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# Adword auction bidding strategies of budget-limited advertisers on competing search engines

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**Abstract**—The economic stakes of advertising on the Internet - and in particular, of auctions for keywords in search engines- are enormous and still increasing. We focus in this paper on situations where bidders (advertisers) on sponsored search auction systems have a limited budget, so that they may not be able to afford to participate in all auctions.

Using a game-theoretical model of the strategic interactions among advertisers, we analyze the equilibrium strategies in terms of bidding frequencies, in the case of one monopoly search engine and when two search engines are in competition. Our results exhibit the importance for search engines to develop their attractiveness to customers, due to the impact this has on auction revenues.

**Index Terms**—Sponsored auctions, Budget limit, Game Theory

## I. INTRODUCTION

While the Internet continues to soar in terms of available bandwidth and number of users, the economics of online services has evolved so that more and more services are proposed for free, the providers basing their business models on advertising. The most famous example is Google, that offers a multiplicity of free services (e-mail, data storage, website hosting, ...) and yet raised US\$10 billion in 2006 in advertising revenues [1].

For search engines, a large part of those revenues comes from the so-called *sponsored search auctions*, or *adword auctions*, that are auctions run among advertisers to have their ad displayed next to -or above- the normal (called organic) results of the user search for some given search words. Indeed, advertisers expect that ads related to the searched words are likely to interest the user and lead to a sale; they are therefore willing to pay to have their ad appear in a good position on the screen, and/or get clicked, when users search for particular keywords. Since the economic stakes of adword auctions are enormous, it can be expected that advertisers rationalize their strategies, possibly taking into account the competitors decisions. Therefore, game-theoretic models and tools [2] seem (highly) appropriate to analyze the strategic decisions of interacting advertisers, and several models have been developed for that context (see [3] and references therein).

Nevertheless, the case when advertisers have a limited budget to devote to advertising is often difficult to analyze,

and only few papers consider that constraint. We focus on that issue in this paper, and on its effect on the bidding strategies of advertisers. In particular, we investigate how the budget limit affects advertisers depending on their willingness-to-pay to have their ad displayed. We consider that advertisers know only the distribution of the willingness-to-pay of their competitors (but not the exact values), and that they use that knowledge to anticipate the bidding behavior of the other advertisers as well as to determine their own bidding strategy. Both the case of one search engine and several competing search engines are considered. In that latter case, search engines are actually in competition since each advertiser decides how to split his budget between them.

The remainder of this paper is organized as follows. Section II presents our general model in terms of auction rules, advertiser valuations, and knowledge. The equilibrium bidding strategies are analyzed for the case of a single search engine in Section III, while the case of two competing search engines is treated in Section IV. Section V draws our conclusions and suggests directions for future work.

## II. THE MODEL

Table I provides an overview of the notation used in this paper. Note that the last four variables in the table will be indexed by the search engine when we consider several competing search engines.

$\lambda$	Arrival rate of user search requests
$K$	Total number of advertisers
$B_i$	Budget of advertiser $i$
$f$	Probability density of valuation repartition among advertisers
$q$	Click-Through Rate of the search engine
$p(v)$	Probability of participating in an auction when one's valuation is $v$
$H(v)$	Probability of winning an auction with bid $v$
$E(v)$	Expected price paid per auction with bid $v$

Table I  
MAIN VARIABLES AND NOTATIONS, FOR THE CASE OF ONE SEARCH ENGINE.

### A. Auction model

In the first part of this paper, we consider only one search engine. We also consider searches for a single keyword along

the paper, user requests on the search engine occurring with an average frequency of  $\lambda$  requests per time unit.

To simplify the analysis, we assume as in [4], [5] that only one slot is available to display ads in the search engine interface. The auction mechanism that we consider the search engine implements is called Generalized Second Price: advertisers submit bids, which are ordered according to a ranking rule, and the ad of the highest ranked bidder is displayed in the ad slot of the search engine. The price that the winner then has to pay is charged only when the ad is clicked on by the user who performed the search, and equals the minimum bid that the winning advertiser could have set to keep his position in the ranking.

Remark that the ranking can be based on the bid value only, or can involve the probability of the ad being clicked on, called its click-through rate (CTR). In the model presented in this paper, we assume that the ads of all advertisers have the same click-through rate  $q$ , so that bid-based and revenue-based rankings are equivalent: the advertiser with the highest bid wins the slots, and pays the value of the second-highest bid when the ad is clicked on.

When only one slot is available to display ads, the Generalized Second Price mechanism is equivalent to the well-known Vickrey-Clarke-Groves mechanism [6], [7], [8], that has good properties in terms of incentives: that scheme is *incentive compatible* (i.e., truthfully bidding one's value for the good -here, the ad slot- is a dominant strategy) and *individually rational* (i.e., when bidding truthfully the price paid is always below the value for the good). As a result, it is expected that advertisers always participate in the auctions and declare the value that the slot has to them. Nevertheless, in our context auctions occur repeatedly over time, and advertiser classically have a limited advertising budget so that participating to all auctions is not always feasible.

