

# AROUND KING'S RANK-ONE THEOREMS: FLOWS AND $\mathbb{Z}^n$ -ACTIONS

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ABSTRACT. We study the generalizations of Jonathan King's rank-one theorems (Weak-Closure Theorem and rigidity of factors) to the case of rank-one  $\mathbb{R}$ -actions (flows) and rank-one  $\mathbb{Z}^n$ -actions. We prove that these results remain valid in the case of rank-one flows. In the case of rank-one  $\mathbb{Z}^n$  actions, where counterexamples have already been given, we prove partial Weak-Closure Theorem and partial rigidity of factors.

## 1. INTRODUCTION

Very important examples in ergodic theory have been constructed in the class of rank-one transformations, which is closely connected to the notion of transformations with fast cyclic approximation [3]: If the rate of approximation is sufficiently fast, then the transformation will be inside the rank-one class. The notion of rank-one transformations has been defined in [8], where mixing examples have appeared. Later, Daniel Rudolph used them for a machinery of counterexamples [12].

Jonathan King contributed to the theory of rank-one transformations by several deep and interesting facts. His Weak-Closure-Theorem (WCT) [4] is now a classical result with applications even out of the range of  $\mathbb{Z}$ -actions (see for example [16]). He also proved the minimal-self-joining (MSJ) property for rank-one mixing automorphisms (see [5]), the rigidity of non-trivial factors [4], and the weak closure property for all joinings for flat-roof rank-one transformations [6].

A natural question is whether the corresponding assertions remain true for flows ( $\mathbb{R}$ -actions) and for  $\mathbb{Z}^n$ -actions. We show that for flows the situation is quite similar: The joining proof of the Weak-Closure Theorem given in [13] (see also [15]) can be adapted to the situation of a rank-one  $\mathbb{R}$ -action (Theorem 5.2). We also give in the same spirit a proof of the rigidity of non-trivial factors of rank-one flows (Theorem 6.2) which, with some simplification, provides a new proof of King's result in the case of  $\mathbb{Z}$ -actions. We prove a flat-roof flow version as well (Theorem 7.1). Note that a proof of the Weak-Closure Theorem for rank-one flows had already been published in [17]. Unfortunately it relies on the erroneous assumption that if  $(T_t)_{t \in \mathbb{R}}$  is a rank-one flow, then there exists a real number  $t_0$  such that  $T_{t_0}$  is a rank-one transformation (see beginning of Section 3.2 in [17]).

Concerning multidimensional rank-one actions, the situation is quite different. The Weak-Closure Theorem is no more true [1], and factors may be non-rigid [2]. Rank-one partially mixing  $\mathbb{Z}$ -actions have MSJ [7], however it is proved in [2] that

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for  $\mathbb{Z}^2$ -actions this is generally not true. We remark that it was an answer for  $\mathbb{Z}^2$ -action to Jean-Paul Thouvenot's question: Whether a mildly mixing rank-one action possesses MSJ, though this interesting problem remains open for  $\mathbb{Z}$ -actions. Regardless these surprising results, there are some partial versions of WCT: Commuting automorphisms can be partially approximated by elements of the action (Corollary 8.4), and non-trivial factors must be partially rigid (Corollary 8.5). We present these results as consequences of A. Pavlova's theorem (Theorem 8.3, see also [14]).

## 2. PRELIMINARIES AND NOTATIONS

**Weak convergence of probability measures.** We are interested in groups of automorphisms of a Lebesgue space  $(X, \mathcal{A}, \mu)$ , where  $\mu$  is a continuous probability measure. The properties of these group actions are independent of the choice of the underlying space  $X$ , and for practical reasons we will assume that  $X = \{0, 1\}^{\mathbb{Z}}$ , equipped with the product topology and the Borel  $\sigma$ -algebra. This  $\sigma$ -algebra is generated by the cylinder sets, that is sets obtained by fixing a finite number of coordinates. On the set  $\mathcal{M}_1(X)$  of Borel probability measures on  $X$ , we will consider the topology of weak convergence, which is characterized by

$$\nu_n \xrightarrow[n \rightarrow \infty]{w} \nu \iff \text{for all cylinder set } C, \nu_n(C) \xrightarrow[n \rightarrow \infty]{} \nu(C),$$

and turns  $\mathcal{M}_1(X)$  into a compact metrizable space.

We will often consider probability measures on  $X \times X$ , with the same topology of weak convergence. We will use the following observation: If  $\nu_n$  and  $\nu$  in  $\mathcal{M}_1(X \times X)$  have their marginals absolutely continuous with respect to our reference measure  $\mu$ , with bounded density, then the weak convergence of  $\nu_n$  to  $\nu$  ensures that for all measurable sets  $A$  and  $B$  in  $\mathcal{A}$ ,  $\nu_n(A \times B) \xrightarrow[n \rightarrow \infty]{} \nu(A \times B)$ .

**Self-joinings.** Let  $T = (T_g)_{g \in G}$  be an action of the Abelian group  $G$  by automorphism of the Lebesgue space  $(X, \mathcal{A}, \mu)$ . A *self-joining* of  $T$  is any probability measure on  $X \times X$  with both marginals equal to  $\mu$  and invariant by  $T \times T = (T_g \times T_g)_{g \in G}$ . For any automorphism  $S$  commuting with  $T$ , we will denote by  $\Delta_S$  the self-joining concentrated on the graph of  $S^{-1}$ , defined by

$$\forall A, B \in \mathcal{A}, \Delta_S(A \times B) := \mu(A \cap SB).$$

In particular, for any  $g \in G$  we will denote by  $\Delta^g$  the self-joining  $\Delta_{T_g}$ . In the special case where  $S = T_0 = \text{Id}$ , we will note simply  $\Delta$  instead of  $\Delta^0$  or  $\Delta_{\text{Id}}$ .

If  $\mathcal{F}$  is a factor (a sub- $\sigma$ -algebra invariant under the action  $(T_g)$ ), we denote by  $\mu \otimes_{\mathcal{F}} \mu$  the *relatively independent joining above  $\mathcal{F}$* , defined by

$$\mu \otimes_{\mathcal{F}} \mu(A \times B) := \int_X \mathbb{E}_{\mu}[\mathbb{1}_A | \mathcal{F}] \mathbb{E}_{\mu}[\mathbb{1}_B | \mathcal{F}] d\mu.$$

Recall that  $\mu \otimes_{\mathcal{F}} \mu$  coincides with  $\Delta$  on the  $\sigma$ -algebra  $\mathcal{F} \otimes \mathcal{F}$ .

**Flows.** A *flow* is a continuous family  $(T_t)_{t \in \mathbb{R}}$  of automorphisms of the Lebesgue space  $(X, \mathcal{A}, \mu)$ , with  $T_t \circ T_s = T_{t+s}$  for all  $t, s \in \mathbb{R}$ , and such that  $(t, x) \mapsto T_t(x)$  is measurable. We recall that the measurability condition implies that for all measurable set  $A$ ,  $\mu(A \triangle T_t A) \xrightarrow[t \rightarrow 0]{} 0$ .

**Lemma 2.1.** *Let  $(T_t)_{t \in \mathbb{R}}$  be an ergodic flow on  $(X, \mathcal{A}, \mu)$ . Let  $Q$  be a dense subgroup of  $\mathbb{R}$ , and  $\lambda$  be an invariant probability measure for the action of  $(T_t)_{t \in Q}$ . Assume further that  $\lambda \ll \mu$ , with  $\frac{d\lambda}{d\mu}$  bounded by some constant  $C$ . Then  $\lambda = \mu$ .*

*Proof.* Let  $t \in \mathbb{R}$ , and let  $(t_n)$  be a sequence in  $Q$  converging to  $t$ . For any measurable set  $A$ , we have

$$\lambda(T_t A \triangle T_{t_n} A) \leq C \mu(T_t A \triangle T_{t_n} A) \xrightarrow{n \rightarrow \infty} 0.$$

Hence  $\lambda(T_t A) = \lim_n \lambda(T_{t_n} A) = \lambda(A)$ . This proves that  $\lambda$  is  $T_t$ -invariant for each  $t \in \mathbb{R}$ . Since  $\mu$  is ergodic under the action of  $(T_t)_{t \in \mathbb{R}}$ , we get  $\lambda = \mu$ .  $\square$

### 3. RANK-ONE FLOWS

**Definition 3.1.** *A flow  $(T_t)_{t \in \mathbb{R}}$  is of rank one if there exists a sequence  $(\xi_j)$  of partitions of the form*

$$\xi_j = \left\{ E_j, T_{s_j} E_j, T_{s_j}^2 E_j, \dots, T_{s_j}^{h_j-1} E_j, X \setminus \bigcup_{i=0}^{h_j-1} T_{s_j}^i E_j \right\}$$

such that  $\xi_j$  converges to the partition into points (that is, for every measurable set  $A$  and every  $j$ , we can find a  $\xi_j$ -measurable set  $A_j$  in such a way that  $\mu(A \triangle A_j) \xrightarrow{j \rightarrow \infty} 0$ ),  $s_j/s_{j+1}$  are integers,  $s_j \rightarrow 0$  and  $s_j h_j \rightarrow \infty$ .

