An axiomatic version of Zariski's patching theorem

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Abstract. We state six axioms concerning any regularity property P in a given birational equivalence class of algebraic threefolds. Axiom 5 states the existence of a local uniformization in the sense of valuations for P. If axioms 1 to 4 are satisfied by P, then the function field has a projective model which is everywhere regular w.r.t. P. Axiom 6 ensures the existence of P-resolution of singularities for any projective model. Applications concern resolution of singularities of vector fields and a weak version of Hironaka's Strong Factorization Conjecture for birational morphisms of nonsingular projective threefolds, both of them in characteristic zero.

1. INTRODUCTION.

Zariski's Patching Theorem ([33], Fundamental Theorem on p. 539) was one of his main achievements in the field of resolution of singularities. The content of this theorem is that local uniformization of valuations in a three-dimensional function field K over a ground field k of characteristic zero implies the existence of a nonsingular projective model of K. Since Zariski had proved the existence of a local uniformization for any valuation and any function field over a ground field of characteristic zero in [31], existence of nonsingular projective models was established for threefolds.

Hironaka's Resolution of Singularities [18] in 1964 closed this approach in characteristic zero. Indeed, Hironaka's theorem proves much more than the existence of nonsingular projective models as considered by Zariski. Moreover, since de Jong's work on alterations [22] some fifteen years ago, a much simpler proof of the existence of nonsingular projective models in characteristic zero and all dimensions is now available [4] [6]. On the other hand, the Patching Theorem has never been extended to dimensions higher than three (without assuming beforehand the existence of nonsingular projective models).

In positive characteristics, the situation is very different. The best result up to date is proved in a joint work of the author with V. Cossart (conjecture 3.1 in [12], stated in the introduction and proved in [13]) for threefolds over any ground field k of characteristic $p > 0$ which is differentially finite over a perfect field. This proof actually uses an extension of Zariski's Patching Theorem (proposition 4.8 in [12]) to arbitrary ground fields of positive characteristics.

Since Zariski's time, several resolution type problems have been considered. Resolution of Singularities of vector fields is considered by F. Cano in [7], and that of

foliations in [8] [9]. Hironaka's Strong Factorization Conjecture for birational morphisms by blowing ups and blowing down regular centers is also a resolution type problem (stated in [18], a weaker form is proved in [5] [29]). All of these problems are open in dimension four or more. In dimension three and characteristic zero, the question for foliations has been settled by F. Cano in [9].

A local formulation of each of these problems can be given in terms of valuation theory, which is called Local Uniformization (axiom 5 in section 2). Specifying a valuation ring V in a function field K, with $k \subset V$ and $QF(V) = K$, is equivalent to specifying a center x_V on each projective model X/k of K, compatible with morphisms. The Local Uniformization problem is in general substantially simpler than the Resolution problem because the concrete resolution strategy is allowed to be adapted to the valuation. Also valuation rings are algebraic objects with a simple combinatorial structure in which terms resolution problems can be tracked more easily.

The main applications of this article concern vector fields and a weak version of Hironaka's Strong Factorization Conjecture in dimension three [18]. Our main result (theorem 2.5 below) reduces (global) Resolution to Local Uniformisation in each of these cases. Combining it with their Local Uniformization theorem, F. Cano, C. Roche and M. Spivakovsky ([10] theorem 2) obtain Resolution for vector fields (restated as corollary 4.9).

Combining theorem 2.5 with K. Karu's Local Uniformization for the Hironaka Strong Factorisation Conjecture, thorem 0.1 [23], we obtain our globalized version in theorem 4.10. We emphasize that our result builds up projective birational morphisms which are *locally* a composition of blowing ups along regular centers, a fairly wider class than that of compositions of blowing ups with regular centers. On the other hand, germs of such morphisms have been used quite systematically to build up local uniformizations: Zariski's Perron transforms [31], Cutkosky's MTSs [14], Karu's star subdivisions in the toric case [23] and F. Cano, C. Roche and M. Spivakovsky's Puiseux packages are examples. From this viewpoint, one can understand the assumptions (axioms 1 to 4 and 6) of theorem 2.5 as a criterion for arbitrary regularity properties in dimension three in order that the only obstruction to Resolution be Local Uniformization.

Our setup consists in studying any abstract regularity property (definition 2.1) in a given function field K of dimension three over any ground field k. In section 2, we introduce six axioms to be verified by P: 1. openness, 2. stability by normalization and normalized blowing ups of points, 3. making curves "permissible" at regular points, 4. principalization of ideals, 5. local uniformization and 6. making curves "permissible" at singular points. The first five axioms ensure the existence of a model of K which is everywhere regular w.r.t. P (theorem 2.4). Adding axiom 6 ensures Resolution of Singularities in the birational equivalence class of K (theorem 2.5).

Section 3 contains those examples of regularity properties we are interested in: reducing multiplicity (property P_μ, example 3.4), getting a vector field to be log-elementary (property P_δ, example 3.6), and solving our local version of Hironaka's Strong Factorization Conjecture (property $P_{RS(Y)}$, definitions 3.8 and 3.9). All of this is developed in a relative context, that is, to any given regularity property P, we associate a new one refining P by requiring furthermore regularity w.r.t. one of the above problems. Our axioms 1, 2, 3 and 6 also have good behaviour by taking the boolean operation $P_1 \wedge P_2$ of two regularities properties P_1 and P_2 (proposition 3.11).

Section 4 contains the main applications: checking that certain regularity properties of P_δ-type and $P_{RS(Y_1)} \wedge P_{RS(Y_2)}$-type verify axioms 1 to 6, we deduce our main theorems (proposition 4.8 and theorem 4.10). The main difficulty consists in proving principalization of ideals for P_δ (axiom 4, proposition 4.2) and making curves "permissible" at singular points for $P_{RS(Y)}$ (axiom 6, proposition 4.7).

Finally, sections 5 and 6 consist in adapting Zariski's proof of the Patching Theorem to any regularity property P satisfying our axioms. We conclude by suggesting some open problems, the most challenging of which is problem 7.1: to get a constructive version of theorem 2.5.

2. P-REGULARITY AND STATEMENT OF MAIN RESULT.

All along this article, we consider a field k of arbitrary characteristic $p \geq 0$ and a function field K of transcendence degree three over k.

Definition 2.1. *A regularity property P in K is an application from the set of all local rings $R \subseteq K$ which are localization at a prime ideal of an integral k-algebra of finite type such that $QF(R) = K$ to the set $\{\text{regular}, \text{singular}\}$. Those R mapping to $\{\text{regular}\}$ (resp. $\{\text{singular}\}$) are then called P-regular (resp. P-singular). Given any proper model X/k of K, we denote*

$$\text{RegP}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is } P - \text{regular}\}, \ \text{SingP}(X) := X \backslash \text{RegP}(X).$$

Remark: although our applications concern regularity properties P which are refinements of the usual regularity property (i.e. $\text{RegP}(X) \subseteq X_{\text{reg}}$ for any projective model X/k of K), we do *not* make this extra assumption unless otherwise stated (example 3.6 and section 4).

We now introduce various axioms to be satisfied by P.

Axiom 1 (Openness). For any projective model X/k of K, the set $\text{RegP}(X)$ is a nonempty open set of X for its Zariski topology.

We are interested in defining a class of projective birational morphisms $\pi : X' \to X$ such that $\pi^{-1}(\text{RegP}(X)) \subseteq \text{RegP}(X')$. A first step is the following axiom:

Axiom 2 (Stability by normalization and normalized blowing up). The following holds for any projective model X/k of K:

  (i) let $n : X_{\text{norm}} \to X$ be the normalization of X. Then $n^{-1}(\text{RegP}(X)) \subseteq \text{RegP}(X_{\text{norm}})$.
  (ii) let $x \in \text{RegP}(X)$ be such that $\mathcal{O}_{X,x}$ is normal, $Z := \overline{\{x\}}$ and $\pi : \tilde{X} \to X$ be the normalized blowing up of X along Z. There exists a nonempty open set $U_Z \subset Z$ such that $\pi^{-1}(U) \subseteq \text{RegP}(\tilde{X})$.

If $\dim \mathcal{O}_{X,x} = 1$, π is an isomorphism above x, so (ii) is always satisfied in this case. If x is a closed point, i.e. $\dim \mathcal{O}_{X,x} = 3$, and (ii) of axiom 2 holds, we have $U_Z = Z$ necessarily. In case $\dim \mathcal{O}_{X,x} = 2$, we need some control on the open set U_Z. This leads to the following definitions.

Definition 2.2. *Assume that the regularity property P verifies axioms 1 and 2 and keep notations and assumptions as in the statement of (ii) of axiom 2.*

We say that Z is P-permissible at some point $z \in Z$ if $z \in \mathrm{RegP}(X)$, z is a regular point of Z in the usual sense, $\mathcal{O}_{X,z}$ is normal, and U_Z can be chosen in such a way that $z \in U_Z$.

We say that Z is stably P-permissible at some point $z \in Z$ if for any finite sequence

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \tag{1}$$

of normalized blowing ups at points $z_{i-1} \in X_{i-1}$ such that $z_0 := z$ and for $1 \leq i \leq n$, z_i maps to z_{i-1} and belongs to the strict transform Z_i of Z in X_i, Z_n is P-permissible at z_n.

Note that Z is stably P-permissible at its generic point x by definition, since $Z_1 = \emptyset$ in (1) when $z = x$. In particular, any closed point in any normal model of K is stably P-permissible if P verifies axioms 1 and 2.

Example 2.3. *Take P_{reg} to be the usual regularity property, i.e. $\mathcal{O}_{X,x}$ has property P_{reg} iff $\mathcal{O}_{X,x}$ is a regular local ring. To avoid confusions, we will denote the corresponding regularity open sets $\mathrm{RegP}_{\mathrm{reg}}(X)$ by X_{reg} for any projective model X/k of K.*

It is immediately seen that P_{reg} verifies axioms 1 and 2. If X/k is a projective model of K, $Z \subset X$ an integral curve and $z \in X_{\mathrm{reg}}$ is a regular point of Z, then Z is stably P-permissible at z.

Axiom 3 (Permissibilizing curves). The following holds for any *normal* projective model X/k of K and for any $x \in \mathrm{RegP}(X)$ such that $\dim \mathcal{O}_{X,x} = 2$: there exists a finite sequence

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n =: X' \tag{2}$$

of normalized blowing ups at closed points such that the strict transform Z' of $Z = \overline{\{x\}}$ in X' is stably P-permissible at all points $z' \in Z' \cap \mathrm{RegP}(X')$.

It is worth noting that, if P verifies axiom 2, we necessarily have $\pi^{-1}(\mathrm{RegP}(X)) \subseteq \mathrm{RegP}(X')$, where $\pi : X' \to X$ is the composed map in (2). In particular, any integral curve in a normal variety whose generic point is P-regular is stably P-permissible at all but finitely many closed points if P verifies axioms 1, 2 and 3.

Note that one gets equivalent notions for axioms 1, 2 and 3 if the word "projective" is replaced with "proper", since the question is local on X.

Axiom 4 (Principalization). For any *normal* projective model X/k of K and any ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, there exists a projective birational morphism $\pi : \tilde{X} \to X$ having the following properties:

(i) $\mathcal{I}\mathcal{O}_{\pi^{-1}(\mathrm{RegP}(X))}$ is locally principal.
(ii) $\pi^{-1}(\mathrm{RegP}(X)) \subseteq \mathrm{RegP}(\tilde{X})$.
(iii) the induced map $\pi^{-1}(\mathrm{RegP}(X) \cap U) \to \mathrm{RegP}(X) \cap U$ is an isomorphism, where $U \subseteq X$ is the open set

$$U := \{x \in X \mid \mathcal{I}_x \text{ is principal}\}.$$

Note that we do not require π to be a composition of elementary blowing ups (for example at integral or at regular centers) in this definition. We obtain an equivalent

formulation of axiom 4 by replacing (i) by the stronger property:

(i)' $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is locally principal.

Namely, since $\mathcal{I}\mathcal{O}_{\pi^{-1}(\mathrm{RegP}(X))}$ is already locally principal, blowing up $\mathcal{I}\mathcal{O}_{\tilde{X}}$ achieves (i)' without affecting neither (ii) nor (iii).

Finally, note that $X \backslash U$ has dimension at most one, since X is assumed to be normal in the definition of axiom 4, and that π is the blowing up along an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that

(iii)' $\mathcal{J}\mathcal{O}_{\mathrm{RegP}(X) \cap U}$ is locally principal.

We now introduce the local uniformization axiom. Let V be a k-valuation ring of K, i.e. $k \subset V$ and $QF(V) = K$. For any proper model X/k of K, V has a unique *center* $x_V \in X$. There is a one-to-one correspondence

$$\{k - \text{valuation rings of } K\} \;\leftrightarrow\; \varprojlim X, \; V \mapsto (x_V)_X,$$

where the inverse limit is taken over all proper models X/k of K. By Chow's lemma, we get an equivalent notion if the word "proper" is replaced with "projective". The following axiom is a weak statement of Local Uniformization for property P. Similarly, we get an equivalent notion if the word "proper" is replaced with "projective", since the question is local on X.

Axiom 5 (Local Uniformization). For each k-valuation ring V/k of K, there exists a proper model X/k of K such that the center $x_V \in X$ of V satisfies $x_V \in \mathrm{RegP}(X)$.

Our axiomatic version of Zariski's patching theorem goes as follows:

Theorem 2.4. *Let P be a regularity property verifying axioms 1 to 5 and X/k be a projective model of K. There exists a birational projective morphism $\pi : X' \to X$, X' normal, such that $\mathrm{RegP}(X') = X'$.*

This result can be strengthened by introducing a new axiom:

Axiom 6. The following holds for any *normal* projective model X/k of K and for any $x \in \mathrm{RegP}(X)$ such that $\dim \mathcal{O}_{X,x} = 2$: there exists a finite sequence

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n =: X'$$

of normalized blowing ups at closed points such that the strict transform Z' of $Z := \overline{\{x\}}$ in X' verifies $Z' \subset \mathrm{RegP}(X')$.

Theorem 2.5. *Let P be a regularity property verifying axioms 1 to 6 and X/k be a normal projective model of K. There exists a birational projective morphism $\pi : X' \to X$, X' normal, such that $\mathrm{RegP}(X') = X'$ and the induced map*

$$\pi^{-1}(\mathrm{RegP}(X)) \to \mathrm{RegP}(X)$$

is an isomorphism.

