

AROUND KING'S RANK-ONE THEOREMS: FLOWS AND \mathbb{Z}^n -ACTIONS

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ABSTRACT. We study the generalizations of Jonathan King's rank-one theorems (Weak-Closure Theorem and rigidity of factors) to the case of rank-one \mathbb{R} -actions (flows) and rank-one \mathbb{Z}^n -actions. We prove that these results remain valid in the case of rank-one flows. In the case of rank-one \mathbb{Z}^n actions, where counterexamples have already been given, we prove partial Weak-Closure Theorem and partial rigidity of factors.

1. INTRODUCTION

Very important examples in ergodic theory have been constructed in the class of rank-one transformations, which is closely connected to the notion of transformations with fast cyclic approximation [3]: If the rate of approximation is sufficiently fast, then the transformation will be inside the rank-one class. The notion of rank-one transformations has been defined in [8], where mixing examples have appeared. Later, Daniel Rudolph used them for a machinery of counterexamples [11].

Jonathan King contributed to the theory of rank-one transformations by several deep and interesting facts. His Weak-Closure-Theorem (WCT) [4] is now a classical result with applications even out of the range of \mathbb{Z} -actions (see for example [15]). He also proved the minimal-self-joining (MSJ) property for rank-one mixing automorphisms (see [5]), the rigidity of non-trivial factors [4], and the weak closure property for all joinings for flat-roof rank-one transformations [6].

A natural question is whether the corresponding assertions remain true for flows (\mathbb{R} -actions) and for \mathbb{Z}^n -actions. We show that for flows the situation is quite similar: The joining proof of the Weak-Closure Theorem given in [12] (see also [14]) can be adapted to the situation of a rank-one \mathbb{R} -action (Theorem 5.2). We also give in the same spirit a proof of the rigidity of non-trivial factors of rank-one flows (theorem 6.2) which, with some simplification, provides a new proof of King's result in the case of \mathbb{Z} -actions. We prove a flat-roof flow version as well (Theorem 7.1).

Concerning multidimensional rank-one actions, the situation is quite different. The Weak-Closure Theorem is no more true [1], and factors may be non-rigid [2]. Rank-one partially mixing \mathbb{Z} -actions have MSJ [7], however it is proved in [2] that for \mathbb{Z}^2 -actions this is generally not true. We remark that it was an answer for \mathbb{Z}^2 -action to Jean-Paul Thouvenot's question: Whether a mildly mixing rank-one action possesses MSJ, though this interesting problem remains open for \mathbb{Z} -actions.

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Regardless these surprising results, there are some partial versions of WCT: Commuting automorphisms can be partially approximated by elements of the action (Corollary 8.4), and non-trivial factors must be partially rigid (Corollary 8.5). We present these results as consequences of A. Pavlova's theorem (Theorem 8.3, see also [13]).

2. PRELIMINARIES AND NOTATIONS

Weak convergence of probability measures. We are interested in groups of automorphisms of a Lebesgue space (X, \mathcal{A}, μ) , where μ is a continuous probability measure. The properties of these group actions are independent of the choice of the underlying space X , and for practical reasons we will assume that $X = \{0, 1\}^{\mathbb{Z}}$, equipped with the product topology and the Borel σ -algebra. This σ -algebra is generated by the cylinder sets, that is sets obtained by fixing a finite number of coordinates. On the set $\mathcal{M}_1(X)$ of Borel probability measures on X , we will consider the topology of weak convergence, which is characterized by

$$\nu_n \xrightarrow[n \rightarrow \infty]{w} \nu \iff \text{for all cylinder set } C, \nu_n(C) \xrightarrow[n \rightarrow \infty]{} \nu(C),$$

and turns $\mathcal{M}_1(X)$ into a compact metrizable space.

We will often consider probability measures on $X \times X$, with the same topology of weak convergence. We will use the following observation: If ν_n and ν in $\mathcal{M}_1(X \times X)$ have their marginals absolutely continuous with respect to our reference measure μ , with bounded density, then the weak convergence of ν_n to ν ensures that for all measurable sets A and B in \mathcal{A} , $\nu_n(A \times B) \xrightarrow[n \rightarrow \infty]{} \nu(A \times B)$.

Self-joinings. Let $T = (T_g)_{g \in G}$ be an action of the Abelian group G by automorphism of the Lebesgue space (X, \mathcal{A}, μ) . A *self-joining* of T is any probability measure on $X \times X$ with both marginals equal to μ and invariant by $T \times T = (T_g \times T_g)_{g \in G}$. For any automorphism S commuting with T , we will denote by Δ_S the self-joining concentrated on the graph of S^{-1} , defined by

$$\forall A, B \in \mathcal{A}, \Delta_S(A \times B) := \mu(A \cap SB).$$

In particular, for any $g \in G$ we will denote by Δ^g the self-joining Δ_{T_g} . In the special case where $S = T_0 = \text{Id}$, we will note simply Δ instead of Δ^0 or Δ_{Id} .

Flows. A *flow* is a continuous family $(T_t)_{t \in \mathbb{R}}$ of automorphisms of the Lebesgue space (X, \mathcal{A}, μ) , with $T_t \circ T_s = T_{t+s}$ for all $t, s \in \mathbb{R}$, and such that $(t, x) \mapsto T_t(x)$ is measurable. We recall that the measurability condition implies that for all measurable set A , $\mu(A \Delta T_t A) \xrightarrow[t \rightarrow 0]{} 0$.

Lemma 2.1. *Let $(T_t)_{t \in \mathbb{R}}$ be an ergodic flow on (X, \mathcal{A}, μ) . Let Q be a dense subgroup of \mathbb{R} , and λ be an invariant probability measure for the action of $(T_t)_{t \in Q}$. Assume further that $\lambda \ll \mu$, with $\frac{d\lambda}{d\mu}$ bounded by some constant C . Then $\lambda = \mu$.*

Proof. Let $t \in \mathbb{R}$, and let (t_n) be a sequence in Q converging to t . For any measurable set A , we have

$$\lambda(T_t A \Delta T_{t_n} A) \leq C \mu(T_t A \Delta T_{t_n} A) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence $\lambda(T_t A) = \lim_n \lambda(T_{t_n} A) = \lambda(A)$. This proves that λ is T_t -invariant for each $t \in \mathbb{R}$. Since μ is ergodic under the action of $(T_t)_{t \in \mathbb{R}}$, we get $\lambda = \mu$. \square

3. RANK-ONE FLOWS

Definition 3.1. A flow $(T_t)_{t \in \mathbb{R}}$ is of rank one if there exists a sequence (ξ_j) of partitions of the form

$$\xi_j = \left\{ E_j, T_{s_j} E_j, T_{s_j}^2 E_j, \dots, T_{s_j}^{h_j-1} E_j, X \setminus \bigsqcup_{i=0}^{h_j-1} T_{s_j}^i E_j \right\}$$

such that ξ_j converges to the partition onto points (that is, for every measurable set A and every j , we can find a ξ_j -measurable set A_j in such a way that $\mu(A \Delta A_j) \xrightarrow{j \rightarrow \infty} 0$), s_j/s_{j+1} are integers, $s_j \rightarrow 0$ and $s_j h_j \rightarrow \infty$.

Several authors have generalized the notion of a rank-one transformation to an \mathbb{R} -action using continuous Rokhlin towers (see e.g. [10]). One can show that the above definition includes all earlier definitions of rank-one flows with continuous Rokhlin towers. The above definition without the requirement that s_j/s_{j+1} be integers was given by the third author in [12].

