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A DECAY ESTIMATE FOR A WAVE EQUATION WITH TRAPPING AND A COMPLEX POTENTIAL

LARS ANDERSSON, PIETER BLUE, AND JEAN-PHILIPPE NICOLAS

Abstract. In this brief note, we consider a wave equation that has both trapping and a complex potential. For this problem, we prove a uniform bound on the energy and a Morawetz (or integrated local energy decay) estimate. The equation is a model problem for certain scalar equations appearing in the Maxwell and linearised Einstein systems on the exterior of a rotating black hole.

1. Introduction

We consider the Cauchy problem

$$\left(-\partial_t^2 + \partial_x^2 + V(\Delta_\omega - N) + i\epsilon W\right) u = 0, \tag{1}$$

$$u(0,x,\omega) = \psi_0(x,\omega), \quad \partial_t u(0,x,\omega) = \psi_1(x,\omega), \tag{2}$$

on $(t,x,\omega) \in M = \mathbb{R}\times\mathbb{R}\times S^2$ with smooth, compactly supported initial data. Here u is a complex function $u = v + iw$,

$$V = \frac{1}{x^2+1},$$

W is a smooth, real-valued, compactly supported function which is nonvanishing at $x = 0$ and uniformly bounded by 1, and $\epsilon > 0$ is a small parameter. Finally, Δ_ω is the Laplacian in the angular variables and N is a number chosen to be sufficiently large to allow us to avoid certain technical problems.

The equation (1) has both trapping, which occurs at $x = 0$, and a complex potential, which does not vanish at the trapped set. The interaction of these creates problems, which appear to frustrate the use of energy and Morawetz estimates at the classical level. By adapting known, pseudodifferential methods, we show how to overcome these problems. We now state our main result in terms of the energy

$$E(t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}\times S^2} |\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)\, \mathrm{d}x\mathrm{d}^2\omega,$$

Theorem 1. *There is a constant C such that, if ψ_0 and ψ_1 are such that $E(0)$ is finite then*

$$\forall t \in \mathbb{R}: \qquad E(t) \le CE(0), \tag{3a}$$

$$\int_M \frac{|\partial_x u|^2}{x^2+1} + \frac{x^2}{1+x^2}\left(\frac{|\nabla_\omega u|^2}{1+|x|^3} + \frac{|\partial_t u|^2}{x^2+1}\right) + \frac{|u|^2}{1+|x|^3}\, \mathrm{d}t\mathrm{d}x\mathrm{d}^2\omega \le CE(0), \tag{3b}$$

$$\int_M \frac{|u||\partial_t u|}{1+|x|^3}\, \mathrm{d}t\mathrm{d}x\mathrm{d}^2\omega \le CE(0). \tag{3c}$$

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Since the equation (1) has t independent coefficients, one might naively think that Noether's theorem provides a positive conserved energy. However, for the Lagrangian $\mathcal{L}_1[u,\partial u] = -(\partial_t u)^2 + (\partial_x u)^2 + V(\nabla_\omega u \cdot \nabla_\omega u + Nu^2) - i\epsilon W u^2$, which has the wave equation (1) as its Euler-Lagrange equation, the conserved quantity associated to the time translation symmetry is indefinite, being approximately the energy of the real component of u minus the energy of the imaginary component (plus ϵ times a term involving Wvw). On the other hand, a Lagrangian of the form $\mathcal{L}_2[u,\partial u] = -|\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)$, which corresponds to the energy expression considered above, does not yield equation (1) as its Euler-Lagrange equation.

The wave equation (1) is a model for equations arising in the study of the Maxwell and linearised Einstein equations outside a Kerr black hole. The Kerr black holes are a family of Lorentzian manifolds arising in general relativity, and they are characterised by a mass parameter M and an angular momentum parameter a. Black holes are believed to be the enormously massive objects at the center of most galaxies. The case $|a| \leq M$ is the physically relevant. The case $a = 0$ is the Schwarzschild class of black holes.