Remark: if a regularity property P verifies some of axioms 1, 2, 3 or 6, so does P_{norm}, where for any projective model X/k of K,

$$\mathrm{RegP}_{\mathrm{norm}}(X) := \mathrm{RegP}(X) \cap \{x \in X \mid \mathcal{O}_{X,x} \text{ is normal}\}.$$

The same fact holds for axioms 4 and 5 provided P verifies (i) of axiom 2. Finally, X' satisfies the conclusion of theorem 2.4 (resp. of theorem 2.5) w.r.t. P_{norm} whenever X' satisfies the conclusion of theorem 2.4 (resp. of theorem 2.5) w.r.t. P.

3. Basic properties of P-regularity and applications.

In this section, we point out some simple consequences of axioms 1 to 6, then introduce several examples of regularity properties which fit into our framework.

Proposition 3.1. *Let P be any regularity property verifying axioms 1, 2 and 5. Let X/k be a normal and projective model of K. Then*

$$\dim(\mathrm{SingP}(X)) \leq 1.$$

Proof. Since $\mathrm{RegP}(X)$ is Zariski open by axiom 1, it is sufficient to prove that $x \in \mathrm{RegP}(X)$ whenever $\dim \mathcal{O}_{X,x} = 1$. Then $V := \mathcal{O}_{X,x}$ is a prime divisor.

By axiom 5, there exists a projective model Y/k of K such that the center $y_V \in Y$ of V belongs to $\mathrm{RegP}(Y)$. Let

$$Y =: Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$$

be the sequence of normalized blowing ups defined as follows: for $i \geq 1$, Y_i is the normalized blowing up along $\overline{\{y_{i-1,V}\}}$, where $y_{i-1,V}$ is the center of V in Y_{i-1}. Using a clasical argument of Zariski ([30] first paragraph in the proof of theorem 10 which works in all dimensions), the sequence of local rings $\mathcal{O}_{Y_i,y_{i,V}}$ gets constant and equal to $V = \mathcal{O}_{X,x}$ for $i \geq i_0$. On the other hand, $y_{i,V} \in \mathrm{RegP}(Y_i)$ for all $i \geq 0$ by (i) and (ii) of axiom 2 and the conclusion follows. $\square$

Proposition 3.2. *Let P be any regularity property verifying axioms 1, 2 and 3 (resp. 1, 2, 3 and 6) and Y/k be a fixed projective model of K. We define a regularity property P_Y as follows: for any projective model X/k of K,*

$$\mathrm{RegP}_Y(X) := \mathrm{RegP}(X) \cap \{x \in X \mid X \cdots \to Y \text{ is defined}\}.$$

Then P_Y verifies axioms 1, 2 and 3 (resp. 1, 2, 3 and 6). If moreover P verifies axiom 4 (resp. 4 and 5), then P_Y also verifies axiom 4 (resp. 4 and 5).

Proof. Let X/k be a projective model of K and Y' be the closure of the graph of the rational map $X \cdots \to Y$, so Y' dominates both of X and Y. Then $Y' \to X$ is projective, hence the blowing up along a certain ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. This gives the following description:

$$\mathrm{RegP}_Y(X) := \mathrm{RegP}(X) \cap \{x \in X \mid \mathcal{I}_x \text{ is locally principal}\}. \tag{3}$$

Hence P_Y verifies axioms 1, 2 and 3, since P does.

We now use axiom 4 for P to the pair $(X, \mathcal{I})$. This produces a projective birational morphism $\pi : \tilde{X} \to X$ with the following properties:

(i') $\mathrm{RegP}_Y(\tilde{X}) = \mathrm{RegP}(\tilde{X})$.
(ii) $\pi^{-1}(\mathrm{RegP}(X)) \subseteq \mathrm{RegP}_Y(\tilde{X})$.
(iii) the induced map $\pi^{-1}(\mathrm{RegP}_Y(X)) \to \mathrm{RegP}_Y(X)$ is an isomorphism.

In other terms, we may substitute X by $\tilde{X}$ in order to prove that P_Y verifies axiom 4. Since $\mathrm{RegP}_Y(X') = \mathrm{RegP}(X')$ for every projective model X'/k of K birationally

dominating $\tilde{X}$, axiom 4 for P_Y is a direct consequence of axiom 4 for P.

Assume that P verifies axiom 6. Let X/k be a normal projective model of K and $Z \subset X$ be an integral curve such that $Z \cap \mathrm{RegP}_Y(X) \neq 0$. Then $\mathcal{IO}_{X,Z}$ is locally principal. To prove that P_Y verifies axiom 6, it can be furthermore assumed that $Z \subset \mathrm{RegP}(X)$ and that Z is a regular curve. Let v' be a fixed discrete k-valuation of K dominating $\mathcal{O}_{X,Z}$ with residue field a finite algebraic extension k' of $k(Z)$, and $\overline{v}$ be any discrete valuation of k'. We denote by v the composed valuation $v = v' \circ \overline{v}$, with group $\mathbf{Z}_{\mathrm{lex}}^2$. Let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be that sequence of normalized blowing ups at closed points $z_i \in X_i$, where z_i is the center of v. Since P_Y verifies axiom 3, it is sufficient to prove that $z_i \in \mathrm{RegP}_Y(X)$ for some $i \geq 0$. Let $I := \mathcal{IO}_{X_0,z_0}$ and $f_1 \in I$ be such that $v(f_1) = (a,b)$, where

$$(a,b) := \min\{v(f) \mid f \in I\}. \tag{4}$$

We claim that $I\mathcal{O}_{X_i,z_i} = (f_1)$ for some $i \geq 0$, which will complete the proof.

To prove the claim, let $I = (f_1, \ldots, f_e)$ be a minimal basis of I. By assumption, $I\mathcal{O}_{X_0,Z} = (f_1)$, hence there exists $g \in \mathcal{O}_{X_0,z_0}$, $v'(g) = 0$, such that $gf_i/f_1 \in \mathcal{O}_{X_0,z_0}$ for $2 \leq i \leq e$. It can be assumed that $m := \overline{v}(g)$ is minimal among all g's satisfying these properties. Assume that $e \geq 2$. Then $m \geq 1$ and $v(gf_i/f_1) \geq (0,m)$ by (4). Moreover $(t) = m_z\mathcal{O}_{X_1,z_1}$ factors out from g and from gf_i/f_1 in $\mathcal{O}_{X_1,z_1}$. Since $0 \leq \overline{v}(g/t) < m$, we get a new invariant $(e_1, m_1) <_{\mathrm{lex}} (e,m)$ w.r.t. $\mathcal{O}_{X_1,z_1}$. Therefore $e_i = 1$ for some $i \geq 0$ and the claim is proved.

Finally, that P_Y verifies axiom 5 is a consequence of the following lemma. $\qquad\square$

Lemma 3.3. (Refined local uniformization). *Let P be any regularity property verifying axioms 1 to 5. Let V be a k-valuation ring of K and X/k be a fixed projective model of K. There exists a projective model Y/k of K such that the center $y_V \in Y$ of V satisfies $y_V \in \mathrm{RegP}(Y)$ and the rational map $Y \cdots \to X$ is everywhere defined.*

Proof. Since P_X verifies axiom 4 and P verifies axiom 5, there exists a projective model X'/k of K such that $X' \cdots \to X$ is defined at the center $x'_V \in X'$ and $x'_V \in \mathrm{RegP}(X')$. Let Y be the closure of the graph of the rational map $X' \cdots \to X$. Then Y dominates X, and Y is locally isomorphic to X' at x'_V, so Y satisfies the conclusion of the proposition.

$\qquad\square$

Example 3.4. *Let P be any regularity property and $\mu \geq 1$ be an integer. For X/k any projective model of K, define*

$$\mathrm{RegP}_\mu(X) := \{x \in \mathrm{RegP}(X) : e_X(x) \leq \mu\},$$

where $e_X(x) \geq 1$ denotes the multiplicity of X at x. Note that $\mathrm{RegP}_1(X) = \mathrm{RegP}(X) \cap X_{\mathrm{reg}}$ with notations as in example 2.3.

Proposition 3.5. *Let P be any regularity property and $\mu \geq 1$ be an integer. If P verifies axioms 1, 2 and 3 (resp. 1, 2, 3 and 6), then P_μ also verifies axioms 1, 2 and 3 (resp. 1, 2, 3 and 6). If X/k is any projective model of K, $x \in X$ is the generic point of an integral curve $Z \subset X$, and $z \in Z \cap \mathrm{RegP}_\mu(X)$ is such that Z is stably P-permissible at z and $e_X(z) = e_X(x)$, then Z is stably P_μ-permissible at z.*

Proof. This is well known for axiom 1. Also, multiplicity does not increase by blowing up a point, nor by normalizing (theorem 14.8 [26]) and thus P_μ verifies axiom 2 if P

does. With assumptions as in the last statement of the proposition, we have

$$e_{X'}(z') \leq e_E(z') \leq \mu = e_X(x),$$

where $\pi : X' \to X$ is the blowing up along Z, $E := \pi^{-1}(Z)$ and $z' \in \pi^{-1}(z)$ is any point. Combining with (i) of axiom 2, this proves the last statement. With respect to axioms 3 and 6, the proposition is a consequence of the following claim: let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots \tag{5}$$

be that sequence of normalized blowing ups at points $z_{i-1} \in X_{i-1}$ such that $z_0 := z$ and for $1 \leq i \leq n$, z_i maps to z_{i-1} and belongs to the strict transform Z_i of Z in X_i. Then $e_{X_n}(z_n) = e_X(x)$ for all $n >> 0$.

To prove this claim, it can be assumed that $\mu' = e_{X_n}(z_n) \geq 2$ for all $n \geq 0$ without loss of generality. Consider the base change $\tilde{X} := X \otimes \kappa(z)$ and pick a preimage $z' \in \tilde{X}$ of z, Z' a reduced irreducible component of $Z \otimes \kappa(z)$ going through z', and X' a reduced irreducible component of $\tilde{X}$ containing Z'. Without loss of generality, it can be assumed that $(\kappa(z), X', Z', z') = (k, X, Z, z)$ using basic properties of multiplicity (theorem 14.7 and 14.8 [26]).

Changing once again k to a finite extension, it can be furthermore assumed that K/k is separably generated. Let now $\pi : X \to \mathbf{A}_k^3$ be a generic projection around z. Since $\mu' \geq 2$, π factors through a hypersurface singularity $(Y, y) \subset \mathbf{A}_{k,0}^4$ with $\mu' = e_Y(y)$ which is birational to X. In particular, X is isomorphic to the normalization of Y in a neighborhood of y. It can be furthermore assumed that π is transverse to Z, i.e. that $e_Y(\pi(x)) = e_X(x)$. We now define from (5) a sequence of maps:

$$\begin{array}{ccccccc}
X =: X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_n \\
\downarrow & & \downarrow & & & & \downarrow \\
Y =: Y_0 & \leftarrow & Y_1 & \leftarrow & \cdots & \leftarrow & Y_n
\end{array} \tag{6}$$

as follows, by induction on $i \geq 1$: let $Y_i \to Y_{i-1}$ be the blowing up along the image y_{i-1} of z_{i-1} by the morphism $\pi_{i-1} : X_{i-1} \to Y_{i-1}$. If the ideal $m_{y_{i-1}} \mathcal{O}_{X_{i-1}, z_{i-1}}$ is a reduction of $m_{z_{i-1}}$ (which holds for $i = 1$), both ideals have isomorphic normalized blowing up; on the other hand, the blowing up of X_{i-1} along $m_{y_{i-1}} \mathcal{O}_{X_{i-1}, z_{i-1}}$ is then finite over Y_i, hence X_i is isomorphic to the normalization of Y_i above a neighborhood of y_i; finally $\mu' = e_{X_i}(z_i) \leq e_{Y_i}(y_i) \leq e_{Y_{i-1}}(y_{i-1}) = \mu'$ which implies that $m_{y_i} \mathcal{O}_{X_i, z_i}$ is a reduction of m_{z_i}.

Let $x_i \in X_i$ be the generic point of Z_i. Since X_i is isomorphic to the normalization of Y_i above a neighborhood of y_i, we have

$$e_X(x) = e_{X_i}(x_i) = e_{Y_i}(\pi_i(x_i)) \leq e_{Y_i}(y_i) = e_{X_i}(z_i) = \mu'$$

for all $i \geq 0$. Since each $Y_i \to Y_{i-1}$ is the blowing up along y_{i-1}, it is well known that $e_{Y_i}(y_i) = e_{Y_i}(\pi_i(x_i))$ for $i >> 0$ and we get $e_{X_i}(z_i) = e_X(x)$ as required. $\qquad\square$

Example 3.6. [10] *Assume that k is algebraically closed, char$k = 0$ and let $\delta \in \mathrm{Der}_k K$ be a nonzero element. Let X/k be a projective model of K and $x \in X$ be a closed point such that $\mathcal{O}_{X,x}$ is a regular local ring. Given a germ of divisor with normal crossings $E_x = \mathrm{div}(u_1 \cdots u_e)$ at x, $0 \leq e \leq 3$, there exists an expression*

$$\delta = \phi_x \left(\sum_{i=1}^{e} f_{i,x} u_i \frac{\partial}{\partial u_i} + \sum_{j=e+1}^{3} f_{j,x} \frac{\partial}{\partial u_j} \right), \tag{7}$$

with $\phi_x \in K$, $f_{1,x}, f_{2,x}, f_{3,x} \in \mathcal{O}_{X,x}$ and $\dim(\mathcal{O}_{X,x}/(f_{1,x}, f_{2,x}, f_{3,x})) \leq 1$.

If $x \in X$ is any closed point, we say that δ is log-elementary at x w.r.t. E_x if x is a regular point of X and

$$\nu_x(\delta, E_x) := \min\{\operatorname{ord}_x f_{i,x}, \ 1 \le i \le 3\} \le 1. \tag{8}$$

We say that δ is log-elementary at x if there exists a germ of divisor with normal crossings E_x at x such that δ is log-elementary at x w.r.t. E_x. These notions extend to arbitrary (not necessarily closed) points: δ is log-elementary at a point x if it is at a general point of $\overline{\{x\}}$.

Let P be any regularity property. We now define the regularity property P_δ as follows: for any projective model X/k of K, let

$$\operatorname{RegP}_\delta(X) := \operatorname{RegP}(X) \cap \{x \in X \mid \delta \text{ is pseudo} - \text{elementary at } x\}.$$

The following proposition is a consequence of the preparatory results in chapter I of [7].

Proposition 3.7. *Assume that k is algebraically closed, char$k = 0$ and let $\delta \in \operatorname{Der}_k K$ be a nonzero element. If P verifies axioms 1, 2 and 3 (resp. 1, 2, 3 and 6), then P_δ also verifies axioms 1, 2 and 3 (resp. 1, 2, 3 and 6).*

Proof. Although the definition of log-elementary singularities is pointwise, the exceptional divisor E_x can be taken to be locally constant on a given variety X and this proves that axiom 1 is verified by P_δ. Note that proposition I.3.4.9 of [7] allows to make blowing up along stably P_δ-permissible compatible with E_x.