Lemma 3.2. Let $(T_t)_{t \in \mathbb{R}}$ be a rank-one flow. Then the sequences (s_j) and (h_j) in the definition can be chosen so that

$$s_j^2 h_j \xrightarrow{j \rightarrow \infty} \infty.$$

Proof. Let (s_j) and (h_j) be given as in the definition. Recall that $h_j s_j \rightarrow \infty$. For each j , let $n_j > j$ be a large enough integer such that $s_j s_{n_j} h_{n_j} > j$. Define $\ell_j := s_j / s_{n_j} \in \mathbb{Z}_+$. We consider the new partition

$$\tilde{\xi}_j := \left\{ \tilde{E}_j, T_{s_j} \tilde{E}_j, \dots, T_{s_j}^{\tilde{h}_j-1} \tilde{E}_j, X \setminus \bigsqcup_{i=0}^{\tilde{h}_j-1} T_{s_j}^i \tilde{E}_j \right\}$$

where

$$\tilde{E}_j := \bigsqcup_{i=0}^{\ell_j-1} T_{s_{n_j}}^i E_{n_j}$$

and $\tilde{h}_j := [h_{n_j} / \ell_j]$. One can easily check that $\tilde{\xi}_j$ still converges to the partition onto points. Moreover we have $s_j^2 \tilde{h}_j = s_j^2 [h_{n_j} s_{n_j} / s_j] \rightarrow \infty$. \square

We assume here that s_j/s_{j+1} is an integer, and that $s_j \xrightarrow{j \rightarrow \infty} 0$.

Lemma 3.3 (Choice Lemma for flows, abstract setting). Let ν be an ergodic invariant measure under the action of the flow $(T_t)_{t \in \mathbb{R}}$. Let a family of measures (ν_j^k) satisfy the conditions:

- There exists a sequence (d_j) of positive numbers with $d_j \xrightarrow{j \rightarrow \infty} 0$, such that for all measurable set A and all k, j

$$(1) \quad |\nu_j^k(T_{s_j} A) - \nu_j^k(A)| < s_j d_j;$$

- There exists a family of positive numbers (a_j^k) with $\sum_k a_j^k = 1$ for all j , such that

$$(2) \quad \sum_k a_j^k \nu_j^k \xrightarrow{j \rightarrow \infty} \nu.$$

Then there is a sequence (k_j) such that $\nu_j^{k_j} \xrightarrow{j \rightarrow \infty} \nu$.

Proof. Given a cylinder set B , an integer $j \geq 1$ and $\varepsilon > 0$, we consider the sets K_j of all integers k such that

$$\nu(B) - \nu_j^k(B) > \varepsilon.$$

Suppose that the (sub)sequence K_j satisfies the condition

$$\sum_{k \in K_j} a_j^k \geq a > 0.$$

Let λ be a limit point for the sequence of measures $(\sum_{k \in K_j} a_j^k)^{-1} \sum_{k \in K_j} a_j^k \nu_j^k$. Then $\lambda \neq \nu$ since $\lambda(B) \leq \nu(B) - \varepsilon$, but by (2), we have $\lambda \ll \nu$, and $d\lambda/d\nu \leq 1/a$. Moreover, the measure λ is invariant by T_{s_p} for all p . Indeed, for $j \geq p$, since s_p/s_j is an integer, we get from (1) that

$$|\nu_j^k(T_{s_p}A) - \nu_j^k(A)| < s_p d_j \xrightarrow{j \rightarrow \infty} 0.$$

By Lemma 2.1, it follows that $\lambda = \nu$. The contradiction shows that

$$\sum_{k \in K_j} a_j^k \rightarrow 0.$$

Thus, for all large enough j , most of the measures ν_j^k satisfy

$$|\nu_j^k(B) - \nu(B)| < \varepsilon.$$

Let $\{B_1, B_2, \dots\}$ be the countable family of all cylinder sets. Using the diagonal method we find a sequence k_j such that for each n

$$|\nu_j^{k_j}(B_n) - \nu(B_n)| \xrightarrow{j \rightarrow \infty} 0,$$

i.e. $\nu_j^{k_j} \xrightarrow{j \rightarrow \infty} \nu$. □

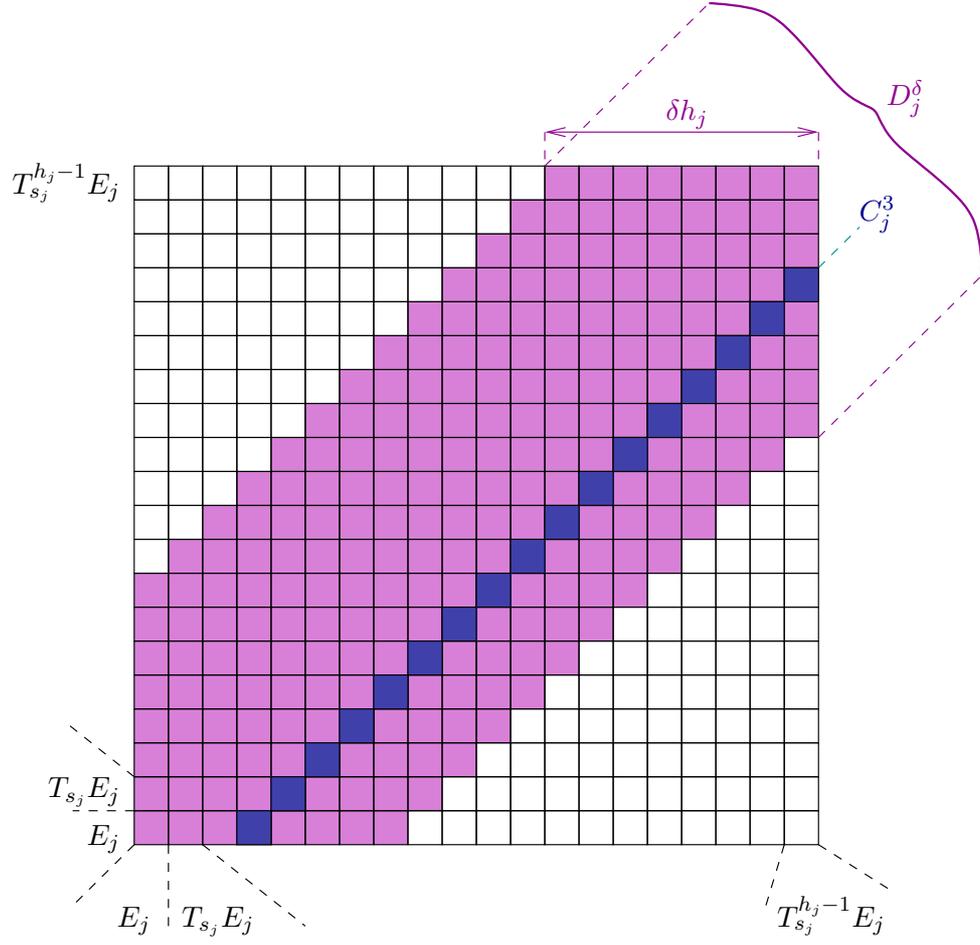
Columns and fat diagonals in $X \times X$. Assume that $(T_t)_{t \in \mathbb{R}}$ is a rank-one flow defined on X , with a sequence (ξ_j) of partitions as in Definition 3.1. For all j and $|k| < h_j - 1$, we define the sets $C_j^k \in X \times X$, called *columns*:

$$C_j^k := \bigsqcup_{\substack{0 \leq r, \ell \leq h_j - 1 \\ r - \ell = k}} T_{s_j}^r E_j \times T_{s_j}^\ell E_j.$$

Given $0 < \delta < 1$, we consider the set

$$D_j^\delta := \bigsqcup_{k = -[\delta h_j]}^{[\delta h_j]} C_j^k.$$

(See Figure 1.)