It is expected that every uncharged black hole will asymptotically approach a Kerr solution under the dynamics generated by the Einstein equations of general relativity. The wave, Maxwell, and linearised Einstein equations on a fixed Kerr geometry are a sequence of increasingly accurate models for these dynamics. By projecting on a null tetrad, the Maxwell and linearised Einstein fields can be decomposed into sets of complex scalars, the Newman-Penrose (NP) scalars [22, 23, 14]. It is well-known that the NP scalars with extreme spin weights satisfy decoupled wave equations, known as the Teukolsky equations, and that the solutions to these reduced equations can be used to reconstruct the full system [29].

For the Maxwell field on the Kerr background, the spin weight 0 NP scalar can be treated in the same way, and the resulting equation is known as the Fackerell-Ipser equation [11]. For linearized gravity on the Schwarzschild background, it is also well known that the imaginary part of the spin weight 0 NP scalar is governed by a wave equation, the Regge-Wheeler equation [26, 24]. The corresponding equation for the real part is more complicated, cf. [33, 21], see also [1].

It was recently shown [1] that in the general ($|a| < M$) Kerr case, by imposing a gauge condition related to the wave coordinates gauge, the equation for both the real and imaginary parts of the spin weight 0 NP scalar of the linearized gravitational field may be put in a form analogous to the Regge-Wheeler and Fackerell-Ipser equations. Explicitely (in the Kerr spacetime with signature $-+++$, working in Boyer-Lindquist coordinates) these take the form

$$\left(\nabla^\alpha \nabla_\alpha + 2s^2 \frac{M}{(r - ia\cos\theta)^3}\right) u = 0, \tag{4}$$

where $s = 0$ corresponds to the free scalar wave equation, $s = 1$ corresponds to the Maxwell (Fackerell-Ipser) case, while $s = 2$ corresponds to the linearized gravity (generalized Regge-Wheeler) case. In particular, for the $a \neq 0$ cases, the analogues of the Regge-Wheeler equations have complex potentials, with the imaginary part depending continuously on a.

For the wave equation in the Schwarzschild case, the use of the energy estimate [31], like (3a) with $C = 1$, and Morawetz estimates[1] are well established [17, 3, 4, 6, 8]. In the Morawetz estimate (3b), there is a loss of control of time and angular derivatives near $x = 0$, in the sense that the integrand cannot control $|x|^p(|u|^{q_t}|\partial_t u|^{2-q_t} + |u|^{q_\omega}|\nabla_\omega u|^{2-q_\omega})$ with both $p = 0$ and either $q_t = 0$ or $q_\omega = 0$. The presence of trapping makes some loss unavoidable [25]. By applying "angular modulation" and "phase space analysis", the range for the angular parameter q_ω can be refined to $p = 0$ and $q > 0$ [5]. This type of refinement is crucial in the current paper. Alternatively, certain pseudodifferential operators have been used to obtain refinements near $x = 0$, to $p > 0$ and $q_t = q_\omega = 0$ [20].

For the wave equation in the general ($|a| < M$) Kerr case, it is possible to apply Fourier transforms first in the ϕ and t variables[2] and then the remaining variables. The individual ϕ modes decay pointwise [12]. Although the problem has a time-translation symmetry, because the generator of time translations fails to be a timelike vector with respect to the Lorentzian inner product of the Kerr geometry, there is no positive, conserved energy. A major advance was the proof that, in the slowly rotating case $|a| \ll M$, there is a uniform energy bound, like estimate (3a). The first proof used an estimate similar to (3b), but with additional restrictions on the support of the Fourier transform [7]. Independent work [28] established estimates similar to (3a) and (3b), but with no restriction on the Fourier support, and there were subsequent pseudodifferential refinements [30]. Also, the first two authors have proved similar results using methods which require two additional levels of regularity but which completely avoid the use of Fourier transforms. Morawetz estimates and refinements are a crucial step in proving pointwise decay estimates [5, 6, 8, 9, 18, 19] and Strichartz estimates [20, 30], including the long-conjectured, inverse-cubic, Price law [16, 27].

