



# On the relationships between copulas of order statistics and marginal distributions

Jorge Navarro, Fabio Spizzichino

## ► To cite this version:

Jorge Navarro, Fabio Spizzichino. On the relationships between copulas of order statistics and marginal distributions. *Statistics and Probability Letters*, 2010, 80 (5-6), pp.473. [⟨10.1016/j.spl.2009.11.025⟩](#). [⟨hal-00612019⟩](#)

**HAL Id: hal-00612019**

**<https://hal.science/hal-00612019v1>**

Submitted on 28 Jul 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

## Accepted Manuscript

On the relationships between copulas of order statistics and marginal distributions

Jorge Navarro, Fabio Spizzichino

PII: S0167-7152(09)00449-0  
DOI: [10.1016/j.spl.2009.11.025](https://doi.org/10.1016/j.spl.2009.11.025)  
Reference: STAPRO 5582

To appear in: *Statistics and Probability Letters*

Received date: 15 May 2009  
Revised date: 30 November 2009  
Accepted date: 30 November 2009

Please cite this article as: Navarro, J., Spizzichino, F., On the relationships between copulas of order statistics and marginal distributions. *Statistics and Probability Letters* (2009), doi:10.1016/j.spl.2009.11.025

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



# On the relationships between copulas of order statistics and marginal distributions

Jorge Navarro<sup>a</sup>, Fabio Spizzichino<sup>b</sup>

<sup>a</sup>Universidad de Murcia, Spain

<sup>b</sup>University of Rome “La Sapienza”, Italy

---

## Abstract

In this paper we study the relationships between copulas of order statistics from heterogeneous samples and the marginal distributions of the parent random variables. Specifically, we study the copula of the order statistics obtained from a general random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . We show that the copula of the order statistics from  $\mathbf{X}$  only depends on the copula of  $\mathbf{X}$  and on the marginal distributions of  $X_1, X_2, \dots, X_n$  through an exchangeable copula and the average of the marginal distribution functions. We study in detail some relevant cases.

*Key words:* copula, order statistic, exchangeable distribution,  $k$ -out-of- $n$  system

---

## 1. Introduction

Consider a vector  $(F_1, F_2, \dots, F_n)$  of  $n$  arbitrarily chosen univariate distribution functions and an arbitrary  $n$ -dimensional copula  $C$ . Then, the celebrated Sklar’s theorem (see, *e.g.*, Nelsen, 2006, p. 18) ensures that there exists an  $n$ -dimensional distribution function  $F$ , that admits  $F_1, F_2, \dots, F_n$  as its marginal distributions and  $C$  as its connecting copula. In fact we can construct  $F$  by simply using the formula

$$F(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)).$$

Thus no condition on the pair  $C, (F_1, F_2, \dots, F_n)$  is requested for their “compatibility”.

Consider now the connecting copula  $K$  and the marginal distributions  $F_{1:n}, F_{2:n}, \dots, F_{n:n}$  of the order statistics obtained from  $F$ . A well known result concerning order statistics

---

\*Corresponding address: Jorge Navarro, Facultad de Matemáticas, Universidad de Murcia, 30100 Murcia, Spain. Telephone number: 34 868883509. Fax number: 34 868884182. E-mail address: jorgenav@um.es

(see Avérous, Genest and Kochar, 2005) is that the connecting copula of the order statistics of  $n$  independent and identically distributed (IID) random variables  $X_1, X_2, \dots, X_n$ , with common marginal distribution  $G$ , is a given copula that does not depend on  $G$ . That is, if  $C$  is the product copula and  $F_1 = F_2 = \dots = F_n = G$ , then  $K$  is a fixed copula (i.e. it does not depend on  $G$ ) called the *order statistics copula*. The picture radically changes if  $F_i \neq F_j$  (even if  $C$  is the product copula). In such a case we find that some relations exist both between  $K$  and  $F_1, F_2, \dots, F_n$  and for  $K$  and  $F_{1:n}, F_{2:n}, \dots, F_{n:n}$ . Of course these relations depend on the choice of  $C$ .

In this note we are interested in discussing the relationships between the connecting copula of the order statistics and the marginal distributions of the parent random variables. In particular we show that for any vector of identically distributed random variables, the connecting copula of the order statistics only depends on the connecting copula of the variables and does not depend on their common marginal distribution. Furthermore, it will also emerge that this property cannot be generally true in the case of variables that have different marginal distributions.