Axioms 2 and 3 are verified by [7], remark I.3.4.5 and proposition I.3.4.9 when $P = P_{\mathrm{reg}}$ and this immediately extends to arbitrary P verifying axioms 1,2,3. So is axiom 6, in view of the corresponding result for P_μ in proposition 3.5 with $\mu := 1$. $\square$

To state our third family of applications, we need the following definition.

Definition 3.8. *Let P be any regularity property. Let X, Y be projective models of K, Y normal, and $x \in Y$. The birational map $\eta : X \cdots \to Y$ is said to be a P-regular sequence at x if the following holds:*

(i) *η is defined at x and $y := \eta(x) \in \operatorname{RegP}(Y)$.*

(ii) *there exists a sequence of projective birational morphisms*

$$Y =: Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_n,$$

where for each i, $1 \le i \le n$, $Y_i \to Y_{i-1}$ is the normalized blowing up along a reduced and irreducible closed subset $Z_{i-1} \subset Y_{i-1}$, $\dim Z_{i-1} \le 1$, whose generic point belongs to $\operatorname{RegP}(Y_{i-1})$, and such that

    (a) the rational map $\eta : X \cdots \to Y_n$ is defined and locally an isomorphism at x;

    (b) for $1 \le i \le n$, Z_{i-1} is stably P-permissible at the point $\pi_i(\eta(x))$, where π_i is the composed map $Y_n \to Y_{i-1}$.

In case $P = P_{\mathrm{reg}}$ is the usual regularity property, we will use the classical terminology of "monoidal sequence" instead of "P_{reg}-regular sequence": in this case each Z_{i-1} appearing in (ii) of the definition can be taken to be either a closed point, or an irreducible curve which contains and is regular at $\pi_i(\eta(x))$. From the above definition of P-regular sequences we associate the following regularity property $P_{RS(Y)}$ to any regularity property P having a globally P-regular model Y.

Definition 3.9. *Let P be a regularity property and Y be a normal projective model of K such that $\mathrm{RegP}(Y) = Y$.*

We define a regularity property $P_{RS(Y)}$ as follows: for any birational projective model X/k of K, $\mathrm{RegP}_{RS(Y)}(X)$ is the set of points $x \in X$ such that the birational map $\eta : X \cdots \to Y$ is a P-regular sequence at x.

Proposition 3.10. *Let P be any regularity property and Y be a normal projective model of K such that $\mathrm{RegP}(Y) = Y$.*

Assume that P veryfies axioms 1,2 and 3. Then $P_{RS(Y)}$ also verifies axioms 1, 2 and 3.

Proof. Let X/k be any projective model of K and $C \subset X$ be an integral curve whose generic point is normal and belongs to $\mathrm{RegP}(X)$. Since P verifies axioms 1,2 and 3, the set

$$V_C := \{z \in C \mid C \text{ is stably permissible at } z\}$$

is a nonempty open set of C (*cf.* section 2, comments after the definition of axiom 3).

Let now $x \in \mathrm{RegP}_{RS(Y)}(X)$ and

$$Y =: Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_n \tag{9}$$

be a sequence of normalized blowing ups at integral varieties Z_{i-1}, $1 \leq i \leq n$ given by (ii) of definition 3.8 and keep notations as in there. For each i, we may choose a nonempty open set $U_{i-1} \subseteq Y_{i-1}$ such that $\pi_i(\eta(x)) \in U_{i-1}$ and Z_{i-1} is stably P-permissible at all points of $Z_{i-1} \cap U_{i-1}$. Then $U := \pi_1^{-1}(U_0) \cap \cdots \cap \pi_n^{-1}(U_{n-1}) \subseteq Y_n$ is Zariski open and $\eta(x) \in U$. After possibly restricting U, it can be assumed that η is defined and is an isomorphism above U. Let now $x' \in \eta^{-1}(U)$. Then $Y_i \to Y_{i-1}$ is an isomorphism at $\pi_{i+1}(\eta(x'))$ if and only if $\pi_i(\eta(x')) \notin Z_{i-1}$. Therefore $x' \in \mathrm{RegP}_{RS(Y)}(X)$ and an associated sequence satisfying the conditions of (ii.b) of definition 3.8 is obtained by erasing from (9) those Y_i's such that $\pi_i(\eta(x')) \notin Z_{i-1}$. This proves that $\mathrm{RegP}_{RS(Y)}(X)$ is Zariski open, hence $\mathrm{RegP}_{RS(Y)}$ verifies axiom 1.

Since $\mathrm{RegP}_{RS(Y)}(X)$ is normal by (ii) of definition 3.8 (Y being normal), $\mathrm{RegP}_{RS(Y)}$ verifies (i) of axiom 2. Now (ii) of axiom 2 is tautological if x is a closed point or is a consequence of the openness of the set V_C defined at the beginning of this proof if x is the generic point of an integral curve C.

To prove that axiom 3 is also verified by $P_{RS(Y)}$, let X/k be any normal projective model of K, $C \subset X$ be an integral curve whose generic point belongs to $\mathrm{RegP}_{RS(Y)}(X)$ and $z \in C$. By definition of $P_{RS(Y)}$, C is stably $P_{RS(Y)}$-permissible at z if and only if C is stably P-permissible at z. Moreover, if $\pi : X_1 \to X$ is the normalized blowing up along z, $C_1 \subset X_1$ the strict transform of C, and $z_1 \in C_1$ satisfies $\pi(z_1) = z$, then $z_1 \in \mathrm{RegP}_{RS(Y)}(X_1)$ since P verifies axiom 2. Therefore $P_{RS(Y)}$ verifies axiom 3 if P does. $\qquad\square$

Another way of constructing regularity properties is by conjunction.

Proposition 3.11. *Let P_1 and P_2 be two regularity properties and let P be the conjunction of $P_1 \wedge P_2$, i.e.*

$$\mathrm{RegP}(X) := \mathrm{RegP}_1(X) \cap \mathrm{RegP}_2(X)$$

for any projective model X/k of K. If each of P_1 and P_2 verifies any of axioms 1, 2 and 3 (resp. verifies axioms 3 and 6), so does P (resp. P verifies axioms 3 and 6).

Proof. This is a direct consequence of the definition of axioms 1, 2 and 3 and 6. $\qquad\square$

Remark: propositions 3.10 and 3.11 need not extend to axioms 4 and 5 in general. The reason is that the definition of these axioms involves no blowing up procedure to build up $\tilde{X}$ (axiom 4) or X (axiom 5) independently of P. See however the next section for results about axioms 4 (propositions 4.1 and 4.2) and 6 (proposition 4.7) when P_1 and P_2 are refinements of the usual regularity property.

4. Refinements of the usual regularity property.

In this section, we assume that P is a refinement of the usual regularity property, i.e. $\mathrm{RegP}(X) \subseteq X_{\mathrm{reg}}$ for any projective model X/k of K.

4.1. Principalization of ideals.

We produce various criteria for a regularity property to verify axiom 4. The main difficulty here is that neither of the two standard algorithms for principalizing ideals on a regular threefold (i.e. Hironaka's [19] and Abhyankar's [3]) does the job. The reason is that regular curves which are permissible for these algorithms need not be P-permissible in the sense of definition 2.2. See also the last section on open problems.

That the usual regularity property $P = P_{reg}$ verifies axiom 4 in all characteristics is proved in proposition 4.2 of [12]. We sum up below the algorithm given in *loc.cit.* One immediately reduces to the idealistic exponent formulation in proposition 4.3 of [12]: given $\mathcal{E} = (\mathcal{I}, \mu)$, an idealistic exponent on $U := \mathrm{RegP}(X)$, with

$$\dim\left(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})\right) \leq 1, \tag{10}$$

there exists a finite composition of blowing ups

$$U =: U(0) \leftarrow U(1) \leftarrow \cdots \leftarrow U(n) \tag{11}$$

with stably P-permissible centers (i.e. regular in the usual sense) mapping to $V(\mathcal{I})$ such that the singular locus $\Sigma(n)$ of the transform $\mathcal{E}(n)$ of $\mathcal{E}$ in $U(n)$ is empty.

Let $\mathcal{E}(i) := (\mathcal{I}(i), \mu)$ be the transform of $\mathcal{E}$ in $U(i)$, $\Sigma(i)$ its singular locus. The existence of (11) with the required properties is proved by applying the following algorithm, denoted (A) in the sequel, and showing that it eventually stops:

1. If $\Sigma(i)$ has an irreducible component C of dimension one which is singular (in the usual sense), let $U(i+1) \to U(i)$ be the blowing up along a singular point of C. Otherwise go to 2.
2. If $\Sigma(i)$ has (regular) components of dimension one which do not intersect transversally, or three such components going through one and the same point, let $U(i+1)$ be the blowing up of $U(i)$ along any such intersection point. Otherwise go to 3.
3. If two components of dimension one of $\Sigma(i)$ do intersect, let $U(i+1)$ be the blowing up of $U(i)$ along any such intersecting curve. Otherwise go to 4.
4. If $\Sigma(i) \neq \emptyset$, let $X(i+1)$ be the blowing up of $X(i)$ along any of its connected components. If $\Sigma(i) = \emptyset$, stop the algorithm.

The following proposition gives a criterion for axiom 4 to be verified.

Proposition 4.1. *Let P be a regularity property which is a refinement of the usual regularity property and verifies axioms 1,2 and 3. Assume that the following holds: for any projective model X/k of K, for any $x \in X$ with $\dim\mathcal{O}_{X,x} = 2$ and for any $z \in Z \cap \mathrm{RegP}(X)$ ($Z := \overline{\{x\}}$) such that Z is regular at z (in the usual sense), Z is P-permissible at z. Then P verifies axiom 4.*

Proof. The assumption implies that any blowing up center $Z(i)$ on some $U(i)$, $i \geq 0$ which is permissible for the idealistic exponent $\mathcal{E}(i)$ is actually P-permissible at each $z(i) \in Z(i)$. So the proof is the same as for $P = P_{\mathrm{reg}}$. $\qquad\square$

Proposition 4.2. *Assume that k is algebraically closed and $\mathrm{char}\,k = 0$ and let $\delta \in \mathrm{Der}_k K$ be a nonzero element. Let P be a regularity property satisfying the assumptions of proposition 4.1. Then P_δ verifies axiom 4.*

Proof. With notations as in the beginning of this section, let $\mathcal{E} = (\mathcal{I}, \mu)$ be an idealistic exponent on $U := \mathrm{RegP}(X)$ satisfying (10). It is sufficient to prove the existence of a sequence (11) of blowing ups with stably P_δ-permissible centers such that the singular locus $\Sigma(n)$ of the transform $\mathcal{E}(n)$ of $\mathcal{E}$ in $U(n)$ is empty. We will assume familiarity with embedded resolution procedures in regular threefolds in this proof, e.g. [19] or proposition 4.2 of [12] (which are characteristic free), as well as with the definitions and preliminary results in chapter I of [7].

Let $x \in U$. By definition of log-elementary singularities (example 3.6), there exists a germ of normal crossings divisor E_x at x such that δ is expressed locally et x by formula (7) with property (8). In particular, there exists an open neighborhood V_x of x and a normal crossings divisor E on V_x such that δ is log-elementary at each $y \in V_x$ w.r.t. the stalk E_y of E at y. One easily deduces that there exists a finite open covering $V_1, \ldots, V_s$ of U and a normal crossings divisor E_j on V_j for each j, $1 \leq j \leq s$ such that the following holds: for any $x \in U$ and any j, $1 \leq j \leq s$ with $x \in V_j$, δ is log-elementary at x w.r.t. $E_{j,x}$. We fix such an open covering to begin with and first reduce to the case $s = 1$.

Definition 4.3. *With notations as above, a blowing up center $Y(i) \subset U(i)$ is said to be $(\mathcal{E}(i), P_\delta)$-permissible at a point $x(i) \in Y(i)$ if it is stably P-permissible, permissible for $\mathcal{E}(i) = (\mathcal{I}(i), \mu)$, the transform of $\mathcal{E}$ in $U(i)$, and "permissible for δ adapted to the total transform of each E_j, $1 \leq j \leq s$" at $x(i)$ in the sense of [7] I.3.4.4. A sequence of blowing ups*

$$U =: U(0) \leftarrow U(1) \leftarrow \cdots \leftarrow U(n) \tag{12}$$

is said to be an $(\mathcal{E}, P_\delta)$-sequence if for each i, $0 \leq i \leq n-1$, the blowing up center $Y(i) \subset U(i)$ is $(\mathcal{E}(i), P_\delta)$-permissible at each of its points.

In the above definition, note that $Y(i)$ is stably P-permissible iff $Y(i)$ is regular, since P satisfies the assumptions of proposition 4.1. Also $Y(i)$ is permissible for $\mathcal{E}(i)$ iff $Y(i) \subseteq \Sigma(i)$. The definition of permissibility in [7] I.3.4.4 is a refinement of P_δ-permissibility in definition 2.2 designed to get compatibility of blowing ups with normal crossings divisors. An immediate consequence of [7] proposition I.3.4.9 is that the strict transform $Y(i+j)$ of any curve $Y(i) \subset \Sigma(i)$ can be made $(\mathcal{E}(i+j), P_\delta)$-permissible after blowing up $j \geq 0$ many times along closed points.

Let $\pi(i): U(i) \to U$ be the induced morphism. Given an $(\mathcal{E}, P_\delta)$-sequence, we let

$$E_j(i) := \pi(i)^{-1}(E_j)_{\mathrm{red}} \subset V_j(i) := \pi(i)^{-1}(V_j).$$

In particular, the following holds for each i, $0 \leq i \leq n$: $\mathrm{RegP}_\delta(U(i)) = U(i)$; $\pi(i)$ is an isomorphism away from Σ; for each j, $1 \leq j \leq s$, $E_j(i)$ is a normal crossings divisor on $V_j(i)$ such that δ is log-elementary at each $x(i) \in V_j(i)$ w.r.t. the stalk of $E_j(i)$ at $x(i)$. Moreover, $Y(i)$ has normal crossings with $E_j(i)$ for each i, j, $0 \leq i \leq n-1$, $1 \leq j \leq s$.

Lemma 4.4. *There exists an $(\mathcal{E}, P_\delta)$-sequence (12) such that $\pi(n)(\Sigma(n))$ is a finite set.*

Proof. Let $C \subset \Sigma$ be an irreducible curve. We apply the following algorithm (B) for $i \geq 0$:

1. If $\Sigma(i)$ has an irreducible component $C(i)$ with $\pi(i)(C(i)) = C$, and a point $x(i) \in C(i)$ at which $C(i)$ is not $(\mathcal{E}, P_\delta)$-permissible, let $U(i+1) \to U(i)$ be the blowing up along $x(i)$. Otherwise go to 2.
2. If $\Sigma(i)$ has an irreducible component $C(i)$ with $\pi(i)(C(i)) = C$, let $U(i+1) \to U(i)$ be the blowing up along $C(i)$ and go back to step 1. Otherwise stop.