The study of the Maxwell and linearised Einstein systems in the Kerr geometry is still in its infancy. For the general Kerr case, a certain transformed, separated version of the Teukolsky system has no exponentially growing modes [32]. In the Schwarzschild case, the ϕ modes of the Teukolsky equation decay pointwise [13]. Recently, improved decay estimates for the Regge-Wheeler type equation (4) on the Schwarzschild background, giving decay rates of t^{-3}, t^{-4} and t^{-6}, respectively, for $s = 0, 1, 2$, have been proved [10]. The Regge-Wheeler equation has been used with the full system to prove energy and Morawetz (and pointwise decay) estimates for the Maxwell system [2] and, more recently, under assumptions on the asymptotic behaviour, for the full (not merely linearised) Einstein equation [15].

For the spin-weight 0 equations arising from the Maxwell and linearised Einstein equations with $a \neq 0$, the presence of complex potentials in the reduced equations (4) prevents the existence of a positive conserved energy and means that an unrefined Morawetz estimate, such as (3b), is insufficient to control the growth of the energy, which is why we also prove estimate (3c). In the Kerr geometry, there is no positive, conserved energy because the generator of the time-translation symmetry fails to be timelike everywhere. In contrast, for the equation (1), there is no positive, conserved energy because the complex potential prevents any variational approach from providing an energy-momentum tensor that satisfies the dominant

[1]These are also called integrated local energy decay estimates.

[2]Here ϕ is the azimuthal angle, which would be one component of ω in the notation of this paper.

energy condition. The method of this paper combines a Fourier-transform-in-time technique (as in [7, 28]) with a "modulation" (or Fourier-rescaling) technique (from [5]).

As is common, C will be used to denote a constant which may vary from line to line, but which is independent of the choice of u or T. The notation $A \lesssim B$ is used to denote that there is some C such that $A < CB$, with C independent of u and T, and similarly for $\gtrsim$.

2. A preliminary energy estimate

We derive an estimate for an energy for the wave equation (1) by integrating by parts against $\partial_t \bar{u}$ and following the standard procedure for getting an energy estimate:

$$\begin{aligned} 0 &= \mathrm{Re}\left((\partial_t \bar{u})(-\partial_t^2 + \partial_x^2 + V(\Delta_\omega - N) + i\epsilon W)u\right) \\ &= -\frac{1}{2}\partial_t |\partial_t u|^2 \\ &\quad + \partial_x \mathrm{Re}((\partial_t \bar{u})\partial_x u) - \frac{1}{2}\partial_t |\partial_x u|^2 \\ &\quad + \nabla_\omega \cdot \mathrm{Re}((\partial_t \bar{u})\nabla_\omega u) - \frac{1}{2}\partial_t \left(V(|\nabla_\omega u|^2 + N|u|^2)\right) \\ &\quad - \epsilon W \mathrm{Im}((\partial_t \bar{u})u). \end{aligned}$$

Introducing an energy which we denote by

$$E(t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}\times S^2} |\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)\ \mathrm{d}x\mathrm{d}^2\omega,$$

assuming that u decays sufficiently rapidly as $|x| \to \infty$, and integrating the previous formula over a region $[t_1, t_2] \times \mathbb{R} \times S^2$, we find

$$E(t_2) - E(t_1) = \int_{[t_1,t_2]\times\mathbb{R}\times S^2} -\epsilon W \mathrm{Im}((\partial_t \bar{u})u)\ \mathrm{d}t\mathrm{d}x\mathrm{d}^2\omega. \tag{5}$$

In particular, note that the energy fails to be conserved and that an estimate of the form (3b) would be insufficient to control the right-hand side. There is, however, a trivial exponential bound:

$$E(t_2) \leq e^{\epsilon(t_2 - t_1)} E(t_1).$$

3. The Morawetz estimate

Following the standard procedure for investigating the wave equation, we derive a Morawetz estimate by multiplying the wave equation by $(f(x)\partial_x \bar{u} + q(x)\bar{u})$, where f and q are real-valued functions.