 FIGURE 1. Columns and fat diagonals in $X \times X$

4. APPROXIMATION THEOREM

Lemma 4.1. *Let ν be an ergodic joining of the rank-one flow $(T_t)_{t \in \mathbb{R}}$. Let $0 < \delta < 1$ be such that*

$$(3) \quad \ell_\delta := \lim_j \nu(D_j^\delta) > 0.$$

Then there exists a sequence (k_j) with $-\delta h_j \leq k_j \leq \delta h_j$ such that

$$\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

Proof. Our strategy is the following: First we prove that the joining ν can be approximated by sums of parts of off-diagonal measures, then applying the Choice Lemma we find a sequence of parts tending to ν .

By definition of D_j^δ , we have

$$\nu(D_j^\delta \triangle (T_{s_j} \times T_{s_j})D_j^\delta) \leq \frac{C}{h_j}.$$

It follows that for any fixed p , the sets D_j^δ are asymptotically $T_{s_p} \times T_{s_p}$ -invariant: Indeed, since $T_{s_p} = T_{s_j}^{s_p/s_j}$ where s_p/s_j is an integer when $j \geq p$, we get

$$\nu\left(D_j^\delta \triangle (T_{s_p} \times T_{s_p})D_j^\delta\right) \leq \frac{s_p}{s_j} \frac{C}{h_j} \xrightarrow{j \rightarrow \infty} 0$$

(recall that $s_j h_j \rightarrow \infty$).

Let λ be a limit measure of $\nu(\cdot | D_j^\delta)$. Then λ is $T_{s_p} \times T_{s_p}$ -invariant for each p , by (3), λ is absolutely continuous with respect to ν , and $\frac{d\lambda}{d\nu} \leq \frac{1}{\ell_s} < \infty$. By Lemma 2.1, it follows that $\lambda = \nu$. Hence we have

$$(4) \quad \nu(\cdot | D_j^\delta) \xrightarrow{j \rightarrow \infty} \nu.$$

We now prove that

$$(5) \quad \sum_{k=-[\delta h_j]}^{[\delta h_j]} \nu(C_j^k | D_j^\delta) \Delta^{ks_j}(\cdot | C_j^k) \xrightarrow{j \rightarrow \infty} \nu.$$

For arbitrary measurable sets A, B we can find ξ_j -measurable sets A_j, B_j such that

$$\varepsilon_j := \mu(A \triangle A_j) + \mu(B \triangle B_j) \rightarrow 0.$$

We have

$$\sum_k \nu(C_j^k | D_j^\delta) \Delta^{ks_j}(A \times B | C_j^k) - \nu(A \times B) = M_1 + M_2 + M_3 + M_4,$$

where

$$\begin{aligned} M_1 &:= \sum_k \nu(C_j^k | D_j^\delta) (\Delta^{ks_j}(A \times B | C_j^k) - \Delta^{ks_j}(A_j \times B_j | C_j^k)), \\ M_2 &:= \sum_k \nu(C_j^k | D_j^\delta) \Delta^{ks_j}(A_j \times B_j | C_j^k) - \nu(A_j \times B_j | D_j^\delta), \\ M_3 &:= \nu(A_j \times B_j | D_j^\delta) - \nu(A \times B | D_j^\delta), \\ M_4 &:= \nu(A \times B | D_j^\delta) - \nu(A \times B). \end{aligned}$$

The density of the projections of the measure $\Delta^{ks_j}(\cdot | C_j^k)$ with respect to μ is bounded by $(1 - \delta)^{-1}$. Hence $M_1 \leq \varepsilon_j / (1 - \delta)$.

Since A_j, B_j are ξ_j -measurable,

$$\nu(A_j \times B_j | C_j^k) = \Delta^{ks_j}(A_j \times B_j | C_j^k),$$

and we get $M_2 = 0$.

The absolute value of the third term M_3 can be bounded above as follows

$$|M_3| \leq \nu(D_j^\delta)^{-1} \nu\left((A_j \times B_j) \triangle (A \times B)\right) \leq \frac{\varepsilon_j}{\nu(D_j^\delta)} \rightarrow 0.$$

The last term M_4 goes to zero as $j \rightarrow \infty$ by (4), and this ends the proof of (5).

To apply the Choice Lemma for the measures $\nu_j^k = \Delta^{ks_j}(\cdot | C_j^k)$ and $a_j^k = \nu(C_j^k | D_j^\delta)$, it remains to check the first hypothesis of the lemma. By construction of the columns C_j^k , we have for any measurable subset $A \in X \times X$ and all $k \in \{-[\delta h_j], \dots, [\delta h_j]\}$,

$$(6) \quad \left| \Delta^{ks_j}(T_{s_j} \times T_{s_j} A | C_j^k) - \Delta^{ks_j}(A | C_j^k) \right| < \frac{C}{h_j}$$

where C is a constant. We get the desired result by setting $d_j := \frac{C}{s_j h_j}$.

The Choice Lemma then gives a sequence (k_j) with $-\delta h_j \leq k_j \leq \delta h_j$ such that $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu$. \square

Theorem 4.2. *Let a flow $T = (T_t)_{t \in \mathbb{R}}$ be of rank-one and ν be an ergodic self-joining of $(T_t)_{t \in \mathbb{R}}$. Then there is a sequence (k_j) such that $\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \frac{1}{2}\nu + \frac{1}{2}\nu'$ for some self-joining ν' : For all measurable sets A, B*

$$\mu(A \cap T_{s_j}^{k_j} B) \rightarrow \frac{1}{2}\nu(A \times B) + \frac{1}{2}\nu'(A \times B).$$

Proof. For any $1/2 < \delta < 1$, we have

$$\lim_{j \rightarrow \infty} \nu(D_j^\delta) > 1 - 2(1 - \delta) = 2\delta - 1 > 0.$$

Hence we can apply Lemma 4.1 for any $1/2 < \delta < 1$. By a diagonal argument, we get the existence of (k_j) and $(\delta_j) \searrow \frac{1}{2}$ with $-\delta_j h_j \leq k_j \leq \delta_j h_j$ such that

$$\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

Let us decompose $\Delta^{k_j s_j}$ as

$$\Delta^{k_j s_j} = \Delta^{k_j s_j}(\cdot | C_j^{k_j}) \Delta^{k_j s_j}(C_j^{k_j}) + \Delta^{k_j s_j}(\cdot | X \times X \setminus C_j^{k_j}) (1 - \Delta^{k_j s_j}(C_j^{k_j})).$$

Since $\liminf_{j \rightarrow \infty} \Delta^{k_j s_j}(C_j^{k_j}) \geq 1/2$, we get the existence of some self-joining ν' such that

$$\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \frac{1}{2}\nu + \frac{1}{2}\nu'.$$

\square

Corollary 4.3. *A mixing rank-one flow has minimal self-joinings of order two.*

Proof. Let ν be an ergodic self-joining of a mixing rank-one flow $(T_t)_{t \in \mathbb{R}}$. Let (k_j) be the sequence given by Theorem 4.2. If $|k_j s_j| \rightarrow \infty$, since T is mixing we have

$$\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \mu \times \mu,$$

hence $\mu \times \mu = \frac{1}{2}\nu + \frac{1}{2}\nu'$ for some self-joining ν' . The ergodicity of $\mu \times \mu$ then implies that $\mu \times \mu = \nu$. Otherwise, along some subsequence we have $k_j s_j \rightarrow s$ for some real number s . Then $\Delta^s = \frac{1}{2}\nu + \frac{1}{2}\nu'$ for some self-joining ν' , and again the ergodicity of Δ^s yields $\nu = \Delta^s$. Thus T has minimal self-joinings of order two.. \square

