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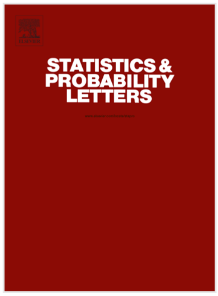

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AN ESTIMATE FOR THE PROBABILITY OF DEPENDENT EVENTS

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ABSTRACT. In this note we prove an estimate for the probability that none of several events will occur provided that some of those events are dependent. This estimate (essentially due to Filaseta, Ford, Konyagin, Pomerance and Yu) can be applied to coverings of $\mathbb{Z}$ by systems of congruences, coverings of $\mathbb{Z}^d$ by lattices and similar problems. Although this result is similar to the Lovász local lemma, it is independent of it. We will also prove a corollary in the style of the local lemma and show that in some situations our lower bound is stronger than that given by the Lovász lemma. As an illustration, we shall make some computations with an example considered earlier by Chen.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Throughout, we shall use the notation $E^c = \Omega \backslash E$ for every event E. If the events $E_1, \dots, E_\ell$ are all independent of one another (more precisely, e.g., for every $j = 2, \dots, \ell$, the events E_j and $\cup_{i=1}^{j-1} E_i$ are independent), then

$$\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) = \mathbb{P}(E_1^c)\mathbb{P}(E_2^c) \dots \mathbb{P}(E_\ell^c) = \prod_{k=1}^{\ell} (1 - \mathbb{P}(E_k)).$$

This implies that if $\mathbb{P}(E_k) < 1$ for each k, then there is a positive probability that none of the events $E_1, \dots, E_\ell$ will occur. The Lovász local lemma (Erdős and Lovász, 1975) gives the same conclusion when the mutual independence of events is replaced by their 'rare' dependence. The 'symmetric' version of the local lemma states that if $\mathbb{P}(E_k) \leqslant p$ for every $k = 1, \dots, \ell$, where each of the events E_k is independent of all other events except for at most d of them, then $\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) > 0$ provided that $ep(d+1) \leqslant 1$. See, for instance, (Alon and Spencer, 2000), (Chen, 1997) for more precise and more general versions of this statement, although, in general, the constant e which occurs in this inequality cannot be replaced by any smaller constant (Shearer, 1985). There are many useful applications of the Lovász local lemma in combinatorics, graph theory, number theory as well as in other fields of mathematics, statistics and computer science (Czumaj and Scheideler, 2000), (Deng, Stinson and Wei, 2004), (Dubickas, 2008), (Grytczuk, 2007), (Scott and Sokal, 2006), (Srinivasan, 2006), (Szabó, 1990).

Recently, in a paper of Filaseta, Ford, Konyagin, Pomerance and Yu (2007) devoted to the study of covering systems of congruences, another lower bound for the quantity $\mathbb{P}(\cap_{k=1}^{\ell} E_k^c)$, where some of the events E_i and E_j are dependent, was obtained. Although

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their lemma (see Lemma 2.1 in (Filaseta et al., 2007)) is given in terms of densities of covering systems, a corresponding probabilistic statement asserts that

$$\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) \geqslant \prod_{k=1}^{\ell}(1-\mathbb{P}(E_k)) - \sum_{\substack{1\leqslant i<j\leqslant \ell \\ E_i,E_j \text{ dependent}}} \mathbb{P}(E_i)\mathbb{P}(E_j) \tag{1}$$

if $E_1,\dots,E_\ell$ are events in a probability space with the property that if E_k is independent individually of the events $E_{j_1},\dots,E_{j_t}$, then it is independent of every event in the sigma algebra generated by $E_{j_1},\dots,E_{j_t}$ (see Remark 2 in (Filaseta et al., 2007)).