The problem considered here is equivalent to study the copula for continuous *ordered* random variables  $Y_1, Y_2, \dots, Y_n$ , i.e., for variables such that

$$P(Y_1 \leq Y_2 \leq \dots \leq Y_n) = 1. \quad (1.1)$$

Note that this problem reduces to the study of the vector of order statistics from a random permutation of  $Y_1, Y_2, \dots, Y_n$ . In fact, we will see that we can find a vector of exchangeable variables whose order statistics are jointly distributed as  $(Y_1, Y_2, \dots, Y_n)$ .

Our paper is then strictly related with the wide literature concerning the distribution of order statistics; in particular we address the reader to basic references such as David and Nagaraja (2003); Arnold, Balakrishnan and Nagaraja (2008). Trivially the condition (1.1) implies that the marginal distributions are ordered in the usual stochastic sense. More details about the conditions to be satisfied by this vector of the marginal distributions have been given in the papers by Rychlik (1994); Durante and Jaworski (2008); Jaworski and Rychlik (2008); Jaworski (2009) and the references therein. Some copula representations for the distributions of order statistics were also obtained in Navarro and Spizzichino (2009).

However, the main purpose of this note is different since we want to understand what are the constraints to be satisfied by the connecting copula of a vector of order statistics.

The rest of the paper is organized as follows. In Section 2 we develop a general analysis and we give the main results. Examples and enlightening details are presented in Section 3, where we study the general results for the bivariate case and some particular cases of special interest.

## 2. Main results

We consider a vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  of random variables with a continuous distribution function  $F(x_1, x_2, \dots, x_n)$ . Let  $C(u_1, u_2, \dots, u_n)$  and  $F_1, F_2, \dots, F_n$ , respectively, denote the connecting copula and the marginal distribution functions. Hence  $C$  is the joint distribution function of the variables  $F_1(X_1), F_2(X_2), \dots, F_n(X_n)$  and, from Sklar's theorem,

$$F(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)).$$

It is well known that the random variables  $X_1, X_2, \dots, X_n$  are independent if and only if the copula  $C$  is equal to the product copula  $C_I(u_1, u_2, \dots, u_n) = u_1 u_2 \dots u_n$  for  $0 < u_i < 1$  and  $i = 1, 2, \dots, n$ .

Furthermore, we denote by  $\mathcal{P}_n$  the group of permutations of  $\{1, 2, \dots, n\}$  and, for  $\pi \in \mathcal{P}_n$ , set

$$\mathbf{X}_\pi = (X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)}).$$

Concerning the joint distribution function  $F_\pi$  and the connecting copula  $C_\pi$  of  $\mathbf{X}_\pi$ , we can write

$$\begin{aligned} F_\pi(x_1, x_2, \dots, x_n) &= P(X_{\pi(1)} \leq x_1, X_{\pi(2)} \leq x_2, \dots, X_{\pi(n)} \leq x_n) \\ &= C_\pi(F_{\pi(1)}(x_1), F_{\pi(2)}(x_2), \dots, F_{\pi(n)}(x_n)), \end{aligned}$$

where

$$C_\pi(u_1, u_2, \dots, u_n) = C(u_{\pi^{-1}(1)}, u_{\pi^{-1}(2)}, \dots, u_{\pi^{-1}(n)}).$$

A random vector  $\mathbf{X}$  has an exchangeable distribution if  $\mathbf{X}$  is equal in law to  $\mathbf{X}_\pi$  for all  $\pi \in \mathcal{P}_n$  or, equivalently, if its distribution function  $F$  is exchangeable, that is, if

$$F(x_1, x_2, \dots, x_n) = F(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

for all  $\pi \in \mathcal{P}_n$ . If  $F$  is exchangeable, then its copula  $C$  is exchangeable and  $F_1 = F_2 = \dots = F_n$ .

Let us construct a random vector  $\tilde{\mathbf{X}}$  associated to  $\mathbf{X}$  with an exchangeable distribution and with order statistics which are equal in law to that of  $\mathbf{X}$ . This random vector  $\tilde{\mathbf{X}}$  can be called the *exchangeable random vector* associated to  $\mathbf{X}$ . Denote by  $\Pi$  a random variable uniformly distributed over  $\mathcal{P}_n$ . We now consider  $\tilde{\mathbf{X}} = \mathbf{X}_\Pi$ , that is,  $\tilde{\mathbf{X}} = \mathbf{X}_\pi$  with probability  $1/n!$ . The exchangeable distribution function  $\tilde{F}$  of the vector  $\tilde{\mathbf{X}}$  is the uniform mixture of  $F_\pi$  for  $\pi \in \mathcal{P}_n$ , that is,