### B. Bidders (advertisers) model

We consider a set  $\mathcal{K}$  of  $K$  advertisers interested in a given keyword. Each advertiser  $i$  knows the value of his ad being clicked on by a customer: this is classically estimated as the probability of the user ending up buying the product proposed by the advertiser, multiplied by the average benefit on that sale. That value is denoted by  $v_i$  for each advertiser  $i \in \mathcal{K}$ . We assume that it is a private knowledge of each advertiser. However, we consider that advertisers have some knowledge regarding their opponents, as formalized below.

*Assumption A:* All advertisers know the total number  $K$  of advertisers interested in the considered keyword. Moreover, all advertisers assume that the valuations of their competitors are drawn from a probability distribution that is common knowledge, with cumulative distribution function  $F$  and density  $f$ , whose support is denoted by  $S_F$ .

In other words, each advertiser knows how many competitors he is facing in the auction game, and can use the distribution of valuations to determine his bidding strategies, possibly by inferring the strategies of the other advertisers.

Due to the incentive compatibility property of the auction scheme, we assume that each advertiser  $i \in \mathcal{K}$  bids truthfully, i.e., declares his valuation  $v_i$ , when he decides to participate in an auction. The bidding strategy of an advertiser then consists in decisions regarding whether to participate or not in any auction when the keyword is searched for by a user. The objective of each advertiser will be to participate in the maximum number of auctions (because of the *individual rationality* property) while not spending more on average than his budget limit  $B_i$  per time unit.

Remark that the more advertisers participate in a given auction, the fewer chances they have of winning the auction and the more likely the winner is to pay a high charge. Therefore advertisers should try to participate in different auctions when it is possible, and thus a strategy consisting in participating to all auctions until the budget is spent is clearly not optimal. Since advertisers are not supposed to be able to coordinate their actions so as to minimize the price paid, the most natural way for them is to randomize their participation: at each auction, each provider  $i \in \mathcal{K}$  chooses to participate with probability  $p_i$ , and stays out of the auction with probability  $1 - p_i$ . The strategic variable of each provider  $i \in \mathcal{K}$  is then his bidding probability  $p_i$ , that has to be computed so as to satisfy the budget constraint. That bidding probability should depend on the value  $v_i$  of a clicked ad, and on the budget  $B_i$  of the advertiser. However, to simplify the analysis we do not consider that latter parameter, by considering advertisers with equal budgets.

*Assumption B:* All advertisers have the same budget limit  $B$ . Formally,  $\forall i \in \mathcal{K}, B_i = B$ .

Even if this assumption restricts the model, it still allows to highlight the impact of other parameters in a strategic game among advertisers, such as for example the click-through-rate, the purpose of this paper.

## III. EQUILIBRIUM STRATEGIES OF THE AUCTION GAME

### A. Bidding function

In our model, advertisers differ only by their valuation  $v_i$ , so that the bidding probability should be a function of  $v_i$ . We denote by  $p$  that function:  $\forall i \in \mathcal{K}, p_i = p(v_i)$ .

The analysis of the game played among advertisers will then consist in determining that bidding function  $p$ , so that bidding with probability  $p(v)$  is the optimal strategy for an advertiser with valuation  $v$ . In other words, we are looking for a function  $p$  that constitutes a Nash equilibrium of the bidding game: for each advertiser  $i \in \mathcal{K}$ , bidding with probability  $p(v_i)$  is the best strategy when all competitors follow the same policy.

In the rest of this section, we derive conditions for  $p$  to be an equilibrium bidding function, and we prove its existence.

### B. Expected distribution of competitor bids

We consider an advertiser  $i \in \mathcal{K}$ , with valuation  $v_i$ . From his point of view (i.e., given the information he has), all  $K - 1$  competitors are identical in distribution: each one has a valuation  $v$  taken from the distribution  $F$  and participates in each auction independently with probability  $p(v)$ .

So advertiser  $i$  could reason by conditioning for a given auction: a given competitor will *a priori*

- not participate in the auction, with probability

$$N := \int_v (1 - p(v))f(v)dv = 1 - \int_v p(v)f(v)dv \quad (1)$$

- participate in the auction with a bid in the interval  $[v, v + dv]$  with probability  $p(v)f(v)dv$ .

Each advertiser can then compute the *a priori* cumulative distribution  $H$  of the bid of one particular competitor,  $H(v)$  being the probability that the competitor does not bid, or bids a value below  $v$ :

$$H(v) = N + \int_{u=0}^v p(u)f(u)du = 1 - \int_{u>v} p(u)f(u)du \quad (2)$$

### C. Expected winning probability and price paid per auction

An advertiser  $i \in \mathcal{K}$  bidding  $v_i$  in a given auction will win the auction if all of his  $K - 1$  competitors:

- either do not bid in that auction,
- or (non-exclusively) have a valuation below  $v_i$ .

