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AROUND QUILLEN'S THEOREM A

BRUNO KAHN

Abstract. We introduce a notion of cellular functor. It allows us to give a variant of Quillen's proof of the Solomon-Tits theorem, which does not use Theorem A. We also use it to generalise some exact sequences of Quillen, and to reformulate them into a rank spectral sequence converging to the homology of the K'-theory space of an integral scheme.

In the memory of Daniel Quillen

Contents

Introduction 1
1. Nerves with coefficients 3
2. A long homology exact sequence 7
3. Cellular functors and the Solomon-Tits theorem 13
4. The rank spectral sequence 15
References 18

Introduction

Let A be a Dedekind domain. Inspection shows immediately that the exact sequences of [12, Th. 3], used by Quillen to prove that the K-groups of A are finitely generated when A is a ring of S-integers in a global field, assemble to define an exact couple, hence a spectral sequence converging to the homology of KA. This gives potentially more power to Quillen's method, which as such yields no information on the ranks of these K-groups.

A natural way to imagine such a spectral sequence is to consider the maps

$$BQ_{n-1}\mathcal{P}(A) \to BQ_n\mathcal{P}(A)$$

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used by Quillen as *homotopy cofibrations* rather than homotopy fibrations. The resulting rank spectral sequence coincides with the abovementioned spectral sequence: this is not immediately obvious, but follows from a simple argument of Vogel, see Remark 4.3.4.

The aim of this note is to construct the rank spectral sequence in a way as functorial as possible. The two operative ingredients are Thomason's theorem on the nerve of a Grothendieck construction (Theorem 1.4.3) and the notion of *cellular functor* (Definition 2.3.2), which is well-adapted to the present context thanks to Theorem 2.3.6. After the first version of this paper was written, Fei Sun observed that Theorem 2.3.6 may also be used to recover part of Quillen's proof of the Solomon-Tits theorem in [12]. We complete Sun's remark in §3, by covering the missing part: unlike in Quillen's argument, Theorem A is not used there. In this light, one might think of Theorem 2.3.6 as "dual" to Theorem A, potentially playing a similar rôle for homotopy cofibrations as Theorem A plays for homotopy fibrations.

The main theorem is Theorem 4.3.3: we apply the theory in the slightly more general case of torsion-free sheaves over an integral scheme, which might be useful elsewhere.

The next step is to compute the d^1 differentials of this rank spectral sequence. This has been done by Fei Sun in his thesis, using the *universal modular symbols* of Ash-Rudolph [1]: see [9] and [13].

Acknowledgements. I thank Pierre Vogel for explaining me the argument of Remark 4.3.4 in 2008, and Georges Maltsiniotis for pointing out Thomason's paper [14]. I also thank Fei Sun for several discussions around these questions and Joël Riou for helpful comments, especially for pointing out a wrong axiomatisation of Proposition 3.1.2 in a previous version of this manuscript.

I don't think Theorem A is used explicitly anywhere. Yet I feel its spirit is prevalent in this text, hence the title.

Notation. We denote by **Set**, **Ab**, **sSet**, **Cat** the category of (small) sets, abelian groups, simplicial sets, categories. For $n \geq 0$, $[n]$ denotes the totally ordered set $\{0, \dots, n\}$, considered as a small category. We write $*$ for the category with one object and one morphism (sometimes for the set with 1 element). Finally, $\mathbf{\Delta}$ denotes the category of simplices (objects: finite nonempty ordinals, morphisms: non-decreasing maps).

We shall use Mac Lane's comma notation [10, p. 47]: if we have a diagram of functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{F'} \mathcal{C}'$$

the comma category $F \downarrow F'$ has for objects the diagram $F(c) \to F'(c')$, and for morphisms the obvious commutative diagrams. We use the following abbreviations: if $c \in \mathcal{C}$, yielding the functor $F_c : * \to \mathcal{C}$, we write $F_c \downarrow F' = c \downarrow F'$; similarly on the right.

1. Nerves with coefficients

Subsections 1.1–1.3 can essentially be found in Goerss-Jardine [6, Ch. IV].

1.1. **Set-valued coefficients.**

1.1.1. Let $\mathcal{D} \in \mathbf{Cat}$ be a small category. The *nerve* of $\mathcal{D}$ is the simplicial set $N(\mathcal{D})$ with

$$N_n(\mathcal{D}) = Ob\,\mathbf{Cat}([n], \mathcal{D}) = \coprod_{d_0 \to \cdots \to d_n} *,\ d_i \in \mathcal{D}$$

cf. [5, II, 4.1].

1.1.2. Let $\mathcal{D} \in \mathbf{Cat}$ and let $F : \mathcal{D} \to \mathbf{Set}$ be a covariant functor. The *nerve of $\mathcal{D}$ with coefficients in F* is the simplicial set $N(\mathcal{D}, F)$ with

$$N_n(\mathcal{D}, F) = \coprod_{d_0 \to \cdots \to d_n} F(d_0),\ d_i \in \mathcal{D}$$

cf. [5, App. II, 3.2]. For F the constant functor with value $*$, we recover the nerve of $\mathcal{D}$.

1.1.3. Let $(\mathcal{D}, F)$ be as in 1.1.2. We have the associated category

$$[\mathcal{D}, F] = \{(d, x) \mid d \in \mathcal{D}, x \in F(d)\}$$

where a morphism $(d, x) \to (d', x')$ is a morphism $f \in \mathcal{D}(d, d')$ such that $F(f)(x) = x'$. This category has two other equivalent descriptions:

(1) [5, II, 1.1] Let $y = \mathcal{D}^{op} \to \mathbf{Cat}(\mathcal{D}, \mathbf{Set})$ be the "coYoneda" embedding: then $[\mathcal{D}, F] \simeq y \downarrow F$, where F is considered as an object of $\mathbf{Cat}(\mathcal{D}, \mathbf{Set})$.
(2) $[\mathcal{D}, F] \simeq * \downarrow F$, where $* \in \mathbf{Cat}(\mathcal{D}, \mathbf{Set})$ is the constant functor with value $*$ and F is considered as a functor.

The following lemma is obvious:

1.1.4. **Lemma.** *There is a canonical isomorphism: $N(\mathcal{D}, F) \simeq N([\mathcal{D}, F])$.*

1.2. **Abelian group-valued coefficients.**

1.2.1. Suppose that F takes its values in the category $\mathbf{Ab}$ of abelian groups. Then $N(\mathcal{D}, F)$ is a simplicial abelian group; it has *homology groups* [5, App. II, 3.2, p. 153]

$$H_i(\mathcal{D}, F) = H_i([N(\mathcal{D}, F)]) = \pi_i(N(\mathcal{D}, F); 0)$$

where $[X]$ is the chain complex associated to a simplicial abelian group X by taking for differentials the alternating sums of the faces.