5. WEAK CLOSURE THEOREM FOR RANK-ONE FLOWS

Lemma 5.1 (Weak Closure Lemma). *If the automorphism S commutes with the rank-one flow $(T_t)_{t \in \mathbb{R}}$, then there exist $1/2 \leq d \leq 1$, a sequence (k_j) of integers and a sequence of measurable sets (Y_j) such that, for all measurable sets A, B*

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \mu(A \cap SB),$$

where Y_j has the form

$$Y_j^{d,-} := \bigsqcup_{0 \leq i < d h_j} T_{s_j}^i E_j \quad \text{or} \quad Y_j^{d,+} := \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j.$$

Proof. This lemma is a consequence of the proof of Theorem 4.2, when the joining ν is equal to Δ_S . Given a sequence $(\delta_j) \searrow \frac{1}{2}$, the proof provides a sequence (k_j) where $-\delta_j h_j \leq k_j \leq \delta_j h_j$, such that $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) \xrightarrow[j \rightarrow \infty]{w} \Delta_S$, and $\Delta^{k_j s_j}(C_j^{k_j})$ converges to some number $d \geq 1/2$. Let $Y_j^{k_j}$ be the projection on the first coordinate of $C_j^{k_j}$, that is

$$Y_j^{k_j} = \begin{cases} \bigsqcup_{i=k_j}^{h_j} T_{s_j}^i E_j & \text{if } k_j \geq 0 \\ \bigsqcup_{i=0}^{h_j+k_j} T_{s_j}^i E_j & \text{if } k_j < 0. \end{cases}$$

We then have $\Delta^{k_j s_j}(\cdot | C_j^{k_j}) = \Delta^{k_j s_j}(\cdot | Y_j^{k_j} \times X)$, and $\mu(Y_j^{k_j}) = \Delta^{k_j s_j}(C_j^{k_j}) \rightarrow d$. Since the marginals of $\Delta^{k_j s_j}(\cdot | C_j^{k_j})$ are absolutely continuous with respect to μ , with bounded density, this yields, for all measurable sets A, B ,

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j^{k_j}) \rightarrow d \mu(A \cap SB).$$

If there exist infinitely many j 's such that $k_j \geq 0$, then along this subsequence, we have

$$\mu\left(Y_j^{k_j} \Delta Y_j^{d,+}\right) \xrightarrow[j \rightarrow \infty]{} 0,$$

since $(h_j - k_j)/h_j \rightarrow d$. A similar result holds along the subsequence of j 's such that $k_j < 0$, with $Y_j^{d,+}$ replaced by $Y_j^{d,-}$. \square

Theorem 5.2 (Weak Closure Theorem for rank-one flows). *If the automorphism S commutes with the rank-one flow $(T_t)_{t \in \mathbb{R}}$, then there exists a sequence of integers (k_j) such that $\Delta^{k_j s_j} \rightarrow \Delta_S$: For all measurable sets A, B ,*

$$\mu(A \cap T^{k_j s_j} B) \rightarrow \mu(A \cap SB).$$

Proof. We fix T and consider the maximal number d for which the statement of Lemma 5.1 is true. (If $d = 1$, the theorem is proved.)

So we start from the following statement: We have a sequence of sets $\{Y_j\}$, of the form given in Lemma 5.1, such that for all measurable A, B

$$\mu(A \cap T_{s_j}^{k_j} B \cap Y_j) \rightarrow d \mu(A \cap SB).$$

Then a similar statement holds when Y_j is replaced by SY_j : Indeed, since S commutes with T and μ is invariant by S , we have

$$\begin{aligned} \mu(A \cap T_{s_j}^{k_j} B \cap SY_j) &= \mu(S^{-1} A \cap T_{s_j}^{k_j} S^{-1} B \cap Y_j) \\ &\xrightarrow[j \rightarrow \infty]{} d \mu(S^{-1} A \cap S S^{-1} B) = d \mu(A \cap SB). \end{aligned}$$

Let λ be a limit point for the sequence of probability measures $\{\nu_j\}$ defined on $X \times X$ by

$$\nu_j(A \times B) := \frac{1}{\mu(Y_j \cup SY_j)} \mu\left(A \cap T_{s_j}^{k_j} B \cap (Y_j \cup SY_j)\right).$$

Then $\lambda \leq 4d \Delta_S$. Moreover, the measure λ is invariant by $T_{s_p} \times T_{s_p}$ for all p . Indeed, for $j \geq p$, we have

$$\mu(T_{s_p} Y_j \Delta Y_j) = \mu(T_{s_j}^{s_p/s_j} Y_j \Delta Y_j)$$

which is of order $\frac{s_p}{s_j h_j}$, hence vanishes as $j \rightarrow \infty$. Since Δ_S is an ergodic measure for the flow $\{T_t \times T_t\}$, we can apply Lemma 2.1, which gives $\lambda = \Delta_S$. We obtain

$$\mu\left(A \cap T_{s_j}^{k_j} B \cap (Y_j \cup SY_j)\right) \rightarrow u \mu(A \cap SB),$$

where $u := \lim_j \mu(Y_j \cup SY_j)$ (if the limit does not exist, then we consider some subsequence of $\{j\}$).

Our aim is to show that $u = 1$, which will end the proof of the theorem. Let us introduce

$$W_j := \left(\bigsqcup_{0 \leq i \leq h_j} T_{s_j}^i E_j \right) \setminus Y_j.$$

Assume that $u < 1$, then (denoting by Y^c the complementary of $Y \subset X$)

$$\lim_j \Delta_S(W_j \times W_j) = \lim_j \mu(W_j \cap SW_j) = \lim_j \mu(Y_j^c \cap SY_j^c) = 1 - u > 0.$$

Let us consider the case where Y_j has the form $Y_j^{d,-} = \bigsqcup_{0 \leq i < dh_j} T_{s_j}^i E_j$. Then $W_j = \bigsqcup_{dh_j \leq i \leq h_j} T_{s_j}^i E_j$, and we define for any $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{(1-\delta')h_j < i \leq h_j} T_{s_j}^i E_j \subset W_j.$$

In the same way, if Y_j has the form $Y_j^{d,+} = \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j$, we set for $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{0 < i < \delta' h_j} T_{s_j}^i E_j \subset W_j.$$

In both cases, note that

$$\Delta_S\left((W_j \times W_j) \setminus (W_j(\delta') \times W_j(\delta'))\right) \leq 2(1 - d - \delta').$$

Thus, for δ' close enough to $1 - d$, we get

$$\limsup_j \Delta_S\left(W_j(\delta') \times W_j(\delta')\right) \geq 1 - u - 2(1 - d - \delta') > 0.$$