$$\begin{aligned}\tilde{F}(x_1, x_2, \dots, x_n) &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} F_\pi(x_1, x_2, \dots, x_n) \\ &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(F_{\pi(1)}(x_1), F_{\pi(1)}(x_2), \dots, F_{\pi(n)}(x_n))\end{aligned}$$

and the common marginal distribution  $G$  is the average of the marginal distributions of  $X_1, X_2, \dots, X_n$ , that is,

$$G(x) = \frac{1}{n} \sum_{i=1}^n F_i(x).$$

Hence, using Sklar's theorem,  $\tilde{F}$  can also be written as

$$\tilde{F}(x_1, x_2, \dots, x_n) = \tilde{C}(G(x_1), G(x_2), \dots, G(x_n)),$$

where  $\tilde{C}$  is the copula of  $\tilde{\mathbf{X}}$ . Thence, if  $G^{-1}(u) = \inf\{x \in [-\infty, +\infty] : G(x) = u\}$  for  $0 \leq u \leq 1$ ,  $\tilde{C}$  can be written as

$$\tilde{C}(u_1, u_2, \dots, u_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(F_{\pi(1)}(G^{-1}(u_1)), F_{\pi(2)}(G^{-1}(u_2)), \dots, F_{\pi(n)}(G^{-1}(u_n)))$$

for  $0 < u_i < 1$  and  $i = 1, 2, \dots, n$ . Note that  $\tilde{C} = \tilde{C}(C, F_1, F_2, \dots, F_n)$ , that is,  $\tilde{C}$  depends both on  $C$  and on the marginal distributions  $F_1, F_2, \dots, F_n$ . Also note that  $\tilde{C}$  is exchangeable but it is not necessarily equal to the exchangeable copula obtained from  $C$ ,

$$C_\Pi(u_1, u_2, \dots, u_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(u_1, u_2, \dots, u_n).$$

Note that  $C_\Pi = C_\Pi(C)$ , that is,  $C_\Pi$  only depends on  $C$ . The exchangeable copulas  $\tilde{C}$  and  $C_\Pi$  can be called the *copula of mixtures* and the *mixture of copulas*, respectively. Now we can state the following properties.

**Lemma 2.1.** *With the notation introduced above, the following properties hold.*

- (i) *If  $F_1 = F_2 = \dots = F_n$ , then  $\tilde{C} = C_\Pi$ .*
- (ii) *If  $C$  is exchangeable, then  $C = C_\Pi$ .*
- (iii) *If  $C$  is exchangeable and  $F_1 = F_2 = \dots = F_n$ , then  $C = C_\Pi = \tilde{C}$ .*

*Proof.* (i) If  $F_1 = F_2 = \dots = F_n$ , then  $G = F_1 = F_2 = \dots = F_n$ . Thence,

$$\begin{aligned}\tilde{C}(u_1, u_2, \dots, u_n) &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(F_{\pi(1)}(G^{-1}(u_1)), F_{\pi(2)}(G^{-1}(u_2)), \dots, F_{\pi(n)}(G^{-1}(u_n))) \\ &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(u_1, u_2, \dots, u_n) \\ &= C_\Pi(u_1, u_2, \dots, u_n).\end{aligned}$$

(ii) If  $C$  is exchangeable, then  $C = C_\pi$  for all  $\pi \in \mathcal{P}_n$ . Thence,

$$\begin{aligned}C_\Pi(u_1, u_2, \dots, u_n) &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C_\pi(u_1, u_2, \dots, u_n) \\ &= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C(u_1, u_2, \dots, u_n) \\ &= C(u_1, u_2, \dots, u_n).\end{aligned}$$

The proof of (iii) is immediate from (i) and (ii). □

Note that if  $\tilde{C} = C_\Pi$  holds, then  $\tilde{C}$  only depends on  $C$  and it does not depend on  $F_1, F_2, \dots, F_n$ .

We now pass to consider the joint distribution of the vector  $\mathbf{X}_{OS} = (X_{1:n}, X_{2:n}, \dots, X_{n:n})$  of order statistics obtained from  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . Analogously, we denote by  $\tilde{\mathbf{X}}_{OS} = (\tilde{X}_{1:n}, \tilde{X}_{2:n}, \dots, \tilde{X}_{n:n})$  the vector of order statistics obtained from  $\tilde{\mathbf{X}}$ . As it is easy to check (see, *e.g.*, Spizzichino, 2001; Navarro, Spizzichino and Balakrishnan, 2009), we have the following property.