Indeed, let r_k, n_k, $k=1,2,\dots,\ell$, be a collection of pairs of integers, where $n_k \geqslant 2$ and $r_k \in \{0,1,\dots,n_k-1\}$. Here n_k, $k=1,2,\dots,\ell$, are not necessarily distinct, but $r_i \neq r_j$ if $n_i = n_j$. Let N be the least common multiple of the numbers $n_1,\dots,n_\ell$. Put $\Omega = \{1,2,\dots,N\}$. Suppose that the sigma algebra $\mathcal{F}$ consists of all 2^N subsets of Ω, and that the distribution is uniform, namely, $\mathbb{P}(n) = 1/N$ for every $n \in \Omega$. Assume that E_k is the event that an integer $n \in \Omega$ belongs to the residue class $r_k \pmod{n_k}$. It is easy see that the events E_i and E_j are dependent if and only if $\gcd(n_i,n_j) > 1$. Note that the density of integers (in $\mathbb{Z}$) which are not of the form $r_k \pmod{n_k}$, where $k=1,\dots,\ell$, is equal to the probability $\mathbb{P}(\cap_{k=1}^{\ell} E_k^c)$. Using $\mathbb{P}(E_k) = |E_k|/|\Omega| = 1/n_k$, one derives from (1) Lemma 2.1 of (Filaseta et al., 2007) stating that this density is at least

$$\prod_{k=1}^{\ell}(1-1/n_k) - \sum_{\substack{1\leqslant i<j\leqslant \ell \\ \gcd(n_i,n_j)>1}} 1/(n_i n_j).$$

In this note, we shall prove the following:

Theorem 1. *Let E_j, $j=1,2,\dots,\ell$, be events in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that if, for every $j=2,\dots,\ell$, the event E_j is independent of each of the events $E_{k_1},\dots,E_{k_t}$, where $k_1,\dots,k_t$ are some indices of the set $\{1,2,\dots,j-1\}$, then the events E_j and $E_{k_1}\cup\dots\cup E_{k_t}$ are also independent. Set $F_j = \cup_{i<j}^{*} E_i$, where the union is taken over every index $i<j$ for which the events E_i and E_j are dependent. Then*

$$\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) \geqslant \prod_{k=1}^{\ell}(1-\mathbb{P}(E_k)) - \sum_{j=2}^{\ell} \mathbb{P}(E_j)(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_\ell))\mathbb{P}(F_j). \tag{2}$$

Clearly, $\mathbb{P}(F_j) \leqslant \sum_{i<j}^{*} \mathbb{P}(E_i)$, where the sum is taken over every $i<j$ for which the events E_i and E_j are dependent. So (2) implies that

$$\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) \geqslant \prod_{k=1}^{\ell}(1-\mathbb{P}(E_k)) - \sum_{\substack{1\leqslant i<j\leqslant \ell \\ E_i,E_j \text{ dependent}}} \mathbb{P}(E_i)\mathbb{P}(E_j)(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_\ell)). \tag{3}$$

From $(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_\ell)) \leqslant 1$, we see that (3) yields (1).

Note that (2) is stronger than (3) and (1). In fact, $F_j^c = \cap_{i<j}^{*} E_i^c$, where the intersection is taken over every index $i<j$ for which the events E_i and E_j are dependent. Writing in

(2)
$$\mathbb{P}(F_j) = 1 - \mathbb{P}(F_j^c) = 1 - \mathbb{P}(\cap_{i<j}^* E_i^c),$$
one can estimate each $\mathbb{P}(F_j)$ from above, using an estimate on $\mathbb{P}(\cap_{i<j}^* E_i^c)$ from below, as in (2). The latter intersection contains strictly less than $j \leqslant \ell$ terms, so one can continue in this way step by step.

Corollary 2. *Let $E_1, \dots, E_\ell$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the condition of Theorem 1, and let α and β be two positive real numbers satisfying*
$$\alpha(e^\beta - 1) \leqslant 1. \tag{4}$$
If each event is independent of all the other events except for at most d of them and $\mathbb{P}(E_j) \leqslant p$ for every $j = 1, \dots, \ell$, where $p \leqslant \min\{\alpha/d, 1 - e^{-\beta/\ell}\}$, then there is a nonzero probability that none of the events occur.

