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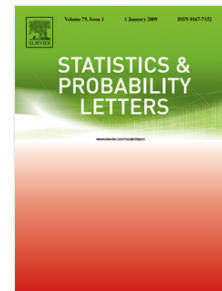
Some remarks on  $L^p$  dispersion orderings

Miguel López-Díaz

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**TITLE:**

Some remarks on  $L^p$  dispersion orderings

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**ABSTRACT:**

New results on the class of  $L^p$  dispersion orderings are stated in this paper. Namely we analyze different properties in relation with the preservation of  $L^p$  dispersion orderings by monotone transformations, truncation at the same quantile and order statistics, making emphasis on the exponential distribution.

**KEYWORDS:**

Exponential distribution,  $L^p$  dispersion ordering, Monotone transformation, Order statistic, Truncation at the same quantile.

# Some remarks on $L^p$ dispersion orderings

López-Díaz, Miguel

## Abstract

New results on the class of  $L^p$  dispersion orderings are stated in this paper. Namely we analyze different properties in relation with the preservation of  $L^p$  dispersion orderings by monotone transformations, truncation at the same quantile and order statistics, making emphasis on the exponential distribution.

**Keywords:** Exponential distribution,  $L^p$  dispersion ordering, Monotone transformation, Order statistic, Truncation at the same quantile.

## 1 Introduction

Different stochastic orderings have been proposed in the statistical literature to order probability distributions in terms of dispersion, in both the univariate and the multivariate cases.

Perhaps the most studied stochastic ordering for comparing dispersion in the univariate case is the ordering referred to as the *dispersive ordering*. So, if  $F$  and  $G$  are two distribution functions, it is said that  $F$  is *less dispersive than*  $G$ , denoted by  $F \preceq_{Disp} G$ , if

$$F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u) \text{ for all } 0 < u < v < 1,$$

with  $F^{-1}$  and  $G^{-1}$  being the quantile functions of  $F$  and  $G$  respectively.

The reader is referred for instance to Lewis and Thompson (1981), Shaked (1982), Rojo and He (1991), Müller and Stoyan (2002), Jeon et al. (2006) or Shaked and Shanthikumar (2007) for different characterizations, properties, examples and references to this concept.

The dispersive ordering has been weakened in Fernández-Ponce and Suárez-Lloréns (2003) by means of the so called *weakly dispersive ordering*. Given  $F$  and  $G$ , which are continuous distribution functions, it is said that  $F$  is *less weakly dispersive than*  $G$ , denoted by  $F \preceq^{wd} G$ , if for all  $\varepsilon > 0$

it holds that

$$\sup_{x \in \mathbb{R}} |F(x + \varepsilon) - F(x)| \geq \sup_{x \in \mathbb{R}} |G(x + \varepsilon) - G(x)|.$$

In terms of continuous random variables the weakly dispersive ordering reads that  $X \preceq^{wd} Y$  if for any  $\varepsilon > 0$ , and for any interval of length  $\varepsilon$  in the support of the variable  $Y$ , let us denote such an interval by  $I_{Y,\varepsilon}$ , it is possible to find  $I_{X,\varepsilon}$ , an interval in the support of  $X$  with length  $\varepsilon$ , such that  $P(X \in I_{X,\varepsilon}) \geq P(Y \in I_{Y,\varepsilon})$ .

The weakly dispersive ordering verifies some desirable properties that the dispersive ordering does not satisfy. In particular, there is an anomaly that the dispersive ordering suffers, that is, if  $X$  is a random variable then  $X$  and  $-X$  are not necessarily equally dispersive or even ordered, which does not occur in the weakly dispersive ordering.

The weakly dispersive order was generalized under mild conditions in Carleos and López-Díaz (2010) by means of the so called  *$L^p$  dispersion orderings*.

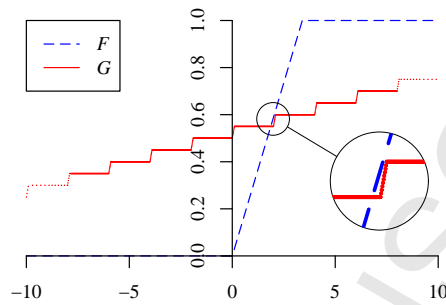
The weakly dispersive ordering is based on the *concentration function* of a probability (see for instance Götze and Zaitsev (1998)). Given  $P$  a probability, its concentration function is defined by  $Q(P, \varepsilon) = \sup_{x \in \mathbb{R}} P([x, x + \varepsilon])$  with  $\varepsilon > 0$ . Note that in some sense the concentration function is a kind of  $L^\infty$ -type norm of the mapping  $x \rightarrow F(x + \varepsilon) - F(x)$ , where  $F$  is the (continuous) distribution function of the probability  $P$ .

The weakly dispersive ordering could be too restrictive to rank distributions in some cases, as is seen in Carleos and López-Díaz (2010) in a problem of genetics, namely in an analysis on the dispersive effect of continuous trait genes. To illustrate such an idea let us consider the following distribution functions given in Figure 1.

From an intuitive point of view it seems that the distribution of  $F$  is less dispersive than that of  $G$ , but the slope of the non-horizontal part of  $G$  is greater than that of  $F$ , and so  $F$  and  $G$  are not ordered by the weakly dispersive ordering. It seems that in cases like this, the weakly dispersive ordering could be too restrictive since it involves an  $L^\infty$ -type norm.

To smooth the behaviour of the weakly dispersive order it is possible to consider  $L^p$  norms with  $p \in [1, \infty]$  instead of the  $L^\infty$  norm to introduce a more versatile family of dispersion orderings. Thus the  *$L^p$  dispersion orderings* are introduced in the following way.