To prove the lemma, it is thus sufficient to prove that algorithm (B) stops, since Σ has finitely many irreducible components of dimension one. As mentioned right after definition 4.3, one reaches step 2 of algorithm (B) after finitely many point blowing ups. If the algorithm does not stop, we get an infinite quadratic sequence of two dimensional regular local rings

$$\mathcal{O}_{U(i_1),C(i_1)} < \mathcal{O}_{U(i_2),C(i_2)} < \cdots < \mathcal{O}_{U(i_n),C(i_n)} < \cdots,$$

with $C(i_n) \subseteq \Sigma(i_n)$ for each $n \geq 1$: a contradiction by e.g. [34] appendix 5, theorem 3 and (E) on p. 391. $\qquad\square$

Choosing $j = 1$ at each of the finitely many points in $\pi(n)(\Sigma(n))$ and working consecutively on each connected component of $\Sigma(n)$, we get:

Corollary 4.5. *With notations as above, it can be assumed that $s = 1$ and $\Sigma \subset E_1$.*

In view of this corollary, we simplify our notations and write E instead of E_1, take $V_1 = U$. For $x(i) \in \Sigma(i)$, let $m(x(i))$, $1 \leq m(x(i)) \leq 3$, be the number of irreducible components of $E(i)_{x(i)}$ going through $x(i)$. Recall the standard definition of the directrix vector space $T_{x(i)}$ and $\tau(x(i)) := \dim_k T_{x(i)}$ of $\mathcal{E}(i)$ (*cf.* [12] section 4).

Lemma 4.6. *There exists an $(\mathcal{E}, P_\delta)$-sequence (12) such that the following holds.*
  (i) *$\Sigma(n) \subset E(n)$ is a union of isolated points and projective lines and has normal crossings with $E(n)$. Any two distinct irreducible components of $\Sigma(n)$ meet at at most one point, any three of them do not meet.*
  (ii) *for each $x(n) \in \Sigma(n)$, there exists $Z_{x(n)} \in T_{x(n)}$ which has normal crossings with $E(n)$ at $x(n)$. In particular $m(x(n)) \leq 2$.*
  Moreover, (i) and (ii) are preserved by any further $(\mathcal{E}, P_\delta)$-permissible blowing ups.

Proof. If $\pi(1) : U(1) \to U$ is the blowing up along an $(\mathcal{E}, P_\delta)$-permissible center $Y \subset U$, with exceptional divisor F, and $x \in Y$, we have $\pi(1)^{-1}(x) \cap \Sigma(1) = \emptyset$ except in the following situations:

(a) $Y = x$ is a closed point, $\tau(x) = 2$ and $F \cap \Sigma(1)$ is the isolated point $x(1)$ corresponding to the projective space associated with T_x which has $\tau(x(1)) \geq 2$.
(b) Y is a curve, $\tau(x) = 1$ and $F \cap \Sigma(1)$ either is a finite set, or is a regular curve mapping isomorphically to Y.
(c) $Y = x$ is a closed point, $\tau(x) = 1$ and $F \cap \Sigma(1)$ is contained in the projective line $L_x \subset F$ associated with T_x.

In view of these facts and of lemma 4.4, we can achieve $\Sigma(n)$ to be a union of isolated points and projective lines. Applying steps 1 and 2 of algorithm (A), we can furthermore assume that any two distinct projective lines in $\Sigma(n)$ intersect transversally on $E(n)$ and at at most one point, any three of them do not have one and the same intersection point. Note that this is preserved by any further $(\mathcal{E}, P_\delta)$-sequence.

Without loss of generality, we let $n = 0$ in the sequel. The last statement in the lemma is an easy consequence of (a)-(c). Finally, (i) is achieved from (ii) by a finite composition of blowing ups at closed points performing embedded resolution of $\Sigma(n) \cup M_2(n)$, where

$$M_2(n) := \{x(n) \in U(n) : m(x(n)) \geq 2\}.$$

There hence remains to prove (ii). We will repeatedly use the following fact: if $x \in \Sigma$ is a closed point with r.s.p. (u_1, u_2, u_3), then

$$Y = V(u_1, u_2) \subseteq \Sigma \implies T_x \subseteq < U_1, U_2 > . \qquad (13)$$

Now consider consecutively the following cases:

Case 1. $\tau(x) = 2$ and x is an isolated point of Σ. By (a), blowing up x gives either $F \cap \Sigma(1) = \emptyset$, or $F \cap \Sigma(1) = \{x(1)\}$ with $\tau(x(1)) \geq 2$. Iterating, the process will stop after finitely many blowing ups at closed points because x is an isolated point of Σ.

Case 1'. $\tau(x) = 2$ and x is not an isolated point of Σ. Since $\tau(x) = 2$, Σ is locally at x an *irreducible* curve C_x by (13). Blowing up gives $\pi(1)^{-1}(x) \cap \Sigma(1) = \emptyset$ unless ($Y = x$ and $\Sigma(1)$ coincides with the strict transform $C_x(1)$ of C_x in a neighborhood of F); then $\tau(x(1)) \geq 2$, where $F \cap C_x(1) =: \{x(1)\}$ by (a). Iterating, it can therefore be assumed w.l.o.g. that C_x is $(\mathcal{E}, P_\delta)$-permissible at x.

Assume now that $\tau(x) = 1$. Pick a r.s.p. (u_1, u_2, u_3) at x such that $E_x = \mathrm{div}(u_1 \cdots u_e)$, where $e = m(x)$. Let $T_x =: < \alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3 >$. Let $\epsilon(x) := 0$ if $\alpha_j \neq 0$ for some $j \geq e + 1$; otherwise define:

$$\epsilon(x) := \sharp\{j : 1 \leq j \leq e \text{ and } \alpha_j \neq 0\} \geq 1.$$

Note that x satisfies (ii) if and only if $\epsilon(x) = 0$.

Case 2. $\epsilon(x) = 3$. If x is not an isolated point of Σ, any component of Σ is locally at x an irreducible curve C, tangent to the line $L := V(\alpha_1 u_1 + \alpha_2 u_2, u_3)$ up to permuting indices by (13). In any case, we get (ii) with $n = 1$ by blowing up x by (c).

Case 3. $\epsilon(x) = 2$. If x is not an isolated point of Σ, any component of Σ is locally at x either (tangent to the line $L := V(\alpha_1 u_1 + \alpha_2 u_2, u_3)$ up to permuting indices with $e = 3$), or tangent to the line $L_1 := V(u_1, u_2)$ by (13). If $Y = C$ is a curve, then $C = L_1$ and (ii) holds with $n = 1$ above x by (b). Otherwise, blowing up x gives (ii) with $n = 1$ above x except possibly at the point $x(1)$ on the strict transform of L_1. The point $x(1)$ is either in one of cases 1, 1' or 2, or again in case 3 and we can iterate. Then we either get (ii) for some $n \geq 1$ above x or $L_1 \subseteq \Sigma$ and its strict transform $L_1(n)$ at $x(n)$ is $(\mathcal{E}(n), P_\delta)$-permissible at $x(n)$. In any case, we get (ii) above x except possibly at $x(n)$.

Case 4. $\epsilon(x) = 1$. Let $R := \{x \in \Sigma : \tau(x) = \epsilon(x) = 1\}$. Then R is a closed subset of Σ and the component $D_x := \mathrm{div}(u_i)$ of E_x such that $T_x = k.U_i$ is constant when x varies in a connected component of R. If $x \in R$ and $\pi(1) : U(1) \to U$ is the blowing up along x, then $F \cap \Sigma(1)$ is contained in the strict transform of D_x by (c). Suppose that there exists an irreducible curve $C \subseteq D \cap \Sigma$ such that $C \nsubseteq M_2 := M_2(0)$. Applying algorithm (B) to C finitely many times and taking into account the analysis of cases 1' and 3 when $Y = C$, we reduce to the case $D \cap \Sigma \subseteq M_2$. In particular, we may

assume w.l.o.g. the following: there exists an irreducible component $D \subseteq E$ such that

$$x \in R \Longrightarrow x \in D \cap M_2 \text{ and } D_x = \operatorname{div}(u_1) \text{ locally at } x. \tag{14}$$

We now use classical results on principalization of ideals. If $x \in R$ and $g \in \mathcal{I}_x$, there is a unique expansion

$$g = g_0 u_1^\mu + \sum_{i=1}^{\mu} g_i u_1^{\mu-i},$$

with $g_0 \in k[[u_1, u_2, u_3]]$; $g_i \in k[[u_2, u_3]]$ and $\operatorname{ord}_{(u_2,u_3)} g_i > i$ for $1 \leq i \leq \mu$. Let $\pi(1) : U(1) \to U$ be the blowing up along x. The ideal

$$c(\mathcal{I}_x) := \left(\{ g_i^{\mu!/i} : 1 \leq i \leq \mu, \ g \in \mathcal{I}_x \} \right),$$

transforms by the following rule at $x(1) \in F \cap \Sigma(1)$: with obvious notations, we have $c(\mathcal{I}(1)_{x(1)}) = u_j^{-\mu!} c(\mathcal{I}_x)$, where (u_j) is the ideal of $F = \pi(1)^{-1}(x)$ at $x(1)$, $j = 2$ or $j = 3$. By embedded resolution of curves in regular surfaces, [17] proposition V.3.9, it can be furthermore assumed that for each $x \in R$,

$$x \in R \Longrightarrow c(\mathcal{I}_x) = (u_2^{a_{2,x}} u_3^{a_{3,x}}),$$

with $a_{2,x}, a_{3,x} \in \mathbf{N}$, $(a_{2,x} + a_{3,x})/\mu! > 1$. By (14), we have for $x \in R$:

$$m(x) = 2 \Longrightarrow 0 \leq a_{3,x}/\mu! < 1. \tag{15}$$

If $Y \subset D \cap M_2$ is an irreducible curve *meeting R* , we let

$$a_Y := a_{i,x}, \ x \in Y \cap R \text{ arbitrary and } Y = V(u_1, u_i) \text{ locally at } x. \tag{16}$$

If $Y \subset D \cap M_2$ is an irreducible curve *not meeting R*, we simply let $a_Y := 0$. If $x \in R$, we let

$$\alpha(x) := \max\{a_{2,x}/\mu!, a_{3,x}/\mu!\}, \ c(x) := (a_{2,x} + a_{3,x})/\mu!. \tag{17}$$

Case 4.1. $x \in R$ is an isolated point of Σ. This is equivalent to $\alpha(x) < 1$. Any $x(1) \in F \cap \Sigma(1)$ is also an isolated point of $\Sigma(1)$, and $c(x(1)) < c(x)$ if $x(1) \in R(1)$. Iterating the argument and taking into account the analysis of cases 1,2 and 3, we may assume furthermore that Σ has equal dimension one.

We now need use the specific properties of log-elementary singularities of vector fields. Recall from (8) the definition of $\nu_x(\delta, E_x)$. We let

$$\Sigma_0 := \{x \in \Sigma : \nu_x(\delta, E_x) = 0\}, \ \Sigma_1 := \{x \in \Sigma : \nu_x(\delta, E_x) = 1\} = \Sigma \backslash \Sigma_0.$$

Note that Σ_0 is open in Σ. The following is the content of I.3.4.1 (with $\nu_x(\delta, E_x) \leq 1$ in our situation), I.3.4.4 (with $m(x) \geq 2$ if $x \in \Sigma$ in our situation) and I.3.4.6 of [7]: if $Y \subseteq \Sigma$ is an irreducible curve, then

$$Y \text{ is } (\mathcal{E}, P_\delta) - \text{permissible at } x \in Y \Leftrightarrow x \in \Sigma_0 \text{ or } Y \subseteq \Sigma_1. \tag{18}$$

Moreover, if a center Y is $(\mathcal{E}, P_\delta)$-permissible at some point $x \in Y$ and $\pi(1) : U(1) \to U$ is the blowing up along Y and $x(1) \in \pi(1)^{-1}(x)$, then

$$\nu_{x(1)}(\delta, E(1)_{x(1)}) \leq \nu_x(\delta, E_x).$$

The vector field δ induces a vector field $\bar{\delta}$ by projection on D. Given a local expansion at x

$$\delta = \phi_x \left(\sum_{i=1}^{e} f_{i,x} u_i \frac{\partial}{\partial u_i} + \sum_{j=e+1}^{3} f_{j,x} \frac{\partial}{\partial u_j} \right), \tag{19}$$

$\phi_x \in K$, $f_{1,x}, f_{2,x}, f_{3,x} \in \mathcal{O}_{U,x}$, $\dim(\mathcal{O}_{U,x}/(f_{1,x}, f_{2,x}, f_{3,x})) \leq 1$ and $\nu_x(\delta, E_x) = \min\{\text{ord}_x f_i, 1 \leq i \leq 3\} \leq 1$, denote $\overline{f}_{j,x}$, $1 \leq j \leq 3$ the image of $f_{j,x}$ in $\mathcal{O}_{D,x}$. Then $\overline{\delta}$ is given at x by

$$\overline{\delta} = \sum_{i=2}^{e} \overline{f}_{i,x} u_i \frac{\partial}{\partial u_i} + \sum_{j=e+1}^{3} \overline{f}_{j,x} \frac{\partial}{\partial u_j},$$

where we still denote u_2, u_3 the images in $\mathcal{O}_{D,x}$ of the corresponding parameters of $\mathcal{O}_{U,x}$. If $U(n) \to U$ is a finite composition of blowing ups at closed points and $x(n) \in D(n)$ is a closed point with r.s.p. $(u_{1,x(n)}, u_{2,x(n)}, u_{3,x(n)})$, we write similarly

$$\delta = \phi_{n,x(n)} \left(\sum_{i=1}^{e} f_{i,x(n)} u_{i,x(n)} \frac{\partial}{\partial u_{i,x(n)}} + \sum_{j=e+1}^{3} f_{j,x(n)} \frac{\partial}{\partial u_{j,x(n)}} \right),$$

where $e = m(x(n))$, $D(n)_{x(n)} = \text{div}(u_{1,x(n)})$ and let

$$\overline{J}_{x(n)} := (\overline{f}_{2,x(n)}, \overline{f}_{3,x(n)}), \; \overline{K}_{x(n)} := \overline{J}_{x(n)} + (\overline{f}_{1,x(n)}).$$

Using two-dimensional results on resolution of vector fields [28], there exists a finite composition of blowing ups at closed points $U(n) \to U$ such that the following holds: for each $x(n) \in D(n) \cap \Sigma(n)$, exactly one of the following holds:

(α) $x(n) \notin \Sigma_1(n)$, or
(β) $\overline{J}_{x(n)} = (u_{2,x(n)}^{b_{2,x(n)}} u_{3,x(n)}^{b_{3,x(n)}})$ is a monomial (possibly zero) ideal, where $b_{3,x(n)} = 0$ if $m(x(n)) = 2$, or
(γ) $\overline{J}_{x(n)} = (u_{2,x(n)}^{b_{2,x(n)}}) \overline{J}'_{x(n)}$, with $m_{x(n)} = 2$ and $(u_{2,x(n)}) + \overline{J}'_{x(n)} = (u_{2,x(n)}, \overline{f}_{3,x(n)}) = (u_{2,x(n)}, u_{3,x(n)})$.