In performing this calculation, it is useful to observe that

$$q'(x)\mathrm{Re}(\bar{u}\partial_x u) = \partial_x\left(\frac{q'}{2}\bar{u}u\right) - \frac{1}{2}q''\bar{u}u.$$

Using this and applying the product rule term-by-term, one finds

$$\begin{aligned}\mathrm{Re}\big(\,(f\partial_x\bar{u}+q\bar{u})\,\big(-\partial_t^2u+\partial_x^2u+V(\Delta_\omega-N)u+i\epsilon Wu\big)\,\big)&\\ =\partial_tp_t+\partial_xp_x+\nabla_\omega p_\omega&\\ +\left(-\frac{1}{2}f'+q\right)|\partial_tu|^2-\left(\frac{1}{2}f'+q\right)|\partial_xu|^2&\\ +\left(\left(\frac{1}{2}f'-q\right)V+\frac{1}{2}f(\partial_xV)\right)|\nabla_\omega u|^2&\\ +\left(N\left(\left(\frac{1}{2}f'-q\right)V+\frac{1}{2}f(\partial_xV)\right)+\frac{1}{2}q''\right)|u|^2&\\ -\epsilon fW\mathrm{Im}((\partial_x\bar{u})u),&\end{aligned}\tag{6}$$

where

$$\begin{aligned}p_t&=p_t(f,q;u)=-\mathrm{Re}((f(\partial_x\bar{u})+q\bar{u})(\partial_tu)),\\ p_x&=p_x(f,q,u)=\frac{1}{2}f|\partial_tu|^2+\frac{1}{2}f|\partial_xu|^2-\frac{1}{2}fV|\nabla_\omega u|^2\\ &\quad+q\mathrm{Re}(\bar{u}\partial_xu)-\frac{1}{2}(NfV+q')|u|^2,\\ p_\omega&=p_\omega(f,q;u)=fV\mathrm{Re}((\partial_x\bar{u})(\nabla_\omega u))+qV\mathrm{Re}(\bar{u}\nabla_\omega u).\end{aligned}$$

We take $f=-\arctan(x)$, for which $f'=-(x^2+1)^{-1}=-V$, $f''=2x(x^2+1)^{-2}$, and $f'''=-2(3x^2-1)(x^2+1)^{-3}$. We take $q=f'/2+\delta(1+x^2)^{-1}\arctan(x)^2$ for some sufficiently small δ.

We use the notation

$$E_{f\partial_x+q}(t)=\int_{\{t\}\times\mathbb{R}\times S^2}\mathrm{Re}(f(\partial_x\bar{u})\partial_tu)+\mathrm{Re}(q\bar{u}(\partial_tu))\;\mathrm{d}x\mathrm{d}^2\omega,$$

and observe that, by a simple Cauchy-Schwarz argument, there is the estimate $|E_{f\partial_x+q}|\leq CE$.

Observing that the left-hand side of (6) vanishes, we have

$$\begin{aligned}0&=\partial_tp_t+\partial_xp_x+\nabla_\omega p_\omega\\ &\quad+\delta\frac{\arctan(x)^2}{1+x^2}|\partial_tu|^2+\frac{1}{1+x^2}(1-\delta\arctan(x)^2)|\partial_xu|^2\\ &\quad+\left(\frac{x\arctan(x)-\delta\arctan(x)^2}{(1+x^2)^2}\right)|\nabla_\omega u|^2\\ &\quad+\left(N\left(\frac{x\arctan(x)-\delta\arctan(x)^2}{(1+x^2)^2}\right)+\frac{1}{2}q''\right)|u|^2\\ &\quad-\epsilon fW\mathrm{Im}((\partial_x\bar{u})u)\end{aligned}$$