Note that if $p = \max_{1 \leqslant j \leqslant \ell} \mathbb{P}(E_j) < 1/\ell$, then
$$\mathbb{P}(\cap_{k=1}^\ell E_k^c) = 1 - \mathbb{P}(\cup_{k=1}^\ell E_k) \geqslant 1 - \sum_{k=1}^{\ell} \mathbb{P}(E_k) \geqslant 1 - \ell p > 0.$$
If $p \leqslant 1/(e(d+1))$, then $\mathbb{P}(\cap_{k=1}^\ell E_k^c) > 0$, by the Lovász local lemma. The same conclusion $\mathbb{P}(\cap_{k=1}^\ell E_k^c) > 0$ follows from Corollary 2, if, say, $\ell = 14$, $d = 5$, $p = 0.08$. In this case, $ep(d+1) > 1.3$ and $\ell p = 1.12 > 1$, so neither the trivial bound nor the Lovász local lemma are applicable. Even the stronger version of Lovász local lemma given by $p(1 + 1/d)^d(d+1) \leqslant 1$ (see (3.1) in (Chen, 1997)) is not applicable, because $0.08(1 + 1/5)^5 6 = 1.1943 \dots > 1$. However, taking $\alpha = 0.43$ and $\beta = 0.2$ in Corollary 2, we see that the conditions (4) and $p \leqslant \min\{\alpha/d, 1 - e^{-\beta/\ell}\}$ hold, because $0.43(e^{1.2} - 1) = 0.9976 \dots < 1$ and $0.08 < \min\{0.43/5, 1 - e^{-1.2/14}\} = 0.0821\dots$. So, for this choice of parameters, we have $\mathbb{P}(\cap_{k=1}^{14} E_k^c) > 0$.

The proofs of Theorem 1 and Corollary 2 are given in the next section. We stress that our proof is just a probabilistic interpretation of a number-theoretic argument given in (Filaseta et al., 2007), although the inequality (2) is slightly stronger than (1) and the condition of the theorem is weaker. One may expect that the inequalities (1) – (3) can be applied not only to covering systems of congruences of $\mathbb{Z}$ as in Theorem 1 of (Filaseta et al., 2007), but also to covering systems of lattices that cover (or do not cover) the lattice $\mathbb{Z}^d$, where d is a positive integer, and also in some situations similar to those, where the Lovász local lemma is applicable.

We remark that in most applications of the local lemma the condition of Theorem 1 (and that of Corollary 2) that the pairwise independence of events implies their union-wise independence is satisfied, so these statements can be applied. In order to establish mutual independence one can use, for example, a so-called Mutual Independence Principle stated on p. 41 in (Molloy and Reed, 2002), which asserts that if $X = X_1, \dots, X_m$ is a sequence of random experiments and $E_1, \dots, E_\ell$ is a set of events, where each E_k is determined by $G_k \subseteq X$, and $G_k \cap (G_{k_1}, \dots, G_{k_t}) = \emptyset$, then E_k is mutually independent of $\{E_{k_1}, \dots, E_{k_t}\}$.

In conclusion, we shall give three simple examples. The first uses covering systems of congruences and shows that in some cases the inequality (2) is sharp, although (1) and (3) are not. The second shows how Theorem 1 can be applied to covering systems of $\mathbb{Z}^d$ by lattices. The third example is taken from Chen's paper (1997). It is given in (Chen, 1997) to demonstrate that a better version of the Lovász local lemma beats all the previous bounds. We show that the bound (3) is even stronger for the same choice of probabilities and the same 'dependency digraph'.

2. Proofs

Proof of Theorem 1: The inequality (2) clearly holds for $\ell = 1$, because $\mathbb{P}(E_1^c) = 1 - \mathbb{P}(E_1)$. Suppose that it holds for $\ell = l-1$, namely,

$$\mathbb{P}(\cap_{k=1}^{l-1} E_k^c) \geqslant \prod_{k=1}^{l-1}(1-\mathbb{P}(E_k)) - \sum_{j=2}^{l-1} \mathbb{P}(E_j)(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_{l-1}))\mathbb{P}(F_j). \quad (5)$$