Several authors have generalized the notion of a rank-one transformation to an  $\mathbb{R}$ -action using continuous Rokhlin towers (see e.g. [10]). One can show that the above definition includes all earlier definitions of rank-one flows with continuous Rokhlin towers. The above definition without the requirement that  $s_j/s_{j+1}$  be integers was given by the third author in [13].

**Lemma 3.2.** *Let  $(T_t)_{t \in \mathbb{R}}$  be a rank-one flow. Then the sequences  $(s_j)$  and  $(h_j)$  in the definition can be chosen so that*

$$s_j^2 h_j \xrightarrow{j \rightarrow \infty} \infty.$$

*Proof.* Let  $(s_j)$  and  $(h_j)$  be given as in the definition. Recall that  $h_j s_j \rightarrow \infty$ . For each  $j$ , let  $n_j > j$  be a large enough integer such that  $s_j s_{n_j} h_{n_j} > j$ . Define  $\ell_j := s_j/s_{n_j} \in \mathbb{Z}_+$ . We consider the new partition

$$\tilde{\xi}_j := \left\{ \tilde{E}_j, T_{s_j} \tilde{E}_j, \dots, T_{s_j}^{\tilde{h}_j-1} \tilde{E}_j, X \setminus \bigcup_{i=0}^{\tilde{h}_j-1} T_{s_j}^i \tilde{E}_j \right\}$$

where

$$\tilde{E}_j := \bigcup_{i=0}^{\ell_j-1} T_{s_{n_j}}^i E_{n_j}$$

and  $\tilde{h}_j := [h_{n_j}/\ell_j]$ . One can easily check that  $\tilde{\xi}_j$  still converges to the partition into points. Moreover we have  $s_j^2 \tilde{h}_j = s_j^2 [h_{n_j} s_{n_j}/s_j] \rightarrow \infty$ .  $\square$

**Lemma 3.3** (Choice Lemma for flows, abstract setting). *Let  $(T_t)_{t \in \mathbb{R}}$  be an arbitrary flow, and let  $\nu$  be an ergodic invariant measure under the action of  $(T_t)_{t \in \mathbb{R}}$ . Let a family of measures  $(\nu_j^k)$  satisfy the conditions:*

- There exist sequences  $(d_j)$  and  $(s_j)$  of positive numbers with  $d_j \xrightarrow{j \rightarrow \infty} 0$ ,  $s_j/s_{j+1}$  is an integer for all  $j$ , and  $s_j \xrightarrow{j \rightarrow \infty} 0$ , such that for all measurable set  $A$  and all  $k, j$

$$(1) \quad |\nu_j^k(T_{s_j} A) - \nu_j^k(A)| < s_j d_j;$$

- There exists a family of positive numbers  $(a_j^k)$  with  $\sum_k a_j^k = 1$  for all  $j$ , such that

$$(2) \quad \sum_k a_j^k \nu_j^k \xrightarrow{j \rightarrow \infty} \nu.$$

Then there is a sequence  $(k_j)$  such that  $\nu_j^{k_j} \xrightarrow{j \rightarrow \infty} \nu$ .

*Proof.* Given a cylinder set  $B$ , an integer  $j \geq 1$  and  $\varepsilon > 0$ , we consider the sets  $K_j$  of all integers  $k$  such that

$$\nu(B) - \nu_j^k(B) > \varepsilon.$$

Suppose that the (sub)sequence  $K_j$  satisfies the condition

$$\sum_{k \in K_j} a_j^k \geq a > 0.$$

Let  $\lambda$  be a limit point for the sequence of measures  $(\sum_{k \in K_j} a_j^k)^{-1} \sum_{k \in K_j} a_j^k \nu_j^k$ . Then  $\lambda \neq \nu$  since  $\lambda(B) \leq \nu(B) - \varepsilon$ , but by (2), we have  $\lambda \ll \nu$ , and  $d\lambda/d\nu \leq 1/a$ . Moreover, the measure  $\lambda$  is invariant by  $T_{s_p}$  for all  $p$ . Indeed, for  $j \geq p$ , since  $s_p/s_j$  is an integer, we get from (1) that

$$|\nu_j^k(T_{s_p} A) - \nu_j^k(A)| < s_p d_j \xrightarrow{j \rightarrow \infty} 0.$$

By Lemma 2.1, it follows that  $\lambda = \nu$ . The contradiction shows that

$$\sum_{k \in K_j} a_j^k \rightarrow 0.$$

Thus, for all large enough  $j$ , most of the measures  $\nu_j^k$  satisfy

$$|\nu_j^k(B) - \nu(B)| < \varepsilon.$$

Let  $\{B_1, B_2, \dots\}$  be the countable family of all cylinder sets. Using the diagonal method we find a sequence  $k_j$  such that for each  $n$

$$|\nu_j^{k_j}(B_n) - \nu(B_n)| \xrightarrow{j \rightarrow \infty} 0,$$

i.e.  $\nu_j^{k_j} \xrightarrow{j \rightarrow \infty} \nu$ . □

**Columns and fat diagonals in  $X \times X$ .** Assume that  $(T_t)_{t \in \mathbb{R}}$  is a rank-one flow defined on  $X$ , with a sequence  $(\xi_j)$  of partitions as in Definition 3.1. For all  $j$  and  $|k| < h_j - 1$ , we define the sets  $C_j^k \in X \times X$ , called *columns*:

$$C_j^k := \bigsqcup_{\substack{0 \leq r, \ell \leq h_j - 1 \\ r - \ell = k}} T_{s_j}^r E_j \times T_{s_j}^\ell E_j.$$

Given  $0 < \delta < 1$ , we consider the set

$$D_j^\delta := \bigsqcup_{k=-[\delta h_j]}^{[\delta h_j]} C_j^k.$$

(See Figure 1.)