Since $W_j(\delta') \times W_j(\delta') \subset D_j^{\delta'}$, this ensures that

$$\limsup \Delta_S(D_j^{\delta'}) > 0.$$

Lemma 4.1 then provides a sequence (k'_j) with $-\delta' h_j \leq k'_j \leq \delta' h_j$, such that

$$\Delta^{k'_j s_j}(\cdot | C_j^{k'_j}) \xrightarrow{j \rightarrow \infty} \Delta_S,$$

and the projections $Y_j^{k'_j}$ of $C_j^{k'_j}$ on the first coordinate satisfy

$$\lim_j \mu(Y_j^{k'_j}) \geq 1 - \delta' > d,$$

which contradicts the maximality of d . Hence $u = 1$. \square

6. RIGIDITY OF FACTORS OF RANK-ONE FLOWS

Lemma 6.1. *Let \mathcal{F} be a non-trivial factor of a rank-one flow $(T_t)_{t \in \mathbb{R}}$. Then there exist $1/2 \leq d \leq 1$, a sequence of integers (k_j) with $|k_j s_j| \rightarrow 0$ and a sequence of measurable sets (Y_j) such that, for all measurable sets $A, B \in \mathcal{F}$*

$$\mu(A \cap T^{k_j s_j} B \cap Y_j) \rightarrow d\mu(A \cap B),$$

where Y_j has the form

$$Y_j^{d,-} := \bigsqcup_{0 \leq i < dh_j} T_{s_j}^i E_j \quad \text{or} \quad Y_j^{d,+} := \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j.$$

Proof. We start with the relatively independent joining above the factor \mathcal{F} and consider an ergodic component ν such that $\nu(\{(x, x), x \in X\}) = 0$. Observe that for any sets $A, B \in \mathcal{F}$, we have $\nu(A \times B) = \mu(A \cap B)$. We repeat the proof of Lemma 5.1 with ν in place of Δ_S . This provides the sequences (k_j) and (Y_j) such that for all measurable sets A, B

$$\mu(A \cap T^{k_j s_j} B \cap Y_j) \rightarrow d\nu(A \times B).$$

If we had $k_j s_j \rightarrow 0$, then $\nu(A \times B) = \mu(A \cap B)$, which would contradict the hypothesis that ν gives measure 0 to the diagonal. \square

Theorem 6.2. *Let \mathcal{F} be a non-trivial factor of a rank-one flow $(T_t)_{t \in \mathbb{R}}$. Then there exists a sequence of integers (k_j) with $|k_j s_j| \rightarrow \infty$ such that, for all measurable sets $A, B \in \mathcal{F}$*

$$\mu(A \cap T^{k_j s_j} B) \rightarrow \mu(A \cap B).$$

Proof. Again we fix some ergodic component ν such that $\nu(\{(x, x), x \in X\}) = 0$. We consider the maximal number d for which the statement of Lemma 6.1 is true. We thus have a sequence of sets $\{Y_j\}$, of the form given in Lemma 6.1, such that

$$(7) \quad \forall A, B \in \mathcal{F}, \quad \frac{1}{\mu(Y_j)} \mathbb{E}_\mu \left[\mathbb{1}_A \mathbb{1}_{T^{k_j s_j} B} \mathbb{1}_{Y_j} \right] \rightarrow \mu(A \cap B).$$

In the above equation, one can replace $\mathbb{1}_{Y_j}$ by $\phi_j(x) := \mathbb{E}_\nu[\mathbb{1}_{Y_j}(x')|x]$: Indeed, since ν coincides with Δ on $\mathcal{F} \otimes \mathcal{F}$, we have $\mathbb{1}_A(x') = \mathbb{1}_A(x)$ and $\mathbb{1}_{T^{k_j s_j} B}(x') = \mathbb{1}_{T^{k_j s_j} B}(x)$ ν -a.s. Hence,

$$\mathbb{E}_\mu \left[\mathbb{1}_A \mathbb{1}_{T^{k_j s_j} B} \mathbb{1}_{Y_j} \right] = \mathbb{E}_\nu \left[\mathbb{1}_A(x) \mathbb{1}_{T^{k_j s_j} B}(x) \mathbb{1}_{Y_j}(x') \right] = \mathbb{E}_\mu \left[\mathbb{1}_A(x) \mathbb{1}_{T^{k_j s_j} B}(x) \phi_j(x) \right].$$

We note that

$$(8) \quad \mathbb{E}_\mu \left[|\phi_j - \phi_j \circ T_{s_j}| \right] \leq \mu(Y_j \Delta T_{s_j} Y_j) = O\left(\frac{1}{h_j}\right).$$

For any $\varepsilon > 0$, let

$$U_j^\varepsilon := \{x : \phi_j(x) > \varepsilon\}.$$

We would like to prove that (7) remains valid with $\mathbb{1}_{Y_j}$ replaced by $\mathbb{1}_{U_j^\varepsilon}$ for ε small enough. To this end, we need almost-invariance of U_j^ε under T_{s_j} , which does not seem to be guaranteed for arbitrary ε . Therefore, we use the following technical argument to find a sequence (ε_j) for which the desired result holds.

Fix $\varepsilon > 0$ small enough so that $\mu(U_j^\varepsilon) > \mu(Y_j)/2$ for all large j . By Lemma 3.2, we can assume that $s_j^2 h_j \rightarrow \infty$. Let $\delta_j = o(s_j)$ such that $(\delta_j h_j)^{-1} = o(s_j)$. We divide the interval $[\varepsilon/2, \varepsilon]$ into $\varepsilon/(4\delta_j)$ disjoint subintervals of length $2\delta_j$. One of these subintervals, called I_j , satisfy

$$(9) \quad \mu(\{x : \phi_j(x) \in I_j\}) \leq \frac{4\delta_j}{\varepsilon}.$$

Let us call ε_j the center of the interval I_j . Observe that

$$\mu(U_j^{\varepsilon_j} \Delta T_{s_j} U_j^{\varepsilon_j}) \leq \mu(\{x : |\phi_j(x) - \varepsilon_j| < \delta_j\}) + \mu(\{x : |\phi_j(x) - \phi_j(T_{s_j}(x))| \geq \delta_j\}).$$

By (9) and (8), we get that

$$(10) \quad \mu(U_j^{\varepsilon_j} \Delta T_{s_j} U_j^{\varepsilon_j}) = O\left(\delta_j + \frac{1}{\delta_j h_j}\right) = o(s_j).$$

Taking a subsequence if necessary, we can assume that the sequence of probability measures λ_j , defined by

$$\forall A, B \in \mathcal{A}, \quad \lambda_j(A \times B) := \frac{1}{\mu(U_j^{\varepsilon_j})} \mathbb{E}_\mu \left[\mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{U_j^{\varepsilon_j}} \right],$$

converges to some probability measure λ , which is invariant by $T_{s_p} \times T_{s_p}$ for all p by (10). Recall that $\mu(U_j^{\varepsilon_j}) > \mu(Y_j)/2$ and that $\mathbb{1}_{U_j^{\varepsilon_j}} \leq \phi_j/\varepsilon_j$. Then, since $\varepsilon_j > \varepsilon/2$, we have $\lambda|_{\mathcal{F} \otimes \mathcal{F}} \leq \frac{4}{\varepsilon} \Delta|_{\mathcal{F} \otimes \mathcal{F}}$. Since $\Delta|_{\mathcal{F} \otimes \mathcal{F}}$ is an ergodic measure for the flow $\{T_t \times T_t\}|_{\mathcal{F} \otimes \mathcal{F}}$, we can apply Lemma 2.1, which gives $\lambda|_{\mathcal{F} \otimes \mathcal{F}} = \Delta|_{\mathcal{F} \otimes \mathcal{F}}$. This means that (7) remains valid with $\mathbb{1}_{Y_j}$ replaced by $\mathbb{1}_{U_j^{\varepsilon_j}}$.