Set $B_j = \cap_{k\leqslant j} E_k^c$. Clearly, $B_l = B_{l-1} \cap E_l^c$, so $B_l \subseteq B_{l-1}$ and $\mathbb{P}(B_{l-1}) = \mathbb{P}(B_l) + \mathbb{P}(B_{l-1}\cap E_l)$. Let $J(j)$ be the subset of all indices of the set $\{1,2,\dots,j-1\}$ for which the events E_j and E_i, where i is a fixed element of $J(j)$, are independent. Let $I(j) = \{1,2,\dots,j-1\} \setminus J(j)$, so that $F_j = \cup_{i\in I(j)} E_j$. By the condition of the theorem, the events E_l and $\cup_{i\in J(l)} E_i$ are independent, so the events E_l and $\Omega \setminus \cup_{i\in J(l)} E_i = \cap_{i\in J(l)} E_i^c$ are also independent. Hence, using $B_{l-1} \subseteq \cap_{i\in J(l)} E_i^c$, we deduce that

$$\mathbb{P}(B_{l-1}\cap E_l) \leqslant \mathbb{P}(\cap_{i\in J(l)} E_i^c \cap E_l) = \mathbb{P}(\cap_{i\in J(l)} E_i^c)\mathbb{P}(E_l).$$

Note that $\cap_{i\in J(l)} E_i^c$ is contained in the union of B_{l-1} and F_l, so $\mathbb{P}(\cap_{i\in J(l)} E_i^c) \leqslant \mathbb{P}(B_{l-1}) + \mathbb{P}(F_l)$. It follows that

$$\mathbb{P}(B_{l-1}) = \mathbb{P}(B_l) + \mathbb{P}(B_{l-1}\cap E_l) \leqslant \mathbb{P}(B_l) + \mathbb{P}(E_l)(\mathbb{P}(B_{l-1}) + \mathbb{P}(F_l)),$$

giving

$$\mathbb{P}(B_l) \geqslant \mathbb{P}(B_{l-1})(1-\mathbb{P}(E_l)) - \mathbb{P}(E_l)\mathbb{P}(F_l).$$

Since $B_{l-1} = \cap_{k=1}^{l-1} E_k^c$, inserting (5) into the right hand side of this inequality, we find that $\mathbb{P}(B_l) = \mathbb{P}(\cap_{k=1}^{l} E_k^c)$ is at least

$$\prod_{k=1}^{l}(1-\mathbb{P}(E_k)) - \mathbb{P}(E_l)\mathbb{P}(F_l) - (1-\mathbb{P}(E_l))\sum_{j=2}^{l-1}\mathbb{P}(E_j)(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_{l-1}))\mathbb{P}(F_j),$$

which is equal to

$$\prod_{k=1}^{l}(1-\mathbb{P}(E_k)) - \sum_{j=2}^{l}\mathbb{P}(E_j)(1-\mathbb{P}(E_{j+1}))\dots(1-\mathbb{P}(E_l))\mathbb{P}(F_j).$$

So (2) holds for $\ell = l$. This completes the proof of the theorem by induction. □

Proof of Corollary 2: Without loss of generality we may increase each $\mathbb{P}(E_j)$ to p, so assume that $\mathbb{P}(E_j) = p$ for every $j = 1,\dots,\ell$. Note that each E_j is independent of the event $\cup_{i<j}^{(d)} E_i$, where (d) means that at most d elements of the union are omitted (those

dependent with E_j), so for each fixed j the sum on the right hand side of (3) contains at most d elements. Hence, in order to show that $\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) > 0$, by (3), it suffices to prove the inequality

$$(1-p)^{\ell} > dp^2(1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{\ell-2}). \tag{6}$$

The right hand side of (6) is strictly smaller than

$$dp^2(1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{\ell-1}) = dp^2 \frac{1-(1-p)^{\ell}}{1-(1-p)} = dp(1-(1-p)^{\ell}).$$

Dividing both sides of (6) by $(1-p)^{-\ell}$, we see that $\mathbb{P}(\cap_{k=1}^{\ell} E_k^c) > 0$ if $dp((1-p)^{-\ell}-1) \leqslant 1$. From $p \leqslant 1-e^{-\beta/\ell}$ it follows that $(1-p)^{-\ell} \leqslant e^{\beta}$, hence $(1-p)^{-\ell}-1 \leqslant e^{\beta}-1$. Moreover, we have $pd \leqslant \alpha$. Multiplying these two inequalities and using (4), we obtain $dp((1-p)^{-\ell}-1) \leqslant \alpha(e^{\beta}-1) \leqslant 1$, which is the required inequality. □