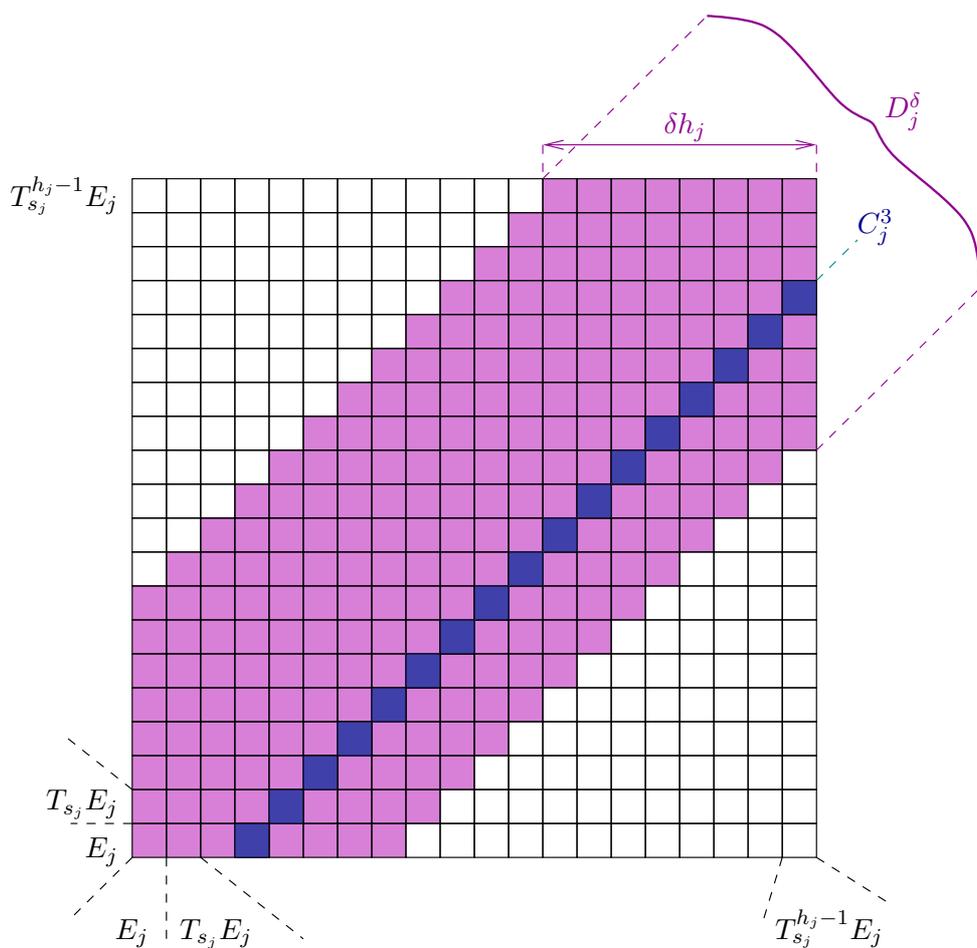


FIGURE 1. Columns and fat diagonals in  $X \times X$

#### 4. APPROXIMATION THEOREM

Recall from Section 2 that, given a flow  $(T_t)_{t \in \mathbb{R}}$ ,  $\Delta^t$  stands for the self-joining supported by the graph of  $T_{-t}$ .

**Lemma 4.1.** *Let  $\nu$  be an ergodic joining of the rank-one flow  $(T_t)_{t \in \mathbb{R}}$ . Let  $0 < \delta < 1$  be such that*

$$(3) \quad \ell_\delta := \lim_j \nu(D_j^\delta) > 0.$$

Then there exists a sequence  $(k_j)$  with  $-\delta h_j \leq k_j \leq \delta h_j$  such that

$$\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

*Proof.* Our strategy is the following: First we prove that the joining  $\nu$  can be approximated by sums of parts of off-diagonal measures, then applying the Choice Lemma we find a sequence of parts tending to  $\nu$ .

By definition of  $D_j^\delta$ , we have

$$\nu(D_j^\delta \Delta (T_{s_j} \times T_{s_j}) D_j^\delta) \leq \frac{C}{h_j}.$$

It follows that for any fixed  $p$ , the sets  $D_j^\delta$  are asymptotically  $T_{s_p} \times T_{s_p}$ -invariant: Indeed, since  $T_{s_p} = T_{s_j}^{s_p/s_j}$  where  $s_p/s_j$  is an integer when  $j \geq p$ , we get

$$\nu(D_j^\delta \Delta (T_{s_p} \times T_{s_p}) D_j^\delta) \leq \frac{s_p}{s_j} \frac{C}{h_j} \xrightarrow{j \rightarrow \infty} 0$$

(recall that  $s_j h_j \rightarrow \infty$ ).

Let  $\lambda$  be a limit measure of  $\nu(\cdot | D_j^\delta)$ . Then  $\lambda$  is  $T_{s_p} \times T_{s_p}$ -invariant for each  $p$ , by (3),  $\lambda$  is absolutely continuous with respect to  $\nu$ , and  $\frac{d\lambda}{d\nu} \leq \frac{1}{\ell_\delta} < \infty$ . By Lemma 2.1, it follows that  $\lambda = \nu$ . Hence we have

$$(4) \quad \nu(\cdot | D_j^\delta) \xrightarrow{j \rightarrow \infty} \nu.$$

We now prove that

$$(5) \quad \sum_{k=-[\delta h_j]}^{[\delta h_j]} \nu(C_j^k | D_j^\delta) \Delta^{k s_j}(\cdot | C_j^k) \xrightarrow{j \rightarrow \infty} \nu.$$

For arbitrary measurable sets  $A, B$  we can find  $\xi_j$ -measurable sets  $A_j, B_j$  such that

$$\varepsilon_j := \mu(A \Delta A_j) + \mu(B \Delta B_j) \rightarrow 0.$$

We have

$$\sum_k \nu(C_j^k | D_j^\delta) \Delta^{k s_j}(A \times B | C_j^k) - \nu(A \times B) = M_1 + M_2 + M_3 + M_4,$$

where

$$M_1 := \sum_k \nu(C_j^k | D_j^\delta) (\Delta^{k s_j}(A \times B | C_j^k) - \Delta^{k s_j}(A_j \times B_j | C_j^k)),$$

$$M_2 := \sum_k \nu(C_j^k | D_j^\delta) \Delta^{k s_j}(A_j \times B_j | C_j^k) - \nu(A_j \times B_j | D_j^\delta),$$

$$M_3 := \nu(A_j \times B_j | D_j^\delta) - \nu(A \times B | D_j^\delta),$$

$$M_4 := \nu(A \times B | D_j^\delta) - \nu(A \times B).$$

The density of the projections of the measure  $\Delta^{k s_j}(\cdot | C_j^k)$  with respect to  $\mu$  is bounded by  $(1 - \delta)^{-1}$ . Hence  $M_1 \leq \varepsilon_j / (1 - \delta)$ .

Since  $A_j, B_j$  are  $\xi_j$ -measurable,

$$\nu(A_j \times B_j | C_j^k) = \Delta^{k s_j}(A_j \times B_j | C_j^k),$$

and we get  $M_2 = 0$ .

The absolute value of the third term  $M_3$  can be bounded above as follows

$$|M_3| \leq \nu(D_j^\delta)^{-1} \nu\left((A_j \times B_j) \Delta (A \times B)\right) \leq \frac{\varepsilon_j}{\nu(D_j^\delta)} \rightarrow 0.$$

The last term  $M_4$  goes to zero as  $j \rightarrow \infty$  by (4), and this ends the proof of (5).

To apply the Choice Lemma for the measures  $\nu_j^k = \Delta^{k s_j}(\cdot | C_j^k)$  and  $a_j^k = \nu(C_j^k | D_j^\delta)$ , it remains to check the first hypothesis of the lemma. By construction of the columns  $C_j^k$ , we have for any measurable subset  $A \in X \times X$  and all  $k \in \{-[\delta h_j], \dots, [\delta h_j]\}$ ,

$$(6) \quad \left| \Delta^{k s_j}(T_{s_j} \times T_{s_j} A | C_j^k) - \Delta^{k s_j}(A | C_j^k) \right| < \frac{C}{h_j}$$

where  $C$  is a constant. We get the desired result by setting  $d_j := \frac{C}{s_j h_j}$ .