1.2.2. Let us recall the Eilenberg-Zilber–Cartier theorem as expounded in [7, p. 7] (see also [4, 2.9 and 2.16]). To a bisimplicial abelian group X, one may associate a double complex $[X]$ as in the simplicial case. To a bisimplicial object X one may associate the diagonal simplicial object δX:

$$(\delta X)_n = X_{n,n}$$

and to a double complex C one may associate the total complex $\operatorname{Tot} C$:

$$(\operatorname{Tot} C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

Then, given the diagram of functors

$$\begin{array}{ccc} (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}}\,\mathbf{Ab} & \xrightarrow{[\ ]} & C_{++}(\mathbf{Ab}) \\ {\scriptstyle\delta}\big\downarrow & & {\scriptstyle\operatorname{Tot}}\big\downarrow \\ \mathbf{\Delta}^{\mathrm{op}}\,\mathbf{Ab} & \xrightarrow{[\ ]} & C_{+}(\mathbf{Ab}) \end{array}$$

there exist two natural transformations

$$\begin{aligned} \text{shuffle}_X : &\operatorname{Tot}[X] \to [\delta X] \\ \text{Alexander-Whitney}_X : &[\delta X] \to \operatorname{Tot}[X] \end{aligned}$$

which are quasi-inverse homotopy equivalences.

1.3. **Simplicial set-valued coefficients.**

1.3.1. If X is a simplicial set, we may associate to it the free simplicial abelian group $\mathbf{Z}X$ generated by X, with $(\mathbf{Z}X)_n = \mathbf{Z}X_n$. Similarly with a bisimplicial set. The *homology* of X is the homotopy of $\mathbf{Z}X$, or equivalently the homology of $[\mathbf{Z}X]$. Similarly with coefficients in an abelian group A, using $\mathbf{Z}X \otimes A$.

We shall usually write $[\mathbf{Z}X] =: C_*(X)$; if $X = N(\mathcal{D})$ for a category $\mathcal{D}$, we abbreviate $C_*(X)$ into $C_*(\mathcal{D})$.

1.3.2. Let $\mathcal{D} \in \mathbf{Cat}$ and let $\mathbf{F} : \mathcal{D} \to \mathbf{sSet}$ be a functor. We may generalise the definition of 1.1.2 to get a *bisimplicial set* $N(\mathcal{D}, \mathbf{F})$:

$$N_{p,q}(\mathcal{D}, \mathbf{F}) = \coprod_{d_0 \to \cdots \to d_p} \mathbf{F}_q(d_0),\ d_i \in \mathcal{D}.$$

We may then take the diagonal $\delta N(\mathcal{D}, \mathbf{F})$, which is a simplicial set. We define

$$\begin{aligned} \pi_i(\mathcal{D}, \mathbf{F}; (d_0, x_0)) &= \pi_i(\delta N(\mathcal{D}, \mathbf{F}); (d_0, x_0)) \\ C_*(\mathcal{D}, \mathbf{F}; A) &= C_*(\delta N(\mathcal{D}, \mathbf{F})) \otimes A \\ H_i(\mathcal{D}, \mathbf{F}; A) &= H_i(C_*(\mathcal{D}, \mathbf{F}; A)) \end{aligned}$$

for $d_0 \in \mathcal{D}_0$ and $x_0 \in F_0(d_0)$ a chosen base point, and for A an abelian group of coefficients.

1.3.3. **Lemma.** *a) Let $X = (X_{p,q}), Y = (Y_{p,q})$ be two bisimplicial sets and $\varphi : X \to Y$ a bisimplicial map. Suppose that, for each $p \geq 0$, $\varphi_{p,*} : X_{p,*} \to Y_{p,*}$ is a weak equivalence. Then $\delta\varphi : \delta X \to \delta Y$ is a weak equivalence.*
b) Let $\mathbf{F}, \mathbf{G} : \mathcal{D} \to \mathbf{sSet}$ be two functors, and let $\varphi : \mathbf{F} \to \mathbf{G}$ be a morphism of functors. Suppose that, for each $d \in \mathcal{D}$, $\varphi(d) : \mathbf{F}(d) \to \mathbf{G}(d)$ is a weak equivalence. Then $\delta N(\mathcal{D}, \varphi) : \delta N(\mathcal{D}, \mathbf{F}) \to \delta N(\mathcal{D}, \mathbf{G})$ is a weak equivalence.

Proof. a) is well-known (for example, it is the special case of the Bousfield-Friedlander theorem of [6, Th. 4.9 p. 229] where $Y = W = *$), and b) follows from a). □

1.3.4. For an abelian group A and for $q \geq 0$, we may consider the additive functor

$$\begin{aligned} H_q(\mathbf{F}, A) : \mathcal{D} &\to \mathbf{Ab} \\ d &\mapsto H_q(\mathbf{F}(d), A). \end{aligned}$$

This gives a meaning to:

1.3.5. **Lemma.** *There is a spectral sequence*

$$E^2_{p,q} = H_p(\mathcal{D}, H_q(\mathbf{F}, A)) \Rightarrow H_{p+q}(\mathcal{D}, \mathbf{F}; A)$$

Proof. We shall use 1.2.2: it implies that $[\delta \mathbf{Z} N(\mathcal{D}, \mathbf{F})]$ is homotopy equivalent to $\mathrm{Tot}[\mathbf{Z} N(\mathcal{D}, \mathbf{F})]$. Therefore

$$\begin{aligned} &H_*(\mathcal{D}, \mathbf{F}; A) := H_*([\delta \mathbf{Z} N(\mathcal{D}, \mathbf{F})] \otimes A) \\ &\simeq H_*(\mathrm{Tot}[\mathbf{Z} N(\mathcal{D}, \mathbf{F})] \otimes A) = H_*(\mathrm{Tot} \bigoplus_{p,q} \bigoplus_{d_0 \to \cdots \to d_p} [\mathbf{Z} F_q(d_0)] \otimes A). \end{aligned}$$

Consider the first spectral sequence associated to this double complex in Cartan-Eilenberg [3, Ch. XV, §6]: the formula (1) of loc. cit, p. 331 shows that it is the desired spectral sequence. □

1.4. **Category-valued coefficients.** (See also [15, IV.3].)