Taking ϵ sufficiently small, N sufficiently large, and δ sufficiently small, the factors in front of $|\partial_xu|^2$ and $|u|^2$ are nonnegative and one can dominate the term involving W using these two terms. (These estimates are uniform, in the sense that, if the estimate holds for choices of ϵ_0, N_0, and δ_0, then it remains valid for $\epsilon<\epsilon_0$, $N=N_0$, and $\delta=\delta_0$.) Thus, by integrating over a time-space slab $M_{[t_1,t_2]}=[t_1,t_2]\times\mathbb{R}\times S^2$,

one can conclude that there is a constant C such that

$$E(t_2)+E(t_1) \gtrsim \int_{M_{[t_1,t_2]}} \frac{|\partial_x u|^2}{x^2+1} + |\arctan(x)|^2 \left(\frac{|\nabla_\omega u|^2}{1+|x|^3} + \frac{|\partial_t u|^2}{x^2+1}\right) + \frac{|u|^2}{1+|x|^3}\, \mathrm{d}t\mathrm{d}x\mathrm{d}^2\omega. \quad (7)$$

4. Pseudodifferential refinements

4.1. **The wave equation for an approximate solution.** We define a smooth characteristic function of an interval $[a,b]$ to be a function which is identically 1 on $[a,b]$, which is supported on $[a-1,b+1]$, and which is monotonic on each of the intervals $[a-1,a]$ and $[b,b+1]$. A smooth characteristic function of a collection of intervals, each of which are separated by distance at least two, is defined to be the sum of the smooth characteristic functions of each interval.

Let $T>0$ be a large constant. (Here large means larger than $-\log|\epsilon|$ and 2.) Let χ_1 be a smooth characteristic function on $[0,T]$, and let χ_2 be a smooth characteristic function of $[-1,0]\cup[T,T+1]$. Let $\chi_{|x|\le 2}$ be a smooth characteristic function of $[-1,1]$. We will use χ_1, χ_2, and $\chi_{|x|\le 2}$ to denote $\chi_1(t)$, $\chi_2(t)$, and $\chi_{|x|\le 2}(x)$ respectively.

Since χ_1 is smooth, there is a uniform bound on its derivative and second derivative, each of which are supported on $[0,1]\cup[T,T+1]$, so that there is a constant C such that $|\partial_t\chi_1|+|\partial_t^2\chi_1|\le C\chi_2$.

The functions

$$\begin{aligned} u_1 &= \chi_1\chi_{|x|\le 2}u,\\ u_2 &= \chi_2\chi_{|x|\le 2}u,\\ u_3 &= \chi_1 u, \end{aligned}$$

satisfies the equation

$$\left(-\partial_t^2+\partial_x^2+V(\Delta_\omega-N)+i\epsilon W\right)u_1 = F(u_2,\nabla u_2,t,x)+G(u_3,\nabla u_3,t,x), \quad (8)$$

where

$$\begin{aligned} F(u_2,\nabla u_2,t,x) &= -2(\partial_t\chi_1)(\partial_t u_2)-(\partial_t^2\chi_1)u_2\\ G(u_3,\nabla u_3,t,x) &= 2(\partial_x\chi_{|x|\le 2})(\partial_x u_3)+(\partial_x^2\chi_{|x|\le 2})u_3. \end{aligned}$$

Since all functions of t in this equation are smooth and supported in $t\in[-2,T+2]$, they are Schwartz class in t, so we may take the Fourier transform in t and remain in the Schwartz class. We will use $\widehat{\ }$ to denote the Fourier transform in t, and τ for the argument of such functions. We will typically use the word "functions" to describe u, u_1, u_2, and u_3 and the words "Fourier transforms" to describe their Fourier transforms. We will use L^2 to denote $L^2(\mathrm{d}\omega\mathrm{d}x\mathrm{d}t)$ for functions and to denote $L^2(\mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau)$ for Fourier transforms. We will use $\|\cdot\|$ for $\|\cdot\|_{L^2}$ unless otherwise specified.

We introduce the following space-time integrals

$$\begin{aligned} I(T) &= \int_{-2}^{T+2}\int_{-2}^{2}\int_{S^2} x^2|\partial_t u|^2+|\partial_x u|^2+|u|^2\mathrm{d}\omega\mathrm{d}x\mathrm{d}t,\\ J(T) &= \int_{\mathbb{R}\times\mathbb{R}\times S^2} |\tau|^{6/5}|\widehat{u_1}|^2\mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

The dependence of J upon T is through the smooth cut-off χ_1 in u_1. Typically, the argument T will be clear from context and will be omitted. From the Morawetz estimate (7) and the exponential bound on the energy, it follows that $I \lesssim E(T) + E(0)$.