The Choice Lemma then gives a sequence  $(k_j)$  with  $-\delta h_j \leq k_j \leq \delta h_j$  such that  $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu$ .  $\square$

**Theorem 4.2.** *Let a flow  $T = (T_t)_{t \in \mathbb{R}}$  be of rank-one and  $\nu$  be an ergodic self-joining of  $(T_t)_{t \in \mathbb{R}}$ . Then there is a sequence  $(k_j)$  such that  $\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \frac{1}{2}\nu + \frac{1}{2}\nu'$  for some self-joining  $\nu'$ : For all measurable sets  $A, B$*

$$\mu(A \cap T_{s_j}^{k_j} B) \rightarrow \frac{1}{2}\nu(A \times B) + \frac{1}{2}\nu'(A \times B).$$

*Proof.* For any  $1/2 < \delta < 1$ , we have

$$\lim_{j \rightarrow \infty} \nu(D_j^\delta) > 1 - 2(1 - \delta) = 2\delta - 1 > 0.$$

Hence we can apply Lemma 4.1 for any  $1/2 < \delta < 1$ . By a diagonal argument, we get the existence of  $(k_j)$  and  $(\delta_j) \searrow \frac{1}{2}$  with  $-\delta_j h_j \leq k_j \leq \delta_j h_j$  such that

$$\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

Let us decompose  $\Delta^{k_j s_j}$  as

$$\Delta^{k_j s_j} = \Delta^{k_j s_j}(\cdot | C_j^{k_j}) \Delta^{k_j s_j}(C_j^{k_j}) + \Delta^{k_j s_j}(\cdot | X \times X \setminus C_j^{k_j}) (1 - \Delta^{k_j s_j}(C_j^{k_j})).$$

Since  $\liminf_{j \rightarrow \infty} \Delta^{k_j s_j}(C_j^{k_j}) \geq 1/2$ , we get the existence of some self-joining  $\nu'$  such that

$$\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \frac{1}{2}\nu + \frac{1}{2}\nu'.$$

$\square$

**Corollary 4.3.** *A mixing rank-one flow has minimal self-joinings of order two.*

*Proof.* Let  $\nu$  be an ergodic self-joining of a mixing rank-one flow  $(T_t)_{t \in \mathbb{R}}$ . Let  $(k_j)$  be the sequence given by Theorem 4.2. If  $|k_j s_j| \rightarrow \infty$ , since  $T$  is mixing we have

$$\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \mu \times \mu,$$

hence  $\mu \times \mu = \frac{1}{2}\nu + \frac{1}{2}\nu'$  for some self-joining  $\nu'$ . The ergodicity of  $\mu \times \mu$  then implies that  $\mu \times \mu = \nu$ . Otherwise, along some subsequence we have  $k_j s_j \rightarrow s$  for some real number  $s$ . Then  $\Delta^s = \frac{1}{2}\nu + \frac{1}{2}\nu'$  for some self-joining  $\nu'$ , and again the ergodicity of  $\Delta^s$  yields  $\nu = \Delta^s$ . Thus  $T$  has minimal self-joinings of order two.  $\square$

## 5. WEAK CLOSURE THEOREM FOR RANK-ONE FLOWS

**Lemma 5.1** (Weak Closure Lemma). *If the automorphism  $S$  commutes with the rank-one flow  $(T_t)_{t \in \mathbb{R}}$ , then there exist  $1/2 \leq d \leq 1$ , a sequence  $(k_j)$  of integers and a sequence of measurable sets  $(Y_j)$  such that, for all measurable sets  $A, B$*

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \mu(A \cap SB),$$

where  $Y_j$  has the form

$$Y_j^{d,-} := \bigsqcup_{0 \leq i < d h_j} T_{s_j}^i E_j \quad \text{or} \quad Y_j^{d,+} := \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j.$$

*Proof.* This lemma is a consequence of the proof of Theorem 4.2, when the joining  $\nu$  is equal to  $\Delta_S$ . Given a sequence  $(\delta_j) \searrow \frac{1}{2}$ , the proof provides a sequence  $(k_j)$  where  $-\delta_j h_j \leq k_j \leq \delta_j h_j$ , such that  $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \Delta_S$ , and  $\Delta^{k_j s_j}(C_j^{k_j})$  converges to some number  $d \geq 1/2$ . Let  $Y_j^{k_j}$  be the projection on the first coordinate of  $C_j^{k_j}$ , that is

$$Y_j^{k_j} = \begin{cases} \bigsqcup_{i=k_j}^{h_j} T_{s_j}^i E_j & \text{if } k_j \geq 0 \\ \bigsqcup_{i=0}^{h_j+k_j} T_{s_j}^i E_j & \text{if } k_j < 0. \end{cases}$$

We then have  $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) = \Delta^{k_j s_j}(\cdot | Y_j^{k_j} \times X)$ , and  $\mu(Y_j^{k_j}) = \Delta^{k_j s_j}(C_j^{k_j}) \rightarrow d$ . This yields, for all measurable sets  $A, B$ ,

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j^{k_j}) \rightarrow d \mu(A \cap SB).$$

If there exist infinitely many  $j$ 's such that  $k_j \geq 0$ , then along this subsequence, we have

$$\mu\left(Y_j^{k_j} \Delta Y_j^{d,+}\right) \xrightarrow{j \rightarrow \infty} 0,$$

since  $(h_j - k_j)/h_j \rightarrow d$ . A similar result holds along the subsequence of  $j$ 's such that  $k_j < 0$ , with  $Y_j^{d,+}$  replaced by  $Y_j^{d,-}$ .  $\square$

**Theorem 5.2** (Weak Closure Theorem for rank-one flows). *If the automorphism  $S$  commutes with the rank-one flow  $(T_t)_{t \in \mathbb{R}}$ , then there exists a sequence of integers  $(k_j)$  such that  $\Delta^{k_j s_j} \rightarrow \Delta_S$ : For all measurable sets  $A, B$ ,*

$$\mu(A \cap T_{s_j}^{k_j} B) \rightarrow \mu(A \cap SB).$$

*Proof.* We fix  $T$  and consider the set of real numbers  $d$  for which the conclusion in the statement of Lemma 5.1 holds. It is easy to show by a diagonal argument that this set is closed. Hence we consider its maximal element, which we still denote by  $d$ . (If  $d = 1$ , the theorem is proved.)

So we start from the following statement: We have a sequence of sets  $\{Y_j\}$ , of the form given in Lemma 5.1, such that for all measurable  $A, B$

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \mu(A \cap SB).$$

Then a similar statement holds when  $Y_j$  is replaced by  $SY_j$ : Indeed, since  $S$  commutes with  $T$  and  $\mu$  is invariant by  $S$ , we have

$$\begin{aligned} \mu(A \cap T_{s_j}^{k_j} B \cap SY_j) &= \mu(S^{-1}A \cap T_{s_j}^{k_j} S^{-1}B \cap Y_j) \\ &\xrightarrow{j \rightarrow \infty} d \mu(S^{-1}A \cap SS^{-1}B) = d \mu(A \cap SB). \end{aligned}$$

Let  $\lambda$  be a limit point for the sequence of probability measures  $\{\nu_j\}$  defined on  $X \times X$  by

$$\nu_j(A \times B) := \frac{1}{\mu(Y_j \cup SY_j)} \mu\left(A \cap T_{s_j}^{k_j} B \cap (Y_j \cup SY_j)\right).$$

Then  $\lambda \leq 2\Delta_S$ . Moreover, the measure  $\lambda$  is invariant by  $T_{s_p} \times T_{s_p}$  for all  $p$ . Indeed, for  $j \geq p$ , we have

$$\mu(T_{s_p} Y_j \triangle Y_j) = \mu(T_{s_j}^{s_p/s_j} Y_j \triangle Y_j)$$

which is of order  $\frac{s_p}{s_j h_j}$ , hence vanishes as  $j \rightarrow \infty$ . Since  $\Delta_S$  is an ergodic measure for the flow  $\{T_t \times T_t\}$ , we can apply Lemma 2.1, which gives  $\lambda = \Delta_S$ . We obtain

$$\mu\left(A \cap T_{s_j}^{k_j} B \cap (Y_j \cup SY_j)\right) \rightarrow u \mu(A \cap SB),$$

where  $u := \lim_j \mu(Y_j \cup SY_j)$  (if the limit does not exist, then we consider some subsequence of  $\{j\}$ ).

