed by W. Raskind, the signs appearing in [22, (1.2.2)] should not be there (cf. [19, p. 10, footnote]).

quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$\sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k}(g_1(c) \otimes \dots \otimes g_n(c)) \tag{6.1}$$

where $(C, f, (g_i)_{i=1,\dots,n})$ runs through all relation data of geometric type. (Here we used the notation $g_i(c) = \iota_c^*(g_i) \in \mathcal{F}(k(c))$, where $\iota_c : c = \operatorname{Spec} k(c) \to C'$ is the closed immersion.)

The rest of this section is devoted to a proof of the following theorem:

6.2. **Theorem.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The homomorphism* (2.10) *induces an isomorphism*

$$K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \dots \otimes \mathcal{F}_n[0]). \tag{6.2}$$

6.3. For a smooth variety X over k, denote as usual by $L(X)$ the Nisnevich sheaf with transfers represented by X. Recall that $L(X)(U) = c(U, X)$ is the group of finite correspondences for any smooth variety U over k, viz. the free abelian group on the set of closed integral subschemes of $U \times X$ which are finite and surjective over some irreducible component of U. A morphism $X \to X'$ of smooth varieties induces a map $L(X) \to L(X')$ of Nisnevich sheaves with transfers.

We recall two facts from [30, p. 206], which are fundamental in the definition of the tensor product in $\mathbf{PST}$.

(1) For any $\mathcal{F} \in \mathbf{PST}$, there is a surjective map $\oplus_X L(X) \to \mathcal{F}$ of presheaves with transfers, where X runs through a (huge) set of smooth varieties over k.
(2) We have (by definition) $L(X) \otimes_{\mathbf{PST}} L(Y) = L(X \times Y)$ for smooth varieties X and Y.

6.4. Let $\mathcal{F} \in \mathbf{PST}$. Suppose that we are given the following data: (i) a smooth projective connected curve C over k, (ii) a surjective morphism $f : C \to \mathbf{P}^1$, (iii) a map $\alpha : L(C') \to \mathcal{F}$ in $\mathbf{PST}$, where $C' = f^{-1}(\Delta)$ and $\Delta = \mathbf{P}^1 \setminus \{1\} (\cong \mathbf{A}^1)$. To such a triple (C, f, α), we associate an element

$$\alpha(\operatorname{div}(f)) \in \mathcal{F}(k), \tag{6.3}$$

where we regard $\operatorname{div}(f)$ as an element of $Z_0(C') = c(\operatorname{Spec} k, C') = L(C')(k)$.

One can rewrite the element (6.3) as follows. The map $\alpha : L(C') \to \mathcal{F}$ can be regarded as a section $\alpha \in \mathcal{F}(C')$. To each closed point $c \in C'$, we write $\alpha(c)$ for the image of α in $\mathcal{F}(k(c))$ by the map induced

by $c = \operatorname{Spec} k(c) \to C'$. Now (6.3) is rewritten as

$$\sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k} \alpha(c). \tag{6.4}$$

6.5. **Proposition.** *Let $\mathcal{F} \in \mathbf{PST}$. We define $\mathcal{F}(k)_{\mathrm{rat}}$ to be the subgroup of $\mathcal{F}(k)$ generated by elements* (6.3) *for all triples (C, f, α) as in §6.4. Then we have*

$$h_0(\mathcal{F})(k) = \mathcal{F}(k)/\mathcal{F}(k)_{\mathrm{rat}}.$$

Proof. By definition we have

$$h_0(\mathcal{F})(k) = \operatorname{Coker}(i_0^* - i_\infty^* : \mathcal{F}(\Delta) \to \mathcal{F}(k)),$$

where $\Delta = \mathbf{P}^1 \setminus \{1\} (\cong \mathbf{A}^1)$ and i_a^* is the pull-back by the inclusion $i_a : \{a\} \to \Delta$ for $a \in \{0, \infty\}$.

Suppose we are given a triple (C, f, α) as in §6.4, and set $C' = f^{-1}(\Delta)$. The graph $\gamma_{f|_{C'}}$ of $f|_{C'}$ defines an element of $c(\Delta, C') = L(C')(\Delta)$. In the commutative diagram

$$\begin{array}{ccc} L(C')(\Delta) & \xrightarrow{\alpha} & \mathcal{F}(\Delta) \\ {\scriptstyle i_0^* - i_\infty^*} \downarrow & & \downarrow {\scriptstyle i_0^* - i_\infty^*} \\ L(C')(k) & \xrightarrow{\alpha} & \mathcal{F}(k), \end{array}$$

the image of $\gamma_{f|_{C'}}$ in $L(C')(k) = Z_0(C')$ is $\operatorname{div}(f)$, which shows the vanishing of $\alpha(\operatorname{div}(f))$ in $h_0(\mathcal{F})(k)$.

Conversely, given $\alpha \in \mathcal{F}(\Delta)$, (6.3) for the triple $(\mathbf{P}^1, \mathrm{id}_{\mathbf{P}^1}, \alpha)$ coincides with $(i_0^* - i_\infty^*)(\alpha)$. This completes the proof. □

6.6. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n$. Let (C, f, α) be a triple considered in §6.4. Then $\alpha \in \mathcal{F}(C')$ is the sum of a finite number of elements of the form*

$$\operatorname{Tr}_h(g_1 \otimes \cdots \otimes g_n), \tag{6.5}$$

where D is a smooth projective curve, $h : D \to C$ is a surjective morphism, $g_i \in \mathcal{F}_i(h^{-1}(C'))$ for $i = 1, \dots, n$, and $\operatorname{Tr}_h : \mathcal{F}(h^{-1}(C')) \to \mathcal{F}(C')$ is the transfer with respect to $h|_{h^{-1}(C')}$.

Proof. By the facts recalled in §6.3, we are reduced to the case $\mathcal{F}_i = L(X_i)$ where X_i is a smooth variety over k for each $i = 1, \dots, n$. Then we have $\mathcal{F} = L(X)$ with $X = X_1 \times \cdots \times X_n$. Let Z be an integral closed subscheme of $C' \times X$ which is finite and surjective over C'. It suffices to show that $Z \in c(C', X) = L(X)(C')$ can be written as (6.5).