**Lemma 2.2.** *The joint distribution of  $\mathbf{X}_{OS}$  coincides with the one of  $\tilde{\mathbf{X}}_{OS}$ .*

The proof is immediate since  $\mathbf{X}_{OS}$  is equal in law to  $(\mathbf{X}_\pi)_{OS}$  for all  $\pi \in \mathcal{P}_n$ . As an immediate consequence we have that the copula of  $\mathbf{X}_{OS}$  is equal to the copula of  $\tilde{\mathbf{X}}_{OS}$  which only depends on  $\tilde{C}$  and  $G$ . Moreover we have the following property.

**Proposition 2.3.** *If  $F_1 = F_2 = \dots = F_n$ , then the copula of  $\mathbf{X}_{OS}$  only depends on  $C$ .*

*Proof.* From Lemma 2.1 (i) we have that if  $F_1 = F_2 = \dots = F_n$ , then  $\tilde{C} = C_{\Pi}$ . Note that  $\tilde{C}$  is the joint distribution of  $U_1 = G(\tilde{X}_1)$ ,  $U_2 = G(\tilde{X}_2)$ ,  $\dots$ ,  $U_n = G(\tilde{X}_n)$ , and that it only depends on  $C$ . Also note that the order statistics  $U_{1:n}, U_{2:n}, \dots, U_{n:n}$  from  $U_1, U_2, \dots, U_n$  satisfy  $U_{i:n} = G(\tilde{X}_{i:n})$  for  $i = 1, 2, \dots, n$ . Hence, the copula of  $(\tilde{X}_{1:n}, \tilde{X}_{2:n}, \dots, \tilde{X}_{n:n})$  is equal to the copula of  $(U_{1:n}, U_{2:n}, \dots, U_{n:n})$  which only depends on  $C$ .  $\square$

Now we can obtain the main result of the paper which can be stated as follow.

**Proposition 2.4.** *The copula of  $\mathbf{X}_{OS}$  only depends on  $\tilde{C} = \tilde{C}(C, F_1, F_2, \dots, F_n)$  and the marginal distributions of  $\mathbf{X}_{OS}$  only depend on  $\tilde{C} = \tilde{C}(C, F_1, F_2, \dots, F_n)$  and on  $G$ .*

*Proof.* From Lemma 2.2 we have that the joint distribution of  $\mathbf{X}_{OS}$  coincides with the one of  $\tilde{\mathbf{X}}_{OS}$ . Hence their copulas and marginal distributions are equal. Moreover, as  $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$  are identically distributed (ID), from the Proposition 2.3 we have that the copula of  $\tilde{\mathbf{X}}_{OS}$  only depends on  $\tilde{C}$ . Hence the copula of  $\mathbf{X}_{OS}$  only depends on  $\tilde{C}$ . Moreover, the marginal distributions of  $\tilde{\mathbf{X}}_{OS}$  only depend on the distribution of  $\tilde{\mathbf{X}}$ , that is, on  $\tilde{C}$  and on  $G$ .  $\square$

Let us now denote by  $K$  and  $F_{i:n}$  ( $i = 1, 2, \dots, n$ ) the copula and the marginal distributions of  $(X_{1:n}, X_{2:n}, \dots, X_{n:n})$ , respectively. Notice that  $K$  only depends on  $\tilde{C}$  and so we can write  $K = K_{\tilde{C}}$ . However, note that  $\tilde{C}$  depends both on  $C$  and on  $F_1, F_2, \dots, F_n$ . Furthermore,

$$F_{i:n}(x) = P(X_{i:n} \leq x) = P(\tilde{X}_{i:n} \leq x) = P(U_{i:n} \leq G(x))$$

for  $i = 1, 2, \dots, n$ , where  $U_{1:n}, U_{2:n}, \dots, U_{n:n}$  are the order statistics from  $U_i = G(\tilde{X}_{i:n})$ ,  $i = 1, 2, \dots, n$ .

From Lemma 2.1 and Proposition 2.4, we can consider four special cases:

(i) If  $F_1 = F_2 = \dots = F_n$ , then  $\tilde{C} = C_{\Pi}$  and hence  $K$  only depends on the exchangeable copula  $C_{\Pi}$ , that is, the mixture of copulas obtained from  $C$ .