Given the information available to advertiser  $i$ , each competitor submits a bid below  $v_i$  or no bid at all with probability  $H(v_i)$ , independently of each other. The *a priori* probability  $P_{\text{win}}(v_i)$  of  $i$  winning an auction where he bids  $v_i$  is thus

$$P_{\text{win}}(v_i) = H(v_i)^{K-1} = \left(1 - \int_{v>v_i} p(v)f(v)dv\right)^{K-1}. \quad (3)$$

We now compute the price that advertiser  $i$  expects to pay when bidding  $v_i$ . To do so, we compute the distribution of the maximum value among the  $K - 1$  potential bids from the competitors. That cumulative distribution, giving the probability that this maximum value is below some value  $v$  (or that no other advertiser bids in that auction) is simply

$$H_{\text{max}}(v) := (H(v))^{K-1}. \quad (4)$$

Indeed, the maximum of the potential bids of competitors is smaller than  $v$  if and only if all bidders place a bid below  $v$  or do not participate.

Finally, an advertiser is charged only if he wins the auction and the user clicks on the ad. Therefore the price  $E(v_i)$  that advertiser  $i$  expects to pay when bidding  $v_i$  in an auction can be expressed as

$$E(v_i) = q \int_{v=0}^{v_i} v dH_{\text{max}}(v) \quad (5)$$

$$= q \int_{v=0}^{v_i} (K-1)(H(v))^{K-2} v h(v) dv, \quad (6)$$

where  $h(v)$  is the right derivative of  $H$  at  $v$ , and equals  $p(v)f(v)$  from (2).

Note that  $E(v_i)$  can be written only in terms of the number  $K$  of advertisers, the bidding function  $p$ , and the density  $f$  of the valuation distribution:

$$E(v_i) = q(K-1) \int_{v=0}^{v_i} \left(1 - \int_{u>v} p(u)f(u)du\right)^{K-2} v p(v)f(v)dv. \quad (7)$$

### D. Equilibrium condition

We now express the condition for  $p$  to be an equilibrium bidding function, i.e., for each advertiser  $i \in \mathcal{K}$  with valuation  $v_i$  to choose to participate in each auction with probability  $p(v_i)$  if he knows that his competitors apply the same strategy.

So consider such an advertiser  $i$ , that assumes that all the other providers  $j \neq i$  participate in each given auction with a probability  $p_j$  linked to their valuation  $v_j$  through the formula  $p_j = p(v_j)$ . Recall that there are on average  $\lambda$  auctions per time unit, and that when advertiser  $i$  bids his valuation  $v_i$  on a given auction, he expects to pay  $E(v_i)$  on average as expressed in (7); consequently his average budget spent per time unit is  $\lambda p_i E(v)$  if advertiser  $i$  bidding probability is  $p_i$ .

Now, if advertiser  $i$  wants to bid as often as possible while conforming to his budget constraint  $B$ , his bidding probability should be

$$p_i = \min\left(\frac{B}{\lambda E(v_i)}, 1\right), \quad (8)$$

which should also equal  $p(v_i)$  if  $p$  is an equilibrium bidding function. Formally, the equilibrium condition is thus expressed by the following (functional) fixed point relation:

$$\forall v, p(v) = \min\left(1, \frac{B/(\lambda q(K-1))}{\int_{w=0}^v (1 - \int_{u>w} p(u)f(u)du)^{K-2} w p(w)f(w)dw}\right) \quad (9)$$

We now prove that, under mild assumptions, such a bidding function exists.

*Proposition 1:* Assume that the valuation distribution  $F$  admits a density  $f$  and has a finite mass, i.e.  $\int f(v)dv < +\infty$ . Then there exists an equilibrium bidding strategy function  $p(\cdot)$ .

*Proof:* Let us denote by  $X$  the topological vector space of functions from  $S_F$  to  $\mathbb{R}$ , with the topology of pointwise convergence (a sequence  $(g_n)$  in  $X$  converges towards  $g \in X$  if and only if  $(g_n(x))$  converges towards  $g(x)$  for all  $x \in S_F$ ). Let  $Y \subset X$  be the set of functions in  $X$  with values in the interval  $[0, 1]$ . That set is convex, and also compact from the Tychonoff theorem.

We consider here the mapping  $T : Y \mapsto Y$  such that for a function  $g \in Y$ ,  $T(g)$  is the function  $T_g \in Y$  defined by

$$T_g(v) = \min\left(1, \frac{B/(\lambda q(K-1))}{\int_{w=0}^v (1 - \int_{u>w} g(u)f(u)du)^{K-2} w g(w)f(w)dw}\right) \quad (10)$$

for all  $v \in S_F$ .

The function  $T_g$  can be interpreted as the bidding strategy that an advertiser would implement if he considers that his  $K - 1$  opponents will apply the bidding strategy  $g$ .

Under the assumptions of the proposition, the mapping  $T$  is continuous: consider a sequence  $(g_n)$  in  $X$  that converges

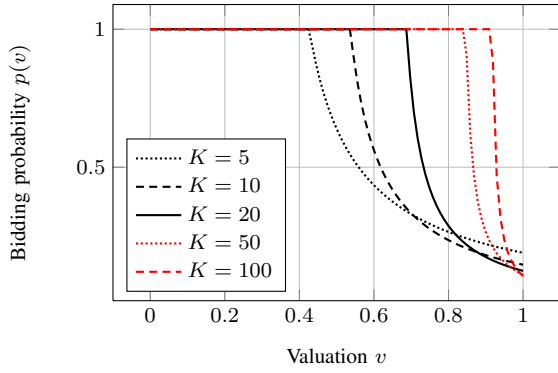


Figure 1. Equilibrium bidding strategies.