The analogue of (7) is also valid when we replace $\mathbb{1}_{Y_j}$ by $\mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}}$: Indeed, we also have the almost-invariance property

$$\mu((Y_j \cup U_j^{\varepsilon_j}) \Delta T_{s_j}(Y_j \cup U_j^{\varepsilon_j})) = o(s_j)$$

and $\mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}} \leq \mathbb{1}_{Y_j} + \mathbb{1}_{U_j^{\varepsilon_j}}$. We conclude by a similar argument.

Since ε can be taken arbitrarily small, we can now use a diagonal argument to show that (7) remains valid with $\mathbb{1}_{Y_j}$ replaced by $\mathbb{1}_{U_j^{\varepsilon_j}}$ where the sequence (ε_j) now satisfies $\varepsilon_j \rightarrow 0$. Hence, taking a subsequence if necessary to ensure that $\mu(Y_j \cup U_j^{\varepsilon_j})$ converges to some number u , we get

$$\forall A, B \in \mathcal{F}, \quad \mathbb{E}_\mu \left[\mathbb{1}_A \mathbb{1}_{T_{s_j}^{k_j} B} \mathbb{1}_{Y_j \cup U_j^{\varepsilon_j}} \right] \rightarrow u\mu(A \cap B).$$

It now remains to prove that $u = 1$, which we do by repeating the end of the proof of Theorem 5.2. Assume that $u < 1$. Let us introduce

$$W_j := \left(\bigsqcup_{0 \leq i \leq h_j} T_{s_j}^i E_j \right) \setminus Y_j.$$

We have

$$\lim_j \nu(W_j \times W_j) = \lim_j \nu(Y_j^c \times Y_j^c) = \lim_j \mathbb{E}_\mu \left[\mathbb{1}_{Y_j^c} (1 - \phi_j) \right].$$

Observe that $(1 - \phi_j) \geq \mathbb{1}_{(U_j^{\varepsilon_j})^c} - \varepsilon_j$. Hence

$$\lim_j \nu(W_j \times W_j) \leq \lim_j \mathbb{E}_\mu \left[\mathbb{1}_{Y_j^c} \mathbb{1}_{(U_j^{\varepsilon_j})^c} \right] = 1 - u > 0.$$

Let us consider the case where Y_j has the form $Y_j^{d,-} = \bigsqcup_{0 \leq i < dh_j} T_{s_j}^i E_j$. Then $W_j = \bigsqcup_{dh_j \leq i \leq h_j} T_{s_j}^i E_j$, and we define for any $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{(1-\delta')h_j < i \leq h_j} T_{s_j}^i E_j \subset W_j.$$

In the same way, if Y_j has the form $Y_j^{d,+} = \bigsqcup_{(1-d)h_j < i \leq h_j} T_{s_j}^i E_j$, we set for $\delta' < 1 - d$

$$W_j(\delta') := \bigsqcup_{0 < i < \delta' h_j} T_{s_j}^i E_j \subset W_j.$$

In both cases, note that

$$\nu((W_j \times W_j) \setminus (W_j(\delta') \times W_j(\delta'))) \leq 2(1 - d - \delta').$$

thus, for δ' close enough to $1 - d$, we get

$$\limsup_j \nu\left(W_j(\delta') \times W_j(\delta')\right) \geq 1 - u - 2(1 - d - \delta') > 0.$$

Since $W_j(\delta') \times W_j(\delta') \subset D_j^{\delta'}$, this ensures that

$$\limsup \nu(D_j^{\delta'}) > 0.$$

Lemma 4.1 then provides a sequence (k'_j) with $-\delta' h_j \leq k'_j \leq \delta' h_j$, such that

$$\Delta^{k'_j s_j}(\cdot | C_j^{k'_j}) \xrightarrow{j \rightarrow \infty} \nu.$$

In particular, $\Delta^{k'_j s_j}(\cdot | C_j^{k'_j})|_{\mathcal{F} \otimes \mathcal{F}} \xrightarrow{j \rightarrow \infty} \Delta|_{\mathcal{F} \otimes \mathcal{F}}$. Since the projections $Y_j^{k'_j}$ of $C_j^{k'_j}$ on the first coordinate satisfy

$$\lim_j \mu(Y_j^{k'_j}) \geq 1 - \delta' > d,$$

this contradicts the maximality of d . Hence $u = 1$. \square

7. KING'S THEOREM FOR FLAT-ROOF RANK-ONE FLOW

We consider a rank-one flow $(T_t)_{t \in \mathbb{R}}$. We say that $(T_t)_{t \in \mathbb{R}}$ has *flat roof* if we can choose the sequence $\xi_j = \{E_j, T_{s_j} E_j, \dots, T_{s_j}^{h_j-1} E_j, X \setminus \bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j\}$ in the definition such that

$$\frac{\mu\left(T_{s_j}^{h_j} E_j \Delta E_j\right)}{\mu(E_j)} \xrightarrow{j \rightarrow \infty} 0.$$

Theorem 7.1. *Let $(T_t)_{t \in \mathbb{R}}$ be a flat-roof rank-one flow, and ν be an ergodic self-joining of $(T_t)_{t \in \mathbb{R}}$. Then there exists a sequence (k_j) such that $\Delta^{k_j s_j} \xrightarrow{j \rightarrow \infty} \nu$.*

Proof. Let us defined, for $0 \leq k \leq h_j - 1$

$$a_k^j := \nu\left(T_{s_j}^k E_j \times E_j\right) \quad \text{and} \quad b_k^j := \nu\left(E_j \times T_{s_j}^{h_j-k} E_j\right).$$

We claim that the flat-roof property implies

$$(11) \quad h_j \sum_{k=1}^{h_j-1} |a_k^j - b_k^j| \xrightarrow{j \rightarrow \infty} 0.$$

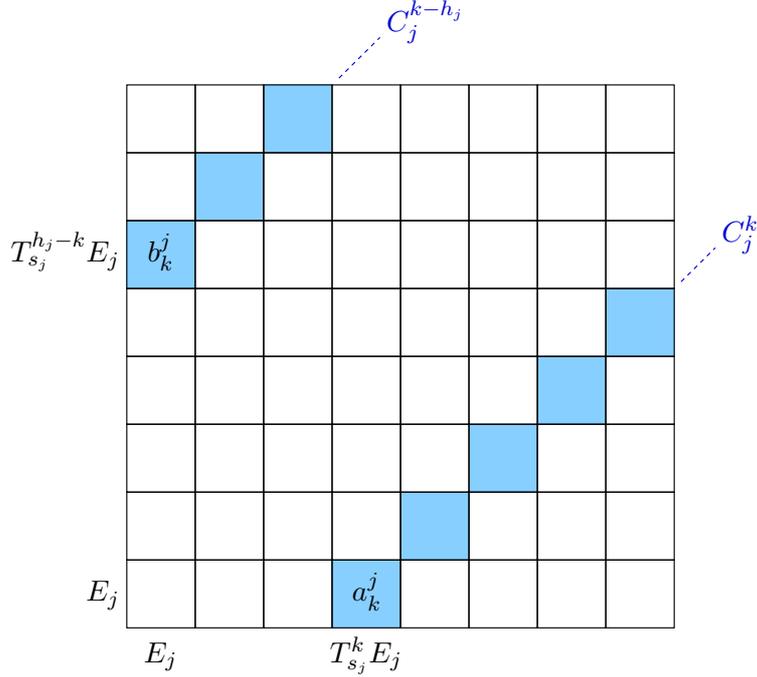
Indeed, by invariance $a_k^j = \nu\left(T_{s_j}^{h_j} E_j \times T_{s_j}^{h_j-k} E_j\right)$. Hence

$$|a_k^j - b_k^j| \leq \nu\left(\left(T_{s_j}^{h_j} E_j \Delta E_j\right) \times T_{s_j}^{h_j-k} E_j\right),$$

and

$$\sum_{k=1}^{h_j-1} |a_k^j - b_k^j| \leq \nu\left(\left(T_{s_j}^{h_j} E_j \Delta E_j\right) \times X\right) = \mu\left(\left(T_{s_j}^{h_j} E_j \Delta E_j\right)\right).$$

The claim follows, since $\mu(E_j) \sim 1/h_j$.