Let $q : D' \to Z$ be the normalization, and let $h : D' \to C'$ be the composition $D' \to Z \to C'$, so that h is a finite surjective morphism. For $i = 1, \dots, n$, we define $g_i \in c(D', X_i) = L(X_i)(D')$ to be the graph of $D' \to X \to X_i$. If we set $g = g_1 \otimes \cdots \otimes g_n \in L(X)(D')$,

then by definition we have $\mathrm{Tr}_h(g) = Z$ in $L(X)(C')$. The assertion is proved. □

6.7. Now it follows from Definition 6.1 b), Proposition 6.5, Lemma 6.6 and (6.4) that (2.9) and (2.6) induce an isomorphism

$$K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \cong h_0(\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$$

for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. If $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$, the right hand side is canonically isomorphic to $\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_-}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0])$ by (3.3) + (2.8). This completes the proof of Theorem 6.2. □

7. Milnor K-theory

7.1. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. We obtained a surjective homomorphism

$$K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \to K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n). \tag{7.1}$$

Our aim is to show that this map is bijective. The first step is the special case of the multiplicative groups.

7.2. **Proposition.** *When $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbf{G}_m$, the map (7.1) is bijective.*

Proof. It suffices to show that relations (6.1) vanish in $K(k; \mathbf{G}_m, \dots, \mathbf{G}_m)$. Because of Somekawa's isomorphism [22, Theorem 1.4]

$$K(k; \mathbf{G}_m, \dots, \mathbf{G}_m) \cong K_n^M(k) \tag{7.2}$$

given by $\{x_1, \dots, x_n\}_{E/k} \mapsto N_{E/k}(\{x_1, \dots, x_n\})$, it suffices to show this vanishing in the usual Milnor K-group $K_n^M(k)$, which follows from Weil reciprocity [3, Ch. I, (5.4)]. □

The following lemmas appear to be crucial in the proof of the main theorem.

7.3. **Lemma.** *Let C be a smooth projective connected curve over k, and let $Z = \{p_1, \dots, p_s\}$ be a finite set of closed points of C. If k is infinite, then we have $K_2^M(k(C)) = \{k(C)^*, \mathcal{O}_{C,Z}^*\}$.*

Proof. Let $\mathfrak{p}_i$ be the maximal ideal of $A = \mathcal{O}_{C,Z}$ corresponding to p_i. Since A is a semi-local PID, we can choose generators $\pi_1, \dots, \pi_s$ of $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Since k is infinite, we can change π_i into $\mu_i \pi_i$ for suitable $\mu_1, \dots, \mu_s \in k^*$ to achieve $\pi_i + \pi_j \not\equiv 0 \pmod{\mathfrak{p}_k}$ for i, j, k all distinct (indeed, the set of bad $(\mu_1, \dots, \mu_s)$ is contained in a finite union of hyperplanes in $\bar{k}^s$). It follows that $\pi_i + \pi_j \in A^*$ for all $i \neq j$.

By the relation $\{f, -f\} = 0$ ($f \in k(C)^*$), we have $K_2^M(k(C)) = \{A^*, A^*\} + \sum_{i<j} \{\pi_i, \pi_j\}$. Now the identity

$$\{\pi_i, \pi_j\} = \{-\pi_i/\pi_j, \pi_i + \pi_j\}$$

proves the lemma. □

7.4. **Lemma.** *Let C be a smooth projective connected curve over k, and $r > 0$. If k is an infinite field, then $K^M_{r+1}k(C)$ is generated by elements of the form $\{a_1, \dots, a_{r+1}\}$ where the $a_i \in k(C)^*$ satisfy* $\mathrm{Supp}(\mathrm{div}(a_i)) \cap \mathrm{Supp}(\mathrm{div}(a_j)) = \emptyset$ *for all $1 \le i < j \le r$.*

Proof. We proceed by induction on r. The assertion is empty when $r = 1$. Suppose $r > 1$. Take $a_1, \dots, a_{r+1} \in k(C)^*$. By induction, there exist $b_{m,i} \in k(C)^*$ such that $\mathrm{Supp}(\mathrm{div}(b_{m,i})) \cap \mathrm{Supp}(\mathrm{div}(b_{m,j})) = \emptyset$ for all $i < j < r$ and m, and

$$\{a_1, \dots, a_r\} = \sum_m \{b_{m,1}, \dots, b_{m,r}\}$$

holds in $K^M_r k(C)$. For each m, the above lemma shows that there exist $c_{m,i}, d_{m,i} \in k(C)^*$ such that

$$\mathrm{Supp}(\mathrm{div}(c_{m,i})) \cap \left(\bigcup_{j=1}^{r-1} \mathrm{Supp}(\mathrm{div}(b_{m,j})) \right) = \emptyset$$

and that

$$\{b_{m,r}, a_{r+1}\} = \sum_i \{c_{m,i}, d_{m,i}\}$$

holds in $K^M_2 k(C)$. Then we have

$$\{a_1, \dots, a_{r+1}\} = \sum_{m,i} \{b_{m,1}, \dots, b_{m,r-1}, c_{m,i}, d_{m,i}\}$$

in $K^M_{r+1}k(C)$, and we are done. □

8. K-groups of Milnor type

We now generalize the notion of Milnor K-groups to arbitrary homotopy invariant Nisnevich sheaves with transfers, although we shall seriously use this generalization only for special, representable, sheaves.

8.1. Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. We shall call a homomorphism $\mathbf{G}_m \to \mathcal{F}$ a *cocharacter* of $\mathcal{F}$. (By Proposition 4.3, the group $\mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{G}_m, \mathcal{F})$ is canonically isomorphic to $\mathcal{F}_{-1}(k)$.)

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Denote by $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ the subgroup of $(\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$ generated by the elements

$$a_1 \otimes \cdots \otimes \chi_i(a) \otimes \cdots \otimes \chi_j(1-a) \otimes \cdots \otimes a_n \tag{8.1}$$

where $\chi_i : \mathbf{G}_m \to \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \to \mathcal{F}_j$ are 2 cocharacters with $i < j$, $a \in k^* \setminus \{1\}$, and $a_m \in \mathcal{F}_m(k)$ $(m \ne i, j)$.

8.2. **Definition.** For $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$, we write $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ for the quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by the subgroup generated by $\mathrm{Tr}_{E/k}\, St(E; \mathcal{F}_1, \dots, \mathcal{F}_n)$, where E runs through all finite extensions of k. This is the *K-group of Milnor type* associated to $\mathcal{F}_1, \dots, \mathcal{F}_n$.