Our aim is to show that  $u = 1$ , which will end the proof of the theorem. Let us introduce

$$W_j := \left( \bigsqcup_{0 \leq i \leq h_j} T_{s_j}^i E_j \right) \setminus Y_j.$$

Assume that  $u < 1$ , then (denoting by  $Y^c$  the complementary of  $Y \subset X$ )

$$\lim_j \Delta_S(W_j \times W_j) = \lim_j \mu(W_j \cap SW_j) = \lim_j \mu(Y_j^c \cap SY_j^c) = 1 - u > 0.$$

Let us consider the case where  $Y_j$  has the form  $Y_j^{d,-} = \bigsqcup_{0 \leq i < d h_j} T_{s_j}^i E_j$ . Then  $W_j = \bigsqcup_{d h_j \leq i \leq h_j} T_{s_j}^i E_j$ , and we define for any  $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{(1-\delta')h_j < i \leq h_j} T_{s_j}^i E_j \subset W_j.$$

In the same way, if  $Y_j$  has the form  $Y_j^{d,+} = \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j$ , we set for  $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{0 < i < \delta' h_j} T_{s_j}^i E_j \subset W_j.$$

In both cases, note that

$$\Delta_S\left((W_j \times W_j) \setminus (W_j(\delta') \times W_j(\delta'))\right) \leq 2(1 - d - \delta').$$

Thus, for  $\delta'$  close enough to  $1 - d$ , we get

$$\limsup_j \Delta_S\left(W_j(\delta') \times W_j(\delta')\right) \geq 1 - u - 2(1 - d - \delta') > 0.$$

Since  $W_j(\delta') \times W_j(\delta') \subset D_j^{\delta'}$ , this ensures that

$$\limsup \Delta_S(D_j^{\delta'}) > 0.$$

Lemma 4.1 then provides a sequence  $(k'_j)$  with  $-\delta' h_j \leq k'_j \leq \delta' h_j$ , such that

$$\Delta^{k'_j s_j}(\cdot | C_j^{k'_j}) \xrightarrow{j \rightarrow \infty} \Delta_S,$$

and the projections  $Y_j^{k'_j}$  of  $C_j^{k'_j}$  on the first coordinate satisfy

$$\lim_j \mu(Y_j^{k'_j}) \geq 1 - \delta' > d,$$

which contradicts the maximality of  $d$ . Hence  $u = 1$ .  $\square$

## 6. RIGIDITY OF FACTORS OF RANK-ONE FLOWS

**Lemma 6.1.** *Let  $\mathcal{F}$  be a non-trivial factor of a rank-one flow  $(T_t)_{t \in \mathbb{R}}$ . Then there exist  $1/2 \leq d \leq 1$ , a sequence of integers  $(k_j)$  with  $|k_j s_j| \rightarrow 0$  and a sequence of measurable sets  $(Y_j)$  such that, for all measurable sets  $A, B \in \mathcal{F}$*

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \mu(A \cap B),$$

where  $Y_j$  has the form

$$Y_j^{d,-} := \bigsqcup_{0 \leq i < d h_j} T_{s_j}^i E_j \quad \text{or} \quad Y_j^{d,+} := \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j.$$

*Proof.* We start with the relatively independent joining above the factor  $\mathcal{F}$  (see Section 2). Since  $\mathcal{F}$  is a non-trivial factor,  $\mu \otimes_{\mathcal{F}} \mu \neq \Delta$ , hence we can consider an ergodic component  $\nu$  such that  $\nu(\{(x, x), x \in X\}) = 0$ . Observe however that for any sets  $A, B \in \mathcal{F}$ , we have  $\nu(A \times B) = \mu(A \cap B)$ .

We repeat the proof of Lemma 5.1 with  $\nu$  in place of  $\Delta_S$ . This provides sequences  $(k_j)$  and  $(Y_j)$  and a real number  $1/2 \leq d \leq 1$ , such that for all measurable sets  $A, B$

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \nu(A \times B).$$

If we had  $k_j s_j \rightarrow 0$ , then the left-hand side would converge to  $d \mu(A \cap B)$ , which would give  $\nu(A \times B) = \mu(A \cap B)$  for all  $A, B \in \mathcal{A}$ , and this would contradict the hypothesis that  $\nu$  gives measure 0 to the diagonal.  $\square$

**Theorem 6.2.** *Let  $\mathcal{F}$  be a non-trivial factor of a rank-one flow  $(T_t)_{t \in \mathbb{R}}$ . Then there exists a sequence of integers  $(k_j)$  with  $|k_j s_j| \rightarrow \infty$  such that, for all measurable sets  $A, B \in \mathcal{F}$*

$$\mu(A \cap T_{s_j}^{k_j} B) \rightarrow \mu(A \cap B).$$

*Proof.* Again we fix some ergodic component  $\nu$  such that  $\nu(\{(x, x), x \in X\}) = 0$ . We consider the maximal number  $d$  for which the statement of Lemma 6.1 is true. We thus have a sequence of sets  $\{Y_j\}$ , of the form given in Lemma 6.1, such that

$$(7) \quad \forall A, B \in \mathcal{F}, \quad \frac{1}{\mu(Y_j)} \mathbb{E}_{\mu} \left[ \mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{Y_j} \right] \rightarrow \mu(A \cap B).$$

In the above equation, one can replace  $\mathbb{1}_{Y_j}$  by  $\phi_j(x) := \mathbb{E}_{\nu}[\mathbb{1}_{Y_j}(x') | x]$ : Indeed, since  $\nu$  coincides with  $\Delta$  on  $\mathcal{F} \otimes \mathcal{F}$ , we have  $\mathbb{1}_A(x') = \mathbb{1}_A(x)$  and  $\mathbb{1}_{T_{s_j}^{k_j} B}(x') = \mathbb{1}_{T_{s_j}^{k_j} B}(x)$   $\nu$ -a.s. Hence,

$$\mathbb{E}_{\mu} \left[ \mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{Y_j} \right] = \mathbb{E}_{\nu} \left[ \mathbb{1}_A(x) \mathbb{1}_{T_{s_j}^{k_j} B}(x) \mathbb{1}_{Y_j}(x') \right] = \mathbb{E}_{\mu} \left[ \mathbb{1}_A(x) \mathbb{1}_{T_{s_j}^{k_j} B}(x) \phi_j(x) \right].$$

We note that

$$(8) \quad \mathbb{E}_{\mu} [|\phi_j - \phi_j \circ T_{s_j}|] \leq \mu(Y_j \Delta T_{s_j} Y_j) = O\left(\frac{1}{h_j}\right).$$

For any  $\varepsilon > 0$ , let

$$U_j^{\varepsilon} := \{x : \phi_j(x) > \varepsilon\}.$$

We would like to prove that (7) remains valid with  $\mathbb{1}_{Y_j}$  replaced by  $\mathbb{1}_{U_j^{\varepsilon}}$  for  $\varepsilon$  small enough. To this end, we need almost-invariance of  $U_j^{\varepsilon}$  under  $T_{s_j}$ , which does not

seem to be guaranteed for arbitrary  $\varepsilon$ . Therefore, we use the following technical argument to find a sequence  $(\varepsilon_j)$  for which the desired result holds.

Fix  $\varepsilon > 0$  small enough so that  $\mu(U_j^\varepsilon) > \mu(Y_j)/2$  for all large  $j$ . By Lemma 3.2, we can assume that  $s_j^2 h_j \rightarrow \infty$ . Let  $\delta_j = o(s_j)$  such that  $(\delta_j h_j)^{-1} = o(s_j)$ . We divide the interval  $[\varepsilon/2, \varepsilon]$  into  $\varepsilon/(4\delta_j)$  disjoint subintervals of length  $2\delta_j$ . One of these subintervals, called  $I_j$ , satisfy

$$(9) \quad \mu(\{x : \phi_j(x) \in I_j\}) \leq \frac{4\delta_j}{\varepsilon}.$$

Let us call  $\varepsilon_j$  the center of the interval  $I_j$ . Observe that

$$\mu(U_j^{\varepsilon_j} \Delta T_{s_j} U_j^{\varepsilon_j}) \leq \mu(\{x : |\phi_j(x) - \varepsilon_j| < \delta_j\}) + \mu(\{x : |\phi_j(x) - \phi_j(T_{s_j}(x))| \geq \delta_j\}).$$

By (9) and (8), we get that

$$(10) \quad \mu(U_j^{\varepsilon_j} \Delta T_{s_j} U_j^{\varepsilon_j}) = O\left(\delta_j + \frac{1}{\delta_j h_j}\right) = o(s_j).$$

Taking a subsequence if necessary, we can assume that the sequence of probability measures  $\lambda_j$ , defined by

$$\forall A, B \in \mathcal{A}, \quad \lambda_j(A \times B) := \frac{1}{\mu(U_j^{\varepsilon_j})} \mathbb{E}_\mu \left[ \mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{U_j^{\varepsilon_j}} \right],$$

converges to some probability measure  $\lambda$ , which is invariant by  $T_{s_p} \times T_{s_p}$  for all  $p$  by (10). Recall that  $\mu(U_j^{\varepsilon_j}) > \mu(Y_j)/2$  and that  $\mathbb{1}_{U_j^{\varepsilon_j}} \leq \phi_j/\varepsilon_j$ . Then, since  $\varepsilon_j > \varepsilon/2$ , we have  $\lambda|_{\mathcal{F} \otimes \mathcal{F}} \leq \frac{4}{\varepsilon} \Delta|_{\mathcal{F} \otimes \mathcal{F}}$ . Since  $\Delta|_{\mathcal{F} \otimes \mathcal{F}}$  is an ergodic measure for the flow  $\{T_t \times T_t\}|_{\mathcal{F} \otimes \mathcal{F}}$ , we can apply Lemma 2.1, which gives  $\lambda|_{\mathcal{F} \otimes \mathcal{F}} = \Delta|_{\mathcal{F} \otimes \mathcal{F}}$ . This means that (7) remains valid with  $\mathbb{1}_{Y_j}$  replaced by  $\mathbb{1}_{U_j^{\varepsilon_j}}$ .