1.4.1. Let $\mathcal{D} \in \mathbf{Cat}$ and let $\mathbf{F} : \mathcal{D} \to \mathbf{Cat}$ be a functor. Composing $\mathbf{F}$ with the nerve functor, we get a functor $N \circ \mathbf{F} : \mathcal{D} \to \mathbf{sSet}$, hence a bisimplicial set as in 1.3.2

$$N(\mathcal{D}, \mathbf{F}) = N(\mathcal{D}, N(\mathbf{F})).$$

1.4.2. We now extend the construction in 1.1.3. This yields the category $\mathcal{D} \int \mathbf{F}$ (Grothendieck construction, [SGA1, Exp. VI, §§8,9]):

- Objects are pairs (d, x), $d \in Ob(\mathcal{D}), x \in Ob(\mathbf{F}(d))$.
- For two objects $(d, x), (d', x')$, a morphism $(d, x) \to (d', x')$ is a morphism $f : d \to d'$ and a morphism $g : \mathbf{F}(f)(x) \to x'$.
- For three objects $(d, x), (d', x'), (d'', x'')$ and two morphisms $(f, g) : (d, x) \to (d', x')$, $(f', g') : (d', x') \to (d'', x'')$, $(f', g') \circ (f, g) := (f' \circ f, g' \circ \mathbf{F}(f')(g))$.

The Grothendieck construction is covariant in $\mathbf{F}$. In particular, there is a canonical functor $\mathcal{D} \int \mathbf{F} \to \mathcal{D}$ induced by the morphism $\mathbf{F} \to *$, where $*$ is the constant functor with value the point category. We call this functor the *augmentation*.

Lemma 1.1.4 then generalises as

1.4.3. **Theorem** (Thomason [14, Th. 1-2])**.** *There is a canonical weak equivalence*

$$\delta N(\mathcal{D}, \mathbf{F}) \to N(\mathcal{D} \int \mathbf{F})$$

sending a cell $(d_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} d_n, x_0 \xrightarrow{g_1} \dots \xrightarrow{g_n} x_n)$ $(x_i \in F(d_0))$ *to the cell*

$$\Big((d_0, y_0) \xrightarrow{h_1} \dots \xrightarrow{h_n} (d_n, y_n)\Big) \quad (y_i \in F(d_i))$$

with $y_i := F(f_i \dots f_1)x_i$ *and* $h_i = (f_i, F(f_i \dots f_1)g_i)$. □

1.4.4. Let $T : \mathcal{C} \to \mathcal{D}$ be a functor between two small categories. To T we associate the functor $\mathbf{F}_T : \mathcal{D} \to \mathbf{Cat}$ sending d to $T \downarrow d$. The category $\mathcal{D} \int \mathbf{F}_T$ may be identified with the comma category $T \downarrow Id_{\mathcal{D}}$. Therefore there are 3 functors:

$$p_1 : \mathcal{D} \int \mathbf{F}_T \to \mathcal{C}, p_2 : \mathcal{D} \int \mathbf{F}_T \to \mathcal{D}, s : \mathcal{C} \to \mathcal{D} \int \mathbf{F}_T$$

with

$$p_1([T(c) \xrightarrow{\varphi} d]) = c, p_2([T(c) \xrightarrow{\varphi} d]) = d, s(c) = [T(c) = T(c)].$$

We have:

1.4.5. **Lemma.** *s is left adjoint to p_1. Hence s and p_1 induce quasi-inverse homotopy equivalences $N(\mathcal{C}) \simeq N(\mathcal{D} \int \mathbf{F}_T)$.* □

From Theorem 1.4.3 and Lemma 1.3.5, we deduce

1.4.6. **Corollary** (cf. [5, App. II])**.** *There is a spectral sequence*

$$E^2_{p,q} = H_p(\mathcal{D}, H_q(\mathbf{F}_T, A)) \Rightarrow H_{p+q}(\mathcal{C}, A)$$

for any abelian group A. □

2. A long homology exact sequence

2.1. Spectral sequences and exact couples.

2.1.1**.** Let $\mathcal{T}$ be a triangulated category with countable direct sums, and let $C_0 \xrightarrow{i_1} \dots \xrightarrow{i_n} C_n \xrightarrow{i_{n+1}} \dots$ be a sequence of objects of $\mathcal{T}$. Let C be a homotopy colimit (mapping telescope) of the C_n [2]. Let $H : \mathcal{T} \to \mathcal{A}$ be a (co)homological functor to some abelian category $\mathcal{A}$: we assume that H commutes with countable direct sums. (Alternately, we could refuse infinite direct sums and assume that i_n is an isomorphism for n large enough.) To get an associated spectral sequence, the simplest is the technique of exact couples [8, pp. 152–153]: for each n, choose a cone $C_{n/n-1}$ of f_n, so that the exact triangles

$$C_{p-1} \xrightarrow{i_p} C_p \xrightarrow{j_p} C_{p/p-1} \xrightarrow{k_p} C_{p-1}[1]$$

yield long homology exact sequences

$$\dots H_n(C_{p-1}) \xrightarrow{i_{p,n}} H_n(C_p) \xrightarrow{j_{p,n}} H_n(C_{p/p-1}) \xrightarrow{k_{p,n}} H_{n-1}(C_{p-1}) \dots$$

where $H_n(X) := H(X[-n])$. The exact couple defined by

$$D_{p,q} = H_{p+q}(C_p), \quad E_{p,q} = H_{p+q}(C_{p/p-1})$$

and the relevant i, j, k define a spectral sequence abutting to $H_{p+q}(C)$. The E^1-terms of this spectral sequence are simply $E^1_{p,q} = E_{p,q}$, and $d^1_{p,q} = j_{p-1,p+q-1}k_{p,p+q}$ is induced by $j_{p-1}[1]k_p$.