8.3. The assignment $k \mapsto \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ inherits the structure of a cohomological Mackey functor, which is natural in $(\mathcal{F}_1, \dots, \mathcal{F}_n)$. In particular, the choice of elements $f_i \in \mathcal{F}_i(k) = \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{Z}, \mathcal{F}_i)$ for $i = 1, \dots, r$ induces a homomorphism

$$\tilde{K}(k; \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) = \tilde{K}(k; \mathbf{Z}, \dots, \mathbf{Z}, \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) \to \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n). \tag{8.2}$$

8.4. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The image of $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we have a surjective homomorphism $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \to K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and a composite surjective homomorphism*

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \twoheadrightarrow K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \tag{8.3}$$

Proof. This is a straightforward generalization of Somekawa's proof of [22, Th. 1.4]. We need to show the image of (8.1) vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. By functoriality, we may assume that $\mathcal{F}_i = \mathcal{F}_j = \mathbf{G}_m$ for some $i < j$ and χ_i, χ_j are the identity cocharacters. Given $a_m \in \mathcal{F}_m(k)$ $(m \neq i, j)$ and $a \in k^* \setminus \{1\}$, we put $a_i = 1 - at^{-1}, a_j = 1 - t \in \mathbf{G}_m(k(\mathbf{P}^1)) = k(t)^*$. Then $(\mathbf{P}^1, t, (a_1, \dots, a_n))$ is a relation datum of Somekawa type and yields the vanishing of (8.1). □

8.5. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ and let $\mathcal{G}' \twoheadrightarrow \mathcal{G}''$ be an epimorphism in $\mathbf{HI}_{\mathrm{Nis}}$. If (8.3) is bijective for $(\mathcal{G}', \mathcal{F}_1, \dots, \mathcal{F}_n)$, it is bijective for $(\mathcal{G}'', \mathcal{F}_1, \dots, \mathcal{F}_n)$.*

Proof. Let $\mathcal{G} = \mathrm{Ker}(\mathcal{G}' \to \mathcal{G}'')$. The induced sequence

$$\tilde{K}(k; \mathcal{G}, \mathcal{F}_1, \dots, \mathcal{F}_n) \to \tilde{K}(k; \mathcal{G}', \mathcal{F}_1, \dots, \mathcal{F}_n) \xrightarrow{(*)} \tilde{K}(k; \mathcal{G}'', \mathcal{F}_1, \dots, \mathcal{F}_n) \to 0$$

is a complex and $(*)$ is surjective. The corresponding sequence for K' is exact because of Theorem 6.2 and Lemma 3.3. The assertion follows by a diagram chase. □

8.6. **Lemma.** *Let E/k be a finite extension. Let $\mathcal{F}_1, \dots, \mathcal{F}_{n-1} \in \mathbf{HI}_{\mathrm{Nis}}$, and let $\mathcal{F}_n$ be a Nisnevich sheaf with transfers over E. We have canonical isomorphisms*

$$K(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) \cong K(E; \mathcal{F}_1, \dots, \mathcal{F}_n),$$
$$K'(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) \cong K'(E; \mathcal{F}_1, \dots, \mathcal{F}_n),$$
$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) \cong \tilde{K}(E; \mathcal{F}_1, \dots, \mathcal{F}_n).$$

Proof. The first isomorphism was constructed in [24, Lemma 4] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semi-abelian varieties. The same construction works for arbitrary $\mathcal{F}_1, \dots, \mathcal{F}_n$ and also for K' and $\tilde{K}$. □

8.7. If $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathbf{G}_m$, (8.3) is bijective by Proposition 7.2. This is false in general, *e.g.* if all the $\mathcal{F}_i$ are proper (Definition 10.1) and $n > 1$. However, we have:

8.8. **Proposition.** *a) Let $\mathcal{F}_1 = \mathcal{F}_1' \oplus \mathcal{F}_1''$. Then the natural map*

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \to \tilde{K}(k; \mathcal{F}_1', \dots, \mathcal{F}_n) \oplus \tilde{K}(k; \mathcal{F}_1'', \dots, \mathcal{F}_n)$$

is bijective.
b) Let $T_1, \dots, T_n$ be tori. Assume that, for each i, there exists an exact sequence

$$0 \to P_i^1 \to P_i^0 \to T_i \to 0$$

where P_i^0 and P_i^1 are invertible tori (i.e. direct summands of permutation tori). Then (8.3) *is bijective.*

Proof. a) This is formal, as $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ is a quotient of the multiadditive multifunctor $(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n)(k)$ (see 8.3).

b) Note that, by Hilbert's theorem 90, the sequences $0 \to P_i^1 \to P_i^0 \to T_i \to 0$ are exact in $\mathbf{HI}_{\mathrm{Nis}}$. Lemma 8.5 reduces us to the case where all T_i are permutation tori. Then Lemma 8.6 reduces us to the case where all T_i are split tori. Finally, we reduce to $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathbf{G}_m$ by a). □

8.9. *Question.* Is proposition 8.8 true for general tori?

8.10. Let $T_1, \dots, T_n$ be as in Proposition 8.8 b); let C/k be a smooth projective connected curve, with function field K. From Proposition 8.8 b), Theorem 6.2, Theorem 4.8 b) and (4.3), we get a residue map

$$\partial_v : \tilde{K}(K; T_1, \dots, T_n, \mathbf{G}_m) \to \tilde{K}(k(v); T_1, \dots, T_n)$$

for any $v \in C$. These maps satisfy the reciprocity law of Proposition 4.6 and the compatibility of Lemma 4.2.

9. Reduction to the representable case

Following [30, p. 207], we write $h_0^{\mathrm{Nis}}(X) := h_0^{\mathrm{Nis}}(L(X))$ for a smooth variety X over k.