3. Examples

Suppose that E_1, E_2, E_3 are the events that $n \in \Omega = \{1, 2, \dots, 15\}$ belongs to the the residue classes $1 \pmod 3$, $1 \pmod 5$ and $8 \pmod{15}$, respectively. Then $E_1^c \cap E_2^c \cap E_3^c$ is the event that $n \in \Omega$ is one of the numbers $2, 3, 5, 9, 12, 14, 15$. It follows that

$$\mathbb{P}(E_1^c \cap E_2^c \cap E_3^c) = 7/15.$$

The events E_1 and E_2 are independent, whereas the events E_1 and E_3 (and also E_2 and E_3) are dependent. Clearly, $\mathbb{P}(E_1) = 1/3$, $\mathbb{P}(E_2) = 1/5$, $\mathbb{P}(E_3) = 1/15$. Note that $F_3 = E_1 \cup E_2$, so

$$\mathbb{P}(F_3) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2) = 1/3 + 1/5 - 1/15 = 7/15.$$

Thus the right hand side of (2) is equal to

$$(1-\mathbb{P}(E_1))(1-\mathbb{P}(E_2))(1-\mathbb{P}(E_3)) - \mathbb{P}(E_3)\mathbb{P}(F_3) = (1-1/3)(1-1/5)(1-1/15) - 7/225 = 7/15.$$

So we have an equality in (2). In other words, $7/15$ is the density of integers which do not belong to the arithmetic progressions $3k_1+1$, $5k_2+1$, $15k_3+8$, where $k_1, k_2, k_3 \in \mathbb{Z}$. The right hand sides of both (1) and (3) are equal to

$$(1-\mathbb{P}(E_1))(1-\mathbb{P}(E_2))(1-\mathbb{P}(E_3)) - \mathbb{P}(E_1)\mathbb{P}(E_3) - \mathbb{P}(E_2)\mathbb{P}(E_3) = 104/225,$$

which is $0.46222\dots < 0.466666\dots = 7/15$.

Next, suppose that m, d and $q_{ij} \geqslant 2$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant d$, are positive integers. Let $r_{ij} \in \{0, 1, \dots, q_{ij}-1\}$, and let $\Lambda_i = (q_{i1}\mathbb{Z} + r_{i1}, \dots, q_{id}\mathbb{Z} + r_{id})$, $i = 1, \dots, m$, be some m lattices in $\mathbb{Z}^d$. Suppose that, for every $j = 1, \dots, d$, the numbers $q_{1j}, \dots, q_{mj}$ are pairwise coprime. Set $Q_k = q_{k1}q_{k2}\dots q_{kd}$ for $k = 1, \dots, m$. We claim that, for any s lattices of the form $L_t = (Q_1\mathbb{Z} + m_{t1}, \dots, Q_d\mathbb{Z} + m_{td})$, $t = 1, \dots, s$, where $m_{tj} \in \{0, 1, \dots, Q_j - 1\}$, the lattices $\Lambda_1, \dots, \Lambda_m, L_1, \dots, L_s$ do not cover $\mathbb{Z}^d$ if

$$s < (Q_1-1)(Q_2-1)\dots(Q_m-1). \tag{7}$$

Indeed, by the same argument as in the introduction, we can describe E_i as the event that a point in $\mathbb{Z}^d$ belongs to the lattice Λ_i for $i = 1, \dots, m$. Let E_{m+1} be the event that a point in

$\mathbb{Z}^d$ belongs to the union $L_1 \cup \cdots \cup L_s$. The events E_i and E_j are independent if $1 \leqslant i < j \leqslant m$, whereas the events E_i and E_{m+1} are dependent for each $i = 1, \dots, m$. Clearly, $\mathbb{P}(E_i) = 1/Q_i$ for $i = 1, \dots, m$ and $\mathbb{P}(E_{m+1}) \leqslant s/(Q_1 \dots Q_m)$. Since $F_{m+1} = \cup_{i=1}^m E_i$, we have

$$1 - \mathbb{P}(F_{m+1}) = 1 - \mathbb{P}(\cup_{i=1}^m E_i) = \mathbb{P}(\cap_{i=1}^m E_i^c) = \prod_{i=1}^m (1 - \mathbb{P}(E_i)) = (1 - 1/Q_1) \dots (1 - 1/Q_m).$$