The analogue of (7) is also valid when we replace  $\mathbb{1}_{Y_j}$  by  $\mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}}$ : Indeed, we also have the almost-invariance property

$$\mu((Y_j \cup U_j^{\varepsilon_j}) \Delta T_{s_j}(Y_j \cup U_j^{\varepsilon_j})) = o(s_j)$$

and  $\mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}} \leq \mathbb{1}_{Y_j} + \mathbb{1}_{U_j^{\varepsilon_j}}$ . We conclude by a similar argument.

Since  $\varepsilon$  can be taken arbitrarily small, we can now use a diagonal argument to show that (7) remains valid with  $\mathbb{1}_{Y_j}$  replaced by  $\mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}}$  where the sequence  $(\varepsilon_j)$  now satisfies  $\varepsilon_j \rightarrow 0$ . Hence, taking a subsequence if necessary to ensure that  $\mu(Y_j \cup U_j^{\varepsilon_j})$  converges to some number  $u$ , we get

$$\forall A, B \in \mathcal{F}, \quad \mathbb{E}_\mu \left[ \mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}} \right] \rightarrow u \mu(A \cap B).$$

It now remains to prove that  $u = 1$ , which we do by repeating the end of the proof of Theorem 5.2. Assume that  $u < 1$ . Let us introduce

$$W_j := \left( \bigsqcup_{0 \leq i \leq h_j} T_{s_j}^i E_j \right) \setminus Y_j.$$

We have

$$\lim_j \nu(W_j \times W_j) = \lim_j \nu(Y_j^c \times Y_j^c) = \lim_j \mathbb{E}_\mu \left[ \mathbb{1}_{Y_j^c} (1 - \phi_j) \right].$$

Observe that  $(1 - \phi_j) \geq \mathbb{1}_{(U_j^{\varepsilon_j})^c} - \varepsilon_j$ . Hence

$$\lim_j \nu(W_j \times W_j) \geq \lim_j \mathbb{E}_\mu \left[ \mathbb{1}_{Y_j^c} \mathbb{1}_{(U_j^{\varepsilon_j})^c} \right] = 1 - u > 0.$$

Let us consider the case where  $Y_j$  has the form  $Y_j^{d,-} = \bigsqcup_{0 \leq i < dh_j} T_{s_j}^i E_j$ . Then  $W_j = \bigsqcup_{dh_j \leq i \leq h_j} T_{s_j}^i E_j$ , and we define for any  $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{(1-\delta')h_j < i \leq h_j} T_{s_j}^i E_j \subset W_j.$$

In the same way, if  $Y_j$  has the form  $Y_j^{d,+} = \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j$ , we set for  $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{0 < i < \delta' h_j} T_{s_j}^i E_j \subset W_j.$$

In both cases, note that

$$\nu\left((W_j \times W_j) \setminus (W_j(\delta') \times W_j(\delta'))\right) \leq 2(1 - d - \delta').$$

thus, for  $\delta'$  close enough to  $1 - d$ , we get

$$\limsup_j \nu\left(W_j(\delta') \times W_j(\delta')\right) \geq 1 - u - 2(1 - d - \delta') > 0.$$

Since  $W_j(\delta') \times W_j(\delta') \subset D_j^{\delta'}$ , this ensures that

$$\limsup_j \nu(D_j^{\delta'}) > 0.$$

Lemma 4.1 then provides a sequence  $(k'_j)$  with  $-\delta' h_j \leq k'_j \leq \delta' h_j$ , such that

$$\Delta^{k'_j s_j}(\cdot | C_j^{k'_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

In particular,  $\Delta^{k'_j s_j}(\cdot | C_j^{k'_j})|_{\mathcal{F} \otimes \mathcal{F}} \xrightarrow{j \rightarrow \infty} \Delta|_{\mathcal{F} \otimes \mathcal{F}}$ . Since the projections  $Y_j^{k'_j}$  of  $C_j^{k'_j}$  on the first coordinate satisfy

$$\lim_j \mu(Y_j^{k'_j}) \geq 1 - \delta' > d,$$

this contradicts the maximality of  $d$ . Hence  $u = 1$ .  $\square$

## 7. KING'S THEOREM FOR FLAT-ROOF RANK-ONE FLOW

We consider a rank-one flow  $(T_t)_{t \in \mathbb{R}}$ . We say that  $(T_t)_{t \in \mathbb{R}}$  has *flat roof* if we can choose the sequence  $\xi_j = \{E_j, T_{s_j} E_j, \dots, T_{s_j}^{h_j-1} E_j, X \setminus \bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j\}$  in the definition such that

$$\frac{\mu\left(T_{s_j}^{h_j} E_j \triangle E_j\right)}{\mu(E_j)} \xrightarrow{j \rightarrow \infty} 0.$$

**Theorem 7.1.** *Let  $(T_t)_{t \in \mathbb{R}}$  be a flat-roof rank-one flow, and  $\nu$  be an ergodic self-joining of  $(T_t)_{t \in \mathbb{R}}$ . Then there exists a sequence  $(k_j)$  such that  $\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \nu$ .*

*Proof.* Let us defined, for  $0 \leq k \leq h_j - 1$

$$a_k^j := \nu \left( T_{s_j}^k E_j \times E_j \right) \quad \text{and} \quad b_k^j := \nu \left( E_j \times T_{s_j}^{h_j-k} E_j \right).$$

We claim that the flat-roof property implies

$$(11) \quad h_j \sum_{k=1}^{h_j-1} |a_k^j - b_k^j| \xrightarrow{j \rightarrow \infty} 0.$$

Indeed, by invariance  $a_k^j = \nu \left( T_{s_j}^{h_j} E_j \times T_{s_j}^{h_j-k} E_j \right)$ . Hence

$$|a_k^j - b_k^j| \leq \nu \left( (T_{s_j}^{h_j} E_j \triangle E_j) \times T_{s_j}^{h_j-k} E_j \right),$$

and

$$\sum_{k=1}^{h_j-1} |a_k^j - b_k^j| \leq \nu \left( (T_{s_j}^{h_j} E_j \triangle E_j) \times X \right) = \mu \left( (T_{s_j}^{h_j} E_j \triangle E_j) \right).$$

The claim follows, since  $\mu(E_j) \sim 1/h_j$ .

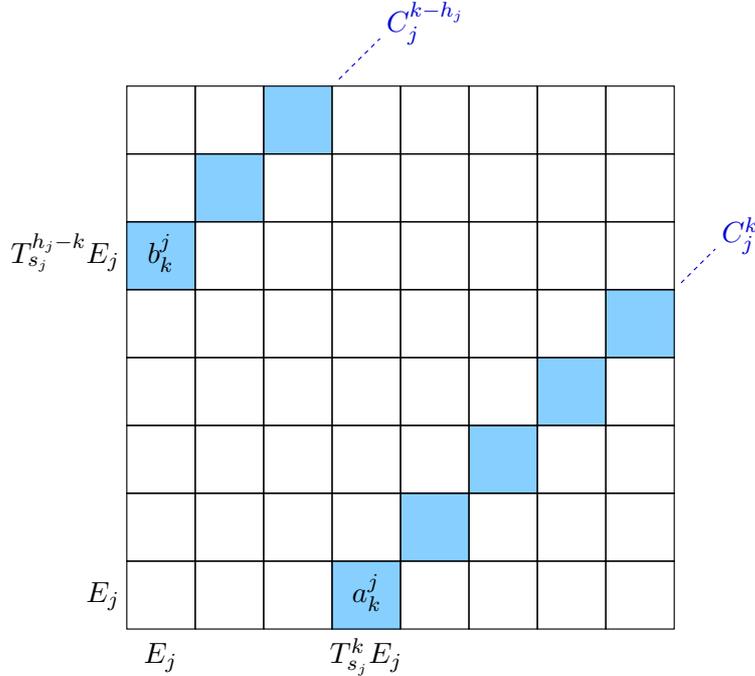


FIGURE 2. The union of  $C_j^k$  and  $C_j^{k-h_j}$  is denoted by  $G_j^k$ .