9.1. **Proposition.** *The following statements are equivalent:*
a) The homomorphism (7.1) *is bijective for any* $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.
b) Let $\mathcal{F}_1 = \dots = \mathcal{F}_n = h_0^{\mathrm{Nis}}(C')$ *for a smooth connected curve* C'/k. *Then* (7.1) *is bijective.*
c) Let C *be a smooth projective connected curve over* k, *and let* $f : C \to \mathbf{P}^1$ *be a surjective morphism. Let* $C' = f^{-1}(\mathbf{P}^1 \setminus \{1\})$ *and let* $\iota : C' \to \mathcal{A}$ *be the tautological morphism, where* $\mathcal{A} = h_0^{\mathrm{Nis}}(C')$. *These data define a relation datum of geometric type* $(C, f, (\iota, \dots, \iota))$ *for* $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathcal{A}$, *and its associated element* (6.1) *is*

$$\sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k}(\iota(c) \otimes \dots \otimes \iota(c)) \in \mathcal{A} \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{A}(k). \tag{9.1}$$

Then the image of (9.1) *in* $K(k; \mathcal{A}, \dots, \mathcal{A})$ *vanishes.*

Proof. Only the implication c) $\Rightarrow$ a) requires a proof. Let $(C, f, (g_i))$ be a relation datum of geometric type for $\mathcal{F}_1, \dots, \mathcal{F}_n$. We need to show the vanishing of (6.1) in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$.

By adjunction, the section $g_i : M(C') \to \mathcal{F}_i[0]$ induces a morphism $\varphi_i : \mathcal{A} \to \mathcal{F}_i$ for all $i = 1, \dots, n$. Then

$$\sum_{c \in C'} v_c(f) \{g_1(c), \dots, g_n(c)\}_{k(c)/k} = 0 \quad \text{in } K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \tag{9.2}$$

because it is the image of (9.1) by the homomorphism $K(k; \mathcal{A}, \dots, \mathcal{A}) \to K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ defined by $(\varphi_1, \dots, \varphi_n)$. □

10. Proper sheaves

10.1. **Definition.** Let $\mathcal{F}$ be a Nisnevich sheaf with transfers. We call $\mathcal{F}$ *proper* if, for any smooth curve C over k and any closed point $c \in C$, the induced map $\mathcal{F}(\mathcal{O}_{C,c}) \to \mathcal{F}(k(C))$ is surjective. We say that $\mathcal{F}$ is *universally proper* if the above condition holds when replacing k by any finitely generated extension K/k, and C by any regular K-curve.

10.2. *Example.* A semi-abelian variety G over k is proper (in the above sense) if and only if G is an abelian variety. A birational sheaf $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ in the sense of [10] is by definition proper. If C is a smooth proper curve, then $h_0^{\mathrm{Nis}}(C)$ is proper. Other examples of birational sheaves will be given in Lemma 11.2 b) below.

In fact:

10.3. **Lemma.** *Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. Then*
a) $\mathcal{F}$ is proper if and only if $\mathcal{F}(C) \xrightarrow{\sim} \mathcal{F}(k(C))$ for any smooth k-curve C.
b) $\mathcal{F}$ is universally proper if and only if it is birational in the sense of [10].

Proof. Let us prove b), as the proof of a) is a subset of it. Let X be a smooth k-variety. By [29, Cor. 4.19], the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective for any dense open subset of X. By definition, $\mathcal{F}$ is birational if one may replace "injective" by "bijective". So birational $\Rightarrow$ universally proper. Conversely, assume $\mathcal{F}$ to be universally proper; let $x \in X^{(1)}$ and let $p : X \to \mathbf{A}^{d-1}$ be a dominant rational map defined at x, where $d = \dim X$. (We may find such a p thanks to Noether's normalization theorem.) Applying the hypothesis to the generic fibre of p, we find that $\mathcal{F}(\mathcal{O}_{X,x}) \to \mathcal{F}(k(X))$ is surjective. Since this is true for all points $x \in X^{(1)}$, we get the surjectivity of $\mathcal{F}(X) \to \mathcal{F}(k(X))$ from Voevodsky's Gersten resolution [29, Th. 4.37]. □

The following proposition is not necessary for the proof of the main theorem, but its proof is much simpler than the general case.

10.4. **Proposition.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Assume that $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ are proper. Then the homomorphism* (7.1) *is bijective.*

Proof. Suppose $(C, f, (g_i))$ is a relation datum of geometric type. It suffices to show the element (6.1) vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Let $\bar{g}_i$ be the image of g_i in $\mathcal{F}(k(C))$. By assumption we have $\bar{g}_i \in \mathrm{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \to \mathcal{F}_i(k(C)))$ for all $c \in C$ and $i = 1, \dots, n-1$. Hence $(C, h, (\bar{g}_i)_{i=1,\dots,n})$ is a relation datum of Somekawa type (with $i(c) = n$ for all $c \in C$). By Corollary 4.11, the element (6.1) coincides with (5.2), hence vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. □

11. Main theorem

11.1. **Definition.** Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. We say that $\mathcal{F}$ is *curve-like* if there exists an exact sequence in $\mathbf{HI}_{\mathrm{Nis}}$

$$0 \to T \to \mathcal{F} \to \bar{\mathcal{F}} \to 0 \tag{11.1}$$

where $\bar{\mathcal{F}}$ is proper (Definition 10.1) and T is a torus for which there exists an exact sequence

$$0 \to R_{E_1/k}\mathbf{G}_m \to R_{E_2/k}\mathbf{G}_m \to T \to 0 \tag{11.2}$$

where E_1 and E_2 are étale k-algebras.

This terminology is justified by the following lemma:

11.2. **Lemma.** *a) If C is a smooth curve over k, then $h_0^{\mathrm{Nis}}(C)$ is the Nisnevich sheaf associated to the presheaf of relative Picard groups*

$$U \mapsto \mathrm{Pic}(\bar{C} \times U, D \times U)$$

where $\bar{C}$ is the smooth projective completion of C, $D = \bar{C} \setminus C$ and U runs through smooth k-schemes.
b) If X is a smooth projective variety over k, then, for any smooth variety U over k, we have

$$h_0^{\mathrm{Nis}}(X)(U) = CH_0(X_{k(U)}), \tag{11.3}$$

where $k(U)$ denotes the total ring of fractions of U. In particular, $h_0^{\mathrm{Nis}}(X)$ is birational.
c) For any smooth curve C, $h_0^{\mathrm{Nis}}(C)$ is curve-like.