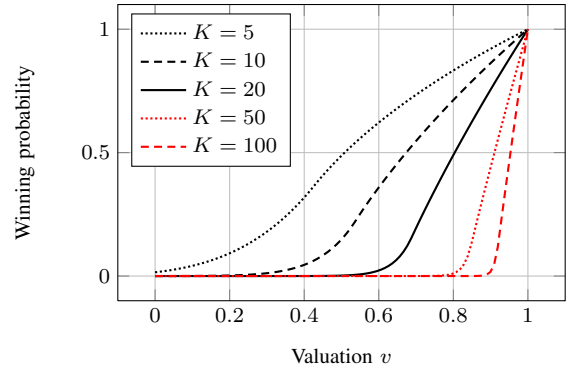


Figure 2. Winning probabilities at equilibrium (knowing that the player bids).

towards  $g \in X$ , then for any  $v \in S_F$  the value  $T_{g_n}(v)$  converges towards  $T_g(v)$ . Indeed,

- for any  $w \in S_F$ , the term  $(1 - \int_{u>w} g_n(u)f(u)du)^{K-2} g_n(w)$  converges towards  $(1 - \int_{u>w} g(u)f(u)du)^{K-2} g(w)$  by dominated convergence;
- those terms being smaller than 1, again by dominated convergence (using here the assumption of  $F$  having a finite mass) the integral in the denominator in (10) for the functions  $(g_n)$  also converges towards the corresponding integral for the limit function  $g$ ;
- finally we apply the continuity of the inverse function and of the function  $x \mapsto \min(1, x)$ .

$T$  being a continuous mapping and the set  $Y$  being convex and compact, the Schauder fixed point theorem [9] ensures the existence of a fixed point of  $T$ . That fixed point is a solution of the equation (9), i.e., an equilibrium bidding strategy function. ■

### E. Numerical analysis

We present here some numerical results that have been obtained for a uniform distribution of valuations over the interval  $[0, 1]$ , i.e.,  $f(v) = 1$  if  $v \in [0, 1]$  and 0 otherwise. Starting with an initial bidding strategy function, we successively computed the corresponding best-reply functions (the functions  $p(v)$  that maximize the utilities), applying the right-hand side of (9), until the bidding function stabilizes. It appeared that this algorithm converged quite rapidly, the pointwise difference between the current function and its best-reply being less than  $10^{-6}$  after less than 25 iterations.

For the same set of parameters, we tried with different initial bidding functions and always reached the same convergence function. Although we were not able to prove the uniqueness of the equilibrium bidding strategy function, those results suggest that the equilibrium is unique, or at least that there is a unique stable equilibrium.

Figure 1 displays that equilibrium bidding function when advertisers have a budget limit  $B = 1$ ,  $\lambda = 10$  user searches for the considered keyword occur on average per time unit, with users clicking on the ad with probability  $q = 0.5$ .

We remark that low-valuation advertisers bid more often than high-valuation advertisers, since the latter are more likely to win the auction and to spend their whole budget within few auctions. For valuations below a given threshold, advertisers simply participate in all auctions since they do not manage to spend their whole budget. This for two reasons: first, since GSP auctions ensure that the price paid is below the valuation, then advertisers with valuation  $v$  will expect to pay less than  $\lambda q v$  per time unit; thus if  $v < B/(\lambda q)$  then on average it is impossible to spend the budget  $B$ . Second, as highlighted in Figure 2 the probabilities of winning the auction are extremely small when submitting a low bid.

Interestingly, we note in Figure 1 that equilibrium bidding functions are not monotone in the number  $K$  of players (here, advertisers): for example advertisers with valuation in the interval  $[0.43, 0.7]$  bid more often when  $K = 10$  than when  $K = 5$ , while it is the opposite for advertisers with valuation above 0.7. This can be interpreted as follows: when the number of bidders is small, then high-valuation advertisers are almost sure to win the auction they participate in, but they face few other bids, whose maximum (the corresponding charge for the winner) is more likely to be small. Therefore to spend the whole budget those advertisers may have to bid more often. On the contrary, for medium-valuation advertisers the difference in the probabilities of winning the auction (the more competitors, the fewer chances of winning) overcomes the effect of the number of competitors on the price paid per auction won (which diminishes when the winner has a low valuation).

The distribution  $H(v)$  expressed in (2) is plotted in Figure 3. It increases linearly for low values of  $v$  since all corresponding advertisers bid with probability 1, and then slows down. Remark that the limit value of  $H$  at 0 is the *a priori* (i.e., not knowing his valuation) probability that a given competitor decides not to bid, that is expressed in (1).