We gather the columns  $C_j^k$  in pairs, defining for  $1 \leq k \leq h_j - 1$ ,  $G_j^k := C_j^k \sqcup C_j^{k-h_j}$ . (See Figure 2.) We also set  $G_j^0 := C_j^0$ . Note that  $\nu(G_j^k) = (h_j - k)a_k^j + kb_k^j$ . Observe also that

$$\nu \left( \bigsqcup_{k=0}^{h_j-1} G_j^k \right) = \nu \left( \bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j \times \bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j \right) \xrightarrow{j \rightarrow \infty} 1.$$

Hence,

$$(12) \quad \sum_{k=0}^{h_j-1} \nu(G_j^k) \nu(\cdot | G_j^k) \xrightarrow{j \rightarrow \infty} \nu.$$

We claim that, using the flat-roof property, we can in the above equation replace  $\nu(\cdot | G_j^k)$  by  $\Delta^{ks_j}$ . Let  $A$  and  $B$  be  $\xi_j$ -measurable sets, which are unions of  $T_{s_j}^i E_j$  ( $0 \leq i \leq h_j - 1$ ). We denote by  $r_k$  (respectively  $\ell_k$ ) the number of elementary cells of the form  $T_{s_j}^{i_1} E_j \times T_{s_j}^{i_2} E_j$  which are contained in  $A \times B$  and which belong to the column  $C_j^k$  (respectively  $C_j^{k-h_j}$ ). We have

$$(13) \quad \nu(A \times B | G_j^k) \nu(G_j^k) = \ell_k b_k^j + r_k a_k^j.$$

Moreover, we will show that the flat-roof property ensures the existence of a sequence  $(\varepsilon_j)$  with  $\varepsilon_j \xrightarrow{j \rightarrow \infty} 0$  such that

$$(14) \quad \left| \Delta^{ks_j}(A \times B) - \frac{\ell_k + r_k}{h_j} \right| \leq \varepsilon_j.$$

Indeed, let us cut  $A$  into  $A_1 := A \cap \bigsqcup_{0 \leq i \leq k-1} T_{s_j}^i E_j$  and  $A_2 := A \cap \bigsqcup_{k \leq i \leq h_j-1} T_{s_j}^i E_j$ . We have

$$\Delta^{ks_j}(A_2 \times B) = r_k \mu(E_j),$$

and

$$\Delta^{ks_j}(A_1 \times B) = \ell_k \Delta^{ks_j}(E_j \times T_{s_j}^{h_j-k} E_j) + \Delta^{ks_j}((A_1 \times B) \setminus C_j^{k-h_j}).$$

Recalling that  $\Delta^{ks_j}(E_j \times T_{s_j}^{h_j-k} E_j) = \mu(E_j \cap T_{s_j}^{h_j} E_j)$ , we get

$$(15) \quad \Delta^{ks_j}(A \times B) = (r_k + \ell_k) \mu(E_j) - \ell_k \mu(E_j \setminus T_{s_j}^{h_j} E_j) + \Delta^{ks_j}((A_1 \times B) \setminus C_j^{k-h_j}).$$

The second term of the right-hand side is bounded by  $h_j \mu(E_j \Delta T_{s_j}^{h_j} E_j)$ , which goes to 0 by the flat-roof property. To treat the last term, we consider the particular case  $A = B = \bigsqcup_{0 \leq i \leq h_j-1} T_{s_j}^i E_j$ , for which this last term is maximized. We have then

$$1 - \Delta^{ks_j}(A \times B) \leq 2\mu \left( X \setminus \bigsqcup_{0 \leq i \leq h_j-1} T_{s_j}^i E_j \right) \xrightarrow{j \rightarrow \infty} 0.$$

On the other hand, (15) gives

$$\Delta^{ks_j}((A_1 \times B) \setminus C_j^{k-h_j}) = \Delta^{ks_j}(A \times B) - h_j \mu(E_j) + k \mu(E_j \setminus T_{s_j}^{h_j} E_j).$$

Since  $h_j \mu(E_j) \rightarrow 1$ , and  $k \mu(E_j \setminus T_{s_j}^{h_j} E_j) \leq h_j \mu(E_j \Delta T_{s_j}^{h_j} E_j) \rightarrow 0$ , we get that the last term of (15) goes to 0 uniformly with respect to  $k$ ,  $A$  and  $B$ . It follows that

$$|\Delta^{ks_j}(A \times B) - (\ell_k + r_k) \mu(E_j)| \xrightarrow{j \rightarrow \infty} 0,$$

uniformly with respect to  $k$ ,  $A$  and  $B$ . This concludes the proof of (14).

Equations (14) and (13) give

$$\begin{aligned} & \sum_{k=0}^{h_j-1} \left| \nu(A \times B | G_j^k) - \Delta^{ks_j}(A \times B) \right| \nu(G_j^k) \\ & \leq \sum_{k=0}^{h_j-1} |a_k^j - b_k^j| \left| \ell_k - \frac{k}{h_j}(\ell_k + r_k) \right| + \varepsilon_j \\ & \leq h_j \sum_{k=0}^{h_j-1} |a_k^j - b_k^j| + \varepsilon_j \end{aligned}$$

which goes to 0 as  $j \rightarrow \infty$  by (11).

Recalling (12), we obtain

$$\sum_{k=0}^{h_j-1} \nu(G_j^k) \Delta^{ks_j} \xrightarrow{j \rightarrow \infty} \nu.$$

It remains to apply the Choice Lemma to conclude the proof of the theorem.  $\square$

## 8. $\mathbb{Z}^n$ -RANK-ONE ACTION

We consider now an action of  $\mathbb{Z}^n$  ( $n \geq 1$ ). For  $k \in \mathbb{Z}^n$ , we denote by  $k(1), \dots, k(n)$  its coordinates.

**Definition 8.1.** A  $\mathbb{Z}^n$ -action  $\{T_k\}_{k \in \mathbb{Z}^n}$  is of rank one if there exists a sequence  $(\xi_j)$  of partitions converging to the partition into points, where  $\xi_j$  is of the form

$$\xi_j = \left\{ (T_k E_j)_{k \in R_j}, X \setminus \bigsqcup_k T_k E_j \right\},$$

and  $R_j$  is a rectangular set of indices:

$$R_j = \{0, \dots, h_j(1) - 1\} \times \dots \times \{0, \dots, h_j(n) - 1\}.$$

Note that the above definition corresponds to so-called  $\mathcal{R}$ -rank one actions defined in [11] with the additional condition that the shapes in the sequence  $\mathcal{R}$  are rectangles. The sequence  $(\xi_j)$  in the above definition being fixed, we define as for the rank-one flows the notions of columns and fat diagonals: For any  $k \in \mathbb{Z}^n$ , we set

$$C_j^k := \bigsqcup_{\substack{r, \ell \in R_j \\ r - \ell = k}} T_r E_j \times T_\ell E_j,$$

and given  $0 < \delta < 1$ ,

$$D_j^\delta := \bigsqcup_{k: \prod_i (h_j(i) - |k(i)|) \geq (1-\delta) \prod_i h_j(i)} C_j^k.$$