The symbol  $\mathcal{F}$  will denote the class of continuous distribution functions on  $\mathbb{R}$ . If  $F \in \mathcal{F}$ , then  $P_F$  will stand for the probability induced by  $F$  on the measurable space  $(\mathbb{R}, \mathcal{B})$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -field on  $\mathbb{R}$ . If  $X$  is a random

Figure 1: Distribution functions  $F$  and  $G$ .

variable,  $P_X$  will indicate the probability induced by  $X$  on  $(\mathbb{R}, \mathcal{B})$ , and  $F_X$  its distribution function.

Given  $F \in \mathcal{F}$  and  $\varepsilon > 0$ , we define the mapping  $W_F^\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  by  $W_F^\varepsilon(x) = F(x + \frac{\varepsilon}{2}) - F(x - \frac{\varepsilon}{2})$ .

Then, if  $F, G \in \mathcal{F}$  and  $p \in [1, \infty]$ ,  $F$  is said to be less dispersive than  $G$  in the  $L^p$  dispersion order, if for all  $\varepsilon > 0$  it holds that

$$\|W_F^\varepsilon\|_{L^p(P_F)} \geq \|W_G^\varepsilon\|_{L^p(P_G)},$$

where  $\|\cdot\|_{L^p(P)}$  denotes the  $L^p$ -norm with respect to a probability  $P$  on  $(\mathbb{R}, \mathcal{B})$ , that is, given  $f : \mathbb{R} \rightarrow \mathbb{R}$ , then

$$\|f\|_{L^p(P)} = \left( \int_{\mathbb{R}} |f|^p dP \right)^{1/p} \quad \text{if } p \in [1, \infty), \quad \text{and}$$

$$\|f\|_{L^\infty(P)} = \inf \{S_f(N) : N \in \mathcal{B}, P(N) = 0\}, \quad \text{where } S_f(N) = \sup_{x \in \mathbb{R} \setminus N} |f(x)|.$$

We will denote the above relation by  $F \preceq_{L^p} G$ . As usual, if  $X$  and  $Y$  are continuous random variables,  $X \preceq_{L^p} Y$  will indicate that  $F_X \preceq_{L^p} F_Y$ .

The reader is referred to Carleos and López-Díaz (2010) for different results, examples and interpretations of the class of  $L^p$ -dispersion orderings and an application of such a class to the field of genetics, namely, to the dispersive effect of continuous trait genes.

We should note that for distributions with convex support, the weakly dispersive ordering and the  $L^\infty$  order are the same (Proposition 7 in Carleos and López-Díaz (2010)). Thus under mild conditions, the class of  $L^p$  dispersive orderings contains the weakly dispersive ordering as a particular case.

In the above reference it is shown that the dispersive order implies the  $L^p$  ordering for any value of  $p \in [1, \infty]$  (Proposition 9 in Carleos and López-Díaz (2010)). The converse of this result is false. It is enough to consider a random variable  $X$  such that the relation  $X \preceq_{Disp} -X$  does not hold. Note that the  $L^p$  orderings are sign-free (Proposition 2 in Carleos and López-Díaz (2010)). As an example of distributions satisfying the  $L^1$  ordering and not the dispersive order we can consider the distribution functions  $F$  and  $G$  given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x < 1, \\ 1 & \text{otherwise,} \end{cases} \quad G(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x}{8} & \text{if } 0 < x < 2, \\ \frac{3x}{2} - \frac{19}{4} & \text{if } 2 \leq x < 2.1, \\ \frac{x}{8} + \frac{1.9}{8} & \text{if } 2.1 \leq x < 6.1, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that  $F$  and  $G$  are not ordered with respect to weakly dispersive ordering since

$$\sup_{x \in \mathbb{R}} |F(x+1) - F(x)| = 1 > \sup_{x \in \mathbb{R}} |G(x+1) - G(x)| \quad \text{and} \\ \sup_{x \in \mathbb{R}} |F(x+0.1) - F(x)| < \sup_{x \in \mathbb{R}} |G(x+0.1) - G(x)| = 1/4.$$

Therefore  $F$  and  $G$  are not ordered by the dispersive order since this implies the weakly dispersive order. On the other hand, it is not hard to see that  $\int_{\mathbb{R}} W_F^\varepsilon(x) dP_F \geq \int_{\mathbb{R}} W_G^\varepsilon(x) dP_G$  for all  $\varepsilon > 0$ , that is,  $F \preceq_{L^1} G$ .

In this paper we present new results on the family of  $L^p$  orderings. The paper is structured as follows: Section 2 is devoted to establishing some results on the preservation of such orderings, mainly by monotone transformations. Section 3 studies the case of truncation at the same quantile. The analysis of order statistics is developed in Section 4, where special emphasis is made on some results of the exponential distribution.

Throughout the paper we will consider random variables with distribution functions in  $\mathcal{F}$ , that is, their distribution functions are continuous.

## 2 Preservation of $L^p$ dispersion orderings by monotone transformations

In this section we analyze some results on  $L^p$  dispersion orderings in relation with the preservation of such orderings, especially by monotone transformations.

First of all we should note that in the same way that the dispersive ordering is not preserved by monotone functions (see for instance Shaked (1982)), the  $L^p$  dispersive orderings are not preserved in general by monotone transformations as the following example shows.

**Example 2.1.** Let  $Y \sim_{st} U_{(0,1)}$  (uniform distribution on the interval  $(0, 1)$ ) and  $X \sim_{st} U_{(10,10.5)}$ . Trivially  $X \sim_{st} g(Y)$ , with  $g : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g(x) = 10 + 0.5x$ . Note that  $g$  is a contraction, thus Theorem 1 in Carleos and López-Díaz (2010) proves that  $X \preceq_{L^p} Y$  for all  $p \in [1, \infty]$ .