Here is a more concrete description of this differential:

2.1.2. **Lemma.** *Let $C_{p/p-2}$ be a cone for $i_p i_{p-1}$, so that we may obtain a commutative diagram*

$$\begin{tikzcd}
C_{p-2} \arrow[r, equal] \arrow[d, "i_{p-1}"'] & C_{p-2} \arrow[d, "i_p i_{p-1}"'] & & & \\
C_{p-1} \arrow[r, "i_p"] \arrow[d, "j_{p-1}"'] & C_p \arrow[r, "j_p"] \arrow[d] & C_{p/p-1} \arrow[r, "k_p"] \arrow[d, "="'] & C_{p-1}[1] \arrow[d, "j_{p-1}[1]"'] \\
C_{p-1/p-2} \arrow[r, "\bar{i}_p"] & C_{p/p-2} \arrow[r, "\bar{j}_p"] & C_{p/p-1} \arrow[r, "\bar{k}_p"] & C_{p-1/p-2}[1]
\end{tikzcd}$$

of exact triangles (from the suitable axiom of triangulated categories). Then $d^1_{p,q}$ is the boundary map $\bar{k}_{p,n}$ in the long exact sequence

$$\dots H_{p+q}(C_{p-1/p-2}) \xrightarrow{\bar{i}_{p,n}} H_{p+q}(C_{p/p-2}) \xrightarrow{\bar{j}_{p,n}} H_{p+q}(C_{p/p-1}) \xrightarrow{\bar{k}_{p,n}} H_{p+q-1}(C_{p-1/p-2}) \dots$$

Proof. The diagram shows that $\bar{k}_p = j_{p-1}[1] \circ k_p$. □

2.1.3. In the usual case of a filtered complex $C_p = F_p C$, we may of course choose $C_{p/p-1} = F_p C / F_{p-1} C$ and $C_{p/p-2} = F_p C / F_{p-2} C$.

2.1.4. Let $\mathcal{Q}_0 \xrightarrow{T_1} \mathcal{Q}_1 \xrightarrow{T_2} \dots \to \mathcal{Q}_n \to \dots$ be a sequence of categories and functors. Let $\mathcal{Q} = \varinjlim \mathcal{Q}_n$. (Since there are no natural transformations involved this is a naïve colimit, defined objectwise and morphismwise.) Considering the corresponding sequence of chain complexes of nerves

$$C_*(\mathcal{Q}_0) \xrightarrow{i_1} C_*(\mathcal{Q}_1) \dots$$

we get from the yoga of 2.1.1 a spectral sequence abutting to $H_*(\mathcal{Q})$ (possibly with coefficients).

If the functors T_n are faithful and injective on objects, the maps i_n are injective and we are in the simpler situation of 2.1.3.

2.2. **Unreduced and reduced homology.**

2.2.1. Let (X, x) be a pointed simplicial set. The *reduced homology* of X with coefficients in an abelian group A is

$$\tilde{H}_i(X, A) = H_i(X, x; A) := \mathrm{Coker}(H_i(x, A) \to H_i(X, A)) \qquad (= H_i(X, A) \text{ if } i \neq 0).$$

This definition apparently depends on the choice of x: if we don't want to choose a basepoint, we may alternately define

$$\tilde{H}_i(X, A) = \mathrm{Ker}(H_i(X, A) \to H_i(*, A)).$$

Any choice of $x \in X_0$ will split the map $X \to *$, realising $\tilde{H}_i(X, A)$ as the above-described direct summand of $H_i(X, A)$.

A homotopy cofibre sequence $X \to Y \to Z$ is equivalent to a homotopy cocartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

hence the corresponding long exact homology sequence may be written (via Mayer-Vietoris!) as

$$\dots H_i(X, A) \to H_i(Y, A) \to \tilde{H}_i(Z, A) \to H_{i-1}(X, A) \to \dots$$

2.2.2. Suppose $\mathbf{F} : \mathcal{D} \to \mathbf{sSet}$ is a functor; suppose that $\mathbf{F}(d) \neq \emptyset$ for any $d \in \mathcal{D}$. We then define

$$C_*(\mathcal{D}, \tilde{\mathbf{F}}; A) = \mathrm{Ker}\,(C_*(\mathcal{D}, \mathbf{F}) \to C_*(\mathcal{D})) \otimes A$$

where the last map is induced by the natural transformation $\mathbf{F} \to *$, and

$$H_i(\mathcal{D}, \tilde{\mathbf{F}}; A) = H_i(C_*(\mathcal{D}, \tilde{\mathbf{F}}; A)).$$

(Here the map $\mathbf{F} \to *$ is not necessarily split, so we have to be more careful. We think of $\tilde{\mathbf{F}}$ as a desuspension of the homotopy cofibre of $\mathbf{F} \to *$.)

More generally, if $\mathbf{F} \to \mathbf{G}$ is a morphism of functors, we define

$$C_*(\mathcal{D}, \mathbf{F} \to \mathbf{G}; A) = \mathit{cone}\,(C_*(\mathcal{D}, \mathbf{F}) \to C_*(\mathcal{D}, \mathbf{G}))\,[-1].$$

If $\mathbf{G} = *$, then $C_*(\mathcal{D}, \mathbf{F} \to *; A)$ is homotopy equivalent to $C_*(\mathcal{D}, \tilde{\mathbf{F}}; A)$ under the nonemptiness assumption on $\mathbf{F}$.

2.2.3. Suppose $\mathbf{F} : \mathcal{D} \to \mathbf{Cat}$ is a functor. We have the projection $\mathbf{F} \to *$, where $*$ is the constant functor with values the category with 1 onject and 1 morphism. As in 2.2.2 we define

$$C_*(\mathcal{D}, \tilde{\mathbf{F}}; A) = \mathrm{Ker}\,(C_*(\mathcal{D}, \mathbf{F}) \to C_*(\mathcal{C})) \otimes A$$
$$H_i(\mathcal{D}, \tilde{\mathbf{F}}; A) = H_i(C_*(\mathcal{D}, \tilde{\mathbf{F}}; A)).$$

provided $\mathbf{F}(d) \neq \emptyset$ for any $d \in \mathcal{D}$. In general, we define $H_*(\mathcal{D}, \tilde{\mathbf{F}}; A)$ as the homology of $C_*(\mathcal{D}, \mathbf{F} \to *; A)$ as before.

2.3. **Cellular functors.**

2.3.1. **Lemma.** *Let $\mathbf{F}_T$ be as in 1.4.4. Suppose T fully faithful. Then, for any $c \in \mathcal{C}$, $\mathbf{F}_T(T(c))$ has a final object.*

Proof. Such a final object is given by $[T(c) = T(c)]$. □

2.3.2. **Definition.** Let $T : \mathcal{C} \to \mathcal{D}$ be a functor. We say that T is *cellular* if

- T is fully faithful.
- For any $d \in \mathcal{D} - \mathcal{C}$ and any $c \in \mathcal{C}$, $\mathcal{D}(d, c) = \emptyset$.