Proof. a) and b) are proven in [27, Th. 3.1] and in [7, Th. 2.2] respectively. With the notation of a), we put $E = H^0(\bar{C}, \mathcal{O}_{\bar{C}})$. Then c) follows from the exact sequence

$$0 \to R_{E/k}\mathbf{G}_m \to R_{D/k}\mathbf{G}_m \to h_0^{\mathrm{Nis}}(C) \to h_0^{\mathrm{Nis}}(\bar{C}) \to 0$$

stemming from the Gysin exact triangle

$$M(D)(1)[1] \to M(C) \to M(\bar{C}) \xrightarrow{+1}$$

of [30, Prop. 3.5.4]. □

11.3. *Remark.* Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ be curve-like. The torus T and proper sheaf $\bar{\mathcal{F}}$ in (11.1) are uniquely determined by $\mathcal{F}$ up to unique isomorphism. Indeed, this amounts to showing that any morphism $T \to \bar{\mathcal{F}}$ is trivial. This is reduced to the case $T = R_{E/k}\mathbf{G}_m$ as in (11.2), and further to $T = \mathbf{G}_m$ by adjunction as in Lemma 8.6. Then we have $\mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{G}_m, \bar{\mathcal{F}}) \cong \bar{\mathcal{F}}_{-1}(k) = 0$ by definition (see (4.1) and Definition 10.1).

We call T and $\bar{\mathcal{F}}$ the *toric* and *proper* part of $\mathcal{F}$ respectively.

11.4. **Lemma.** *a) Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ be curve-like with toric part T, and let C be a smooth proper connected k-curve. Let Z be a closed subset of C, $A = \mathcal{O}_{C,Z}$ and $K = k(C)$. Then the sequence*

$$0 \to T(A) \to T(K) \oplus \mathcal{F}(A) \to \mathcal{F}(K) \to 0$$

is exact.
b) Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be curve-like with toric parts $T_1, \dots, T_n$, and let C, Z, A, K be as in a). Then the group $\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K)$ has the following presentation:

Generators: *for each subset $I \subseteq \{1, \dots, n\}$, elements $[I; f_1, \dots, f_n]$ with $f_i \in \mathcal{F}_i(A)$ if $i \in I$ and $f_i \in T_i(K)$ if $i \notin I$.*

Relations:

- *Multilinearity:*

$$[I; f_1, \dots, f_i + f_i', \dots, f_n] = [I; f_1, \dots, f_i, \dots, f_n] + [I; f_1, \dots, f_i', \dots, f_n].$$

- *Let $I \subsetneq \{1, \dots, n\}$ and let $i_0 \notin I$. Let $[I; f_1, \dots, f_n]$ be a generator. Suppose that $f_{i_0} \in T_{i_0}(A)$. Then $[I; f_1, \dots, f_n] = [I \cup \{i_0\}; f_1, \dots, f_n]$.*

Proof. a) Consider the commutative diagram of 0-sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(A) & \longrightarrow & \mathcal{F}(A) & \longrightarrow & \bar{\mathcal{F}}(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & T(K) & \longrightarrow & \mathcal{F}(K) & \longrightarrow & \bar{\mathcal{F}}(K) & \longrightarrow & 0. \end{array}$$

By [the proof of] [29, Cor. 4.18], the top sequence is a direct summand of the bottom one, which is clearly exact. Thus the top sequence is exact as well, and the lemma follows from a diagram chase. Then b) follows from a). □

11.5. **Proposition.** *Let C/k be a smooth proper connected curve, and let $v \in C, K = k(C)$. Then there exists a unique law associating to a system $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ of n curve-like sheaves a homomorphism*

$$\partial_v : \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* \to \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n)$$

such that

(i) *If σ is a permutation of $\{1, \dots, n\}$, the diagram*

$$\begin{array}{ccc} \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n) \\ \sigma\downarrow & & \sigma\downarrow \\ \mathcal{F}_{\sigma(1)}(K) \otimes \cdots \otimes \mathcal{F}_{\sigma(n)}(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_{\sigma(1)}, \dots, \mathcal{F}_{\sigma(n)}) \end{array}$$

commutes.

(ii) *If $[I, f_1, \dots, f_n]$ is a generator of $\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K)$ as in Lemma 11.4 b) for some Z containing v, with $I = \{1, \dots, i\}$, then*

$$\partial_v(f_1 \otimes \cdots \otimes f_n \otimes f) = \{f_1(v), \dots f_i(v), \partial_v(\{f_{i+1}, \dots, f_n, f\}_{K/K})\}_{k(v)/k}$$

where $\partial_v(\{f_{i+1}, \dots, f_n, f\}_{K/K})$ is the residue of 8.10.

Proof. By Lemma 11.4 b), it suffice to check that ∂_v agrees on relations. Up to permutation, we may assume $I = \{1, \dots, i\}$ and $i_0 = i + 1$. The claim then follows from Proposition 4.10. □

11.6. **Lemma.** *a) Keep the notation of Proposition 11.5. Let L/K be a finite extension; write D for the smooth projective model of L and $h : D \to C$ for the corresponding morphism. Let $Z = h^{-1}(v)$. Write $\mathcal{F}_{n+1} = \mathbf{G}_m$. Then, for any $i \in \{1, \dots, n+1\}$, the diagram*

$$\begin{tikzcd}
\mathcal{F}_1(L) \otimes \cdots \otimes \mathcal{F}_{n+1}(L) \arrow[r, "(\partial_w)"] & \bigoplus_{w \in Z} \tilde{K}(k(w); \mathcal{F}_1, \dots, \mathcal{F}_n) \arrow[dd, "(\mathrm{Tr}_{k(w)/k(v)})"] \\
\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_i(L) \otimes \cdots \otimes \mathcal{F}_{n+1}(K) \arrow[u, "u"] \arrow[d, "d"'] & \\
\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_{n+1}(K) \arrow[r, "\partial_v"] & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n)
\end{tikzcd}$$

commutes, where u is given componentwise by functoriality for $j \neq i$ and by the identity for $j = i$, and d is given componentwise by the identity for $j \neq i$ and by $\mathrm{Tr}_{L/K}$ for $j = i$.
b) The homomorphisms ∂_v induce residue maps

$$\partial_v : \left(\mathcal{F}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{F}_n \overset{M}{\otimes} \mathbf{G}_m\right)(K) \to \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n).$$

which verify the compatibility of Lemma 4.2 b).