(ii) If  $C$  is exchangeable, then  $C = C_{\Pi}$  and

$$\tilde{C}(u_1, u_2, \dots, u_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} C(F_{\pi(1)}(G^{-1}(u_1)), F_{\pi(2)}(G^{-1}(u_2)), \dots, F_{\pi(n)}(G^{-1}(u_n))).$$



(iii) If  $F$  is exchangeable, then  $K$  only depends on  $C = C_{\Pi} = \tilde{C}$ .

(iv) If  $F_1 = F_2 = \dots = F_n$  and  $C$  is the product copula, then  $K$  is a fixed copula known as the order statistics copula. This last result was given by Avérous, Genest and Kochar (2005).

### 3. The bivariate case

In this section we analyze in detail the special case of  $n = 2$  random variables  $X_1, X_2$  with a bivariate distribution function  $F$ , connecting copula  $C$  and with marginal distribution functions  $F_1$  and  $F_2$ , respectively. Of course, from Sklar's theorem, we have

$$F(x, y) = C(F_1(x), F_2(y)).$$

Now consider the exchangeable random vector  $(\tilde{X}_1, \tilde{X}_2)$  obtained from  $(X_1, X_2)$  with distribution function

$$\begin{aligned} \tilde{F}(x, y) &= \frac{1}{2}F(x, y) + \frac{1}{2}F(y, x) \\ &= \frac{1}{2}C(F_1(x), F_2(y)) + \frac{1}{2}C(F_1(y), F_2(x)) \\ &= \tilde{C}(G(x), G(y)), \end{aligned} \tag{3.1}$$

where  $\tilde{C}$  is the connecting copula and  $G = (F_1 + F_2)/2$  is the common marginal distribution. Hence  $\tilde{C}$  can be written as

$$\tilde{C}(u, v) = \frac{1}{2}C(F_1(G^{-1}(u)), F_2(G^{-1}(v))) + \frac{1}{2}C(F_1(G^{-1}(v)), F_2(G^{-1}(u))). \tag{3.2}$$

The joint distribution function  $F_{OS}$  of the pair of order statistics  $(X_{1:2}, X_{2:2})$  can be written as

$$F_{OS}(x, y) = F(x, y) + F(y, x) - F(x, x) \quad \text{for } x \leq y \tag{3.3}$$

and

$$F_{OS}(x, y) = F(y, y) \quad \text{for } y < x \tag{3.4}$$

and the marginal distributions as

$$F_{1:2}(x) = F_1(x) + F_2(x) - F(x, x) = F_1(x) + F_2(x) - C(F_1(x), F_2(x))$$

and

$$F_{2:2}(y) = F(y, y) = C(F_1(y), F_2(y)).$$

Notice that  $(X_{1:2}, X_{2:2})$  and  $(\tilde{X}_{1:2}, \tilde{X}_{2:2})$  are equal in law. Hence these equations can be respectively rewritten as

$$F_{1:2}(x) = 2G(x) - \tilde{C}(G(x), G(x)) \quad (3.5)$$

and

$$F_{2:2}(y) = \tilde{C}(G(y), G(y)). \quad (3.6)$$

It is then self-evident that  $F_{1:2}$  and  $F_{2:2}$  only depend on the pair  $\tilde{C}$  and  $G$ . Analogously, the joint distribution can be rewritten as

$$F_{OS}(x, y) = 2\tilde{C}(G(x), G(y)) - \tilde{C}(G(x), G(x)) \quad \text{for } x \leq y, \quad (3.7)$$

and

$$F_{OS}(x, y) = \tilde{C}(G(y), G(y)) = F_{2:2}(y) \quad \text{for } y < x. \quad (3.8)$$

On the other hand, if  $K$  is the connecting copula of the order statistics, then from Sklar's theorem, we have

$$F_{OS}(x, y) = K(F_{1:2}(x), F_{2:2}(y)).$$

Hence,

$$K(u, v) = F_{OS}(F_{1:2}^{-1}(u), F_{2:2}^{-1}(v)).$$

Thence, for  $F_{1:2}^{-1}(u) \leq F_{2:2}^{-1}(v)$ ,

$$K(u, v) = 2\tilde{C}(G(F_{1:2}^{-1}(u)), G(F_{2:2}^{-1}(v))) - \tilde{C}(G(F_{1:2}^{-1}(u)), G(F_{1:2}^{-1}(u))). \quad (3.9)$$