If T is cellular, then it defines a "stratification" $\mathcal{D} = \mathcal{C} \coprod (\mathcal{D} - \mathcal{C})$ in the following sense:

2.3.3. **Lemma.** *Let $T : \mathcal{C} \to \mathcal{D}$ be a cellular functor; let $F_1, F_2 : \mathcal{D} \to \mathcal{E}$ be two functors and let $\varphi : F_1 \Rightarrow F_2$ be a natural transformation. Then there is a unique factorisation of φ as a composition*

$$F_1 \overset{\varphi_1}{\Rightarrow} F_\varphi \overset{\varphi_2}{\Rightarrow} F_2$$

such that $F_\varphi(d) = F_2(d)$, $\varphi_2(d) = 1_{F_2(d)}$ for $d \in \mathcal{D} - T(\mathcal{C})$, and $F_\varphi T(c) = F_1 T(c)$, $\varphi_1(T(c)) = 1_{F_1 T(c)}$ for $c \in \mathcal{C}$.

Proof. For simplicity, we drop the notation T in this proof. Let us define a functor structure on F_φ as follows: if $f : d \to d$ is a morphism in $\mathcal{D}$, then three cases may occur:

- $d, d' \in \mathcal{D} - \mathcal{C}$. We define $F_\varphi(f)$ as $F_2(f)$.
- $d \in \mathcal{C}$, $d' \in \mathcal{D} - \mathcal{C}$. We define $F_\varphi(f)$ as $\varphi_d F_1(f) = F_2(f)\varphi_c$.
- $d, d' \in \mathcal{C}$. We define $F_\varphi(f)$ as $F_1(f)$.

Similarly, we define $\varphi_1(d)$ as $\varphi(d)$ for $d \in \mathcal{D} - \mathcal{C}$ and $\varphi_2(c) = \varphi(c)$ for $c \in \mathcal{C}$.

These definitions are the only possible ones if the lemma is to be correct. Checking that F_φ respects composition of morphisms and that φ_1, φ_2 do define natural transformations is a matter of case-by-case bookkeeping. □

2.3.4. **Proposition.** *Let $T : \mathcal{C} \to \mathcal{D}$ and $\varphi : F_1 \Rightarrow F_2$ be as in Lemma 2.3.3, with $\mathcal{E} = \mathbf{sSet}$, and let $\mathcal{D} - \mathcal{C}$ be the full subcategory of $\mathcal{D}$ given by the objects not in $\mathcal{C}$. Assume that $\varphi_{T(c)}$ is a weak equivalence for any $c \in \mathcal{C}$. Then the commutative diagram of bisimplicial sets*

$$\begin{array}{ccc} N(\mathcal{D} - \mathcal{C}, F_1) & \longrightarrow & N(\mathcal{D}, F_1) \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\varphi}\big\downarrow \\ N(\mathcal{D} - \mathcal{C}, F_2) & \longrightarrow & N(\mathcal{D}, F_2) \end{array}$$

is homotopy cocartesian, i.e. *becomes so after applying the diagonal δ.*

Proof. Applying Lemma 2.3.3, we can enlarge the above diagram as follows:

$$\begin{array}{ccc} N(\mathcal{D}-\mathcal{C},F_1) & \longrightarrow & N(\mathcal{D},F_1) \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_1}\downarrow \\ N(\mathcal{D}-\mathcal{C},F_\varphi) & \longrightarrow & N(\mathcal{D},F_\varphi) \\ {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_2}\downarrow \\ N(\mathcal{D}-\mathcal{C},F_2) & \longrightarrow & N(\mathcal{D},F_2) \end{array} \tag{2.1}$$

It suffices to show that the top and bottom squares in (2.1) are both homotopy cocartesian.

By hypothesis $\varphi_2(d)$ is a weak equivalence for $d \in \mathcal{C}$, and it is trivially a weak equivalence for $d \in \mathcal{D}-\mathcal{C}$. Therefore, by Lemma 1.3.3 b), the two vertical maps in the bottom square of (2.1) become weak equivalences after applying δ; *a fortiori*, this bottom square is homotopy cocartesian.

On the top right of (2.1), a typical term is

$$\coprod_{d_0\to\cdots\to d_p} F_1(d_0).$$

We split this coproduct in two parts: one is where all the d_i are in $\mathcal{D}-\mathcal{C}$ (call it A_1) and the other is the rest (call it B_1). Similarly on the middle right of (2.1) (call them A_φ and B_φ).

Now the cellularity assumption implies that all cells in B_1 and B_φ begin with $d_0 \in \mathcal{C}$. But for such a d_0, $F_1(d_0) \to F_\varphi(d_0)$ is a bijection. Thus $B_1 \to B_\varphi$ is bijective. On the other hand, by definition of the cofibre, A_1 and A_φ become a point in the cofibres. Thus the induced map on cofibres is bijective, and the top square of (2.1) is homotopy cocartesian as well. □

2.3.5. **Definition.** A (naturally) commutative square of categories and functors *is homotopy cocartesian* if it is after applying the nerve functor.

2.3.6. **Theorem.** *Let $\mathcal{D}-\mathcal{C}$ be the full subcategory of $\mathcal{D}$ given by the objects not in $\mathcal{C}$. If T is cellular, the naturally commutative diagram of categories*

$$\begin{array}{ccc} (\mathcal{D}-\mathcal{C})\int \mathbf{F}_T & \xrightarrow{p} & \mathcal{C} \\ {\scriptstyle\varepsilon}\downarrow & & {\scriptstyle T}\downarrow \\ \mathcal{D}-\mathcal{C} & \xrightarrow{\iota} & \mathcal{D} \end{array}$$

is homotopy cocartesian, where ε is the augmentation (see §1.4.2), p is induced by the first projection p_1 of Lemma 1.4.5 and ι is the inclusion.