Proof. a) For clarity, we distinguish two cases: $i < n+1$ and $i = n+1$. In the former case, up to permutation we may assume $i = n$. It is enough to check commutativity on generators in the style of Lemma 11.4 b). Let T_l denote the toric part of $\mathcal{F}_l$. In view of Lemma 11.4 a) and Proposition 11.5 (i), it suffices to check the commutativity for $x = f_1 \otimes \cdots \otimes f_n \otimes f$ when one of the following two conditions is satisfied:

(i) for some $j \in \{0, \dots, n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ $(1 \leq l \leq j)$, $f_l \in T_l(K)$ $(j+1 \leq j \leq n-1)$, $f_n \in T_n(L)$ and $f \in K^*$.
(ii) for some $j \in \{0, \dots, n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ $(1 \leq l \leq j)$, $f_l \in T_l(K)$ $(j+1 \leq j \leq n-1)$, $f_n \in \mathcal{F}_n(\mathcal{O}_{D,Z})$ and $f \in K^*$.

Let $w \in Z$. If (i) holds, we have

$$\partial_w(u(x)) = \{f_1(w), \dots, f_j(w), \partial_w(\{f_{j+1}, \dots, f_n, f\}_{L/L})\}_{k(w)/k(w)}$$

and

$$\partial_v(d(x)) = \{f_1(v), \dots, f_j(v), \partial_v(\{f_{j+1}, \dots, \mathrm{Tr}_{L/K}(f_n), f\}_{K/K})\}_{k(v)/k(v)}.$$

Observe that the restriction of $f_l(v)$ to $k(w)$ is $f_l(w)$ for every $w \in Z$ and $l = 1, \dots, j$. Since the residue maps (∂_w) of §8.10 verify the

compatibility of Lemma 4.2, the commutativity for x follows. (Recall that $\mathrm{Tr}_{k(w)/k(v)}(\{a_1,\dots,a_n\}_{k(w)/k(w)}) = \{a_1,\dots,a_n\}_{k(w)/k(v)}$.)

If (ii) holds, we have

$$\partial_w(u(x)) = \{f_1(w),\dots,f_j(w),\partial_w(\{f_{j+1},\dots,f_{n-1},f\}_{L/L}),f_n(w)\}_{k(w)/k(w)}$$

and

$$\partial_v(d(x)) = \{f_1(v),\dots,f_j(v),\partial_v(\{f_{j+1},\dots f_{n-1},f\}_{K/K}),\mathrm{Tr}_{L/K}(f_n)(v)\}_{k(v)/k(v)}.$$

In addition to the observation mentioned in (i), we remark that the restriction of $\partial_v(\{f_{j+1},\dots,f_{n-1},f\}_{K/K})$ to $k(w)$ is $\partial_w(\{f_{j+1},\dots,f_{n-1},f\}_{L/L})$ for every $w \in Z$. The commutativity for x follows from Lemma 4.2 b) applied to $\mathcal{F}_n$.

If $i = n+1$ the check is similar, the projection formula working on the last variable.

Now b) follows from a) and the definition of $\overset{M}{\otimes}$ as in [8, p. 84]. □

11.7. **Lemma.** *The homomorphisms ∂_v of Lemma 11.6 induce residue maps*

$$\partial_v : \tilde{K}(K;\mathcal{F}_1,\dots,\mathcal{F}_n,\mathbf{G}_m) \to \tilde{K}(k(v);\mathcal{F}_1,\dots,\mathcal{F}_n).$$

which verify the compatibility of Lemma 4.2 b).

Proof. Set $\mathcal{F}_{n+1} = \mathbf{G}_m$. Let $i < j$ be two elements of $\{1,\dots,n+1\}$ and let $\chi_i : \mathbf{G}_m \to \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \to \mathcal{F}_j$ be two cocharacters. Let $f \in K^* - \{1\}$. We must show that ∂_v vanishes on

$$x = f_1 \otimes \cdots \otimes \chi_i(f) \otimes \cdots \otimes \chi_j(1-f) \otimes \cdots \otimes f_{n+1}$$

for any $(f_1,\dots,f_{n+1}) \in \mathcal{F}_1(K) \times \cdots \times \mathcal{F}_{n+1}(K)$ (product excluding (i,j)). By functoriality, we may assume that χ_i, χ_j are the identity cocharacters. We distinguish two cases for clarity: $j < n+1$ and $j = n+1$. But exactly the same argument works for both cases. Presently we suppose $j < n+1$.

Up to permutation, we may assume $i = n-1$, $j = n$. Let us say that an element $(x_1,\dots,x_{n-2}) \in \mathcal{F}_1(K) \times \cdots \times \mathcal{F}_{n-2}(K)$ is *in normal form* if, for each i, either $x_i \in \mathcal{F}_i(\mathcal{O}_v)$ or $x_i \in T_i(K)$. Then Lemma 11.4 reduces us to the case where $(f_1,\dots,f_{n-2})$ is in normal form. Up to permutation, we may assume that $f_r \in \mathcal{F}_r(\mathcal{O}_v)$ for $r \le r_0$ and $f_r \in T_r(K)$ for $r_0 < r \le n-2$. Then

$$\partial_v x = \{f_1(v),\dots,f_{r_0}(v),\partial_v(\{f_{r_0+1},\dots,f_{n-2},f,(1-f),f_{n+1}\}_{K/K})\}_{k(v)/k(v)}.$$

Let $\varphi_v : \tilde{K}(k(v),T_{r_0+1},\dots,T_n) \to \tilde{K}(k(v),\mathcal{F}_1,\dots,\mathcal{F}_n)$ be the homomorphism induced by $(f_1(v),\dots,f_{r_0}(v))$ via (8.2), and let φ_K :

$T_{r_0+1}(K) \otimes \cdots \otimes T_n(K) \otimes K^* \to \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^*$ be the analogous homomorphism defined by $(f_1, \dots, f_{r_0})$. The diagram

$$\begin{array}{ccc} T_{r_0+1}(K) \otimes \cdots \otimes T_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); T_{r_0}, \dots, T_n) \\ \varphi_K \big\downarrow & & \varphi_v \big\downarrow \\ \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n) \end{array}$$

commutes. But the top map factors through $\tilde{K}(K; T_{r_0+1}, \dots, T_n, \mathbf{G}_m)$, hence the desired vanishing.