Consider now the random variables  $e^X$  and  $e^Y$  which obviously are continuous with bounded supports. In accordance with Proposition 6 in Carleos and López-Díaz (2010), if  $e^X \preceq_{L^p} e^Y$  for some  $p \in [1, \infty]$ , then  $m(\text{supp } e^X) \leq m(\text{supp } e^Y)$ , where  $m$  denotes the Borel measure on  $\mathbb{R}$  and  $\text{supp}$  stands for the support of a distribution. However,  $m(\text{supp } e^X) = e^{10.5} - e^{10}$  and  $m(\text{supp } e^Y) = e - 1$ .

We should note that this example also shows that the weakly dispersive ordering is not preserved under monotone transformations since, in this case, the orderings  $\preceq^{wd}$  and  $\preceq_{L^\infty}$  are the same, note that the supports of  $X$  and  $Y$  are convex.

In the first place we state a result on the preservation of  $L^p$  dispersion orderings by the product of scalars that we will apply later in the paper.

**Proposition 2.2.** *Let  $X, Y$  be random variables and  $a \in \mathbb{R}$ . For any  $p \in [1, \infty]$  the relation  $X \preceq_{L^p} Y$  implies  $aX \preceq_{L^p} aY$ .*

*Proof.* The case  $a = 0$  is obvious. Let  $a > 0$  and  $p < \infty$ . Then for all  $\varepsilon > 0$  we have that

$$\|W_{F_{aX}}^\varepsilon\|_{L^p(P_{aX})}^p = \int_{\mathbb{R}} W_{F_{aX}}^\varepsilon(x)^p dP_{aX} = \int_{\mathbb{R}} (F_{aX}(x + \frac{\varepsilon}{2}) - F_{aX}(x - \frac{\varepsilon}{2}))^p dP_{aX}.$$

Let us consider the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  with  $T(x) = ax$ . Then the above integral is equal to (see for instance Theorem 16.13 in Billingsley (1995))

$$\begin{aligned} & \int_{\mathbb{R}} (F_{aX}(ax + \frac{\varepsilon}{2}) - F_{aX}(ax - \frac{\varepsilon}{2}))^p dP_X \\ &= \int_{\mathbb{R}} (F_X(x + \frac{\varepsilon}{2a}) - F_X(x - \frac{\varepsilon}{2a}))^p dP_X = \|W_{F_X}^{\frac{\varepsilon}{2a}}\|_{L^p(P_X)}^p. \end{aligned}$$

That is,  $\|W_{F_{aX}}^\varepsilon\|_{L^p(P_{aX})} = \|W_{F_X}^{\frac{\varepsilon}{2a}}\|_{L^p(P_X)}$ , and the same formula holds for the random variable  $Y$ .



Since  $X \preceq_{L^p} Y$ , we have that  $\|W_{F_X}^\varepsilon\|_{L^p(P_X)} \geq \|W_{F_Y}^\varepsilon\|_{L^p(P_Y)}$  for all  $\varepsilon > 0$ , and so  $\|W_{F_{aX}}^\varepsilon\|_{L^p(P_{aX})} \geq \|W_{F_{aY}}^\varepsilon\|_{L^p(P_{aY})}$ , which proves the result when  $a > 0$  and  $p \in [1, \infty)$ .

If  $a < 0$ , the above development leads to  $-aX \preceq_{L^p} -aY$ . Proposition 2 in Carleos and López-Díaz (2010) states that the  $L^p$  orderings are sign-free, which concludes the proof when  $p \in [1, \infty)$ .

Now, let us analyze the case  $p = \infty$ . We have proved that for all  $p \in [1, \infty)$  and  $\varepsilon > 0$ , it holds that  $\|W_{F_{aX}}^\varepsilon\|_{L^p(P_{aX})} = \|W_{F_X}^\varepsilon\|_{L^p(P_X)}$ . As a consequence we obtain that

$$\|W_{F_{aX}}^\varepsilon\|_{L^\infty(P_{aX})} = \lim_{p \rightarrow \infty} \|W_{F_{aX}}^\varepsilon\|_{L^p(P_{aX})} = \lim_{p \rightarrow \infty} \|W_{F_X}^\varepsilon\|_{L^p(P_X)} = \|W_{F_X}^\varepsilon\|_{L^\infty(P_X)}$$

for all  $\varepsilon > 0$ , and we have the same relation for the random variable  $Y$ . This obviously leads to the result when  $p = \infty$ .  $\square$

Now we analyze the behavior of the  $L^p$  dispersion orderings when random variables are transformed by means of monotone functions satisfying a Lipschitz type condition.

**Proposition 2.3.** *Let  $X$  be a random variable, let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a monotone mapping such that  $|h(y) - h(x)| \leq k|y - x|$  for all  $x, y \in \mathbb{R}$ , with  $k \in (0, \infty)$ . Then for all  $p \in [1, \infty]$  it holds that  $h(X) \preceq_{L^p} kX$ .*

*Proof.* It is a consequence of Theorem 2.1 in Rojo and He (1991). Such a result reads that  $h(X) \preceq_{Disp} kX$  if  $h$  is nondecreasing and  $h(X) \preceq_{Disp} -kX$  when  $h$  is nonincreasing. Since the dispersive order implies the  $L^p$  ordering for any value of  $p \in [1, \infty]$  (Proposition 9 in Carleos and López-Díaz (2010)) and the  $L^p$  orderings are sign-free (Proposition 2 in Carleos and López-Díaz (2010)), we obtain the result.  $\square$

In a similar way, an analogous result can be deduced when the inequality of the Lipschitz condition is reversed.