Analogously, for  $F_{1:2}^{-1}(u) > F_{2:2}^{-1}(v)$ , we have

$$K(u, v) = v. \quad (3.10)$$

Recall that from (3.5) and (3.6),  $F_{1:2}$  and  $F_{2:2}$  only depend on  $\tilde{C}$  and  $G$ . Then equations (3.9) and (3.10) not only show that  $K$  depends on  $C, F_1, F_2$  through  $\tilde{C}$  and  $G$ , but also point out the relationships between  $K$ , on one side, and  $F_{1:2}, F_{2:2}$  on the other side. Also note

that the support of  $K$  is included in the region  $\{(u, v) \in (0, 1)^2 : F_{1:2}^{-1}(u) \leq F_{2:2}^{-1}(v)\}$ . Note that (3.6) can be rewritten as

$$F_{2:2}(y) = \delta_{\tilde{C}}(G(y)),$$

where  $\delta_{\tilde{C}}(u) = \tilde{C}(u, u)$  is the diagonal section of the copula  $\tilde{C}$ . Hence, if  $\delta_{\tilde{C}}^{-1}(z) = \inf\{u \in [0, 1] : \delta_{\tilde{C}}(u) = z\}$ , then

$$F_{2:2}^{-1}(z) = G^{-1}(\delta_{\tilde{C}}^{-1}(z)).$$

Analogously,  $F_{1:2}^{-1}$  can be related to the diagonal section of the survival copula associated to  $\tilde{C}$  as follows. The survival copula of  $(\tilde{X}_1, \tilde{X}_2)$  is a copula  $\tilde{S}$  such that the survival function can be written as

$$P(\tilde{X}_1 > x, \tilde{X}_2 > y) = \tilde{S}(1 - G(x), 1 - G(y)).$$

It is well known that  $\tilde{S}$  can be written as

$$\tilde{S}(u, v) = u + v - 1 + \tilde{C}(1 - u, 1 - v).$$

Also note that

$$F_{1:2}(x) = 1 - P(\tilde{X}_1 > x, \tilde{X}_2 > x) = 1 - \tilde{S}(1 - G(x), 1 - G(x)).$$

Thence

$$F_{1:2}^{-1}(z) = G^{-1}(1 - \delta_{\tilde{S}}^{-1}(1 - z)),$$

where  $\delta_{\tilde{S}}(u) = \tilde{S}(u, u)$  is the diagonal section of the survival copula  $\tilde{S}$  and  $\delta_{\tilde{S}}^{-1}(z) = \inf\{u \in [0, 1] : \delta_{\tilde{S}}(u) = z\}$ . Note that  $\tilde{S}$  and  $\delta_{\tilde{S}}$  only depend on  $\tilde{C}$ .

Next we consider some special cases and examples.

**Example 3.1** ( $C = C_I$ ). If  $X_1$  and  $X_2$  are independent, with marginal distributions functions  $F_1$  and  $F_2$ , respectively, then  $F(x, y) = F_1(x)F_2(y)$  and  $C(u, v) = C_I(u, v) = uv$  is exchangeable. Hence, from (3.1) and (3.2), we have

$$\tilde{F}(x, y) = \frac{1}{2}F_1(x)F_2(y) + \frac{1}{2}F_1(y)F_2(x) = \tilde{C}(G(x), G(y)),$$

where  $G = (F_1 + F_2)/2$  and

$$\tilde{C}(u, v) = \frac{1}{2}F_1(G^{-1}(u))F_2(G^{-1}(v)) + \frac{1}{2}F_2(G^{-1}(u))F_1(G^{-1}(v)).$$

Of course,  $\tilde{C}$  only depends on  $F_1$  and  $F_2$ . Analogously, from (3.3) and (3.4), the joint distribution of the order statistics is given by

$$F_{OS}(x, y) = F_1(x)F_2(y) + F_1(y)F_2(x) - F_1(x)F_2(x) \quad \text{for } x \leq y$$

and  $F_{OS}(x, y) = F_1(y)F_2(y)$  for  $y < x$ . Alternatively, from (3.7) and (3.8), it can be written as

$$F_{OS}(x, y) = 2\tilde{C}(G(x), G(y)) - \tilde{C}(G(x), G(x)) \quad \text{for } x \leq y$$

and  $F_{OS}(x, y) = \tilde{C}(G(y), G(y))$  for  $y < x$ . The marginal distributions of the order statistics can be written as

$$F_{1:2}(x) = F_1(x) + F_2(x) - F_1(x)F_2(x) = 2G(x) - \tilde{C}(G(x), G(x)) = H_{1:2}(G(x))$$