FIGURE 2. The union of C_j^k and $C_j^{k-h_j}$ is denoted by G_j^k .

We gather the columns C_j^k in pairs, defining for $1 \leq k \leq h_j - 1$, $G_j^k := C_j^k \sqcup C_j^{k-h_j}$. (See Figure 2.) We also set $G_j^0 := C_j^0$. Note that $\nu(G_j^k) = (h_j - k)a_k^j + kb_k^j$. Observe also that

$$\nu \left(\bigsqcup_{k=0}^{h_j-1} G_j^k \right) = \nu \left(\bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j \times \bigsqcup_{k=0}^{h_j-1} T_{s_j}^k E_j \right) \xrightarrow{j \rightarrow \infty} 1.$$

Hence,

$$(12) \quad \sum_{k=0}^{h_j-1} \nu(G_j^k) \nu(\cdot | G_j^k) \xrightarrow{j \rightarrow \infty} \nu.$$

We claim that, using the flat-roof property, we can in the above equation replace $\nu(\cdot | G_j^k)$ by Δ^{ks_j} . Let A and B be ξ_j -measurable sets, which are unions of $T_{s_j}^i E_j$ ($0 \leq i \leq h_j - 1$). We denote by r_k (respectively ℓ_k) the number of elementary cells of the form $T_{s_j}^{i_1} E_j \times T_{s_j}^{i_2} E_j$ which are contained in $A \times B$ and which belong to the column C_j^k (respectively $C_j^{k-h_j}$). We have

$$(13) \quad \nu(A \times B | G_j^k) \nu(G_j^k) = \ell_k b_k^j + r_k a_k^j.$$

Moreover, we will show that the flat-roof property ensures the existence of a sequence (ε_j) with $\varepsilon_j \xrightarrow{j \rightarrow \infty} 0$ such that

$$(14) \quad \left| \Delta^{ks_j}(A \times B) - \frac{\ell_k + r_k}{h_j} \right| \leq \varepsilon_j.$$

Indeed, let us cut A into $A_1 := A \cap \bigsqcup_{0 \leq i \leq k-1} T_{s_j}^i E_j$ and $A_2 := A \cap \bigsqcup_{k \leq i \leq h_j-1} T_{s_j}^i E_j$. We have

$$\Delta^{ks_j}(A_2 \times B) = r_k \mu(E_j),$$

and

$$\Delta^{ks_j}(A_1 \times B) = \ell_k \Delta^{ks_j}(E_j \times T_{s_j}^{h_j-k} E_j) + \Delta^{ks_j}\left((A_1 \times B) \setminus C_j^{k-h_j}\right).$$

Recalling that $\Delta^{ks_j}(E_j \times T_{s_j}^{h_j-k} E_j) = \mu(E_j \cap T_{s_j}^{h_j} E_j)$, we get

$$(15) \quad \Delta^{ks_j}(A \times B) = (r_k + \ell_k) \mu(E_j) - \ell_k \mu(E_j \setminus T_{s_j}^{h_j} E_j) + \Delta^{ks_j}\left((A_1 \times B) \setminus C_j^{k-h_j}\right).$$

The second term of the right-hand side is bounded by $h_j \mu(E_j \Delta T_{s_j}^{h_j} E_j)$, which goes to 0 by the flat-roof property. To treat the last term, we consider the particular case $A = B = \bigsqcup_{0 \leq i \leq h_j-1} T_{s_j}^i E_j$, for which this last term is maximized. We have then

$$1 - \Delta^{ks_j}(A \times B) \leq 2\mu\left(X \setminus \bigsqcup_{0 \leq i \leq h_j-1} T_{s_j}^i E_j\right) \xrightarrow{j \rightarrow \infty} 0.$$

On the other hand, (15) gives

$$\Delta^{ks_j}\left((A_1 \times B) \setminus C_j^{k-h_j}\right) = \Delta^{ks_j}(A \times B) - h_j \mu(E_j) + k \mu(E_j \setminus T_{s_j}^{h_j} E_j).$$

Since $h_j \mu(E_j) \rightarrow 1$, and $k \mu(E_j \setminus T_{s_j}^{h_j} E_j) \leq h_j \mu(E_j \Delta T_{s_j}^{h_j} E_j) \rightarrow 0$, we get that the last term of (15) goes to 0 uniformly with respect to k , A and B . It follows that

$$\left| \Delta^{ks_j}(A \times B) - (\ell_k + r_k) \mu(E_j) \right| \xrightarrow{j \rightarrow \infty} 0,$$

uniformly with respect to k , A and B . This concludes the proof of (14).

Equations (14) and (13) give

$$\begin{aligned} & \sum_{k=0}^{h_j-1} \left| \nu(A \times B | G_j^k) - \Delta^{ks_j}(A \times B) \right| \nu(G_j^k) \\ & \leq \sum_{k=0}^{h_j-1} |a_k^j - b_k^j| \left| \ell_k - \frac{k}{h_j} (\ell_k + r_k) \right| + \varepsilon_j \\ & \leq h_j \sum_{k=0}^{h_j-1} |a_k^j - b_k^j| + \varepsilon_j \end{aligned}$$

which goes to 0 as $j \rightarrow \infty$ by (11).

Recalling (12), we obtain

$$\sum_{k=0}^{h_j-1} \nu(G_j^k) \Delta^{ks_j} \xrightarrow{j \rightarrow \infty} \nu.$$

It remains to apply the Choice Lemma to conclude the proof of the theorem. \square

8. \mathbb{Z}^n -RANK-ONE ACTION

We consider now an action of \mathbb{Z}^n ($n \geq 1$). For $k \in \mathbb{Z}^n$, we denote by $k(1), \dots, k(n)$ its coordinates.

Definition 8.1. A \mathbb{Z}^n -action $\{T_k\}_{k \in \mathbb{Z}^n}$ is of rank one if there exists a sequence (ξ_j) of partitions converging to the partition onto points, where ξ_j is of the form

$$\xi_j = \left\{ (T_k E_j)_{k \in R_j}, X \setminus \bigsqcup_k T_k E_j \right\},$$

and R_j is a rectangular set of indices:

$$R_j = \{0, \dots, h_j(1) - 1\} \times \dots \times \{0, \dots, h_j(n) - 1\}.$$

The sequence (ξ_j) in the above definition being fixed, we define as for the rank-one flows the notions of columns and fat diagonals: For any $k \in \mathbb{Z}^n$, we set

$$C_j^k := \bigsqcup_{\substack{r, \ell \in R_j \\ r - \ell = k}} T_r E_j \times T_\ell E_j,$$

and given $0 < \delta < 1$,

$$D_j^\delta := \bigsqcup_{k: \prod_i (h_j(i) - |k(i)|) \geq (1 - \delta) \prod_i h_j(i)} C_j^k.$$