Finally, Figure 4 displays the price  $E(v)$  that an advertiser expects to pay each time he bids  $v$ . In accordance with the observations from Figure 1, low-valuation advertisers pay less per auction when the number of competitors increases (since they have fewer chances to win the auction), while high-valuation advertisers will pay more (since they still frequently win the auctions they participate in, but tend to face higher

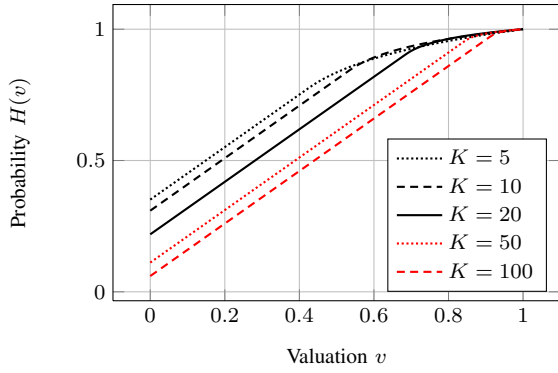


Figure 3. Probability  $H(v)$  of some given opponent to place no bid, or a bid below  $v$ .

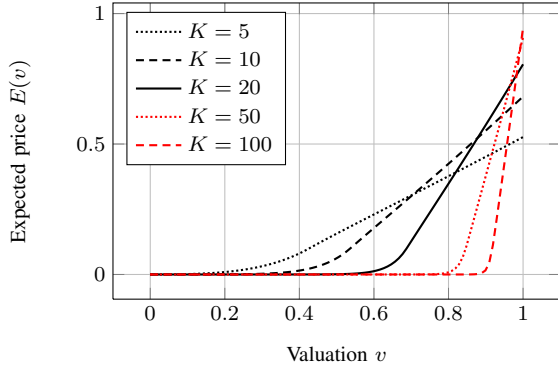


Figure 4. Expected price  $\mathbb{E}_v[c]$  paid by an advertiser who bids  $v$ .

bids).

#### IV. BUDGET REPARTITION BETWEEN TWO SEARCH ENGINES

We now consider that two search engines (e.g., Google and Yahoo!) run auctions among advertisers for the considered keyword, and we investigate the budget repartition of advertisers between those two search engines.

##### A. Model

As in the previous section, we analyze the symmetric case in terms of budgets and click-through rates: all advertisers are assumed to have the same total budget  $B$ , and the same click-through rate  $q_\ell$  on search engine  $\ell$ .

Consequently, at an equilibrium (if any), all advertisers will apply the same bidding strategy (that will be a function of their valuation  $v$ ). Note that we implicitly make the assumption here that advertiser valuations are identical on both search engines, i.e., the probability of a click on the ad to lead to a sale is independent of the search engine chosen by the user.

Both search engines differ by their success that is characterized by the number of requests per time unit  $\lambda_\ell$ , and by the probability  $q_\ell$  of users clicking the ad. Differences in  $q_\ell$  may stem from differences in the populations choosing each search engine, and/or in the interfaces that search engines propose. From the previous analysis, actually only the product  $\lambda_\ell q_\ell$

matters: it indeed represents the average number of clicked ads per time unit, which is what advertisers want to “buy” through the auction.

Each advertiser  $i$  has a total advertising budget  $B$  to spend, and decides, depending on his valuation, how to spread that budget between the two search engines. Equivalently, that budget repartition decision can be interpreted in terms of bidding probabilities  $p_{\ell,i}$  on each search engine  $\ell = 1, 2$ .

##### B. Equilibrium analysis

Using the same principles as before, we denote by  $p_\ell(v)$  the probability that an advertiser with valuation  $v$  bids, on each auction of search engine  $\ell$ , for  $\ell = 1, 2$ .

Then for a given search engine  $\ell$ , the probability  $H_\ell(v)$  that some (random) advertiser does not bid or bids below the value  $v$  is

$$H_\ell(v) = 1 - \int_{u>v} p_\ell(u) f(u) du. \quad (11)$$

Likewise, when there are  $K$  advertisers, the probability  $P_{\text{win},\ell}(v)$  that an advertiser bidding  $v$  wins the auction is given by

$$P_{\text{win},\ell}(v) = (H_\ell(v))^{K-1} = \left(1 - \int_{u>v} p_\ell(u) f(u) du\right)^{K-1}$$

, which is also the distribution of the maximum bid among the  $K - 1$  competitors.

Consequently, the expected price  $E_\ell(v)$  that an advertiser pays when he decides to bid  $v$  on search engine  $\ell$  while his competitors follow the strategy  $p_\ell$  is

$$\begin{aligned} E_\ell(v) &= q_\ell \int_{u=0}^v u dH_{\text{max},\ell}(u) \\ &= q_\ell \int_{u=0}^v (K-1)(H_\ell(u))^{K-2} u h_\ell(u) du \end{aligned}$$

where  $h_\ell$  is the right derivative of  $H_\ell$ . The expected cost per time unit of the bidding policy  $(p_1, p_2)$  is therefore

$$C = \lambda_1 p_1(v) E_1(v) + \lambda_2 p_2(v) E_2(v), \quad (12)$$

where  $E_\ell(v)$  can be rewritten as a function of  $p_\ell$  and  $f$  only:

$$E_\ell(v) = (K-1) q_\ell \int_{u=0}^v \left(1 - \int_{w>u} p_\ell(w) f(w) dw\right)^{K-2} u p_\ell(u) f(u) du.$$