**Proposition 2.4.** *Let  $Y$  be a random variable, let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a monotone mapping such that  $|h(y) - h(x)| \geq k|y - x|$  for all  $x, y \in \mathbb{R}$ , with  $k \in (0, \infty)$ . Then it holds that  $kY \preceq_{L^p} h(Y)$  for all  $p \in [1, \infty]$ .*

As a consequence of the preceding results we can make the following statement.

**Theorem 2.5.** *Let  $X, Y$  be random variables with  $X \preceq_{L^p} Y$  for some  $p \in [1, \infty]$ . Let  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  be monotone mappings such that  $|h_1(y) - h_1(x)| \leq$*

$k_1|y - x|$  and  $|h_2(y) - h_2(x)| \geq k_2|y - x|$  for all  $x, y \in \mathbb{R}$ , with  $0 < k_1 \leq k_2$ . Then it holds that

$$h_1(X) \preceq_{L^p} k_1 X \preceq_{L^p} k_2 Y \preceq_{L^p} h_2(Y).$$

*Proof.* The first and the third relationships are direct consequences of Proposition 2.3 and 2.4 respectively.

With respect to the second relationship, Proposition 2.2 implies that  $k_1 X \preceq_{L^p} k_1 Y$ , but  $k_1 Y \sim_{st} (k_1/k_2)k_2 Y$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g(x) = (k_1/k_2)x$  is a contraction, so Theorem 1 in Carleos and López-Díaz (2010) leads to  $k_1 Y \preceq_{L^p} k_2 Y$ . The transitivity of the  $L^p$  dispersion orderings concludes the result.  $\square$

Now we analyze a result on random variables with symmetric distribution.

**Lemma 2.6.** *Let  $X, Y$  be random variables with symmetric probability distribution about  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$  respectively. Then for any  $p \in [1, \infty]$  it holds that  $X \preceq_{L^p} Y$  if and only if  $|X - a| \preceq_{L^p} |Y - b|$ .*

*Proof.* Since for any  $p \in [1, \infty]$  the  $L^p$  dispersion ordering is invariant under translations (see Proposition 3 in Carleos and López-Díaz (2010)), we can assume without loss of generality that  $a = b = 0$ .

Trivially, the symmetry condition of the probability distributions implies that  $F_{|X|}(x) = 2F_X(x) - 1$  for all  $x \in \mathbb{R}$ , and the same relation is satisfied by  $F_{|Y|}$  and  $F_Y$ .

Thus for all  $p \in [1, \infty)$  and  $\varepsilon > 0$  it holds that

$$\begin{aligned} \|W_{F_{|X|}}^\varepsilon\|_{L^p(P_{|X|})}^p &= \int_{\mathbb{R}} (F_{|X|}(x + \frac{\varepsilon}{2}) - F_{|X|}(x - \frac{\varepsilon}{2}))^p dP_{|X|} \\ &= 2^p \int_{\mathbb{R}} (F_X(x + \frac{\varepsilon}{2}) - F_X(x - \frac{\varepsilon}{2}))^p dP_{|X|}. \end{aligned}$$

Let us take the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  with  $T(x) = |x|$ . We should note that for all  $B \in \mathcal{B}$  it holds that  $P_X(T^{-1}(B)) = P_{|X|}(B)$ . Thus the last integral is equal to

$$2^p \int_{\mathbb{R}} (F_X(|x| + \frac{\varepsilon}{2}) - F_X(|x| - \frac{\varepsilon}{2}))^p dP_X.$$

Now, the symmetry of the distribution of  $X$  with respect to 0 implies that for all  $x \in \mathbb{R}$  it holds that

$$F_X(|x| + \frac{\varepsilon}{2}) - F_X(|x| - \frac{\varepsilon}{2}) = F_X(x + \frac{\varepsilon}{2}) - F_X(x - \frac{\varepsilon}{2}),$$

and so the last integral equals

$$2^p \int_{\mathbb{R}} \left( F_X(x + \frac{\varepsilon}{2}) - F_X(x - \frac{\varepsilon}{2}) \right)^p dP_X = 2^p \|W_{F_X}^\varepsilon\|_{L^p(P_X)}^p.$$

Therefore, for all  $\varepsilon > 0$  we have that  $\|W_{F_{|X|}}^\varepsilon\|_{L^p(P_{|X|})} = 2\|W_{F_X}^\varepsilon\|_{L^p(P_X)}$ , which leads to the result when  $p \in [1, \infty)$ .

Now, since for all  $p \in [1, \infty)$  and  $\varepsilon > 0$  we have that  $\|W_{F_{|X|}}^\varepsilon\|_{L^p(P_{|X|})} = 2\|W_{F_X}^\varepsilon\|_{L^p(P_X)}$ , the same result holds when  $p = \infty$ , which proves the lemma in this case.  $\square$

### 3 $L^p$ dispersion orderings and truncation at the same quantile

In this section we study the behaviour of the  $L^p$  dispersion orderings in relation to truncation at the same quantile.

Throughout this section, we will consider distribution functions having unique quantiles.

The  $\alpha$  truncation of a distribution function  $F$ , denoted by  $F_\alpha$ , is determined by the distribution function

$$F_\alpha(x) = \begin{cases} 0 & \text{if } x < x_{\alpha/2}, \\ (1 - \alpha)^{-1}(F(x) - \frac{\alpha}{2}) & \text{if } x \in [x_{\alpha/2}, x_{1-\alpha/2}], \\ 1 & \text{if } x > x_{1-\alpha/2}, \end{cases}$$

where  $x_\beta$  denotes the  $\beta$ -quantile of  $F$ , thus  $F(x_{\alpha/2}) = \alpha/2$  and  $F(x_{1-\alpha/2}) = 1 - \alpha/2$ .