Lemma 8.2. For any self-joining ν of the rank-one action $\{T_k\}_{k \in \mathbb{Z}^n}$, for any $\delta > 1 - 1/2^n$, we have

$$\liminf_{j \rightarrow \infty} \nu(D_j^\delta) > 0.$$

Proof. We can find $\varepsilon > 0$, small enough such that

$$\left(\frac{1}{2} - \varepsilon\right)^n > 1 - \delta.$$

Let $r \in \mathbb{Z}^n$ be such that

$$\forall i, \left(\frac{1}{2} - \varepsilon\right) h_j(i) < r(i) < \left(\frac{1}{2} + \varepsilon\right) h_j(i).$$

Then, for any $\ell \in R_j$, we have for all i : $|r(i) - \ell(i)| < (\frac{1}{2} + \varepsilon) h_j(i)$. Hence

$$\prod_i (h_j(i) - |r(i) - \ell(i)|) > (1 - \delta) \prod_i h_j(i),$$

which means that for any $\ell \in R_j$, the column $C_j^{r-\ell}$ is contained in D_j^δ . It follows that

$$\left(\bigsqcup_{r: \forall i, |r(i) - h_j(i)/2| < \varepsilon h_j(i)} T_r E_j \right) \times \left(\bigsqcup_{\ell \in R_j} T_\ell E_j \right) \subset D_j^\delta.$$

We then get

$$\liminf_{j \rightarrow \infty} \nu(D_j^\delta) \geq \liminf_{j \rightarrow \infty} \mu \left(\bigsqcup_{r: \forall i, |r(i) - h_j(i)/2| < \varepsilon h_j(i)} T_r E_j \right) = (2\varepsilon)^n.$$

□

We can now state the analogue of Theorem 4.2 for \mathbb{Z}^n -rank-one action, which was first proved by A.A. Pavlova in [9].

Theorem 8.3. *Let ν be an ergodic self-joining of the \mathbb{Z}^n -rank-one action $\{T_k\}_{k \in \mathbb{Z}^n}$. Then we can find a sequence (k_j) in \mathbb{Z}^n and some self-joining ν' such that $\Delta^{k_j} \xrightarrow[j \rightarrow \infty]{w} \frac{1}{2^n} \nu + \left(1 - \frac{1}{2^n}\right) \nu'$: For all measurable sets A, B*

$$\mu(A \cap T_{k_j} B) \rightarrow \frac{1}{2^n} \nu(A \times B) + \left(1 - \frac{1}{2^n}\right) \nu'(A \times B).$$

Proof. The proof follows the same lines as for Theorem 4.2. First note that Lemma 4.1 can be easily adapted to the \mathbb{Z}^n -situation. Hence, by Lemma 8.2, using a diagonal argument, we get the existence of (k_j) and $(\delta_j) \searrow 1 - \frac{1}{2^n}$ with $C_j^{k_j} \subset D_j^{\delta_j}$ such that

$$\Delta^{k_j} \left(\cdot | C_j^{k_j} \right) \xrightarrow[j \rightarrow \infty]{w} \nu.$$

To conclude, it remains to prove that $\liminf \Delta^{k_j}(C_j^{k_j}) \geq 1/2^n$. To this aim, we count the number of pairs (r, ℓ) such that $T_r E_j \times T_\ell E_j \subset C_j^{k_j}$. We can easily check that these are exactly the pairs (r, ℓ) such that, for all $1 \leq i \leq n$, there exists $m(i) \in \{0, \dots, h_j(i) - 1 - |k_j(i)|\}$ with

$$(r(i), \ell(i)) = \begin{cases} (k_j(i) + m(i), m(i)) & \text{if } k_j(i) \geq 0 \\ (m(i), -k_j(i) + m(i)) & \text{otherwise.} \end{cases}$$

Hence $\Delta^{k_j}(C_j^{k_j}) = \prod_i (h_j(i) - 1 - |k_j(i)|) \mu(E_j)$. Using the fact that $C_j^{k_j} \subset D_j^{\delta_j}$, we get the desired result. \square

When $n \geq 2$, it is known that the Weak Closure Theorem fails (counterexamples have been given in [1, 2]). However, as a consequence of Theorem 8.3, we get the following:

Corollary 8.4 (Partial Weak Closure Theorem for \mathbb{Z}^n -rank-one action). *Let S be an automorphism commuting with the \mathbb{Z}^n -rank-one action $\{T_k\}_{k \in \mathbb{Z}^n}$. Then we can find a sequence (k_j) in \mathbb{Z}^n and some self-joining ν' such that*

$$\Delta^{k_j} \xrightarrow[j \rightarrow \infty]{} \frac{1}{2^n} \Delta_S + \left(1 - \frac{1}{2^n}\right) \nu'.$$

Moreover, if $S \notin \{T_k \mid k \in \mathbb{Z}^n\}$, then $\{T_k\}_{k \in \mathbb{Z}^n}$ is partially rigid: There exists a sequence (k'_ℓ) in \mathbb{Z}^n with $|k'_\ell| \rightarrow \infty$ such that for all measurable sets A and B

$$\liminf_{\ell \rightarrow \infty} \mu(A \cap T_{k'_\ell} B) \geq \frac{1}{2^{2n}} \mu(A \cap B).$$

Proof. The first part is a direct application of Theorem 8.3 with $\nu = \Delta_S$. If moreover $S \notin \{T_k \mid k \in \mathbb{Z}^n\}$, then the sequence (k_j) of the theorem must satisfy $|k_j| \rightarrow \infty$. Let us enumerate the cylinder sets as $\{A_0, A_1, \dots, A_\ell, \dots\}$. Let (ε_ℓ) be a sequence of positive numbers decreasing to zero. For any ℓ , we can find a large enough integer $j_1(\ell)$ such that for all cylinder sets $A, B \in \{A_0, A_1, \dots, A_\ell\}$,

$$\mu\left(T_{k_{j_1(\ell)}} A \cap SB\right) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(SA \cap SB) = \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(A \cap B).$$

Then, we can find a large enough integer $j_2(\ell)$ with $|j_2(\ell)| > 2|j_1(\ell)|$ such that for all cylinder sets $A, B \in \{A_0, A_1, \dots, A_\ell\}$,

$$\mu\left(T_{k_{j_1(\ell)}}A \cap T_{k_{j_2(\ell)}}B\right) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right) \mu(T_{k_{j_1(\ell)}}A \cap SB).$$

It follows that for all $\ell \geq 0$ and all cylinder sets $A, B \in \{A_0, A_1, \dots, A_\ell\}$,

$$\mu\left(A \cap T_{k_{j_2(\ell)} - k_{j_1(\ell)}}B\right) \geq \left(\frac{1}{2^n} - \varepsilon_\ell\right)^2 \mu(A \cap B).$$

This proves the result announced in the corollary when A and B are cylinder sets with $k'_\ell := k_{j_2(\ell)} - k_{j_1(\ell)}$, and this extends in a standard way to all measurable sets. \square

The counterexample given in [2] also shows that the rigidity of factors is no more valid when $n \geq 2$. Theorem 8.3 only ensures the partial rigidity of factors of \mathbb{Z}^n -rank-one actions.

Corollary 8.5 (Partial rigidity of factors of \mathbb{Z}^n -rank-one action). *Let \mathcal{F} be a non-trivial factor of the \mathbb{Z}^n -rank-one action $\{T_k\}_{k \in \mathbb{Z}^n}$. Then there exists a sequence (k_j) in \mathbb{Z}^n with $|k_j| \rightarrow \infty$ such that, for all measurable sets $A, B \in \mathcal{F}$*

$$\liminf \mu(A \cap T_{k_j}B) \geq \frac{1}{2^n} \mu(A \cap B).$$

Proof. This is a direct application of Theorem 8.3 where ν is an ergodic component of the relatively independent joining above the factor \mathcal{F} . \square

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