Proof. Note first the natural transformation

$$u : T \circ p \Rightarrow \iota \circ \varepsilon$$
$$u_{[T(c) \xrightarrow{f} d]} = f$$

which explains "naturally commutative" (here in the weak sense). By Theorem 1.4.3 and Lemma 1.4.5, it suffices to prove that the (commutative) diagram of bisimplicial sets

$$\begin{array}{ccc} N(\mathcal{D} - \mathcal{C}, \mathbf{F}_T) & \longrightarrow & N(\mathcal{D}, \mathbf{F}_T) \\ \downarrow & & \downarrow \\ N(\mathcal{D} - \mathcal{C}) & \longrightarrow & N(\mathcal{D}) \end{array}$$

is homotopy cocartesian, *i.e.* becomes so after applying the diagonal δ. In view of Lemma 2.3.1, this is a special case of Proposition 2.3.4. □

2.3.7. **Corollary.** *Let $T : \mathcal{C} \to \mathcal{D}$ be a cellular functor. Then the mapping cone of*

$$C_*(T) : C_*(\mathcal{C}) \to C_*(\mathcal{D})$$

is quasi-isomorphic to $C_*(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T)[1]$ (cf. 2.2.2). In particular, we have a long exact sequence

$$\cdots \to H_i(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T; A) \to H_i(\mathcal{C}, A)$$
$$\to H_i(\mathcal{D}, A) \to H_{i-1}(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T; A) \to \dots$$

for any abelian group A.

Proof. This follows from Theorems 2.3.6 and 1.4.3. □

2.4. **Cellular filtrations.**

2.4.1. **Theorem.** *Let $\mathcal{Q}_1 \to \mathcal{Q}_2 \to \cdots \to \mathcal{Q}_n \to \cdots \to \mathcal{Q}$ be a sequence of categories. We assume:*

- *The functors $T_n : \mathcal{Q}_{n-1} \to \mathcal{Q}_n$ are cellular (2.3.2).*
- *$\mathcal{Q} = \varinjlim \mathcal{Q}_n$.*

Write $\mathbf{F}_n$ for $\mathbf{F}_{T_n}$. Then, for any abelian group A, there is a spectral sequence of homological type

$$E^1_{p,q} = H_{p+q-1}(\mathcal{Q}_p - \mathcal{Q}_{p-1}, \tilde{\mathbf{F}}_p; A) \Rightarrow H_{p+q}(\mathcal{Q}, A).$$

Proof. This is the spectral sequence of 2.1.4, taking Corollary 2.3.7 into account. □

3. Cellular functors and the Solomon-Tits theorem

3.1. **An abstract version of the Solomon-Tits theorem (for GL_n).** This section was catalysed by Fei Sun's insight that one can use Theorem 2.3.6 to understand part of Quillen's proof of the Solomon-Tits theorem in [12]. We start with an abstraction of his argument.

3.1.1. **Proposition** (Sun, essentially)**.** *Let $\mathbf{V}$ be an poset (considered as a category), $\mathbf{H}$ a subset of maximal elements in $\mathbf{V}$ and $\mathbf{Y} = \mathbf{V} - \mathbf{H}$. For any $W \in \mathbf{V}$, write $\mathbf{V}_W = \{X \in \mathbf{V} \mid X < W\}$. Then the naturally commutative diagram*

$$\begin{array}{ccc} \coprod_{H\in\mathbf{H}} \mathbf{V}_H & \xrightarrow{j} & \mathbf{Y} \\ {\scriptstyle k}\downarrow & & {\scriptstyle i}\downarrow \\ \coprod_{H\in\mathbf{H}} * & \xrightarrow{l} & \mathbf{V} \end{array}$$

is homotopy cocartesian. Here i is the inclusion, j is componentwise the inclusion, k is the tautological projection and l is the inclusion of the discrete set $\coprod_{H\in\mathbf{H}} * = \mathbf{H}$ into $\mathbf{V}$; a natural transformation $ij \Rightarrow lk$ is defined by the inequality $W \leq H$ for $W \in \mathbf{V}_H$.

Proof. The hypotheses imply that i is a cellular functor, so it suffices to compute $\mathbf{H} \int \mathbf{F}_i = \coprod_{H\in\mathbf{H}} i \downarrow H$. Clearly, $i \downarrow H = \mathbf{V}_H$. □

Sun's second insight was that one can replace the simplicial complex Y in the proof of Quillen's Claim in [12, §2] by the poset of its vertices. We make use of this observation now.

3.1.2. **Proposition.** *Keep the notation of Proposition 3.1.1. Assume that there exists $L \in \mathbf{V}$ such that*

(i) $\mathbf{H} = \{W \in \mathbf{V} \mid W$ *maximal and* $L \not\leq W\}$.
(ii) *For any $W \in \mathbf{Y}$, the supremum $L \vee W$ exists in $\mathbf{V}$.*

Then $\mathbf{Y}$ is contractible.

Proof. It is directily inspired by Quillen's proof of his claim (*loc. cit.*), but avoids the use of Theorem A.

For $W \in \mathbf{Y}$, $L \vee W$ cannot be in $\mathbf{H}$ by (i), hence $\varphi(X) = L \vee X$ defines an endofunctor $\varphi : \mathbf{Y} \to \mathbf{Y}$, and the inequality $X \leq L \vee X$ defines a natural transformation $Id_{\mathbf{Y}} \Rightarrow \varphi$. Now

$$\varphi(\mathbf{Y}) = \{X \in \mathbf{Y} \mid L \leq X\}$$

has the minimal element L, hence is contractible. Thus φ factors through a contractible poset; since φ is homotopic to $Id_{\mathbf{Y}}$, this proves the claim. $\square$

3.1.3. **Corollary.** *Under the assumptions of Proposition 3.1.2, there is a zigzag of homotopy equivalences*

$$\bigvee_{H\in\mathbf{H}} \Sigma N(\mathbf{V}_H) \xrightarrow{\sim} N(\mathbf{V})/N(\mathbf{Y}) \xleftarrow{\sim} N(\mathbf{V}).$$

Proof. For any poset X, write $\bar{X}$ for the poset $X \cup \{+\}$, where $+$ is a new element larger than all elements in X: thus $\bar{X}$ is contractible. Consider the strictly commutative diagram of functors

$$\begin{array}{ccc}
\coprod_{H\in\mathbf{H}} \mathbf{V}_H & \xrightarrow{j} & \mathbf{Y} \\
{\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow \\
\coprod_{H\in\mathbf{H}} \bar{\mathbf{V}}_H & \xrightarrow{\bar{\jmath}} & \mathbf{V} \\
{\scriptstyle \varepsilon}\uparrow & & || \\
\coprod_{H\in\mathbf{H}} * & \xrightarrow{l} & \mathbf{V}.
\end{array}$$

Here i' is componentwise the natural inclusion, $\bar{\jmath}$ sends $\mathbf{V}_H$ to itself and $+_H$ to H while ε sends $*_H$ to $+_H$. The latter functor has a retraction $\bar{k}$ which extends the functor k of Proposition 3.1.1. From this and the latter proposition, we deduce that the top square is homotopy cocartesian. We deduce a homotopy equivalence

$$\bigvee_{H\in\mathbf{H}} N(\bar{\mathbf{V}}_H)/N(\mathbf{V}_H) \xrightarrow{\sim} N(\mathbf{V})/N(\mathbf{Y})$$

induced by $\bar{\jmath}$. But $N(\bar{\mathbf{V}}_H)/N(\mathbf{V}_H)$ is canonically equivalent to $\Sigma N(\mathbf{V}_H)$; the conclusion then follows from Proposition 3.1.2. $\square$

3.2. **From abstract to concrete.** Given a poset $\mathbf{V}$, write $\mathrm{Simpl}(\mathbf{V})$ for the abstract simplicial complex associated to $\mathbf{V}$: the simplices of $\mathrm{Simpl}(\mathbf{V})$ are by definition the finite totally ordered subsets of $\mathbf{V}$.