Moreover, ((α) or (β)) is preserved by further blowing ups at closed points. Any (finite or infinite) chain of points $x(n), x(n+1), \dots$ satisfying (γ) belongs to the strict transform of a regular, possibly formal curve on $D(n)$. Therefore $x(n+m) \notin \Sigma(n+m)$ for some $m \geq 0$, and this implies that there exists $n_1 \geq 0$ such that ((α) or (β)) holds for all $n \geq n_1$. We take $n_1 = 0$ w.l.o.g. For $x \in \Sigma_1$, let $\pi(1) : U(1) \to U$ be the blowing up along x and $x(1) \in F \cap \Sigma(1)$. We get the following transformation law from the blowing up formulæ [7] I.2.2.5:

$$\overline{K}_{x(1)} = (u_j^{-\epsilon} \overline{f}_{1,x}) + \overline{J}_{x(1)}, \tag{20}$$

where (u_j) is the ideal of $F = \pi(1)^{-1}(x)$ at $x(1)$, $j = 2$ or $j = 3$, and $\epsilon = 0$ (resp. $\epsilon = 1$) if $(m(x) = 2$ and $\text{ord}_x f_{3,x} = 1)$ (resp. otherwise).

Using once more the blowing up formulæ [7] I.2.2.5 and controlling the order of the transform of $\overline{K}_x$ by (20), there exists a finite composition of blowing ups at closed points $U(n_2) \to U$ such that the following holds: for each $x(n_2) \in D(n_2) \cap \Sigma(n_2)$, exactly one of the following holds:

(α) $x(n_2) \notin \Sigma_1(n_2)$, or
(β') $\overline{K}_{x(n_2)} = (u_{2,x(n_2)}^{b'_{2,x(n_2)}} u_{3,x(n_2)}^{b'_{3,x(n_2)}})$ is a monomial (nonzero) ideal, where $b'_{3,x(n_2)} = 0$ if $m(x(n_2)) = 2$.

Once again, we assume w.l.o.g. that $n_2 = 0$ and define the following invariants from (16):

$$A := \max\{a_Y : Y \subseteq D \cap M_2\}, \ N := \sharp\{Y \subseteq \Sigma : a_Y = A\}.$$

Note that the case $A = 0$, i.e. $R = \emptyset$ is dealt with by cases 1' and 3. Assume now $A > 0$ and we construct an $(\mathcal{E}, P_\delta)$-sequence $\pi(n) : U(n) \to U$ such that $(A(n), N(n)) <_{\text{lex}} (A, N)$ with obvious notations, which will conclude the proof by descending induction on (A, N).

Pick an irreducible component $Y \subseteq \Sigma$ with $a_Y = A$. If Y is $(\mathcal{E}, P_\delta)$-permissible at all points, let $\pi(1) : U(1) \to U$ be the blowing up along Y. There is an isomorphism $Y' := \pi(1)^{-1}(Y) \cap D(1) \simeq Y$ and we have $a_{Y'} = a_Y - \mu!$ if Y' meets $R(1)$. Therefore $(A(1), N(1)) <_{\text{lex}} (A, N)$ in any case. There remains to reduce to this situation.

Assume now that Y is not $(\mathcal{E}, P_\delta)$-permissible at closed points $x_1, \ldots, x_r \in Y$, $r \geq 1$. It can be assumed that $x_j \in R$ by the analysis in cases 1' and 3 above, $1 \leq j \leq r$. By (18), we have $\nu_x(\delta, E_x) = 0$ for $x \in Y \setminus \{x_1, \ldots, x_r\}$, $\nu_{x_j}(\delta, E_{x_j}) = 1$ for $1 \leq j \leq r$.

Pick now $x \in \{x_1, \ldots, x_r\}$. We assume w.l.o.g. that $Y = V(u_1, u_2)$ locally at x. Comparing with (β'), we get a numerical criterion: Y is not $(\mathcal{E}, P_\delta)$-permissible at x if and only if:

$$x \text{ satisfies } (\beta') \text{ with } m(x) = 3 \text{ and } b'_{2,x} = 0. \tag{21}$$

Case 4.2. $a_{3,x}/\mu! < 1$. Let $\pi(1) : U(1) \to U$ be the blowing up along x, $Y' := D(1) \cap \pi(1)^{-1}(x)$, $Y(1)$ be the strict transform of Y and $\{x(1)\} := Y' \cap Y(1)$. We have $a_{Y'} = a_Y + a_{3,x} - \mu! < A$ if $x(1) \in R(1)$. Therefore $(A(1), N(1)) \leq_{\text{lex}} (A, N)$ in any case and we may substitute (U, Y) by $(U(1), Y(1))$.

Case 4.3. $a_{3,x}/\mu! \geq 1$. Let $Z \subset D \cap M_2$ be the Zariski closure of $V(u_1, u_3)$, so $Z \subset \Sigma$. Since $b'_{3,x} \geq 1$, Z is $(\mathcal{E}, P_\delta)$-permissible at all points and let $\pi(1) : U(1) \to U$ be the blowing up along Z, $Z' := D(1) \cap \pi(1)^{-1}(Z)$, $Y(1)$ be the strict transform of Y and $\{x(1)\} := Z' \cap Y(1)$. We have $a_{3,x(1)} = a_Z - \mu! < A$. Therefore $(A(1), N(1)) \leq_{\text{lex}} (A, N)$ in any case and we may substitute as well (U, Y) by $(U(1), Y(1))$.

We now analyze the structure of δ at $x(1)$ by (20) for both cases 4.2 and 4.3: if $b'_{3,x} = 1$, we have $\nu_{x(1)}(\delta, E(1)_{x(1)}) = 0$, i.e. $Y(1)$ is $(\mathcal{E}, P_\delta)$-permissible at $x(1)$; if $b'_{3,x} \geq 2$, we have $\nu_{x(1)}(\delta, E(1)_{x(1)}) = 1$, and $x(1)$ again satisfies (21) with $b'_{3,x(1)} = b'_{3,x} - 1$. Hence there exists an $(\mathcal{E}, P_\delta)$-sequence $\pi(b'_{3,x}) : U(b'_{3,x}) \to U$ such that $(A(b'_{3,x}), N(b'_{3,x})) \leq_{\text{lex}} (A, N)$ and the strict transform $Y(b'_{3,x})$ of Y is $(\mathcal{E}, P_\delta)$-permissible at $\{x(b'_{3,x})\} := \pi(b'_{3,x})^{-1}(x) \cap Y(b'_{3,x})$. Working in this way at each $x \in \{x_1, \ldots, x_r\}$, this concludes the proof of lemma 4.6. $\square$

We now prove proposition 4.2. We may assume w.l.o.g. that $n = 0$ in lemma 4.6. To begin with, considerations similar to those in cases 1 and 1' in the previous proof show that it can be assumed that any point $x \in \Sigma$ with $\tau(x) \geq 2$ actually satisfies $\tau(x) = 2$, and that there is a unique one-dimensional irreducible component C_x of Σ going through x, C_x is $(\mathcal{E}, P_\delta)$-permissible at x. From now on, assume that $\tau(x) = 1$.

We now apply lemma 4.4 and pick $x \in \pi(n)(\Sigma(n))$ with notations as in there. Since char$k = 0$, there exists a *maximal contact* surface D for the idealistic exponent $\mathcal{E}$ around x. Explicitly, pick $g \in \mathcal{I}_x$ with ord$_x g = \mu$. Given a r.s.p. (u_1, u_2, u_3) of

$\mathcal{O}_{U,x}$ with $E_x = \mathrm{div}(u_1 \cdots u_e)$, $1 \le e = m(x) \le 2$, we let

$$z := \frac{1}{(\mu-1)!} \frac{\partial^{\mu-1} g}{\partial u_3^{\mu-1}},$$

and take $D := \mathrm{div}(z)$. Then any $(\mathcal{E}, P_\delta)$-sequence above x is centered on the strict transform of D. Without loss of generality, we add $u_3 := z$ to the convention $E_x = \mathrm{div}(u_1 \cdots u_e)$. Note in particular that $Z_x = U_3$, with notations as in lemma 4.6.

The proof now runs parallel to that of lemma 4.6 and we only indicate those parts where the proof differs in what follows. If $x \in \Sigma$ and $g \in \mathcal{I}_x$, there is a unique expansion

$$g = g_0 u_3^\mu + \sum_{i=1}^\mu g_i u_3^{\mu-i},$$

$g_0 \in k[[u_1, u_2, u_3]]$; $g_i \in k[[u_1, u_2]]$ and $\mathrm{ord}_{(u_1,u_2)} g_i > i$ for $1 \le i \le \mu$ and let

$$c(\mathcal{I}_x) := \left(\{ g_i^{\mu!/i} : 1 \le i \le \mu, \ g \in \mathcal{I}_x \} \right).$$

It can be furthermore assumed w.l.o.g. that $c(\mathcal{I}_x) = (u_1^{a_{1,x}} u_2^{a_{2,x}})$, with $a_{1,x}, a_{2,x} \in \mathbf{N}$, $(a_{1,x} + a_{2,x})/\mu! > 1$ for each $x \in R := \{ x \in \Sigma : \tau(x) = 1 \}$. Similarly, it can be assumed that Σ has equal dimension one and $\Sigma \subseteq M := M(0)$, where

$$M(n) := D(n) \cap E(n).$$

With notations as in (19), let

$$\overline{\delta} = \sum_{i=1}^e \overline{f}_{i,x} u_i \frac{\partial}{\partial u_i} + \sum_{j=e+1}^2 \overline{f}_{j,x} \frac{\partial}{\partial u_j},$$

where we still denote u_1, u_2 the images in $\mathcal{O}_{D,x}$ of the corresponding parameters of $\mathcal{O}_{U,x}$. Similarly, if $U(n) \to U$ is a finite composition of blowing ups at closed points and $x(n) \in D(n)$ is a closed point with r.s.p. $(u_{1,x(n)}, u_{2,x(n)}, u_{3,x(n)})$, $D(n)_{x(n)} = \mathrm{div}(u_{3,x(n)})$, we now let

$$\overline{J}_{x(n)} := (\overline{f}_{1,x(n)}, \overline{f}_{2,x(n)}), \ \overline{K}_{x(n)} := \overline{J}_{x(n)} + (\overline{f}_{3,x(n)}).$$

For $x \in \Sigma_1$, let $\pi(1): U(1) \to U$ be the blowing up along x and $x(1) \in \pi(1)^{-1}(x) \cap \Sigma(1)$. We get the following transformation law from the blowing up formulæ [7] I.2.2.5:

$$\overline{f}_{3,x(1)} = u_j^{-1-\epsilon} \overline{f}_{3,x} \tag{22}$$

where (u_j) is the ideal of $\pi(1)^{-1}(x)$ at $x(1)$, $j = 1$ or $j = 2$, and $\epsilon = 0$ (resp. $\epsilon = 1$) if $(m(x) = 1$ and $\mathrm{ord}_x(f_{2,x}, f_{3,x}) = 1)$ or if $(m(x) = 2$ and $\mathrm{ord}_x f_{3,x} = 1)$ (resp. otherwise).

As in the proof of lemma 4.6, there exists a finite composition of blowing ups at closed points $U(n) \to U$ such that the following holds: for each $x(n) \in \Sigma(n) \cap D(n)$, exactly one of the following holds:

(α) $x(n) \notin \Sigma_1(n)$, or
(β') $\overline{K}_{x(n)} = (u_{1,x(n)}^{b'_{1,x(n)}} u_{2,x(n)}^{b'_{2,x(n)}})$ and $(\overline{f}_{3,x(n)}) = (u_{1,x(n)}^{c_{1,x(n)}} u_{2,x(n)}^{c_{2,x(n)}})$ are monomial ideals, $1 \le b'_{1,x(n)} + b'_{2,x(n)} < +\infty$, with $b'_{2,x(n)} = c_{2,x(n)} = 0$ if $m(x(n)) = 1$.

Moreover, $(($\alpha$)$ or $($\beta'$))$ is preserved by further blowing ups at closed points. We take $n = 0$ w.l.o.g. If $Y \subseteq M = M(0)$ is an irreducible curve *meeting* R, we let

$$a_Y := a_{i,x}, \ x \in Y \cap R \text{ arbitrary with } \tau(x) = 1, \ Y = V(u_i, u_3) \text{ locally at } x.$$

If $Y \subset D \cap M_2$ is an irreducible curve *not meeting* R, we simply let $a_Y := 0$. Define the following invariants:

$$A := \max\{a_Y : Y \subseteq \Sigma\}, \ N := \sharp\{Y \subseteq \Sigma : a_Y = A\}.$$

The case $A = 0$, i.e. $\tau(x) \geq 2$ for each $x \in \Sigma$, is dealt with by blowing up each irreducible component of Σ as in case 1' in the proof of lemma 4.6. Assume now that $A > 0$. We construct an $(\mathcal{E}, P_\delta)$-sequence $\pi(n) : U(n) \to U$ such that $(A(n), N(n)) <_{\text{lex}} (A, N)$ with obvious notations and thus conclude the proof by descending induction.

Pick an irreducible component $Y \subseteq \Sigma$ with $a_Y = A$. If Y is $(\mathcal{E}, P_\delta)$-permissible at all points, we blow up Y and get $(A(1), N(1)) <_{\text{lex}} (A, N)$ as in the proof of lemma 4.6. We reduce below to this situation.