We now aim to prove a Morawetz estimate using the Fourier transform. We take

$$f = -\arctan(|\tau|^\alpha x),$$
$$q = \frac{f'}{2} = \frac{1}{2}\frac{|\tau|^\alpha}{1+|\tau|^{2\alpha}x^2},$$

with $\alpha \in [0, 1/2]$. We multiply the Fourier transform of equation (8) by $(f\partial_x + q)\overline{\widehat{u_1}}$, and integrate the real part over $\mathbb{R}\times\mathbb{R}\times S^2$. This integral is convergent because all the functions are compactly supported in time, so the Fourier transforms are Schwartz class.

4.2. **Controlling the terms arising from the cut-off.** We consider first the integral arising from the right-hand side of (8). This is

$$\int_{\mathbb{R}\times\mathbb{R}\times S^2} \mathrm{Re}\left(\left((f\partial_x + q)\overline{\widehat{u_1}}\right)\left(\widehat{F} + \widehat{G}\right)\right) \mathrm{d}\omega \mathrm{d}x \mathrm{d}\tau \leq \|(f\partial_x + q)\overline{\widehat{u_1}}\|\|\widehat{F} + \widehat{G}\|.$$

The terms on the right can be estimated by

$$\|(f\partial_x + q)\overline{\widehat{u_1}}\| \leq \|f\partial_x\widehat{u_1}\| + \|q\widehat{u_1}\|,$$
$$\|f\partial_x\widehat{u_1}\| \lesssim \|\partial_x\widehat{u_1}\| \lesssim I^{1/2},$$
$$\|q\widehat{u_1}\| \lesssim \||\tau|^\alpha\widehat{u_1}\| \lesssim \|\widehat{u_1}\| + \||\tau|^{1/2}\widehat{u_1}\| \lesssim I^{1/2} + J^{1/2},$$

and

$$\|\widehat{F} + \widehat{G}\| \leq \|\widehat{F}\| + \|\widehat{G}\|.$$

Because G is supported only for $t \in [-1, T+1]$ and $x \in [-2, 2]$, we have

$$\|\widehat{G}\| \lesssim I^{1/2}.$$

Similarly, because F is supported only for $t \in [-1, 0] \cup [T, T+1]$ and $x \in [-2, 2]$, we have that at each instant in t, the function F is bounded in $L^2(\mathrm{d}x\mathrm{d}\omega)$ by either $CE(0)^{1/2}$ or $CE(T)^{1/2}$. Since we are considering two intervals in t of length 1, we have

$$\|\widehat{F}\| = \|F\| \lesssim C(E(0)^{1/2} + E(T)^{1/2}).$$

Thus, the terms on the right-hand side of the Fourier transform of (8) are bounded by

$$\int_{\mathbb{R}\times\mathbb{R}\times S^2} \mathrm{Re}\left((f\partial_x + q)\overline{\widehat{u_1}}\right)\left(\widehat{F} + \widehat{G}\right) \mathrm{d}\omega \mathrm{d}x \mathrm{d}\tau$$
$$\leq C(E(0)^{1/2} + E(T)^{1/2} + J^{1/2})(E(0)^{1/2} + E(T)^{1/2}). \tag{9}$$