Let V be a finite-dimensional vector space over a [possibly skew] field k. Its *Tits building* $\mathbf{T}(V)$ is [canonically isomorphic to] the flag complex of V: elements of $\mathbf{T}(V)$ are flags of proper subspaces of V [12, §2]. It follows that $\mathbf{T}(V) = \mathrm{Simpl}(\mathbf{V})$, where $\mathbf{V}$ is the poset of proper subspaces of V. Thus $\mathbf{T}(V) = \emptyset$ if $\dim V \leq 1$; we now assume

$\dim V \geq 2$. Choosing a line $L \subset V$, we get from Corollary 3.1.3 Quillen's decomposition

$$\mathbf{T}(V) \sim \bigvee_{H \in \mathbf{H}} \Sigma \mathbf{T}(H)$$

where $\mathbf{H}$ is the set of hyperplanes in V which do not contain L. It follows inductively that $\mathbf{T}(V)$ has the homotopy type of a bouquet of $(n-2)$-spheres.

4. The rank spectral sequence

4.1. **K-theory of schemes.** The first example of application of Theorem 2.4.1 is to Quillen's Q-construction $\mathcal{Q}(X)$ on the exact category of locally free sheaves of finite rank over a scheme X. Let $\mathcal{Q}_n = \mathcal{Q}_n(X)$ be the full subcategory of $\mathcal{Q}(X)$ consisting of locally free sheaves of rank $\leq n$. Then the assumptions of Theorem 2.4.1 are satisfied because, in $\mathcal{Q}(X)$, there are no morphisms from a locally free sheaf of rank n to a locally free sheaf of rank $< n$. The resulting spectral sequence may be called the *rank spectral sequence* (for the homology of $\mathcal{Q}(X)$).

Note that $\mathcal{Q}_n - \mathcal{Q}_{n-1}$ is a groupoid, hence we get

$$E^1_{p,q} = \bigoplus_{E_\alpha} H_{p+q-1}(\mathrm{Aut}(E_\alpha), \tilde{\mathbf{F}}_p) \tag{4.1}$$

where E_α runs through the set of isomorphism classes of locally free sheaves of rank p. We took coefficients $\mathbf{Z}$, for simplicity.

4.2. **K'-theory of integral schemes.** Let X be an integral scheme, with function field K. If $\eta = \operatorname{Spec} K$ is the generic point of X, we have the inclusion $j = \eta \to X$. If E is a sheaf of $\mathcal{O}_X$-modules, we write E_K for j^*E.

4.2.1. **Definition.** a) A coherent sheaf E on X is *torsion-free* if the map $E \to j_*E_K$ is a monomorphism.
b) A subsheaf E' of a coherent sheaf E is *pure* if E/E' is torsion-free.

4.2.2. Inside the exact category of coherent sheaves, the full subcategory of torsion-free sheaves is closed under extensions and subobjects. A monomorphism $E' \to E$ between torsion-free sheaves is admissible (within the exact category of torsion-free sheaves) if and only if E' is pure in E.

4.2.3. **Lemma.** *Let $\mathcal{Q}^{\mathrm{coh}}(X)$ be Quillen's Q-construction on the category of coherent sheaves of $\mathcal{O}_X$-Modules, and let $\mathcal{Q}^{\mathrm{tf}}(X)$ be the full subcategory of torsion-free sheaves. Then the inclusion $\mathcal{Q}^{\mathrm{tf}}(X) \to \mathcal{Q}^{\mathrm{coh}}(X)$ is a weak equivalence.*

Proof. The conditions of the resolution theorem [11, Th. 3] are verified since any locally free sheaf is torsion-free. □

4.2.4. **Proposition.** *Let E be a (coherent) torsion-free sheaf on X, with generic fibre E_K. Then the map*

$$F \mapsto F_K$$

defines a bijection from the set $Gr(E)$ of pure subsheaves of E to the set $Gr(E_K)$ of subvector spaces of E_K.

Proof. Let V be a sub-vector space of $j^*E = E_K$. Define

$$E \cap V = E \times_{j_*j^*E} j_*V.$$

Then $E \cap V \in Gr(E)$, because the map $E/(E \cap V) \to j_*j^*(E/(E \cap V))$ is a monomorphism (by definition of $E \cap V$). So we have two maps:

$$j^* : Gr(E) \to Gr(E_K); \quad E \cap - : Gr(E_K) \to Gr(E).$$

We have

$$(E \cap V)_K = j^*(E \times_{j_*j^*E} j_*V) = j^*E \times_{j^*j_*j^*E} j^*j_*V = j^*E \times_{j^*E} V = V$$

so that $j^* \circ (E \cap -) = Id$. On the other hand, if F is a pure subsheaf of E, then $F \subseteq E \cap j^*F$, and $j^*E = j^*(E \cap j^*F)$, hence (by exactness of j^*), $j^*(E \cap j^*F/F) = 0$. Thus $(E \cap j^*F)/F$ is a torsion subsheaf of the torsion-free sheaf E/F, hence is 0, and our two maps are inverse to each other. □

4.2.5**.** We write $\mathcal{Q}_n^{\mathrm{tf}}(X)$ for the full subcategory of $\mathcal{Q}^{\mathrm{tf}}(X)$ of torsion-free sheaves E such that $\dim_K E_K \leq n$. We get another rank spectral sequence

$$E^1_{p,q} = \bigoplus_{E_\alpha} H_{p+q-1}(\mathrm{Aut}(E_\alpha), \tilde{\mathbf{F}}_p) \Rightarrow H_{p+q}(\mathcal{Q}^{\mathrm{coh}}(X)) \tag{4.2}$$

cf. Lemma 4.2.3.