When all advertisers have the same total budget  $B$  per time unit, they should fix their bidding strategy so as to maximize the expected payoff under the constraint that  $C \leq B$ . For an advertiser with valuation  $v$ , that expected payoff is

$$\begin{aligned} P &:= \lambda_1 p_1(v) (v q_1 P_{\text{win},1}(v) - E_1(v)) \\ &\quad + \lambda_2 p_2(v) (v q_2 P_{\text{win},2}(v) - E_2(v)). \end{aligned} \quad (13)$$

The question becomes how to distribute the budget among the two search engines: if for an advertiser with valuation  $v$  an optimal solution is such that  $p_1(v) > 0$  and  $p_2(v) < 1$ , then this means that transferring one infinitesimal amount of

budget from search engine 1 to search engine 2 could only decrease the payoff. From (12), to represent the same total budget such a change in the budget repartition should translate into probability changes  $dp_1 < 0$  and  $dp_2 > 0$  such that

$$dp_2 \lambda_2 E_2(v) = -dp_1 \lambda_1 E_1(v).$$

From (13), the optimality of  $(p_1, p_2)$  thus implies that such a change would correspond to a revenue decrease:

$$0 \geq dP = \lambda_2 dp_2 \left( (v q_2 P_{\text{win},2}(v) - E_2(v)) - (v q_1 P_{\text{win},1}(v) - E_1(v)) \frac{E_2(v)}{E_1(v)} \right),$$

which gives

$E_1(v)(v q_2 P_{\text{win},2}(v) - E_2(v)) \leq E_2(v)(v q_1 P_{\text{win},1}(v) - E_1(v))$ , i.e.,  $\frac{E_1(v)}{q_1 P_{\text{win},1}(v)} \leq \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}$ . This is natural, since the ratios compared are the average prices for a won (clicked) auction: if it is cheaper to have a customer click one's ad on search engine 1 than on search engine 2, then transferring some budget from SE1 to SE2 is not beneficial. Inverting the roles of the search engines, we obtain the following optimality conditions, that are necessary conditions for  $(p_1, p_2)$  to describe an equilibrium:

$$p_1(v) > 0, p_2(v) < 1 \Rightarrow \frac{E_1(v)}{q_1 P_{\text{win},1}(v)} \leq \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}, (14)$$

$$p_1(v) < 1, p_2(v) > 0 \Rightarrow \frac{E_2(v)}{q_2 P_{\text{win},2}(v)} \leq \frac{E_1(v)}{q_1 P_{\text{win},1}(v)}. (15)$$

We can therefore infer the behavior  $(\bar{p}_1(v), \bar{p}_2(v))$  of a profit-maximizing advertiser with total budget  $B$  that assumes that the opponents follow the bidding strategy  $(p_1(v), p_2(v))$ , as described in Alg. 1.

Relation (16) corresponds to the case where the advertiser budget is large enough to participate in all auctions. On the other hand, (17)-(20) give the best-reply strategy when the budget does not allow to always bid: advertisers then prefer the most efficient search engine in terms of the metric  $\frac{E_\ell(v)}{q_\ell P_{\text{win},\ell}(v)}$ : they put as much budget as they can on it, and devote the possible remaining budget to the other search engine. For the special case when both search engines have the exact same value of the efficiency metric, then any splitting of the budget  $B$  among the two search engines is a best reply.

Now, applying iteratively that best-reply algorithm to a given starting bidding strategy does not converge in general. Indeed, (19)-(20) describe an infinity of best replies, while the choice of one of those best-replies affects the next iterations of the algorithm: imagine that in case 2c the choice favors search engine 1, then at the next iteration the bidding probabilities will be such that search engine 2 becomes more interesting, hence an oscillation.

Therefore, we look for *symmetric* equilibrium strategies, i.e., equilibrium strategies  $(p_1, p_2)$  that verify the property  $p_1 = p_2$ : each advertiser bids on both auctions with the same probability. Indeed, such strategies will ensure that  $\frac{E_1(v)}{q_1 P_{\text{win},1}(v)} = \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}$ , which means that each advertiser

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**Alg. 1** Best-reply bidding strategy  $(\bar{p}_1(v), \bar{p}_2(v))$ 


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Input:

- distribution  $F$  of valuations of advertisers
- bidding strategy  $(p_1(\cdot), p_2(\cdot))$  of competing advertisers
- valuation  $v$  and budget  $B$  of the considered advertiser.

For each search engine  $\ell = 1, 2$ , compute the expected cost per auction according to (7). and the probability of winning an auction with bid  $v$

$$P_{\text{win},\ell}(v) = \left( 1 - \int_{u>v} p_\ell(u) f(u) du \right)^{K-1}.$$