It is trivial that if  $F$  is continuous, then also  $F_\alpha$ .

In general, the  $L^p$  dispersion orderings are not preserved by truncation at the same quantile. That is, if  $F$  and  $G$  are distribution functions satisfying that  $F \preceq_{L^p} G$  for some  $p \in [1, \infty]$ , then the condition  $F_\alpha \preceq_{L^p} G_\alpha$  is not necessarily true as the example below shows.

Although the  $L^p$  dispersion orderings are not preserved in general by truncation at the same quantile, the converse result holds as is proved later in the paper. That is, if we have the condition  $F_\alpha \preceq_{L^p} G_\alpha$  for values of  $\alpha$  close enough to 0, then  $F \preceq_{L^p} G$  is satisfied.

Now we show that the  $L^p$  dispersion orderings are not preserved in general by truncation.

**Example 3.1.** Let us consider the distribution functions  $F$  and  $G$  given by the graphics in Figure 2, where  $a, a', b$  and  $b'$  satisfy that  $b - a > b' - a'$ ,

$F(a) = G(a') = \alpha/2$  and  $F(b) = G(b') = 1 - \alpha/2$ . Let us see that  $F \preceq_{L^\infty} G$ , but the relation  $F_\alpha \preceq_{L^\infty} G_\alpha$  is false.

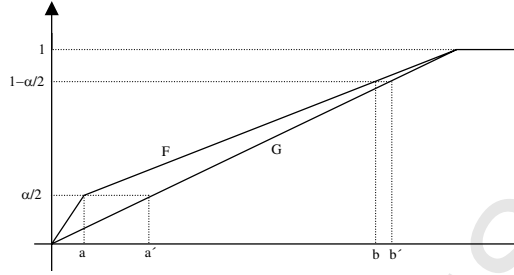


Figure 2: Distribution functions  $F$  and  $G$  in Example 3.1.

Since  $F$  and  $G$  are distribution functions with convex support, Proposition 7 in Carleos and López-Díaz (2010) reads that  $F \preceq_{L^\infty} G$  if and only if  $F \preceq^{wd} G$ .

By means of Theorem 4 in Fernández-Ponce and Suárez-Lloréns (2003), we trivially obtain that  $F \preceq_{L^\infty} G$  since  $F \preceq_{st} G$  ( $\preceq_{st}$  denotes the usual stochastic ordering),  $G$  is concave and the infima of the supports of both distribution are equal and finite.

On the other hand, it is easy to see that the  $\alpha$ -truncation of both distributions satisfy that  $F_\alpha \sim_{st} U_{(a,b)}$  and  $G_\alpha \sim_{st} U_{(a',b')}$ . The relation  $F_\alpha \preceq_{L^\infty} G_\alpha$  is false, observe that  $F_\alpha \preceq_{L^\infty} G_\alpha$  would imply that the Borel measure of the support of  $F_\alpha$  should not be greater than the one of the support of  $G_\alpha$  (Proposition 6 in Carleos and López-Díaz (2010)), but  $b - a > b' - a'$ .

Moreover, on the basis of this example we can guarantee that there exist infinite values of  $p \in [1, \infty)$  such that the relation  $F_\alpha \preceq_{L^p} G_\alpha$  does not hold, otherwise  $F_\alpha \preceq_{L^\infty} G_\alpha$  should be true.

However if an  $L^p$  dispersive ordering holds for the truncations of two distributions, such a dispersion ordering is satisfied for the distributions as is proved now.

The cases  $p \in [1, \infty)$  and  $p = \infty$  are analyzed in different statements.

**Proposition 3.2.** *Let  $F, G \in \mathcal{F}$  and let  $p \in [1, \infty)$ . If there exists  $\delta \in (0, 1/2)$  such that for all  $\alpha \in (0, \delta)$  it holds that  $F_\alpha \preceq_{L^p} G_\alpha$ , then we have that  $F \preceq_{L^p} G$ .*

*Proof.* By hypothesis we have that  $F_\alpha \preceq_{L^p} G_\alpha$  for all  $\alpha \in (0, \delta)$ . In particular for all  $n \geq n_0$  with  $1/n_0 < \delta$  we have that  $F_{1/n} \preceq_{L^p} G_{1/n}$ .

Note that for all  $x \in \mathbb{R}$  it holds that  $\lim_n F_{1/n}(x) = F(x)$ . That implies that the sequence of distribution functions  $\{F_{1/n}\}_n$  tends to  $F$  in the weak

convergence. Obviously the same result holds when we consider  $\{G_{1/n}\}_n$  and  $G$ .

Proposition 4 in Carleos and López-Díaz (2010) reads that the  $L^p$  dispersion orderings with  $p \in [1, \infty)$  are preserved by the weak convergence. Therefore we obtain that  $F \preceq_{L^p} G$ .  $\square$

Now the case  $p = \infty$  is analyzed.

**Proposition 3.3.** *Let  $F, G \in \mathcal{F}$  with convex support. If there exists  $\delta \in (0, 1/2)$  such that for all  $\alpha \in (0, \delta)$  it holds that  $F_\alpha \preceq_{L^\infty} G_\alpha$ , then  $F \preceq_{L^\infty} G$ .*

*Proof.* The proof is similar to the above one. We only have to notice that since  $F$  has convex support,  $F_{1/n}$  verifies the same condition for any  $n \in \mathbb{N}$ , and the same holds for  $G$  and  $G_{1/n}$ .

Proposition 5 in Carleos and López-Díaz (2010) reads that the  $L^\infty$  dispersion ordering is preserved by the weak convergence when we consider distributions with convex supports, which leads to  $F \preceq_{L^\infty} G$  in the same way as in Proposition 3.2.  $\square$

Particular conditions which guarantee the preservation of the  $L^p$  dispersion orderings by truncation at the same quantile are now stated. The following technical lemma is a supporting result which allows us to find such conditions.