and

$$F_{2:2}(y) = F_1(y)F_2(y) = \tilde{C}(G(y), G(y)) = H_{2:2}(G(y)),$$

where  $H_{1:2}(u) = 2u - \tilde{C}(u, u)$  and  $H_{2:2}(v) = \tilde{C}(v, v)$  depend on  $F_1$  and  $F_2$ . Actually they only depend on the diagonal section of  $\tilde{C} = \tilde{C}(F_1, F_2)$ . Hence,  $F_{i:2}^{-1}(u) = G^{-1}(H_{i:2}^{-1}(u))$  for  $i = 1, 2$ . Thence the connecting copula of the order statistics can be written as

$$\begin{aligned} K(u, v) &= F_1(F_{1:2}^{-1}(u))F_2(F_{2:2}^{-1}(v)) + F_1(F_{2:2}^{-1}(v))F_2(F_{1:2}^{-1}(u)) - F_1(F_{1:2}^{-1}(u))F_2(G_{1:2}^{-1}(u)) \\ &= 2\tilde{C}(H_{1:2}^{-1}(u), H_{2:2}^{-1}(v)) - \tilde{C}(H_{1:2}^{-1}(u), H_{1:2}^{-1}(u)) \end{aligned}$$

for  $F_{1:2}^{-1}(u) \leq F_{2:2}^{-1}(v)$  and  $K(u, v) = v$  for  $F_{2:2}^{-1}(v) < F_{1:2}^{-1}(u)$ . Note that in this case,  $F_{1:2}^{-1}(u) \leq F_{2:2}^{-1}(v)$  (resp.  $>$ ) is equivalent to  $H_{1:2}^{-1}(u) \leq H_{2:2}^{-1}(v)$  ( $>$ ). Of course,  $\tilde{C}$  and  $K$  are proper copulas that only depend on  $F_1$  and  $F_2$ , so they can be denoted by  $\tilde{C}_{F_1, F_2}$  and  $K_{F_1, F_2}$ .

**Example 3.2** ( $F_1 = F_2$ ). If  $X_1$  and  $X_2$  are identically distributed, then  $G = F_1 = F_2$  and  $F(x, y) = C(G(x), G(y))$ . Hence, from (3.1) and (3.2), we have

$$\tilde{F}(x, y) = \frac{1}{2}C(G(x), G(y)) + \frac{1}{2}C(G(y), G(x)) = \tilde{C}(G(x), G(y)), \quad (3.11)$$

where

$$\tilde{C}(u, v) = C_{\Pi}(u, v) = \frac{1}{2}C(u, v) + \frac{1}{2}C(v, u).$$

Thus  $\tilde{C}$  only depends on  $C$  and does not depend on  $G = F_1 = F_2$ . Analogously, from (3.7) and (3.8), the joint distribution of the order statistics is given by

$$F_{OS}(x, y) = 2\tilde{C}(G(x), G(y)) - \tilde{C}(G(x), G(x)) \quad \text{for } x \leq y \quad (3.12)$$

and  $F_{OS}(x, y) = \tilde{C}(G(y), G(y))$  for  $y < x$  and, from (3.5) and (3.6), the marginal distributions by

$$F_{1:2}(x) = 2G(x) - C(G(x), G(x)) = 1 - \delta_S(1 - G(x)) \quad (3.13)$$

and

$$F_{2:2}(y) = C(G(y), G(y)) = \delta_C(G(y)), \quad (3.14)$$

where  $\delta_C(x) = C(x, x)$  and  $\delta_S(x) = S(x, x)$  are the diagonal sections of the connecting and survival copulas of  $(X_1, X_2)$ . Thence

$$F_{1:2}^{-1}(u) = G^{-1}(1 - \delta_S^{-1}(1 - u))$$

and

$$F_{2:2}^{-1}(v) = G^{-1}(\delta_C^{-1}(v)).$$

Finally, the connecting copula of the order statistics can be written as

$$K(u, v) = 2\tilde{C}(1 - \delta_S^{-1}(1 - u), \delta_C^{-1}(v)) - \delta_C(1 - \delta_S^{-1}(1 - u)) \quad (3.15)$$

for  $v \geq \delta_C(1 - \delta_S^{-1}(1 - u))$  and  $K(u, v) = v$  for  $v < \delta_C(1 - \delta_S^{-1}(1 - u))$ .