1) **if**  $\lambda_1 E_1(v) + \lambda_2 E_2(v) \leq B$  **set**

$$\bar{p}_1(v) = \bar{p}_2(v) = 1. \quad (16)$$

2) **else**

a) **if**  $\frac{E_1(v)}{q_1 P_{\text{win},1}(v)} < \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}$  **set**

$$\begin{cases} \bar{p}_1(v) = \min \left( 1, \frac{B}{\lambda_1 E_1(v)} \right) \\ \bar{p}_2(v) = \max \left( 0, \frac{B - \lambda_1 E_1(v)}{\lambda_2 E_2(v)} \right) \end{cases} \quad (17)$$

b) **else if**  $\frac{E_1(v)}{q_1 P_{\text{win},1}(v)} > \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}$  **set**

$$\begin{cases} \bar{p}_1(v) = \max \left( 0, \frac{B - \lambda_2 E_2(v)}{\lambda_1 E_1(v)} \right) \\ \bar{p}_2(v) = \min \left( 1, \frac{B}{\lambda_2 E_2(v)} \right) \end{cases} \quad (18)$$

c) **else if**  $\frac{E_1(v)}{q_1 P_{\text{win},1}(v)} = \frac{E_2(v)}{q_2 P_{\text{win},2}(v)}$  **then choose any**  $\bar{p}_1(v)$  **such that**

$$\max \left( 0, \frac{B - \lambda_2 E_2(v)}{\lambda_1 E_1(v)} \right) \leq \bar{p}_1(v) \leq \min \left( 1, \frac{B}{\lambda_1 E_1(v)} \right) \quad (19)$$

**and set**

$$\bar{p}_2 = \frac{B - \bar{p}_1 \lambda_1 E_1(v)}{\lambda_2 E_2(v)} \quad (20)$$


---

is indifferent between both search engines, but conforms to  $(p_1, p_2)$  so that his competitors are also indifferent between the search engines (typical of mixed equilibrium outcomes). Furthermore, with the model we have defined it is natural to look for symmetric strategies since search engines only differ by their numbers of requests per time unit, and the click-through rate on the ads they display: because only clicked ads are charged, advertisers can consider that there are a total of  $\lambda_1 q_1 + \lambda_2 q_2$  clicked ads per time unit to compete for. The question of selecting which auctions to participate in is exactly the same as for a single search engine; we had considered symmetric solutions (the probability of participating being the same at each auction) based on intuitive arguments, and we only apply here the same reasoning.

Remark that based on those arguments, the results presented here can easily be generalized to the case of more than two search engines in competition.

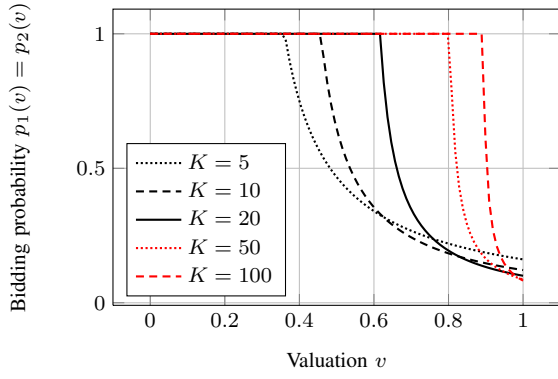


Figure 5. Equilibrium bidding strategies at each SE.

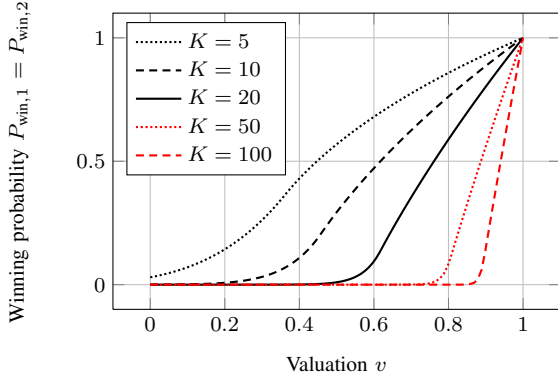


Figure 6. Winning probabilities at equilibrium (knowing that the player bids) at SE  $i$ .

### C. Numerical results

We present here the results that have been obtained when applying successive (symmetric) best-replies to an initial symmetric bidding strategy  $(p_1, p_2) = (p, p)$ . As in the previous section, the bidding strategies converge rapidly (about 20 iterations for a  $10^{-6}$  pointwise distance between the current strategy and its best-reply). The curves displayed here are for the set of parameters  $\lambda_1 = \lambda_2 = 10$ ,  $q_1 = 0.8$ ,  $q_2 = 0.5$ ,  $B = 1$  and valuations uniformly distributed on  $[0, 1]$ . The results are shown in Figures 5 to 8.

Note that the shapes of the bidding probability functions, the winning probabilities, and the expected price paid per bid, are similar to what we obtained for the case of a single search engine. This is because from a mathematical point of view only the total number of clicked ads per time unit has changed (it was  $\lambda q = 5$  in Section III, and now equals  $\lambda_1 q_1 + \lambda_2 q_2 = 13$ ). Since that number has increased, advertisers tend to bid less frequently on each auction. Also, note that the expected prices paid per auction on both search engines, displayed on Figure 7, only differ by proportionality coefficients that are their click-through rates, in accordance with (7). The budgets spent by advertisers on both search engines differ as well by the same coefficients.