**Lemma 3.4.** *Let  $F \in \mathcal{F}$  and let  $f : \mathbb{R} \rightarrow [0, \infty)$  be a continuous mapping. It holds that*

$$\int_{\mathbb{R}} f dP_{F_\alpha} = (1 - \alpha)^{-1} \int_{\mathbb{R}} f I_{[x_{\alpha/2}, x_{1-\alpha/2}]} dP_F.$$

*Proof.* In the first place consider the map  $g = \sum_{i=1}^n a_i I_{(\underline{l}_i, \bar{l}_i]}$  with  $a_i \in [0, \infty)$ ,  $\underline{l}_i, \bar{l}_i \in \mathbb{R}$ ,  $\underline{l}_i < \bar{l}_i$ ,  $1 \leq i \leq n$ , where  $I_C$  stands for the indicator function of the set  $C \subset \mathbb{R}$ .

Let us denote by  $\Lambda$  the set of indexes given by  $\{1, \dots, n\} \setminus \{i : \bar{l}_i \leq x_{\alpha/2} \text{ or } \underline{l}_i \geq x_{1-\alpha/2}, 1 \leq i \leq n\}$ .

Thus,

$$\begin{aligned} \int_{\mathbb{R}} g dP_{F_\alpha} &= \sum_{i=1}^n a_i P_{F_\alpha}((\underline{l}_i, \bar{l}_i]) = \sum_{i=1}^n a_i (F_\alpha(\bar{l}_i) - F_\alpha(\underline{l}_i)) \\ &= \sum_{i \in \Lambda} a_i (F_\alpha(\min\{\bar{l}_i, x_{1-\alpha/2}\}) - F_\alpha(\max\{\underline{l}_i, x_{\alpha/2}\})) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in \Lambda} a_i (1 - \alpha)^{-1} (F(\min \{\bar{l}_i, x_{1-\alpha/2}\}) - F(\max \{l_i, x_{\alpha/2}\})) \\
&= \sum_{i \in \Lambda} a_i (1 - \alpha)^{-1} P_F((\max \{l_i, x_{\alpha/2}\}, \min \{\bar{l}_i, x_{1-\alpha/2}\})) \\
&= (1 - \alpha)^{-1} \int_{x_{\alpha/2}}^{x_{1-\alpha/2}} \sum_{i=1}^n a_i I_{(l_i, \bar{l}_i)}(x) dP_F = (1 - \alpha)^{-1} \int_{x_{\alpha/2}}^{x_{1-\alpha/2}} g(x) dP_F \\
&= (1 - \alpha)^{-1} \int_{\mathbb{R}} g I_{[x_{\alpha/2}, x_{1-\alpha/2}]} dP_F.
\end{aligned}$$

Now given  $f : \mathbb{R} \rightarrow [0, \infty)$  a continuous mapping, it is sufficient to consider a sequence  $\{s_n\}_n$  with  $s_n : \mathbb{R} \rightarrow [0, \infty)$ , measurable mappings such that  $\lim_{n \rightarrow \infty} s_n(x) = f(x)$  for all  $x \in \mathbb{R}$ ,  $s_n \leq s_{n+1}$  for all  $n \in \mathbb{N}$ ,  $s_n$  verifying a functional expression similar to that of  $g$  in the above part of the proof, and applying the monotone convergence theorem to derive the result. Note that the existence of such a sequence is well-known when  $f$  is continuous.  $\square$

The above lemma allows us to obtain conditions on the original distributions under which the  $L^p$  dispersion orderings are preserved by truncation at the same quantile.

**Proposition 3.5.** *Let  $F, G \in \mathcal{F}$ . Given  $p \in [1, \infty]$ , it holds that  $F_\alpha \preceq_{L^p} G_\alpha$  if and only if  $\|I_{[x_{\alpha/2}, x_{1-\alpha/2}]} W_{F_\alpha}^\varepsilon\|_{L^p(P_F)} \geq \|I_{[x_{\alpha/2}, x_{1-\alpha/2}]} W_{G_\alpha}^\varepsilon\|_{L^p(P_G)}$  for all  $\varepsilon > 0$ .*

*Proof.* In the first place let  $p \in [1, \infty)$ . In accordance with Lemma 3.4 we have that

$$\begin{aligned}
&\|W_{F_\alpha}^\varepsilon\|_{L^p(P_{F_\alpha})}^p = \int_{\mathbb{R}} (F_\alpha(x + \frac{\varepsilon}{2}) - F_\alpha(x - \frac{\varepsilon}{2}))^p dP_{F_\alpha} \\
&= (1 - \alpha)^{-1} \int_{\mathbb{R}} I_{[x_{\alpha/2}, x_{1-\alpha/2}]}(x) (F_\alpha(x + \frac{\varepsilon}{2}) - F_\alpha(x - \frac{\varepsilon}{2}))^p dP_F \\
&= (1 - \alpha)^{-1} \|I_{[x_{\alpha/2}, x_{1-\alpha/2}]} W_{F_\alpha}^\varepsilon\|_{L^p(P_F)}^p,
\end{aligned}$$

which easily leads to the result in this case.