Assume that $\mathbb{Z}^d \subseteq \Lambda_1 \cup \dots \Lambda_m \cup L_1 \cup \dots L_s$. Then $\mathbb{P}(E_1^c \cap \cdots \cap E_{m+1}^c) = 0$. Thus (2) with $\ell = m + 1$ implies that

$$\begin{aligned}(1 - 1/Q_1)(1 - 1/Q_2) \dots (1 - 1/Q_m)(1 - \mathbb{P}(E_{m+1})) &\leqslant \mathbb{P}(E_{m+1})\mathbb{P}(F_{m+1}) \\ &= \mathbb{P}(E_{m+1})(1 - (1 - 1/Q_1)(1 - 1/Q_2) \dots (1 - 1/Q_m)).\end{aligned}$$

Hence $(1 - 1/Q_1)(1 - 1/Q_2) \dots (1 - 1/Q_m) \leqslant \mathbb{P}(E_{m+1})$. Using $\mathbb{P}(E_{m+1}) \leqslant s/(Q_1 \dots Q_m)$, we deduce that $s \geqslant (Q_1 - 1)(Q_2 - 1) \dots (Q_m - 1)$, contrary to (7).

Finally, we shall consider Example 1 of (Chen, 1997), where the events $E_1, \dots, E_7$ have a dependency digraph

$$D = \{(1,2), (1,3), (2,1), (2,4), (2,5), (3,1), (3,6), (3,7), (4,2), (5,2), (6,3), (7,3)\}$$
$$\cup \{(i,i) \ : \ 1 \leqslant i \leqslant 7\}.$$

This means that, for each $i = 1, \dots, 7$, the event E_i is mutually independent of the events $\{E_j \ : \ (i,j) \notin D\}$. Assume, in addition, that E_i is independent of $\cup_{j<i,(i,j)\notin D} E_j$, so that Theorem 1 is applicable.

As in (3.13) of (Chen, 1997), suppose first that $p_i = \mathbb{P}(E_i)$ are given by

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (1/9, 9/64, 9/64, 1/8, 1/8, 1/8, 1/8).$$

It was observed in (Chen, 1997) that the above mentioned symmetric version of the Lovász lemma is inapplicable. The trivial bound $\mathbb{P}(\cap_{k=1}^7 E_k^c) \geqslant 1 - \sum_{k=1}^7 p_i$ gives

$$\mathbb{P}(\cap_{k=1}^7 E_k^c) \geqslant 31/288 = 0.1076 \dots.$$

The improved version of the Lovász lemma (see Theorem 1 and (3.16) in (Chen, 1997)) yields

$$\mathbb{P}(\cap_{k=1}^7 E_k^c) \geqslant 0.1456 \dots.$$

However, our inequality (3) gives

$$\begin{aligned}\mathbb{P}(\cap_{k=1}^7 E_k^c) \geqslant \prod_{k=1}^7 (1 - p_k) &- p_1 p_2 (1 - p_3)(1 - p_4)(1 - p_5)(1 - p_6)(1 - p_7) \\ - p_1 p_3 (1 - p_4)(1 - p_5)(1 - p_6)(1 - p_7) &- p_2 p_4 (1 - p_5)(1 - p_6)(1 - p_7) \\ - p_2 p_5 (1 - p_6)(1 - p_7) &- p_3 p_6 (1 - p_7) - p_3 p_7 \\ &= 46745849/150994944 = 0.3095 \dots,\end{aligned}$$

which is better.

Another choice $p_1 = p_2 = \cdots = p_7 = p = 27/256$ (see (3.17) in (Chen, 1997)) with the same dependency digraph D as above gave the following lower bounds on $\mathbb{P}(\cap_{k=1}^7 E_k^c)$:

$0.1334\dots$, $0.2617\dots$ and $0.3507\dots$ (see (3.18) and (3.19) in (Chen, 1997)). Once again, a simple computation shows that our bound (3)

$$\mathbb{P}(\cap_{k=1}^{7} E_k^c) \geqslant (1-p)^7 - p^2((1-p)^5 + (1-p)^4 + (1-p)^3 + (1-p)^2 + (1-p) + 1)$$
$$= 114529002852169/281474976710656 = 0.4068\dots$$

gives a slightly better numerical value.

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