We finally observe the effect of those coefficients, by studying the impact of some variations in the click-through

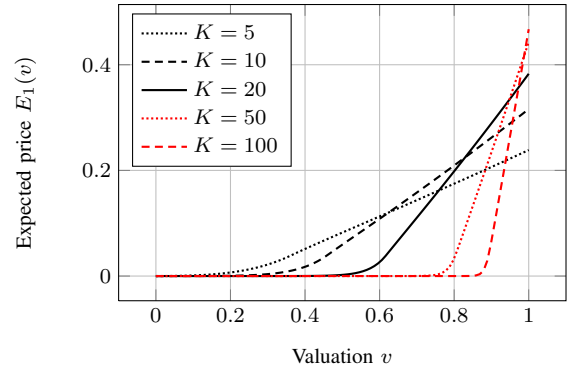


Figure 7. Expected price  $E_1(v)$  paid by a user who bids  $v$  on SE 1 (multiply by  $q_2/q_1$  to get  $E_2(v)$ ).

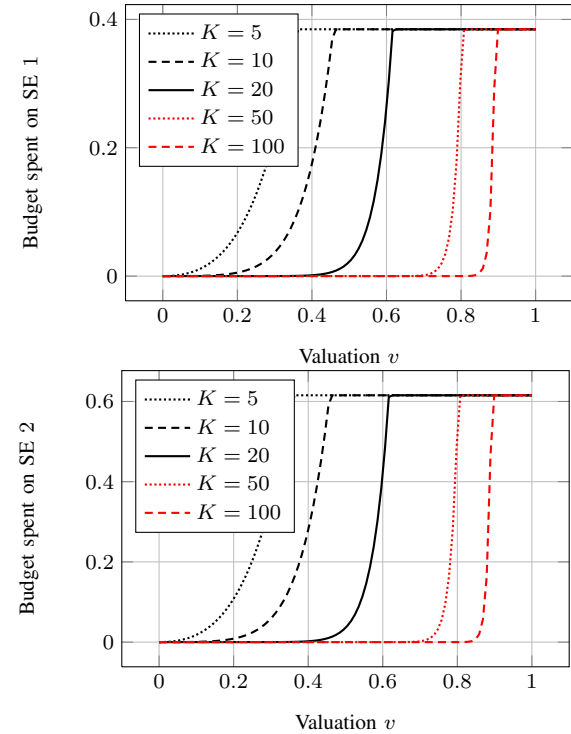


Figure 8. Budget spent on each SE.

rate of search engine 2, while the other parameter values are unchanged ( $\lambda_1 = \lambda_2 = 10$ ,  $q_1 = 0.5$ , and  $B = 1$ ). The expected revenue of search engine 2 is plotted in Figure 9. Interestingly, that total revenue appears to be a concave nondecreasing function of its click-through rate. Similarly, it would also be a concave nondecreasing function of its user request frequency since advertisers are only sensitive to the product request\_frequency  $\times$  CTR.

This suggests that, when search engines can improve their request rates (and possibly CTR) through some investments to attract more customers and/or to target specific consumer segments, anticipating the advertisers' reaction should be taken into account to make the optimal investment decisions.



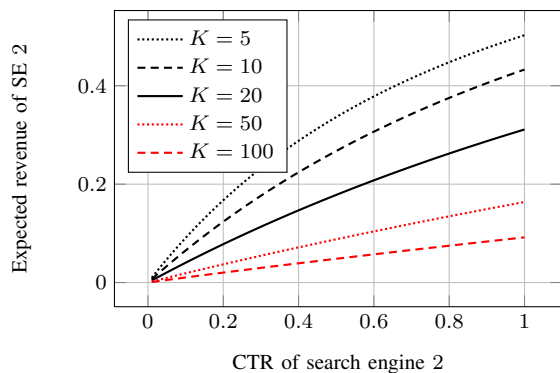


Figure 9. Expected revenue of search engine 2, per time unit, per advertiser.

## V. CONCLUSIONS AND FUTURE WORK

Based on a game-theoretic model to analyze the bidding decisions of information-limited and budget-limited advertisers, we have shown that a bidding strategy equilibrium exists, where each advertiser participates in each auction with a fixed probability that depends on his valuation. At such an equilibrium, high-valuation advertisers tend to bid less often than low-valuation ones, because they win the auction more often and reach their budget limit faster. When several search engines run auctions to make revenue, they actually compete to attract more advertisers and more users so as to raise higher profits from ad auctions. We have observed that the expected profit of a search engine is a nondecreasing concave function of its number of requests per time unit, and of the click-through rate on its ads. Consequently, a higher-level game played among search engines who would anticipate advertiser bidding strategies would be interesting to analyze.

Other directions for future work include relaxations of some of the assumptions made in this paper: advertiser budgets could differ among advertisers, and the case of several displayed ad slots could be studied. The model could also be extended to consider advertiser-related click-through rates, which would moreover add another strategic decision for search engines, that is to base their ranking scheme on bids only or on the products  $\text{bid} \times \text{CTR}$  (the so-called revenue-based mechanism). Analyzing the game then played between search engines, as has been done in [5], [10] for different advertiser models, would be of interest.

Finally, we have assumed that advertisers bid truthfully, based on the incentive properties of the (one-shot) auction scheme. Nevertheless, here we have repeated auctions, which opens the possibility for advertisers to bid below their valuation so as to reduce the price paid when they win an auction. Advertisers may have an interest to follow such a strategy when their budget limit does not allow them to bid truthfully on all auctions. That approach deserves some attention.

## ACKNOWLEDGMENTS

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