4.3. **The Morawetz estimate for the approximate solution.** If we multiply the left-hand side of the Fourier transform of the wave equation (8) by $(f\partial_x + q)\widehat{u_1}$ and take the real part, then we have the analogue of (6)

$$\begin{aligned}\mathrm{Re}\big(\,(f\partial_x\bar{\widehat{u_1}} + q\bar{\widehat{u_1}})\,(\tau^2\widehat{u_1} + \partial_x\widehat{u_1} + V(\Delta_\omega - N)\widehat{u_1} + i\epsilon W\widehat{u_1})\,\big)\\ &= \partial_x p_x + \nabla_\omega p_\omega\\ &\quad + \left(-\frac{1}{2}f' + q\right)|\tau\widehat{u_1}|^2 - \left(\frac{1}{2}f' + q\right)|\partial_x\widehat{u_1}|^2\\ &\quad + \left(\left(\frac{1}{2}f' - q\right)V + \frac{1}{2}f(\partial_x V)\right)|\nabla_\omega\widehat{u_1}|^2\\ &\quad + \left(N\left(\left(\frac{1}{2}f' - q\right)V + \frac{1}{2}f(\partial_x V)\right) + \frac{1}{2}q''\right)|\widehat{u_1}|^2\\ &\quad - \epsilon f W\mathrm{Im}((\partial_x\bar{\widehat{u_1}})\widehat{u_1}),\end{aligned} \tag{10}$$

where

$$\begin{aligned}p_x = p_x(f,q,\widehat{u_1}) &= \frac{1}{2}f|\partial_t\widehat{u_1}|^2 + \frac{1}{2}f|\partial_x\widehat{u_1}|^2 - \frac{1}{2}fV|\nabla_\omega\widehat{u_1}|^2\\ &\quad + q\mathrm{Re}(\bar{\widehat{u_1}}\partial_x\widehat{u_1}) - \frac{1}{2}(NfV + q')|\widehat{u_1}|^2,\\ p_\omega = p_\omega(f,q;\widehat{u_1}) &= fV\mathrm{Re}((\partial_x\bar{\widehat{u_1}})(\nabla_\omega\widehat{u_1})) + qV\mathrm{Re}(\bar{\widehat{u_1}}\nabla_\omega\widehat{u_1}).\end{aligned}$$

Note that there is no p_t term because, for Fourier transforms, the analogue of the product rule is simply $-\overline{i\tau\widehat{u_1}}\widehat{u_1} = \bar{\widehat{u_1}}i\tau\widehat{u_1}$.

When this equality is integrated over a space-time slab, the p_x and p_ω terms integrate to zero, and the remaining terms are all non-negative except for those arising from q'' and from W. The integral of the term involving W is bounded by I.

We now consider the term involving q'':

$$\frac{1}{2}q''|\widehat{u_1}|^2 = |\tau|^{3\alpha}\frac{1 - 3|\tau|^{2\alpha}x^2}{(1 + |\tau|^{2\alpha}x^2)^3}|\widehat{u_1}|^2.$$

From the positivity of the remaining terms and the bound (9) of the terms coming from the right-hand side of the wave equation (8) for u_1, we have that

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}\times S^2}&|\tau|^{3\alpha}\frac{1 - 3|\tau|^{2\alpha}x^2}{(1 + |\tau|^{2\alpha}x^2)^3}|\widehat{u_1}|^2\mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau\\ &\le C(E(0)^{1/2} + E(T)^{1/2} + J^{1/2})(E(0)^{1/2} + E(T)^{1/2}).\end{aligned}$$

This can be combined with an additional factor of MI, where M is a large constant. (700 is sufficient.) The integral I dominates the integral of $(|\tau|^2 + 1)x^2|\widehat{u_1}|^2$ and is bounded by $C(E(T) + E(0))$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}\times S^2}&\left(|\tau|^{3\alpha}\frac{1 - 3|\tau|^{2\alpha}x^2}{(1 + |\tau|^{2\alpha}x^2)^3} + M(|\tau|^2 + 1)x^2\right)|\widehat{u_1}|^2\mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau\\ &\le C(E(0)^{1/2} + E(T)^{1/2} + J^{1/2})(E(0)^{1/2} + E(T)^{1/2}).\end{aligned}$$