**Lemma 8.2.** For any self-joining  $\nu$  of the rank-one action  $\{T_k\}_{k \in \mathbb{Z}^n}$ , for any  $\delta > 1 - 1/2^n$ , we have

$$\liminf_{j \rightarrow \infty} \nu(D_j^\delta) > 0.$$

*Proof.* We can find  $\varepsilon > 0$ , small enough such that

$$\left(\frac{1}{2} - \varepsilon\right)^n > 1 - \delta.$$

Let  $r \in \mathbb{Z}^n$  be such that

$$\forall i, \left(\frac{1}{2} - \varepsilon\right) h_j(i) < r(i) < \left(\frac{1}{2} + \varepsilon\right) h_j(i).$$

Then, for any  $\ell \in R_j$ , we have for all  $i$ :  $|r(i) - \ell(i)| < \left(\frac{1}{2} + \varepsilon\right) h_j(i)$ . Hence

$$\prod_i \left(h_j(i) - |r(i) - \ell(i)|\right) > (1 - \delta) \prod_i h_j(i),$$

which means that for any  $\ell \in R_j$ , the column  $C_j^{r-\ell}$  is contained in  $D_j^\delta$ . It follows that

$$\left( \bigsqcup_{r: \forall i, |r(i) - h_j(i)/2| < \varepsilon h_j(i)} T_r E_j \right) \times \left( \bigsqcup_{\ell \in R_j} T_\ell E_j \right) \subset D_j^\delta.$$

We then get

$$\liminf_{j \rightarrow \infty} \nu(D_j^\delta) \geq \liminf_{j \rightarrow \infty} \mu \left( \bigsqcup_{r: \forall i, |r(i) - h_j(i)/2| < \varepsilon h_j(i)} T_r E_j \right) = (2\varepsilon)^n.$$

□

We can now state the analogue of Theorem 4.2 for  $\mathbb{Z}^n$ -rank-one action, which was first proved by A.A. Pavlova in [9].

**Theorem 8.3.** *Let  $\nu$  be an ergodic self-joining of the  $\mathbb{Z}^n$ -rank-one action  $\{T_k\}_{k \in \mathbb{Z}^n}$ . Then we can find a sequence  $(k_j)$  in  $\mathbb{Z}^n$  and some self-joining  $\nu'$  such that  $\Delta^{k_j} \xrightarrow[j \rightarrow \infty]{w} \frac{1}{2^n} \nu + \left(1 - \frac{1}{2^n}\right) \nu'$ : For all measurable sets  $A, B$*

$$\mu(A \cap T_{k_j} B) \rightarrow \frac{1}{2^n} \nu(A \times B) + \left(1 - \frac{1}{2^n}\right) \nu'(A \times B).$$

*Proof.* The proof follows the same lines as for Theorem 4.2. First note that Lemma 4.1 can be easily adapted to the  $\mathbb{Z}^n$ -situation. Hence, by Lemma 8.2, using a diagonal argument, we get the existence of  $(k_j)$  and  $(\delta_j) \searrow 1 - \frac{1}{2^n}$  with  $C_j^{k_j} \subset D_j^{\delta_j}$  such that

$$\Delta^{k_j} \left( \cdot | C_j^{k_j} \right) \xrightarrow[j \rightarrow \infty]{w} \nu.$$

To conclude, it remains to prove that  $\liminf \Delta^{k_j}(C_j^{k_j}) \geq 1/2^n$ . To this aim, we count the number of pairs  $(r, \ell)$  such that  $T_r E_j \times T_\ell E_j \subset C_j^{k_j}$ . We can easily check that these are exactly the pairs  $(r, \ell)$  such that, for all  $1 \leq i \leq n$ , there exists  $m(i) \in \{0, \dots, h_j(i) - 1 - |k_j(i)|\}$  with

$$(r(i), \ell(i)) = \begin{cases} (k_j(i) + m(i), m(i)) & \text{if } k_j(i) \geq 0 \\ (m(i), -k_j(i) + m(i)) & \text{otherwise.} \end{cases}$$

Hence  $\Delta^{k_j}(C_j^{k_j}) = \prod_i \left(h_j(i) - 1 - |k_j(i)|\right) \mu(E_j)$ . Using the fact that  $C_j^{k_j} \subset D_j^{\delta_j}$ , we get the desired result. □

When  $n \geq 2$ , it is known that the Weak Closure Theorem fails (counterexamples have been given in [1, 2]). However, as a consequence of Theorem 8.3, we get the following:

**Corollary 8.4** (Partial Weak Closure Theorem for  $\mathbb{Z}^n$ -rank-one action). *Let  $S$  be an automorphism commuting with the  $\mathbb{Z}^n$ -rank-one action  $\{T_k\}_{k \in \mathbb{Z}^n}$ . Then we can find a sequence  $(k_j)$  in  $\mathbb{Z}^n$  and some self-joining  $\nu'$  such that*

$$\Delta^{k_j} \xrightarrow{j \rightarrow \infty} \frac{1}{2^n} \Delta_S + \left(1 - \frac{1}{2^n}\right) \nu'.$$

Moreover, if  $S \notin \{T_k \mid k \in \mathbb{Z}^n\}$ , then  $\{T_k\}_{k \in \mathbb{Z}^n}$  is partially rigid: There exists a sequence  $(k'_\ell)$  in  $\mathbb{Z}^n$  with  $|k'_\ell| \rightarrow \infty$  such that for all measurable sets  $A$  and  $B$

$$\liminf_{\ell \rightarrow \infty} \mu(A \cap T_{k'_\ell} B) \geq \frac{1}{2^{2n}} \mu(A \cap B).$$

*Proof.* The first part is a direct application of Theorem 8.3 with  $\nu = \Delta_S$ . If moreover  $S \notin \{T_k \mid k \in \mathbb{Z}^n\}$ , then the sequence  $(k_j)$  of the theorem must satisfy  $|k_j| \rightarrow \infty$ . Let us enumerate the cylinder sets as  $\{A_0, A_1, \dots, A_\ell, \dots\}$ . Let  $(\varepsilon_\ell)$  be a sequence of positive numbers decreasing to zero. For any  $\ell$ , we can find a large enough integer  $j_1(\ell)$  such that for all cylinder sets  $A, B \in \{A_0, A_1, \dots, A_\ell\}$ ,

$$\mu(T_{k_{j_1(\ell)}} A \cap SB) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(SA \cap SB) = \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(A \cap B).$$

Then, we can find a large enough integer  $j_2(\ell)$  with  $|j_2(\ell)| > 2|j_1(\ell)|$  such that for all cylinder sets  $A, B \in \{A_0, A_1, \dots, A_\ell\}$ ,

$$\mu(T_{k_{j_1(\ell)}} A \cap T_{k_{j_2(\ell)}} B) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(T_{k_{j_1(\ell)}} A \cap SB).$$

It follows that for all  $\ell \geq 0$  and all cylinder sets  $A, B \in \{A_0, A_1, \dots, A_\ell\}$ ,

$$\mu(A \cap T_{k_{j_2(\ell)} - k_{j_1(\ell)}} B) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right)^2 \mu(A \cap B).$$

This proves the result announced in the corollary when  $A$  and  $B$  are cylinder sets with  $k'_\ell := k_{j_2(\ell)} - k_{j_1(\ell)}$ , and this extends in a standard way to all measurable sets.  $\square$

The counterexample given in [2] also shows that the rigidity of factors is no more valid when  $n \geq 2$ . Theorem 8.3 only ensures the partial rigidity of factors of  $\mathbb{Z}^n$ -rank-one actions.

**Corollary 8.5** (Partial rigidity of factors of  $\mathbb{Z}^n$ -rank-one action). *Let  $\mathcal{F}$  be a non-trivial factor of the  $\mathbb{Z}^n$ -rank-one action  $\{T_k\}_{k \in \mathbb{Z}^n}$ . Then there exists a sequence  $(k_j)$  in  $\mathbb{Z}^n$  with  $|k_j| \rightarrow \infty$  such that, for all measurable sets  $A, B \in \mathcal{F}$*

$$\liminf \mu(A \cap T_{k_j} B) \geq \frac{1}{2^n} \mu(A \cap B).$$

*Proof.* This is a direct application of Theorem 8.3 where  $\nu$  is an ergodic component of the relatively independent joining above the factor  $\mathcal{F}$ .  $\square$

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