Thus we have shown that the map ∂_v of Proposition 11.5 vanishes on $St(K; \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. The conclusion now follows from Lemma 11.6 b). □

11.8. Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ and let C be a smooth proper k-curve. The *support* of a section $f \in \mathcal{F}(k(C))$ is the finite set

$$\mathrm{Supp}(f) = \{c \in C \mid \partial_c f \neq 0\}.$$

The following proposition generalizes Lemma 7.4:

11.9. **Proposition.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be n curve-like sheaves, and let C be a smooth proper k-curve. Put $\mathcal{F}_{n+1} = \mathbf{G}_m$. If the field k is infinite, the group $\tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$ is generated by elements $\{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$ where D is another curve, $D \to C$ is a finite surjective morphism and $f_i \in \mathcal{F}_i(k(D))$ satisfy*

$$\mathrm{Supp}(f_i) \cap \mathrm{Supp}(f_j) = \emptyset \quad \textit{for all } 1 \leq i < j \leq n. \tag{11.4}$$

Proof. Lemma 11.4 b) reduces us to the case where all $\mathcal{F}_i$ are $R_{E_i/k}\mathbf{G}_m$ for some étale k-algebras E_i/k. Using the formula

$$(R_{E_1/k}\mathbf{G}_{m,E_1})_{E_2} \cong R_{E_1 \otimes_k E_2/E_2}\mathbf{G}_{m,E_1 \otimes E_2}$$

and Lemma 8.6 repeatedly, we are further reduced to the case all $\mathcal{F}_i$ are $\mathbf{G}_m$. Then it follows from Lemma 7.4. □

11.10. **Lemma.** *Let $C, D, \mathcal{F}_1, \dots, \mathcal{F}_n$ be as in Proposition 11.9. Let $f_i \in \mathcal{F}_i(k(D))$ and $v \in D$. Put $\xi := \{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$, regarded as an element of $\tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$.*

1. *If $v(f_{n+1} - 1) > 0$, then we have $\partial_v(\xi) = 0$.*
2. *Suppose (11.4) holds. If $v \in \mathrm{Supp}(f_i)$ for some $1 \leq i \leq n$, then we have*

$$\partial_v(\xi) = \{f_1(v), \dots, \partial_v(f_i, f_{n+1}), \dots, f_n(v)\}_{k(v)/k}.$$

Proof. This follows from Corollary 4.11 and Proposition 4.10. □

11.11. **Proposition.** *Let C be a smooth projective connected curve, and let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be curve-like. The composition*

$$\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v : \tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m) \to \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \\ \to K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$$

is the zero-map.

Proof. a) Assume first k infinite. If $\xi = \{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$ satisfies (11.4), then we have $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\xi) = 0$ by Definition 5.1 and Lemma 11.10 (2). Hence the claim follows from Proposition 11.9.

b) If k is finite, we use a classical trick: let p_1, p_2 be two distinct prime numbers, and let k_i be the $\mathbf{Z}_{p_i}$-extension of k. Let $x \in \tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. By a), the image of x in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k_1; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and $K(k_2; \mathcal{F}_1, \dots, \mathcal{F}_n)$, hence is 0 by a transfer argument. □

Finally, we arrive at:

11.12. **Theorem.** *The homomorphism* (1.1) *is an isomorphism for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.*

Proof. It suffices to show the statement in Proposition 9.1 (3). With the notation therein, the image of (9.1) in $K(k; \mathcal{A}, \dots, \mathcal{A})$ is seen to be $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\{\iota, \dots, \iota, f\}_{k(C)/k(C)})$ by Lemma 11.10, hence trivial by Proposition 11.11. □

12. Application to algebraic cycles

12.1. We assume k is of characteristic zero. Let X be a k-scheme of finite type, and let $M^c(X) := C^c_*(X) \in \mathbf{DM}^{\mathrm{eff}}_-$ be the motive of X with compact supports [30, §4.1]. Then the sheaf $\underline{CH}_0(X)$ of §1.4 agrees with $H_0(M^c(X))$ by [7, Th. 2.2]. If X is quasi-projective, we have an isomorphism

$$CH_{-i}(X, j + 2i) \cong \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_-}(\mathbf{Z}, M^c(X))(i)[-j])$$

for all $i \in \mathbf{Z}_{\geq 0}, j \in \mathbf{Z}$ by [30, Prop. 4.2.9].[4]

[4]The proof of *loc. cit.* is written for equidimensional schemes but is the same in general. Moreover, the assumption "quasi-projective" can be removed if one replaces higher Chow groups by the Zariski hypercohomology of the cycle complex as in [14, after Theorem 1.7].

Proof of Theorem 1.5. Using Lemma 3.3, we see

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \underline{CH}_0(X_1)[0] \otimes \cdots \otimes \underline{CH}_0(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}) \\ \cong \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M^c(X_1) \otimes \cdots \otimes M^c(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}). \\ \cong \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M^c(X)(r)[r]) \cong CH_{-r}(X, r).\end{aligned}$$

(Here we used $\mathbf{G}_m[0] \cong \mathbf{Z}(1)[1]$.) Now the theorem follows from Theorem 11.12. □

12.2. Let X be a k-scheme of finite type. Recall that for $i \in \mathbf{Z}_{\geq 0}, j \in \mathbf{Z}$ the motivic homology of X is defined by [6, Def. 9.4]

$$H_j(X, \mathbf{Z}(-i)) := \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M(X)(i)[-j]). \tag{12.1}$$

When $i = 0$, $H_j(X, \mathbf{Z}(0))$ agrees with Suslin homology [27].

12.3. **Theorem.** *Let $X_1, \ldots, X_n$ be k-schemes of finite type. Suppose either the X_i are smooth or* $\mathrm{char}\, k = 0$. *Put $X = X_1 \times \cdots \times X_n$. For any $r \geq 0$, we have an isomorphism*

$$K(k; h_0^{\mathrm{Nis}}(X_1), \ldots, h_0^{\mathrm{Nis}}(X_n), \mathbf{G}_m, \ldots, \mathbf{G}_m) \xrightarrow{\sim} H_{-r}(X, \mathbf{Z}(r)).$$

Proof. Using Lemma 3.3, we see

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, h_0^{\mathrm{Nis}}(X_1)[0] \otimes \cdots \otimes h_0^{\mathrm{Nis}}(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}) \\ \cong \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M(X_1) \otimes \cdots \otimes M(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}). \\ \cong \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M(X)(r)[r]) \cong H_{-r}(X, \mathbf{Z}(-r)).\end{aligned}$$

Now the theorem follows from Theorem 11.12. □

12.4. *Remark.* If $X_1, \ldots, X_n$ are smooth projective varieties, then (1.3) is valid in any characteristic. Indeed, we have $M(X_i) = M^c(X_i)$ and hence $\underline{CH}_0(X_i) = h_0^{\mathrm{Nis}}(X_i)$. Moreover, [31] and [7, Appendix B] show $H_{-r}(X, \mathbf{Z}(-r)) \cong CH_{-r}(X, r)$. Thus (1.3) follows from Theorem 12.3.