**Example 3.3** ( $F$  exchangeable). This is a special case of Example 3.2 when  $C$  is exchangeable. Then  $\tilde{C} = C_\Pi = C$ . Hence, from (3.11), we have

$$\tilde{F}(x, y) = C(G(x), G(y)).$$

Analogously, from (3.12), the joint distribution of the order statistics is given by

$$F_{OS}(x, y) = 2C(G(x), G(y)) - C(G(x), G(x)) \quad \text{for } x \leq y$$

and  $F_{OS}(x, y) = C(G(y), G(y))$  for  $y < x$ . Analogously, the marginal the marginal distributions are given by (3.13) and (3.14). Thence, from (3.15), the connecting copula of the order statistics can be written as

$$K(u, v) = 2C(1 - \delta_S^{-1}(1 - u), \delta_C^{-1}(v)) - \delta_C(1 - \delta_S^{-1}(1 - u)) \quad (3.16)$$

for  $v \geq \delta_C(1 - \delta_S^{-1}(1 - u))$  and  $K(u, v) = v$  for  $v < \delta_C(1 - \delta_S^{-1}(1 - u))$ .

**Example 3.4** ( $F_1 = F_2$  and  $C = C_I$ ). This is a well known example (Avérous, Genest and Kochar, 2005) which is a special case of Examples 3.1 and 3.2. Then we have  $\tilde{C}(u, v) = C(u, v) = uv$ . Hence  $\tilde{F}(x, y) = G(x)G(y)$ , where  $G = F_1 = F_2$ . Analogously, the joint distribution of the order statistics is given by

$$F_{OS}(x, y) = 2G(x)G(y) - G^2(x) \quad \text{for } x \leq y$$

and

$$F_{OS}(x, y) = G^2(y) \quad \text{for } y < x$$

and the marginal distributions by  $F_{1:2}(x) = 2G(x) - G^2(x)$  and  $F_{2:2}(y) = G^2(y)$ . Thence, from (3.16), the connecting copula of the order statistics from IID random variables can be written as

$$K(u, v) = 2(1 - (1 - u)^{1/2})(v^{1/2}) - (1 - (1 - u)^{1/2})^2$$

for  $0 < u < 1$  and  $(1 - (1 - u)^{1/2})^2 \leq v < 1$  and  $K(u, v) = v$  for  $0 < u < 1$  and  $0 < v < (1 - (1 - u)^{1/2})^2$ . Of course,  $K$  is a fixed copula that does not depend on  $G = F_1 = F_2$ .

#### Acknowledgements

The authors wish to thank the anonymous reviewers for several helpful comments. We are specially grateful to a reviewer who suggested the names of the copulas  $\tilde{C}$  and  $C_{\Pi}$ . JN is partially Supported by Ministerio de Ciencia y Tecnología under grant MTM2006-12834 and Fundación Séneca under grant 08627/PI/08. FS is partially supported by Progetto di Ricerca Università Sapienza 2008 “Interazione e Dipendenza nei Modelli Stocastici”.

#### References

- Arnold, B.C., Balakrishnan, N., Nagaraja, H.N. 2008. A first course in order statistics. Unabridged republication of the 1992 original. Classics in Applied Mathematics, 54. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.
- Avérous, J., Genest, C., Kochar, S.C., 2005. On the dependence structure of order statistics. J. Multivariate Anal. 94, 159–171.

- David, H.A., Nagaraja, H.N., 2003. Order Statistics, Third edition. John Wiley & Sons, Hoboken, New Jersey.
- Durante, F., Jaworski, P., 2008. Absolutely continuous copulas with given diagonal sections. *Comm. Statist. Theory Methods* 37, 2924–2942.
- Jaworski, P. 2009. On copulas and their diagonals. *Information Sciences* 179, 2863–2871.
- Jaworski, P., Rychlik, T., 2008. On distributions of order statistics for absolutely continuous copulas with applications to reliability. *Kybernetika* 44, 757–776.
- Navarro, J., Spizzichino, F., 2009. Comparisons of series and parallel systems with components sharing the same copula. To appear in: *Appl. Stoch. Models Buss. Industry*.
- Navarro, J., Spizzichino, F., Balakrishnan, N., 2009. Average Systems and their Role in the study of Coherent Systems. Submitted.
- Nelsen, R.B., 2006. An introduction to copulas (2nd ed). Springer, New York.
- Rychlik, T., 1994. Distributions and expectations of order statistics for possibly dependent random variables. *J. Multivariate Anal.* 48, 31–42.
- Spizzichino, F., 2001. Subjective probability models for lifetimes. *Monographs on Statistics and Applied Probability*, 91. Chapman & Hall/CRC, Boca Raton, FL.