4.2.6. **Corollary.** *Let $E \in \mathcal{Q}_n^{\mathrm{tf}}(X)$, with generic fibre $E_K \in \mathcal{Q}_n(K)$. Then the functor*

$$j^* : \mathcal{Q}_{n-1}^{\mathrm{tf}}(X) \downarrow E \to \mathcal{Q}_{n-1}(K) \downarrow E_K$$

is an equivalence of categories.

Proof. These categories are equivalent to the ordered sets of proper layers of torsion-free subsheaves of E and j^*E (compare [11, top p. 102]). Thus the result directly follows from Proposition 4.2.4. □

4.2.7. *Example.* Let X be an integral Dedekind scheme (= noetherian, regular of Krull dimension ≤ 1), with function field K. As is well-known, a coherent sheaf F over a Dedekind scheme is torsion-free if and only if it is locally free. Thus the above generalises the remark of [12, pp. 191–192].

4.3. **The Tits building.**

4.3.1. In Corollary 4.2.6, suppose $n \geq 2$. By [12, Prop. p. 188], the classifying space of the poset $\mathcal{Q}_{n-1}(K) \downarrow E_K = J(E_K)$ is $GL(E_K)$-weakly equivalent to the suspension of nerve of the Tits building of E_K, which in turn is weakly equivalent to a wedge of $(n-2)$-spheres by the Solomon-Tits theorem ([12, Th. 2 p. 180], see §3.2). Hence $(\mathbf{F}_n)_{|E}$ is $\mathrm{Aut}(E)$-weakly equivalent to a wedge of $(n-1)$-spheres.

4.3.2. In Corollary 4.2.6, suppose $n = 1$. Then $\mathcal{Q}_{n-1}(K) \downarrow E_K$ has two elements: $0 \to E_K$ and $E_K \to 0$. Hence the conclusion of 4.3.1 is still true.

4.3.3. **Theorem.** *If X is an integral scheme, then the E^1-terms of the rank spectral sequence* (4.2) *(with $\mathbf{Z}$-coefficients) are*

$$E^1_{p,q} = \bigoplus_{E_\alpha} H_q(\mathrm{Aut}(E_\alpha), st(E_\alpha))$$

where E_α runs through the isomorphism classes of torsion-free sheaves of rank p, and $st(E_\alpha) = \tilde{H}_{p-1}((\mathbf{F}_p)_{|E_\alpha})$ is the [reduced] Steinberg module of j^*E_α.

Proof. This follows from Corollary 4.2.6, 4.3.1, 4.3.2 and Lemma 1.3.5. □

4.3.4. *Remark.* By an argument of Vogel, the exact sequences from Corollary 2.3.7 then coincide with those of Quillen in [12, Th. 3 p. 181]. In general, consider a map $f : E \to B$ whose homotopy fibre F has the homotopy type of a bouquet of n-spheres, with $n > 0$. So the Leray-Serre spectral sequence yields a long exact sequence

(4.3)
$$\cdots \to H_p(E) \to H_p(B) \to H_{p-n-1}(B, H_n(F)) \to H_{p-1}(E) \to \dots$$

If C is the homotopy cofibre of f, we have another long exact sequence

$$\cdots \to H_p(E) \to H_p(B) \to \tilde{H}_p(C) \to H_{p-1}(E) \to \dots \tag{4.4}$$

and we want to know that the two sequences coincide.

Here is Vogel's argument. We may assume that f is a Serre fibtration. Let E' be its mapping cylinder and CF the cone over F, so that we have a fibration of pairs

$$\begin{array}{ccc} (CF,F) & \longrightarrow & (E',E) \\ & & \downarrow \\ & & (B,B). \end{array}$$

Since $H_q(CF,F) = \begin{cases} H_n(F) & \text{for } q=n+1 \\ 0 & \text{else} \end{cases}$, the Leray-Serre spectral sequence for the pair

$$H_p(B, H_q(CF,F)) \Rightarrow H_{p+q}(E',E)$$

yields isomorphisms

$$\tilde{H}_p(C) \simeq H_p(E',E) \simeq H_{p-n-1}(B, H_{n+1}(CF,F)) \simeq H_{p-n-1}(B, H_n(F))$$

which commute with the differentials of (4.3) and (4.4) by functoriality.

References

[1] A. Ash, L. Rudolph *The Modular Symbol and Continued Fractions in Higher Dimensions*, lnvent. math. **55** (1979), 241–250.

[2] M. Bökstedt, A. Neeman *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), 209–234.

[3] H. Cartan, S. Eilenberg Homological algebra, Princeton, 1950.

[4] A. Dold, D. Puppe *Homologie nicht-additiver Funktoren. Anwendungen*, Ann. Inst. Fourier 11 1961 201–312.

[5] P. Gabriel, M. Zisman Calculus of fractions and homotopy theory, Ergeb. Math. **35**, Springer, 1967.

[6] P. Goerss, J. F. Jardine Simplicial homotopy theory, Progress in Math. **174**, Birkhäuser, 1999.

[7] L. Illusie Complexe cotangent et déformations, I, Lect. Notes in Math. **239**, Springer, 1971.

[8] B. Kahn *Motivic cohomology of smooth geometrically cellular varieties*, Proc. Symp. Pure Math. **67**, Amer. Math. Soc., 1999, 149–174.

[9] B. Kahn and F. Sun *On the universal modular symbols*, preprint, 2014.

[10] S. Mac Lane Categories for the working mathematician, Grad. Texts in Math. **5**, Springer (2nd ed.), 1998.

[11] D. Quillen *Higher algebraic K-theory, I*, Lect. Notes in Math. **341**, Springer, 1972, 85–147.

[12] D. Quillen *Finite generation of the groups K_i of rings of algebraic integers* (notes by H. Bass), Lect. Notes in Math. **341**, Springer, 1972, 179–198.

[13] F. Sun *The d^1-differential of the rank spectral sequence for algebraic K-theory*, thesis, to appear.

[14] R. W. Thomason *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. **85** (1979), 91–109.

[15] C. Weibel The K-book: An introduction to algebraic K-theory, Grad. Studies in Math. **145**, AMS, 2013.

[SGA1] Revêtements étales et groupe fondamental (SGA1), dirigé par A. Grothendieck, Lect. Notes in Math **224**, Springer, 1971. Recomposed and annotated edition: Doc. Math. **3**, Soc. Math. France, 2003.

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