Appendix A. Extending monoidal structures

A.1. Let $\mathcal{A}$ be an additive category. We write $\mathcal{A}$–Mod for the category of contravariant additive functors from $\mathcal{A}$ to abelian groups. This is a Grothendieck abelian category. We have the additive Yoneda embedding

$$y_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}\text{–Mod}$$

sending an object to the corresponding representable functor.

A.2. Let $f : \mathcal{A} \to \mathcal{B}$ be an additive functor. We have an induced functor $f^* : \mathcal{B}\text{–}\operatorname{Mod} \to \mathcal{A}\text{–}\operatorname{Mod}$ ("composition with f"). As in [SGA4, Exp. 1, Prop. 5.1 and 5.4], the functor f^* has a left adjoint $f_!$ and a right adjoint f_* and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \mathcal{A}\text{–}\operatorname{Mod} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_!}\big\downarrow \\ \mathcal{B} & \xrightarrow{y_{\mathcal{B}}} & \mathcal{B}\text{–}\operatorname{Mod} \end{array}$$

is naturally commutative.

A.3. If f is fully faithful, then $f_!$ and f_* are fully faithful and f^* is a localization, as in [SGA4, Exp. 1, Prop. 5.6].

A.4. Suppose that f has a left adjoint g. Then we have natural isomorphisms

$$g^* \simeq f_!, \quad g_* \simeq f^*$$

as in [SGA4, Exp. 1, Prop. 5.5].

A.5. Suppose further that f is fully faithful. Then $g^* \simeq f_!$ is fully faithful. From the composition

$$g^* g_* \Rightarrow Id_{\mathcal{A}\text{–}\operatorname{Mod}} \Rightarrow g^* g_!$$

of the unit with the counit, one then deduces a canonical morphism of functors

$$g_* \Rightarrow g_!.$$

A.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. Their *tensor product* is the category $\mathcal{A} \boxtimes \mathcal{B}$ whose objects are finite collections (A_i, B_i) with $(A_i, B_i) \in \mathcal{A} \times \mathcal{B}$, and

$$(\mathcal{A} \boxtimes \mathcal{B})((A_i, B_i), (C_j, D_j)) = \bigoplus_{i,j} \mathcal{A}(A_i, C_j) \otimes \mathcal{B}(B_i, D_j).$$

We have a "cross-product" functor

$$\boxtimes : \mathcal{A}\text{–}\operatorname{Mod} \times \mathcal{B}\text{–}\operatorname{Mod} \to (\mathcal{A} \boxtimes \mathcal{B})\text{–}\operatorname{Mod}$$

given by

$$(M \boxtimes N)((A_i, B_i)) = \bigoplus_i M(A_i) \otimes N(B_i).$$

A.7. Let $\mathcal{A}$ be provided with a biadditive bifunctor $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We may view $\bullet$ as an additive functor $\mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$. We may then extend $\bullet$ to $\mathcal{A}\text{–}\operatorname{Mod}$ by the composition

$$\mathcal{A}\text{–}\operatorname{Mod} \times \mathcal{A}\text{–}\operatorname{Mod} \xrightarrow{\boxtimes} (\mathcal{A} \boxtimes \mathcal{A})\text{–}\operatorname{Mod} \xrightarrow{\bullet_!} \mathcal{A}\text{–}\operatorname{Mod}.$$

This is an extension in the sense that the diagram

$$\begin{tikzcd} \mathcal{A}\times\mathcal{A} \arrow[r, "y_{\mathcal{A}}\times y_{\mathcal{A}}"] \arrow[d, "\bullet\times\bullet"'] & \mathcal{A}\text{–Mod}\times\mathcal{A}\text{–Mod} \arrow[d, "\bullet"'] \\ \mathcal{A} \arrow[r, "y_{\mathcal{A}}"] & \mathcal{A}\text{–Mod} \end{tikzcd}$$

is naturally commutative.

If $\bullet$ is monoidal (resp. monoidal symmetric), then its associativity and commutativity constraints canonically extend to $\mathcal{A}$–Mod.

A.8**.** Let $\mathcal{A},\mathcal{B}$ be two additive symmetric monoidal categories, and let $f:\mathcal{A}\to\mathcal{B}$ be an additive symmetric monoidal functor. The above definition shows that the functor $f_!:\mathcal{A}\text{–Mod}\to\mathcal{B}\text{–Mod}$ is also symmetric monoidal.

A.9**.** In §A.7, let us write $\bullet_! = \int$ for clarity. Let $P\in(\mathcal{A}\boxtimes\mathcal{A})\text{–Mod}$. Then $\int P$ is the *left Kan extension of P along* $\bullet$ in the sense of [15, X.3]. This gives a formula for $\int P$ as a *coend* (ibid., Theorem X.4.1); for $A\in\mathcal{A}$:

$$\int P(A) = \int^{(B,B')} \mathcal{A}(A, B\bullet B')\otimes P(B,B'). \tag{A.1}$$

In particular:

A.10. **Proposition.** *Suppose $\mathcal{A}$ rigid. Then* (A.1) *simplifies as*

$$\int P(A) = \int^{B} P(B, A\bullet B^*)$$

where B^* is the dual of $B\in\mathcal{A}$. In particular, if $P = M\boxtimes N$ for $M,N\in\mathcal{A}$–Mod, we have for $A\in\mathcal{A}$:

$$(M\bullet N)(A) = \int^{B} M(B)\otimes N(A\bullet B^*) \tag{A.2}$$

which describes $M\bullet N$ as a "convolution".

Proof. Applying (A.1) and rigidity, we have

$$\begin{aligned}\int P(A) &= \int^{(B,B')} \mathcal{A}(A, B\bullet B')\otimes P(B,B') \\ &= \int^{(B,B')} \mathcal{A}(A\bullet B^*, B')\otimes P(B,B') \\ &= \int^{B} P(B, A\bullet B^*)\end{aligned}$$

because in the third formula, the variable B' is dummy (this simplification is not in Mac Lane!). □

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