Let $x \in Y$ be such that Y is not $(\mathcal{E}, P_\delta)$-permissible at x. We assume w.l.o.g. that $Y = V(u_1, u_3)$ locally at x. Using formulæ I.3.4.4.2 and I.3.4.4.3 of [7], our (nonnumerical) criterion is now: Y is not $(\mathcal{E}, P_\delta)$-permissible at $x \in \Sigma$ if and only if:

$$x \in \Sigma_1 \text{ and } \left(\text{either } c_{1,x} = 0, \text{ or } (f_{3,x} \in (u_1, u_3)^2 \text{ and } b'_{1,x} = 0)\right). \qquad (23)$$

These conditions imply in particular that $m(x) = 2$ and $b'_{1,x} = 0$ by (β'), so $b'_{2,x} \geq 1$ since $x \in \Sigma_1$. Letting $Z \subset M$ to be the Zariski closure of $V(u_2, u_3)$, it follows that Z is $(\mathcal{E}, P_\delta)$-permissible at all points provided $Z \subset \Sigma$.

We now consider two cases analogues of 4.2 (here $a_{2,x}/\mu! < 1$) and 4.3 (here $a_{2,x}/\mu! \geq 1$) in the proof of lemma 4.6. Let $\pi(1) : U(1) \to U$ be the blowing up along x (resp. along Z) in the former case, i.e. $Z \not\subset \Sigma$ (resp. in the latter case, i.e. $Z \subset \Sigma$). Let $Y(1)$ be the strict transform of Y and $x(1) := \pi(1)^{-1}(x) \cap Y(1)$. By (22) and (23), $Y(1)$ is $(\mathcal{E}, P_\delta)$-permissible at $x(1)$ if $(b'_{1,x}, b'_{2,x}) = (0, 1)$; otherwise, $x(1)$ satisfies again one of the properties in (23) with $b'_{2,x(1)} \leq b'_{2,x} - 1$. In any case $(A(1), N(1)) \leq_{\text{lex}} (A, N)$ and we may substitute U, Y by $U(1), Y(1)$. Arguing by descending induction on $b'_{2,x}$, we will get the strict transform $Y(n)$ to be $(\mathcal{E}, P_\delta)$-permissible at all points for some $n \geq 1$, together with $(A(n), N(n)) \leq_{\text{lex}} (A, N)$, which completes the proof. $\square$

4.2. Quadratic sequences along curves.

We do not know if the following proposition remains valid for a regularity property which is not the refinement of the usual regularity property.

Proposition 4.7. *Let P be a regularity property which is a refinement of P_{reg} and veryfies axioms 1,2,3 and 6. Let Y/k be a (regular) projective model of K such that $\text{RegP}(Y) = Y$. Then $P_{RS(Y)}$ verifies axiom 6.*

Proof. Let X/k be a normal projective model of K, $x \in \text{RegP}_{RS(Y)}(X)$ be such that $\dim \mathcal{O}_{X,x} = 2$ and let $C := \overline{\{x\}}$. Since P verifies axioms 3 and 6, it can be assumed w.l.o.g. that $C \subset \text{RegP}(X)$. By proposition 3.2, it can be furthermore assumed that $C \subset \text{RegP}_Y(X)$.

Let v' be a fixed discrete k-valuation of K dominating $\mathcal{O}_{X,C}$ with residue field a finite algebraic extension k' of $k(C)$, and $\overline{v}$ be any discrete valuation of k'. We denote by v the composed valuation $v = v' \circ \overline{v}$, with group $\mathbf{Z}_{\text{lex}}^2$. Let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be that sequence of blowing ups at closed points $z_i \in X_i$, where z_i is the center of v. It is sufficient to prove that $\eta_i : X_i \to Y$ is a P-regular sequence at z_i for some $i \geq 0$.

Note that η_0 is defined at all points of C, since $C \subset \mathrm{RegP}_Y(X)$.

Assume that π_0 is not an isomorphism at x. We now apply definition 3.8 at the point $x \in X$. There exists a certain $Z_0 \subset Y$, Z_0 either a closed point or an integral curve which is stably P-permissible at $\eta_0(x)$, the stalk I of whose ideal at $\eta_0(x)$ is such that $I\mathcal{O}_{X,x}$ is principal.

If $\eta_0(x)$ is a closed point, then $I\mathcal{O}_{X_i,z_i}$ is principal for all $i >> 0$ by proposition 3.2 applied to the blowing up of Y along Z_0.

If $\eta_0(x)$ is the generic point of a curve, we necessarily have $Z_0 = \overline{\{\eta_0(x)\}}$. Since P verifies axiom 3, there exists a composition of blowing ups at closed points $Y' \to Y$ such that the strict transform Z'_0 of Z_0 in Y' is stably P-permissible at all points. Let $M \subset \mathcal{O}_{Y,\eta_0(z_0)}$ be the maximal ideal. By proposition 3.2 applied to the blowing up of Y along $\eta_0(z_0)$, $M\mathcal{O}_{X_i,z_i}$ is principal for all $i >> 0$. Arguing by induction on the number of closed points blown up from Y to Y', we may substitute Y by Y', X by X_n for some $n >> 0$, and thus assume that Z_0 is stably P-permissible at $\eta_0(z_0)$. Then it can be furthermore assumed that η_0 factors locally at z_0 through the blowing up along Z_0, applying once more proposition 3.2 to the blowing up of Y along Z_0.

By induction on the number n_x of blowing ups appearing in definition 3.8 w.r.t. x, we thus reduce w.l.o.g. to the case $n_x = 0$, i.e. η_0 is an isomorphism at x.

As pointed out above, it can now be assumed that $\eta_0(C)$ is stably P-permissible at $\eta_0(z_0)$. In particular, $\eta_0(C)$ is regular at $\eta_0(z_0)$ and this implies that $\kappa(z_i) = \kappa(z_0)$ for all $i \geq 0$. Pick a r.s.p. (u, v, w) of $\mathcal{O}_{Y,\eta_0(z_0)}$ in such a way that (u, v) is the ideal of C at $\eta_0(z_0)$. By standard arguments, there exists $i_0 \geq 0$ such that the following holds for all $i \geq i_0$:

(*) the strict transform of $\mathrm{div}(w)$ at z_i is empty; the exceptional locus E_i of η_i at z_i is irreducible, transverse to the strict transform of C and maps to $\eta_0(z_0)$.

Pick $i \geq i_0$ and let $w_i \in \mathcal{O}_{X_i,z_i}$ be a local equation of E_i. By construction, $\mathcal{O}_{X_i,z_i}$ has a r.s.p. (u_i, v_i, w_i) where $w = \gamma_i w_i^{c_i}$ for some $c_i > 0$ and $\gamma_i \in \mathcal{O}_{X_i,z_i}$ invertible; $u_i = u/w_i^{a_i}, v_i = v/w_i^{b_i}$ for some $a_i, b_i \geq 0$. We define a discrete k-valuation of K as follows: given a nonzero series in $\widehat{\mathcal{O}_{X_i,z_i}} \simeq \kappa(z_0)[[u_i, v_i, w_i]]$, we let

$$\hat{\nu}\left(\sum_{(\alpha,\beta,\gamma)\in\mathbf{N}^3} \lambda_{\alpha\beta\gamma} u_i^\alpha v_i^\beta w_i^\gamma\right) := \min\{\alpha + \beta + \gamma : \lambda_{\alpha\beta\gamma} \neq 0\} \in \mathbf{N}.$$

Then $\hat{\nu}$ extends by linearity to a discrete k-valuation of $QF(\widehat{\mathcal{O}_{X_i,z_i}})$ and we let ν be the restriction of $\hat{\nu}$ to K.

Now, the value group $\mathbf{Z}$ of ν is generated by the values $\nu(u) = 1 + a_i, \nu(v) = 1 + b_i, \nu(w) = c_i$, so we have

$$\mathrm{g.c.d}(1 + a_i, 1 + b_i, c_i) = 1. \tag{24}$$

On the other hand, the residue field $\kappa(\nu)$ of ν is equal to

$$\kappa(z_0)\left(\mathrm{cl}(u_i/w_i), \mathrm{cl}(v_i/w_i)\right), \tag{25}$$

where cl denotes the residue map of the valuation ring R_ν of ν. Doing similar computations in $\mathcal{O}_{Y,\eta(z_0)}$, we get

$$\kappa(\nu) = \kappa(z_0)\left(\{\mathrm{cl}(u^\alpha v^\beta w^\gamma) : (1 + a_i)\alpha + (1 + b_i)\beta + c_i\gamma = 0\}\right). \tag{26}$$

Let $\mathbf{v} := (1 + a_i, 1 + b_i, c_i) \in (\mathbf{Z}^3)^\vee$. By (24), the map $\mathbf{Z}^3 \xrightarrow{\;\;\mathbf{v}\;\;} \mathbf{Z}$ induced by $\mathbf{v}$ is surjective; by (25) and (26), its kernel $K \simeq \mathbf{Z}^2$ projects isomorphically to the first two coordinates lattice $\mathbf{Z}^2 \subset \mathbf{Z}^3$. This implies that $c_i = 1$, and we will consequently take $\gamma_i = 1$ in what follows.

After possibly switching u and v, it can be assumed that $b_i \geq a_i$. If equality holds, η_i obviously factors as a composition of point blowing ups at z_i and the proposition is proved in this case.

Assume now that $d_i := b_i - a_i > 0$ and note that d_i is independent of $i \geq i_0$ since (*) above holds. Let $Z \subset Y$ be the Zariski closure of the germ of curve $V(v, w)$. Once more, it can be assumed that Z is stably P-permissible at $\eta_0(z_0)$ while preserving condition (*) for all $i \geq i_1$ for some $i_1 \geq i_0$. Substituting Y by its blowing up along Z preserves these assumptions and changes (a_i, b_i) to $(a_i, b_i - 1)$, hence d_i to $d_i - 1$. Iterating d_i times this step reduces to the situation of the previous paragraph, thus completes the proof. $\qquad\square$

4.3. **Applications.** Applying our theorem 2.5, F. Cano, C. Roche and M. Spivakovsky obtain corollary 4.9 below (case $P = P_{\mathrm{reg}}$ of the next proposition) from their local uniformization theorem for vector fields (see part 3 of [10]). A slight refinement of their argument yields the following:

Proposition 4.8. *Assume that k is algebraically closed and $\mathrm{char}\, k = 0$. Let X/k be a regular projective model of K and $\delta \in \mathrm{Der}_{K/k}$ be a nonzero element. Let P be a regularity property which satisfies the assumptions of proposition 4.1 and verifies axioms 5 and 6.*

There exists a projective model Y/k of K such that $\mathrm{RegP}_\delta(Y) = Y$, together with a birational projective morphism $\pi: Y \to X$ such that π is an isomorphism above $\mathrm{RegP}_\delta(X)$.

Proof. This proposition will be a consequence of theorem 2.5 applied to the property P_δ and to the model X once we have proved that P_δ verifies axioms 1 to 6.

Since P verifies axioms 1,2,3 and 6, so does P_δ by proposition 3.7. By proposition 4.2, P_δ verifies axiom 4.

The hard part is to prove that P_δ verifies axiom 5 (local uniformization). This is because, for a given k-valuation ring V/k of K, the local uniformization (M, Y) produced by theorem 1 of [10] is obtained from any given model M_0/k of K which is regular (in the usual sense) at the center of V by blowing up centers which are locally regular at the center of V. Since P verifies axiom 5, it can be assumed that M_0 is P-regular at the center of V to begin with. Since P satisfies the assumptions of proposition 4.1, M will consequently be P_δ-regular at the center Y of V. $\qquad\square$

Since the usual regularity property satisfies the assumptions of the proposition (case $\mu = 1$ of proposition 3.5), we get:

Corollary 4.9. [10] **(Cano, Roche, Spivakovsky).** *Assume that k is algebraically closed and $\mathrm{char}\, k = 0$. Let X/k be a regular projective model of K and $\delta \in \mathrm{Der}_{K/k}$ be a nonzero element.*

There exists a projective model Y/k of K such that δ is log-elementary at all points $y \in Y$, together with a birational projective morphism $\pi: Y \to X$. Moreover, π is an isomorphism above each $x \in X$ such that δ is log-elementary at x.

Theorem 2.5 can also be used to prove the following local form of Hironaka's Strong Factorization Conjecture [18], since Local Uniformization has been proved in this context [14] [23]. Note that we get an optimal control on the fundamental loci.

Theorem 4.10. *Assume that* $\mathrm{char}\,k = 0$. *Let* $\pi : Y_2 \to Y_1$ *be a birational (projective) morphism of regular projective models of* K.

There exists a regular projective model Y/k *of* K, *together with birational projective morphism* $\pi_i : Y \to Y_i$, $i = 1, 2$, *such that* π_i *is locally a monoidal sequence at each* $y \in Y$ *and for both* $i = 1, 2$ *and the following holds:*

  (i) *the fundamental locus* $\mathcal{F}_2$ *of* π_2^{-1} *is given by*

$$\mathcal{F}_2 := \{y \in Y_2 : \pi \text{ is not a monoidal sequence at } y\}.$$

  (ii) $\pi(\mathcal{F}_2)$ *is a finite set contained in the fundamental locus* $\mathcal{F}$ *of* π. *In particular, the fundamental locus* $\mathcal{F}_1$ *of* π_1^{-1} *is equal to* F.

Proof. Consider the regularity properties $P_i := P_{\mathrm{reg},RS(Y_i)}$, $i = 1, 2$ from definition 3.9. Let $P := P_1 \wedge P_2$ be the conjunction of P_1 and P_2. We claim that $Y := X'$ satisfies the conclusion of the theorem if and only if X' satisfies the conclusion of theorem 2.5 w.r.t. P and for $X := Y_2$.

Clearly π_i is a monoidal sequence at each $y \in Y$ and for both $i = 1, 2$ if and only if $\mathrm{Reg}P(Y) = Y$. By definition of P, we have $\mathrm{Sing}P(Y_2) = \mathcal{F}_2$, so (i) is a mere reformulation of the content of theorem 2.5. We now prove that (ii) is a consequence of (i). Let $Z \subset \pi^{-1}(\mathcal{F})$ be an integral curve with generic point x such that $\pi(Z)$ is a curve. Then

$$\mathcal{O}_{Y_1,\pi(x)} < \mathcal{O}_{Y_2,x}$$

is an iterated quadratic transform by Abhyankar's theorem 3 [2], i.e. $x \in \mathrm{Reg}P(Y_2)$ and the conclusion easily follows.

There hence remains to prove that P verifies axioms 1 to 6. Since the usual regularity property P_{reg} verifies axioms 1,2,3 and 6 by proposition 3.5 with $\mu = 1$, so does P by propositions 3.10, 4.7 and 3.11. By proposition 4.1, P verifies axiom 4. Finally, that P verifies axiom 5 is the content of theorem 0.1 [23] (this is where the characteristic zero assumption is used). $\qquad \square$

The following example was brought to our attention by S.D. Cutkosky. It shows that not every birational morphism of regular projective threefolds which is everywhere a monoidal sequence is *globally* a composition of blowing ups with regular centers.