If  $p = \infty$ , then

$$\begin{aligned}
&\|W_{F_\alpha}^\varepsilon\|_{L^\infty(P_{F_\alpha})} = \lim_{p \rightarrow \infty} \|W_{F_\alpha}^\varepsilon\|_{L^p(P_{F_\alpha})} \\
&= \lim_{p \rightarrow \infty} (1 - \alpha)^{-1/p} \|I_{[x_{\alpha/2}, x_{1-\alpha/2}]} W_{F_\alpha}^\varepsilon\|_{L^p(P_F)} = \|I_{[x_{\alpha/2}, x_{1-\alpha/2}]} W_{F_\alpha}^\varepsilon\|_{L^\infty(P_F)},
\end{aligned}$$

which trivially proves the proposition.  $\square$

#### 4 $L^p$ dispersion orderings and the order statistics: the case of the exponential distribution.

This section studies the behaviour of  $L^p$  dispersion orderings in relation to order statistics. Special emphasis is placed on the exponential distribution.

Given  $X$  a random variable and a random sample of size equal to  $n$  from such a variable,  $X_{r:n}$  will stand for the  $r^{\text{th}}$  order statistic.

$L^p$  dispersion orderings are not preserved in general by order statistics. The following example shows this affirmation.

**Example 4.1.** Let us consider the random variables  $X$  and  $Y$  with distribution functions

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \sqrt{x} & \text{if } x \in (0, 1), \\ 1 & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad F_Y(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \in (0, 1), \\ 1 & \text{if } x \geq 1. \end{cases}$$

Theorem 4 in Fernández-Ponce and Suárez-Lloréns (2003) leads easily to the condition  $X \preceq^{wd} Y$ . Since the support of  $X$  and  $Y$  are convex this is equivalent to  $X \preceq_{L^\infty} Y$ . On the other hand, it is trivial to see that the relation  $Y \preceq_{L^\infty} X$ , equivalently  $Y \preceq^{wd} X$ , is false.

Let us consider random samples of sizes equal to two from both distributions. Then

$$F_{X_{2:2}}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \in (0, 1), \\ 1 & \text{if } x \geq 1, \end{cases} \quad F_{Y_{2:2}}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^2 & \text{if } x \in (0, 1), \\ 1 & \text{if } x \geq 1. \end{cases}$$

Theorem 5 in Fernández-Ponce and Suárez-Lloréns (2003) proves that  $Y_{2:2} \preceq^{wd} X_{2:2}$ , or equivalently  $Y_{2:2} \preceq_{L^\infty} X_{2:2}$ , while  $X_{2:2} \preceq_{L^\infty} Y_{2:2}$ , equivalently  $X_{2:2} \preceq^{wd} Y_{2:2}$ , is clearly false. Thus, the  $\preceq_{L^\infty}$  order is not preserved by order statistics.

Note that this example shows that the weakly dispersive ordering is not preserved in general by order statistics.

Moreover, this example also shows that the  $L^p$  dispersion orderings are not preserved in general by order statistics even for values of  $p \in [1, \infty)$ . Note that if the  $L^p$  dispersion orderings were preserved when  $p \in [1, \infty)$ , then the same property should be verified when  $p = \infty$ .

In spite of this behavior, it is possible to study in depth some particular distributions whose order statistics preserve  $L^p$  orderings as is the case of the exponential distribution.

In order to make the total proof of this result more readable, we now state the following theorem.

**Theorem 4.2.** *Let  $X$  and  $Y$  be exponential random variables with parameters  $\lambda_X$  and  $\lambda_Y$  respectively. Then, the following conditions are equivalent:*

- i)  $X \preceq^{wd} Y$ ,*
- ii)  $\lambda_X \geq \lambda_Y$ ,*
- iii)  $X \preceq_{Disp} Y$ ,*
- iv)  $X \preceq_{L^p} Y$  for all  $p \in [1, \infty]$ ,*
- v) there exists  $p \in [1, \infty]$ , such that  $X \preceq_{L^p} Y$ .*

*Proof.* We should note that for all  $\varepsilon > 0$  it holds that

$$\begin{aligned} \sup_{x \in \mathbb{R}} |F_X(x + \varepsilon) - F_X(x)| &= \sup_{x \in (0, \infty)} |e^{-\lambda_X x} - e^{-\lambda_X(x + \varepsilon)}| \\ &= \sup_{x \in (0, \infty)} |e^{-\lambda_X x}(1 - e^{-\lambda_X \varepsilon})| = (1 - e^{-\lambda_X \varepsilon}), \end{aligned}$$

and analogously with the random variable  $Y$ . As a consequence, the conditions *i)* and *ii)* are equivalent.

By Lemma 4.1 in Rojo and He (1991) we have that  $X \preceq_{Disp} Y$  if and only if

$$\frac{f_X(x)}{1 - F_X(x)} \geq \lambda_Y,$$

where  $f_X$  stands for the density of  $X$ . Note that

$$\frac{f_X(x)}{1 - F_X(x)} = \lambda_X,$$

and so *ii)* and *iii)* are equivalent.

Regarding the condition *iii)*, we should note that Proposition 9 in Carleos and López-Díaz (2010) states that the dispersive ordering implies the  $L^p$  ordering for any value of  $p \in [1, \infty]$ , therefore *iii)* implies *iv)*.

Obviously *iv)* implies *v)*.

Now, let us suppose that the statement *v)* holds.

In the first place consider the case  $p = \infty$ . Since the supports of  $X$  and  $Y$  are convex, the orderings  $\preceq^{wd}$  and  $\preceq_{L^\infty}$  are the same, and so we obtain that *v)* implies *i)*, which concludes the proof in this case.

Now let us consider the case  $p \in [1, \infty)$ . By simple integration arguments we have that

$$\|W_{F_X}^\varepsilon\|_{L^p(P_{F_X})} = \left( \frac{e^{\lambda_X \frac{\varepsilon}{2}}}{p+1} \left( (1 - e^{-\lambda_X \varepsilon})^p - (1 - e^{-\lambda_X \frac{\varepsilon}{2}})^{p+1} \right) \right)^{1/p},$$



and obviously, the same formula holds for the random variable  $Y$ .