By considering the two cases $|\tau|^\alpha|x| < M^{-1/2}$ and $|\tau|^\alpha|x| \geq M^{-1/2}$, one can see that if $2-2\alpha = 3\alpha$ (i.e. $\alpha = 2/5$), then

$$\left(|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3} + M(|\tau|^2+1)x^2\right)\chi_{|x|\leq 2} \geq C|\tau|^{3\alpha}\chi_{|x|\leq 2}$$

and therefore we find

$$\begin{aligned} &C(E(0)^{1/2}+E(T)^{1/2}+J^{1/2})(E(0)^{1/2}+E(T)^{1/2}) \\ &\qquad \geq \int_{\mathbb{R}\times\mathbb{R}\times S^2}\left(|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3} + M(|\tau|^2+1)x^2\right)|\widehat{u_1}|^2 \mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau \geq J, \end{aligned}$$

which implies

$$J \leq C(E(T)+E(0)). \tag{11}$$

4.4. **Closing the energy estimate.** It is now possible to estimate the integral on the right-hand side of the energy estimate (5). For $|x| \geq 1$, the right-hand side (using the compact support of W and the Cauchy-Schwarz estimate) can be dominated by $I \leq C(E(T)+E(0))$. For $|x| \leq 1$, we would like to dominate the integral over $\mathbb{R}\times\mathbb{R}\times S^2$ of $|W\mathrm{Im}(\bar{u_1}\partial_t u_1)|$ by the integral J. However, this is not entirely correct, because in J there is a contribution arising from the support of u in the region $t \in [-1,0]\cup[T,T+1]$. The error in this approximation is bounded by $C(E(T)+E(0))$. Thus, we have

$$E(T)-E(0) \leq C\epsilon\left(E(T)+E(0)+\left|\int_{\mathbb{R}\times\mathbb{R}\times S^2} W\mathrm{Im}(\bar{u_1}\partial_t u_1)\mathrm{d}\omega\mathrm{d}x\mathrm{d}t\right|\right).$$

We can also take the Fourier transform to obtain an estimate by

$$\begin{aligned} &C\epsilon\left(E(T)+E(0)+\left|\int_{\mathbb{R}\times\mathbb{R}\times S^2} W\,\mathrm{Im}(\bar{\widehat{u_1}}\widehat{\partial_t u_1})\mathrm{d}\omega\mathrm{d}x\mathrm{d}t\right|\right) \\ &\qquad \leq C\epsilon\left(E(T)+E(0)+\int_{\mathbb{R}\times\mathbb{R}\times S^2}|\tau||\widehat{u_1}|^2\mathrm{d}\omega\mathrm{d}x\mathrm{d}t\right). \end{aligned}$$

The integrand is now controlled by $I+J$, which, from estimates (7) and (11), we know can be estimated also by the sum of the initial and final energies. This leaves the estimate

$$E(T)-E(0) \leq C\epsilon(E(T)+E(0)).$$

By taking ϵ sufficiently small relative to the constant, we obtain a uniform bound on the energy

$$E(T) \leq CE(0).$$

We note that, since all the constants were independent of T, the estimate holds uniformly in T. This proves the first statement (3a), in theorem 1. Combining this with estimate (7) (and estimating $x^2(1+x^2) \lesssim \arctan(x)^2$) gives the second, (3b). Finally, the arguments of this section and the bound on $I+J$, from estimates (7) and (11), give the third result (3c).

Remark 2. *Using the method given in* [5]*, the stronger Morawetz estimate* (11) *can be improved to control the integral of* $|\tau|^{2-\varepsilon}|\widehat{u}|^2$ *for any* $\varepsilon > 0$*. Because of the presence of trapping, it is not possible to improve this to* $|\tau|^2|\widehat{u}|^2$*.*

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E-mail address: laan@aei.mpg.de

Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany

E-mail address: P.Blue@ed.ac.uk

The School of Mathematics and the Maxwell Institute, University of Edinburgh,James Clerk Maxwell Building, The King's Buildings, Mayfield Road, Edinburgh, Scotland EH9 3JZ, UK

E-mail address: Jean-Philippe.Nicolas@univ-brest.fr

Laboratoire de Mathématiques, Université de Brest, 6 avenue Victor Le Gorgeu, 29200 Brest, France