Example 4.11. [27] *Take* $P = P_{\mathrm{reg}}$ *the usual regularity property. Let* $\pi : Y \to X := \mathbf{P}_k^3$ *be the toric projective morphism corresponding to the following regular subdivision of* $\mathbf{R}_{\geq 0}^3$ *at the origin* $x = (1 : 0 : 0 : 0)$:
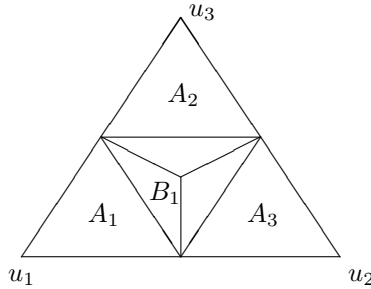
The regular variety Y is covered by six charts above the origin. To prove that π is everywhere locally a monoidal sequence, i.e. that $\mathrm{RegP}_{RS(X)}(Y) = Y$, it is sufficient by symmetry to check it on each of the charts A_1 and B_1 in the above figure.

Chart A_1 corresponds to the inclusion $k[u_1, u_2, u_3] \subset A_1 := k[u_1/u_2u_3, u_2, u_3]$ of affine algebras, which factors as a monoidal sequence as follows:

$$k[u_1, u_2, u_3] \subset k[u_1/u_2, u_2, u_3] \subset A_1.$$

Chart B_1 corresponds to the inclusion $k[u_1, u_2, u_3] \subset B_1 := k[u_1/u_2, u_1/u_3, u_2u_3/u_1]$ of affine algebras, which factors as a monoidal sequence as follows:

$$k[u_1, u_2, u_3] \subset k[u_1/u_2, u_2, u_3] \subset k[u_1/u_2, u_2, u_2u_3/u_1] \subset B_1.$$

On the other hand, π is an isomorphism away from the union of all three coordinate axes $l_1 : x_2 = x_3 = 0$, $l_2 : x_1 = x_3 = 0$, $l_3 : x_1 = x_2 = 0$ going through x. But π does not factor through the blowing up at x, nor through the blowing up of any of the lines l_1, l_2 or l_3. This can be easily checked: for example, none of the corresponding ideals to these closed subsets is principal on all three charts A_1, A_2 and A_3. Therefore π is not a composition of blowing ups with regular centers.

5. Constructing P-regular models: proof of theorem 2.4.

From now on, P is a regularity property of local rings $R \subseteq K$, essentially of finite type over k and such that $QF(R) = K$, verifying axioms 1 to 5. Let X/k be a projective model of K. In order to prove theorem 2.4, the main point is to prove the following:

Proposition 5.1. *Let X_1/k and X_2/k be two projective models of K. There exists a normal and projective model Y/k of K, together with morphisms $\pi_i : Y \to X_i$, such that $\pi_i^{-1}(\mathrm{RegP}(X_i)) \subseteq \mathrm{RegP}(Y)$ for $i = 1, 2$.*

Proof. The proof is long and essentially an adaptation of Zariski's proof of his Patching Theorem ([33], Fundamental Theorem on p. 539). We split it into various steps.

Step 1. It can be assumed that X_1 and X_2 are normal by (i) of axiom 2.

Step 2. It can be assumed that the rational map $\eta : X_2 \cdots \to X_1$ is a morphism. Namely, let Z be the normalization of the closure of the graph of η, so Z dominates both X_1 and X_2. Note that $Z \to X_2$ is projective, hence the blowing up along a certain ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X_2}$.

By (i)' and (ii) of axiom 4 applied to the pair $(X_2, \mathcal{I})$, there exists a projective birational morphism $\pi : Z_1 \to X_2$ such that $\mathcal{I}\mathcal{O}_{Z_1}$ is locally principal (hence π factorize through Z) and $\pi^{-1}(\mathrm{RegP}(X_2)) \subseteq \mathrm{RegP}(Z_1)$. Therefore we may substitute X_2 by the normalization of Z_1, applying once again (i) of axiom 2.

Step 3. Given a birational morphism $\eta : X_2 \to X_1$ of normal projective varieties, we denote by $\mathcal{F}_\eta \subset X_1$ the fundamental locus of η^{-1}. Since X_1 is normal, $\mathcal{F}_\eta$ has codimension at least two, so it contains finitely many distinct irreducible curves. We need the following definition.

Definition 5.2. *With notations as above, let $x \in \mathrm{RegP}(X_1)$ be such that $\dim\mathcal{O}_{X_1,x} = 2$ and let $Z_x := \overline{\{x\}}$. We say that η is factorizable above x if the localization map of η at x*

$$\eta_x : X_{2,x} := X_2 \times \mathrm{Spec}\mathcal{O}_{X_1,x} \to \mathrm{Spec}\mathcal{O}_{X_1,x}$$

is a composition of normalized blowing ups of (closed) points. In this case, we let $n_x \geq 0$ be the number of such normalized blowing ups of points.

We say that η is factorizable if for each $x \in \mathrm{RegP}(X_1)$ such that $\dim \mathcal{O}_{X_1,x} = 2$, η is factorizable at x.

Note that for x as in the definition such that η is factorizable above x, there is an equivalence
$$n_x > 0 \Leftrightarrow Z_x \subseteq \mathcal{F}_\eta.$$

By the following lemma, it can be assumed that $\eta : X_2 \to X_1$ is factorizable, since $\mathcal{F}_\eta$ contains finitely many distinct irreducible curves.

Lemma 5.3. *Let $\eta : X_2 \to X_1$ be a birational morphism of normal projective models of K and $x \in \mathrm{RegP}(X_1)$ be such that $\dim \mathcal{O}_{X_1,x} = 2$. There exists a projective birational morphism $\pi' : X_2' \to X_2$, X_2' normal such that*

  (i) *$\mathcal{F}_{\eta'} = \mathcal{F}_\eta$, where $\eta' : X_2' \to X_1$ is the composed map;*
  (ii) *$\pi'^{-1}(\mathrm{RegP}(X_2)) \subseteq \mathrm{RegP}(X_2')$;*
  (iii) *η' is factorizable above x.*

Proof. By Zariski's theorem on resolution of surface singularities by successive normalized blowing ups (p.688 [30]) and theorem 3 [2], there exists a composition of, say $n_x \geq 0$, normalized blowing ups of points $Z_x \to X_{2,x}$ such that the composed map $Z_x \to \mathrm{Spec} \mathcal{O}_{X_1,x}$ is a composition of normalized blowing ups of points. The lemma holds with $X_2' = X_2$ if $n_x = 0$.

Assume that $n_x \geq 1$ and let $X_{2,x}' \to X_{2,x}$ be the first of these normalized blowing ups, centered at $x' \in X_{2,x}$. Let $Z_{x'}$ be the Zariski closure of x' in X_2, so $Z_{x'}$ is a curve mapping surjectively to Z_x; we consider two cases:

Case 1: $x' \notin \mathrm{RegP}(X_2)$. Then $Z_{x'} \cap \mathrm{RegP}(X_2) = \emptyset$, since $\mathrm{RegP}(X_2)$ is Zariski open (axiom 1). Let $\pi' : X_2' \to X_2$ be the normalized blowing up along $Z_{x'}$ and $\eta' : X_2' \to X_1$ be the composed map. We have $\mathcal{F}_{\eta'} = \mathcal{F}_\eta$ and $\pi'^{-1}(\mathrm{RegP}(X_2)) \subseteq \mathrm{RegP}(X_2')$. On the other hand, $X_{2,x}' = X_2' \times \mathrm{Spec} \mathcal{O}_{X_1,x}$, so replacing X_2 by X_2' yields a reduction in n_x.

Case 2: $x' \in \mathrm{RegP}(X_2)$. We use axiom 3 to x'. There exists a composition $X_{2,0} \to X_2$ of normalized blowing ups of closed points belonging to the successive strict transforms of $Z_{x'}$ such that the strict transform Z_0' of $Z_{x'}$ in $X_{2,0}$ is (stably) P-permissible at all points. Let $X_2' \to X_{2,0}$ be the normalized blowing up along Z_0', $\pi' : X_2' \to X_2$ and $\eta' : X_2' \to X_1$ be the composed map. We have $\mathcal{F}_{\eta'} = \mathcal{F}_\eta$ and $\pi'^{-1}(\mathrm{RegP}(X_2)) \subseteq \mathrm{RegP}(X_2')$ by axiom 2 and definition 2.2. On the other hand, $X_{2,x}' = X_2' \times \mathrm{Spec} \mathcal{O}_{X_1,x}$, so replacing X_2 by X_2' also yields a reduction in n_x. $\qquad\square$

Note that if $\eta : X_2 \to X_1$ is factorizable, an immediate consequence of axiom 2 is the following:
$$\mathrm{RegP}(X_1) \cap \eta(\mathrm{SingP}(X_2)) \text{ is a finite set.} \tag{27}$$

Step 4. The following definition is essential. This step consists in removing bad points for $\mathrm{SingP}(X_1)$.

Definition 5.4. *Let $\eta : X_2 \to X_1$ be a birational morphism of normal projective models of K and $x \in \mathrm{RegP}(X_1)$ be such that $\dim \mathcal{O}_{X_1,x} = 2$. Let $\Sigma_1 := \mathrm{SingP}(X_1)$. We say that a point $x \in \mathrm{RegP}(X_1)$ such that $\dim \mathcal{O}_{X_1,x} = 2$ is bad for Σ_1 if $Z_x := \overline{\{x\}} \subseteq \mathcal{F}_\eta$ and Z_x meets Σ_1.*

Assume that $\eta: X_2 \to X_1$ is factorizable above each $x \in \mathrm{RegP}(X_1)$, $\dim \mathcal{O}_{X_1,x} = 2$, which is bad for Σ_1. We now define our main invariant, with notations as in definition 5.2: let $N_\eta := \sum_x n_x \geq 0$, where the sum runs over all $x \in \mathrm{RegP}(X_1)$, $\dim \mathcal{O}_{X_1,x} = 2$ which are *bad* for Σ_1. Let

$$\Phi := \Sigma_1 \cup \Sigma_b \cup \eta(\mathrm{SingP}(X_2)), \tag{28}$$

where

$$\Sigma_b := \bigcup_{x \text{ bad}} Z_x.$$

Consider now a commutative diagram of birational projective morphisms

$$
\begin{array}{ccc}
X_2 & \xleftarrow{\;f_2\;} & X_2' \\
\eta \downarrow & & \downarrow \eta' \\
X_1 & \xleftarrow{\;f_1\;} & X_1',
\end{array}
\tag{29}
$$

X_1' and X_2' normal, such that

(i) $f_i^{-1}(\mathrm{RegP}(X_i)) \subseteq \mathrm{RegP}(X_i')$, $i = 1, 2$, and
(ii) f_1 induces an isomorphism $X_1' \backslash f_1^{-1}(\Phi) \simeq X_1 \backslash \Phi$.

We let $\Sigma_1' := f_1^{-1}(\Sigma_1)$ and extend the definition of bad points as follows:

Definition 5.5. *With notations as above, let $x' \in f_1^{-1}(\mathrm{RegP}(X_1))$ be such that $\dim \mathcal{O}_{X_1',x'} = 2$. We say that x' is bad for Σ_1' if $Z_{x'} := \overline{\{x'\}} \subseteq \mathcal{F}_{\eta'}$ and $Z_{x'}$ meets Σ_1'.*

If moreover η' is factorizable above each point $x' \in f_1^{-1}(\mathrm{RegP}(X_1))$ which is bad for Σ_1', we let $N_{\eta'} := \sum_{x'} n_{x'} \geq 0$, where the sum runs over all $x' \in f_1^{-1}(\mathrm{RegP}(X_1))$, $\dim \mathcal{O}_{X_1',x'} = 2$ which are bad for Σ_1'.

Lemma 5.6. *Assume that $\eta: X_2 \to X_1$ is factorizable above each $x \in \mathrm{RegP}(X_1)$, $\dim \mathcal{O}_{X_1,x} = 2$, which is bad for Σ_1. There exists a diagram (29), satisfying properties (i) and (ii) such that there exists no $x' \in f_1^{-1}(\mathrm{RegP}(X_1))$, $\dim \mathcal{O}_{X_1',x'} = 2$, which is bad for Σ_1'.*

Proof. Assume that $N_\eta \geq 1$ and let $x \in \mathrm{RegP}(X_1)$, $\dim \mathcal{O}_{X_1,x} = 2$ be bad for Σ_1. By axiom 3, there exists a composition of normalized blowing ups

$$X_1 =: X_{1,0} \leftarrow \cdots \leftarrow X_{1,m} =: X_1'$$

at closed points $z_i \in \mathrm{RegP}(X_{1,i})$, $1 \leq i \leq m-1$, such that the strict transform Z_x' of Z_x in X_1' is (stably) P-permissible at all points in $Z_x' \cap \mathrm{RegP}(X_1')$. By axiom 4 applied to the pair (X_2, M_{z_1}), there exists a projective birational morphism $f_{2,1}: X_{2,1} \to X_2$ such that $M_{p_1} \mathcal{O}_{X_{2,1}}$ is locally principal (hence $X_{2,1} \to X_1$ factorize through $X_{1,1}$) and $f_{2,1}^{-1}(\mathrm{RegP}(X_2)) \subseteq \mathrm{RegP}(X_{2,1})$. By (i) of axiom 2, it can be furthermore assumed that $X_{2,1}$ is normal. By (iii) of axiom 4 and (27), $f_{2,1}$ is an isomorphism above $\eta^{-1}(X_1 \backslash \Phi)$ and above each $x \in \mathrm{RegP}(X_1)$ which is bad for Σ_1.

By induction on i, $1 \leq i \leq m$, we construct a sequence of birational projective morphisms

$$X_2 =: X_{2,0} \leftarrow \cdots \leftarrow X_{2,m} =: X_2',$$

with X_2' normal, such that $X_2' \to X_1$ factorizes through X_1'. Let $f_i: X_i' \to X_i$, $i = 1, 2$ and $\eta': X_2' \to X_1'$. We have thus constructed a diagram (29) with the above properties (i) and (ii) and get defined $\Sigma_1' := f_1^{-1}(\Sigma_1)$. Moreover, f_2 is an isomorphism above each $x \in \mathrm{RegP}(X_1)$ which is bad for Σ_1. Therefore f_1 induces a one-to-one correspondence between bad points for Σ_1' and bad points for Σ_1 and f_1 is an isomorphism at

x' whenever x' is bad for Σ_1'. This proves that $\eta'_{x'} = \eta_x$ whenever x' is bad for Σ_1' and in particular $N_{\eta'} = N_\eta$. Arguing by induction on the number of bad x's for Σ_1, we get a diagram (29) satisfying (i) and (ii), $N_{\eta'} = N_\eta$, and such that the following holds:

(iii) $Z_{x'}$ is (stably) P-permissible at all points in $Z_{x'} \cap \mathrm{RegP}(X_1')$ and for each $x' \in \mathrm{RegP}(X_1')$, $\dim \mathcal{O}_{X_1', x'} = 2$, which is bad for Σ_1'.