Since the mappings  $g_\varepsilon : (0, \infty) \rightarrow \mathbb{R}$  with

$$g_\varepsilon(z) = \frac{e^{z\frac{\varepsilon}{2}}}{p+1} \left( (1 - e^{-z\varepsilon})^p - (1 - e^{-z\frac{\varepsilon}{2}})^{p+1} \right)$$

are nondecreasing for all  $\varepsilon > 0$ , we deduce that the condition  $v)$  is equivalent to the condition  $ii)$ , and so we conclude the proof of the Theorem.  $\square$

On the basis of the above theorem, some results for order statistics and spacings on the  $L^p$  dispersion orderings and the exponential distribution are immediately proved.

**Theorem 4.3.** *Let  $X$  and  $Y$  be exponential random variables with parameters  $\lambda_X$  and  $\lambda_Y$  respectively. Then for any  $p \in [1, \infty]$ , it holds that  $X \preceq_{L^p} Y$  if and only if  $X_{r:n} \preceq_{L^p} Y_{r:n}$  for all  $1 \leq r \leq n$ .*

*Proof.* Let us suppose that  $X \preceq_{L^p} Y$  for some  $p \in [1, \infty]$ . In accordance with Theorem 4.2 we have that  $X \preceq_{Disp} Y$ .

Since the dispersive ordering is preserved by order statistics (see Lemma 3 in Bartoszewicz (1986)), we can conclude that  $X_{r:n} \preceq_{Disp} Y_{r:n}$  for all  $1 \leq r \leq n$ . On the other hand, the dispersive ordering implies the  $L^p$  orderings for any  $p \in [1, \infty]$  (Proposition 9 in Carleos and López-Díaz (2010)), and so  $X_{r:n} \preceq_{L^p} Y_{r:n}$  for all  $1 \leq r \leq n$ .

Let us see the converse result. If  $X_{r:n} \preceq_{L^p} Y_{r:n}$  for all  $1 \leq r \leq n$  with  $p \in [1, \infty]$ , in particular  $X_{1:n} \preceq_{L^p} Y_{1:n}$ . It is well-known that  $X_{1:n}$  and  $Y_{1:n}$  have exponential distributions with parameters  $n\lambda_X$  and  $n\lambda_Y$  respectively, so Theorem 4.2 implies that  $n\lambda_X \geq n\lambda_Y$ , which leads to  $X \preceq_{L^p} Y$ .  $\square$

A similar result for the exponential distribution can be obtained when we consider spacings.

If  $X$  is a random variable and we consider a random sample of size equal to  $n$  from such a distribution,  $V_{r:n}^X$  will denote the  $r^{th}$  sample spacing, that is,  $V_{r:n}^X = X_{r:n} - X_{r-1:n}$ , with  $1 \leq r \leq n$ , where  $X_{0:n}$  is equal to  $\inf \{x \in \mathbb{R} : F_X(x) > 0\}$  if this value is finite, and  $X_{0:n} = 0$  otherwise.

**Theorem 4.4.** *Let  $X$  and  $Y$  be exponential random variables with parameters  $\lambda_X$  and  $\lambda_Y$  respectively. Let  $p \in [1, \infty]$ , then  $X \preceq_{L^p} Y$  if and only if  $V_{r:n}^X \preceq_{L^p} V_{r:n}^Y$  for all  $1 \leq r \leq n$ .*

*Proof.* Theorem 4.2 in Rojo and He (1991) reads that  $X \preceq_{Disp} Y$  if and only if  $V_{r:n}^X \preceq_{Disp} V_{r:n}^Y$  for all  $1 \leq r \leq n$  when  $X$  and  $Y$  have exponential

distribution. On the other hand it is well-known that if  $X$  and  $Y$  have exponential distribution, then  $V_{r:n}^X$  and  $V_{r:n}^Y$ ,  $1 \leq r \leq n$ , satisfy the same property (see for instance Theorem 4.6.1 in Arnold *et al.* (1992)). Thus Theorem 4.2 leads to the result.  $\square$

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## References

- Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1992. A first course in order statistics. John Wiley and Sons, New York.
- Bartoszewicz, J., 1986. Dispersive ordering and the total time on test transformation. *Statist. Probab. Lett.* 4, 285-288.
- Billingsley, P., 1995. Probability and measure. John Wiley and Sons, New York.
- Carleos, C., López-Díaz, M., 2010. A new family of dispersive orderings. *Metrika* (in press).
- Fernández-Ponce, J.M., Suárez-Lloréns, A., 2003. An aging concept based on majorization. *Probab. Engrg. Inform. Sci.* 17, 107-117.
- Götze F., Zaitsev A.Y., 1998. Estimates for the rapid decay of concentration functions of  $n$ -fold convolutions. *J. Theoret. Probab.* 11, 715-731.
- Jeon, J., Kochar, S., Park, Chul G., 2006. Dispersive ordering-some applications and examples. *Statist. Papers* 47, 227-247.
- Lewis, T., Thompson, J.W., 1981. Dispersive distributions, and the connection between dispersivity and strong unimodality. *J. Appl. Probab.* 18, 76-90.
- Müller, A., Stoyan, D., 2002. Comparison methods for stochastic models and risks. John Wiley and Sons, Chichester.
- Rojo, J., He G.Z., 1991. New properties and characterizations of the dispersive ordering. *Statist. Probab. Lett.* 11, 365-372.

Shaked, M., 1982. Dispersive ordering of distributions. *J. Appl. Probab.* 19, 310-320.

Shaked, M., Shanthikumar, J.G., 2007. *Stochastic Orders*. Springer, New York.