Let x' be bad for Σ_1' and $\mathcal{I}_{x'} \subset \mathcal{O}_{X_1'}$ be the ideal of $Z_{x'}$. Since η' is factorizable above x', the complement $C_{x'}$ of the Zariski open set

$$U_{x'} := \{z' \in X_2' \mid \mathcal{I}_{x'} \mathcal{O}_{X_2'} \text{ is principal}\}$$

satisfies $\eta'(C_{x'}) \subset Z_{x'}$ and $\eta'(C_{x'})$ is a finite set.

By axiom 4 applied to the pair $(X_2', \mathcal{I}_{x'} \mathcal{O}_{X_2'})$ and (i) of axiom 2, there exists a birational projective morphism $f_2' : X_2'' \to X_2'$, X_2'' normal, such that

$$f_2'^{-1}(\mathrm{RegP}(X_2')) \subseteq \mathrm{RegP}(X_2''),$$

the ideal $\mathcal{I}_{x'} \mathcal{O}_{X_2''}$ is locally principal and f_2' is an isomorphism above $U_{x'} \cap \mathrm{RegP}(X_2')$. Let $\eta'' : X_2'' \to X_1'$ be the composed map. By construction,

$$\mathcal{F}_{\eta''} \cap f_1^{-1}(\mathrm{RegP}(X_1)) = \mathcal{F}_{\eta'} \cap f_1^{-1}(\mathrm{RegP}(X_1))$$

and we have $\eta''_{x'} = \eta'_{x'}$ for each bad x' for Σ_1'. Replacing X_2' by X_2'' and arguing by induction on the number of bad x'''s for Σ_1', we achieve a diagram (29) satisfying the above properties (i), (ii) and (iii), $N_{\eta'} = N_\eta$, and such that the following holds:

(iv) $\mathcal{I}_{x'} \mathcal{O}_{X_2'}$ is locally principal for each bad x' for Σ_1'.

Let x' be bad for Σ_1'. By (iv), η' then factors through the normalized blowing up $f_1' : X_1'' \to X_1'$ along $Z_{x'}$. By axiom 2 and (iii), we have $f_1'^{-1}(\mathrm{RegP}(X_1')) \subseteq \mathrm{RegP}(X_1'')$. Replacing X_1' by X_1'' therefore gives a new diagram (29) with $N_{\eta'} = N_\eta - 1$. Arguing by induction on N_η, we eventually get a diagram (29) having property (i) and (ii), and such that $N_{\eta'} = 0$. $\qquad\square$

Step 5. Pick a diagram (29) satisfying the conclusion of lemma 5.6. Since there is no bad x' for Σ_1', any irreducible component of $\mathcal{F}_{\eta'}$ is either identically contained in $f_1^{-1}(\mathrm{RegP}(X_1))$ or is contained in Σ_1'. Let $\mathcal{F}_1 := \mathcal{F}_{\eta'} \cap f_1^{-1}(\mathrm{RegP}(X_1))$, $U_1 := X_1' \backslash \mathcal{F}_1$ and $U_2 := f_1^{-1}(\mathrm{RegP}(X_1))$. We have $X_1' = U_1 \cup U_2$. Let $U := U_1 \cap U_2$. By (i) and definition of $\mathcal{F}_1$, we have

$$U \subseteq \mathrm{RegP}(X_1') \text{ and } \eta'^{-1}(U) \simeq U. \tag{30}$$

Since η' is birational and projective, it is the blowing up along a certain ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X_1'}$. Then $\mathcal{I}\mathcal{O}_U$ is locally principal by (30). By axiom 4 and (i), there exists a birational projective morphism $\epsilon : Y_1 \to X_1'$ such that $\mathcal{I}\mathcal{O}_{\epsilon^{-1}(U_2)}$ is locally principal and

$$\epsilon^{-1}(U_2) \subseteq \mathrm{RegP}(Y_1) \text{ and } \epsilon^{-1}(U) \simeq U. \tag{31}$$

By (i) of axiom 2, it can be furthermore assumed that Y_1 is normal. Let $V_2 := \epsilon^{-1}(U_2)$ and $V_1 := \eta'^{-1}(U_1)$. By (30) and (31), V_1 and V_2 glue along $\epsilon^{-1}(U) \simeq \eta'^{-1}(U)$ to a proper model Y/k of K. On the other hand, Y is projective since X_1' is projective and both ϵ and η' are projective morphisms.

We claim that Y satisfies the conclusion of the proposition. Since X_2' and Y_1 are normal, Y is also normal. The rational map $Y \cdots \to X_2'$ is defined and is an isomorphism on V_1; by the universal property of blowing up, $Y \cdots \to X_2'$ is defined on

V_2; therefore there exists a projective morphism $Y \to X_2'$, hence projective morphisms $\pi_i : Y \to X_i$, $i = 1, 2$. By (i) and (31), we have $\pi_1^{-1}(\text{RegP}(X_1)) \subseteq V_2 \subseteq \text{RegP}(Y)$; $\pi_2^{-1}(\text{RegP}(X_2)) \cap V_1 \simeq f_2^{-1}(\text{RegP}(X_2)) \cap V_1$, so by (i) $\pi_2^{-1}(\text{RegP}(X_2)) \cap V_1 \subseteq \text{RegP}(Y)$ and this concludes the proof of proposition 5.1.

Corollary 5.7. *Let X/k be a given projective model of K. There exists a normal and projective model Y/k of K, together with a morphism $\pi : Y \to X$ such that $Y = \text{RegP}(Y)$.*

Proof. By axiom 5, for each k-valuation ring V/k of K, there exists a projective model X_V/k of K such that the center $x_V \in X_V$ of V satisfies $x_V \in \text{RegP}(X_V)$. It can be assumed that $X_V \cdots \to X$ is everywhere defined by proposition 3.3. By axiom 1 and quasi-compactness of the space of k-valuations of K, there exists a finite number of projective models $X_1, \ldots, X_n$ of K such that the following holds: for any k-valuation ring V/k of K, there exists i, $1 \le i \le n$ such that the center $x_{i,V}$ of V in X_i belongs to $\text{RegP}(X_i)$. By applying $n - 1$ consecutive times proposition 5.1, we construct a normal and projective model Y of K such that $Y = \text{RegP}(Y)$ and $Y \to X$ is a morphism. $\square$

6. ELIMINATING THE FUNDAMENTAL LOCUS: PROOF OF THEOREM 2.5.

In this section, P is a regularity property of local rings $R \subseteq K$, essentially of finite type over k and such that $QF(R) = K$, verifying axioms 1 to 6. The proof of theorem 2.5 is merely an adaptation of V. Cossart's refinement of Zariski's Patching Theorem ([11] or proposition 4.8 in [12]).

Let X/k be a given normal and projective model of K. By corollary 5.7, there exists a normal and projective model Y_0 of K, together with a morphism $\pi_0 : Y_0 \to X$ such that $Y_0 = \text{RegP}(Y_0)$.

Since X is normal, the fundamental locus $\mathcal{F}_{\pi_0}$ of π_0^{-1} has dimension at most one; we have $\text{SingP}(X) \subseteq \mathcal{F}_{\pi_0}$. Let $\mathcal{G}_0$ be the union of all irreducible components of $\mathcal{F}_{\pi_0}$ which are not identically contained in $\text{SingP}(X)$, so $\mathcal{F}_{\pi_0} = \mathcal{G}_0 \cup \text{SingP}(X)$. Let $Z_1, \ldots, Z_m$ be those one-dimensional irreducible components of $\mathcal{G}_0$ such that $Z_i \cap \text{SingP}(X) \neq \emptyset$. Let $g_1 : X_1 \to X$ be the composition of all those morphisms $X' \to X$ given by axiom 6 when applied consecutively to each of $Z_1, \ldots, Z_m$. It can be assumed w.l.o.g. that the fundamental locus of g_1^{-1} is contained in $\text{SingP}(X)$ and in particular we have

$$g_1^{-1}(\text{RegP}(X)) \simeq \text{RegP}(X). \tag{32}$$

Let $z \in \mathcal{F}_{\pi_0}$ be a closed point whose normalized blowing up has been performed in the process of building up g_1, so $\pi_0(z) \in \text{SingP}(X)$, and let $M_z \subset \mathcal{O}_X$ be the ideal of z. By axiom 4 applied to the pair (X, M_z), there exists a projective birational morphism $Y_1 \to Y_0$ such that $M\mathcal{O}_{Y_1}$ is locally principal and $\text{RegP}(Y_1) = Y_1$. Note that (iii) of axiom 4 implies that $Y_1 \to Y_0$ is an isomorphism above $Y_0 \backslash \pi_0^{-1}(\text{SingP}(X))$. By (i) of axiom 2, it can be furthermore assumed that Y_1 is normal, whence $Y_1 \to X$ factors through the normalized blowing up along z.

Arguing by induction on the number of normalized blowing up performed in g_1, we obtain a projective morphism $\eta : X_2 \to X_1$ such that $\text{RegP}(X_2) = X_2$. Let Σ_0 (resp. Z_i') be the strict transform of $\mathcal{G}_0$ (resp. Z_i, $1 \le i \le m$) in X_1. We have

$$\Sigma_0 \subseteq \text{RegP}(X_1). \tag{33}$$

By construction, the fundamental locus $\mathcal{F}_\eta$ of η^{-1} satisfies

$$\mathcal{F}_\eta \subseteq \Sigma_0 \cup g_1^{-1}(\text{SingP}(X)). \tag{34}$$

Let $\Sigma_1 := \mathrm{SingP}(X_1)$, so by (33) we have

$$\Sigma_0 \cap \Sigma_1 = \emptyset. \tag{35}$$

The proof is now a variation of steps 3 to 5 of the proof of proposition 5.1. We indicate how to adapt the argument to our case. Recall definition 5.4 of bad points for Σ_1. By lemma 5.3, it can be assumed without loss of generality that f_1 is factorizable above each $x \in \mathrm{RegP}(X_1)$, $\dim\mathcal{O}_{X_1,x} = 2$, which is bad for Σ_1. With notations as in (27), we deduce from (33), (34) and (35) that

$$\Phi \subseteq g_1^{-1}(\mathrm{SingP}(X)) \tag{36}$$

with notations as in (27) (note that $\Phi = \Sigma_1 \cup \Sigma_b$ since $\mathrm{RegP}(X_2) = X_2$).

By lemma 5.6, (32) and (36), there exists a diagram (29) such that there is no $x' \in f_1^{-1}(\mathrm{RegP}(X_1))$, $\dim\mathcal{O}_{X_1',x'} = 2$, which is bad for Σ_1' and the following holds:

(i) $\mathrm{RegP}(X_2') = X_2'$;
(ii) $g_1 \circ f_1$ induces an isomorphism $X_1' \backslash (g_1 \circ f_1)^{-1}(\mathrm{SingP}(X)) \simeq \mathrm{RegP}(X)$.

Since there is no bad x' for Σ_1', any irreducible component of $\mathcal{F}_{\eta'}$ is either identically contained in $f_1^{-1}(\mathrm{RegP}(X_1))$ or is contained in Σ_1'. Let $\mathcal{F}_1 := \mathcal{F}_{\eta'} \cap f_1^{-1}(\mathrm{RegP}(X_1))$, $U_1 := X_1' \backslash \mathcal{F}_1$ and $U_2 := f_1^{-1}(\mathrm{RegP}(X_1)) \subseteq \mathrm{RegP}(X_1')$. We have $X_1' = U_1 \cup U_2$. Let $U := U_1 \cap U_2$, so by definition of $\mathcal{F}_{\eta'}$, we have $\eta'^{-1}(U) \simeq U$. Therefore we can glue the open sets $V_1 := \eta'^{-1}(U_1)$ and U_2 along $\eta'^{-1}(U) \simeq U$ to a proper model Y/k of K such that $\mathrm{RegP}(Y) = Y$ by (i) above. As in step 5 of the proof of proposition 5.1, there is actually a projective morphism $\pi : Y \to X$. Finally, we have $\pi^{-1}(\mathrm{RegP}(X)) \simeq \mathrm{RegP}(X)$, since $Y \to X_1'$ is an isomorphism above U_2 and $\mathrm{RegP}(X) \simeq (g_1 \circ f_1)^{-1}(\mathrm{RegP}(X)) \subseteq U_2$ by (ii) above.

7. Open problems.

In this section, we summarize the main problem and open questions.

Problem 7.1. *Let P be a regularity property verifying axioms 1 to 6 and X/k be a normal projective model of K. Under which extra conditions does there exist a sequence of morphisms*

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n =: X',$$

where $X_{i-1} \leftarrow X_i$ is the normalized blowing up along a regular integral subvariety $Z_{i-1} \subseteq \mathrm{SingP}(X_{i-1})$ for $1 \leq i \leq n$, such that $\mathrm{RegP}(X') = X'$?

If $\mathrm{char}k = 0$ and $P = P_{\mathrm{reg}}$ is the usual regularity property, the problem has an affirmative answer for those properties P_μ, $\mu \geq 1$ of example 3.4, in particular for P itself (with $\mu = 1$). This is easily deduced from Hironaka's theorem [18] using the construction given in (6) along the proof of proposition 3.5. For as far as we know, the same problem is open if $\mathrm{char}k > 0$, or in the situation of corollary 4.9 (vector fields), or in that of theorem 4.10 (our weak version of Hironaka's conjecture).

Other open questions have to do with the redundance of our axioms 1 to 6. If K/k is a function field of dimension two, any regularity property P in K verifying axioms 1, 2 and 5 satisfies the conclusion of (the dimension two version of) theorem 2.5. In

this case, a variety X' as in *loc.cit* can be constructed by Zariski's algorithm: let X/k be a normal projective model of K; any sequence of morphisms

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \cdots,$$

where $X_{i-1} \leftarrow X_i$ is the normalized blowing up of a closed point $z_{i-1} \subseteq \mathrm{SingP}(X_{i-1})$ for $1 \le i \le n$, is finite. This fact is a simple consequence of Zariski's theorem p.688 [30] on resolution of surface singularities by successive normalized blowing ups along the singular locus and of Abhyankar's theorem 2 [2]. This suggests the following problem:

Problem 7.2. *Let K/k be a function field of dimension three. Construct a regularity property P in K verifying axioms 1, 2 and 5 and satisfying (i) below (resp. (ii); (iii)).*

  (i) *P does not verify axiom 3.*
  (ii) *P does not verify axiom 6.*
  (iii) *P verifies axiom 3, but does not verify axiom 4.*

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