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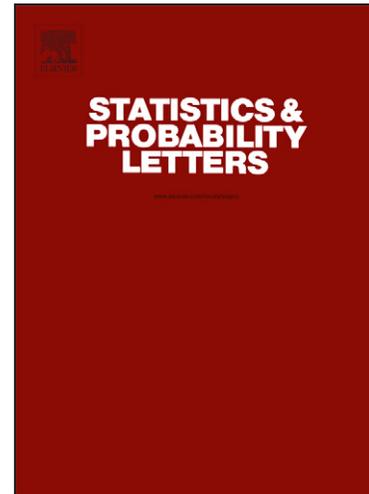
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A continuous non-Brownian motion martingale  
with Brownian motion marginal distributions

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**Abstract**

We construct a continuous martingale that has the same univariate marginal distributions as Brownian motion, but that is not Brownian motion.

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**Keywords and phrases:** Brownian motion; diffusion process; martingale; martingale problem.

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# 1 Introduction

Recently Fima Klebaner brought to our attention that it is an open problem whether there exists a continuous martingale that has the same univariate marginal distributions as standard Brownian motion, but which is not Brownian motion. In this note we make an explicit construction of such a martingale.

Let  $\{X_1(t)\}_{t \geq 0}$  and  $\{X_2(t)\}_{t \geq 0}$  be independent stochastic processes that are weak solutions to the time homogeneous diffusion type stochastic differential equation

$$dX(t) = \frac{dB(t)}{2X(t)}, \quad X(0) = 0, \quad (1)$$

where  $B$  is standard Brownian motion. Let  $Y$  be a random variable that is independent of  $X_1$  and  $X_2$ , and that has probability density function

$$f_Y(y) = \frac{4(\Gamma(\frac{3}{4}))^2}{2\pi^{3/2}\sqrt{1-(y/\sqrt{2})^4}} \quad \text{for } y \in (0, \sqrt{2}). \quad (2)$$

Our main result is the following affirmative answer to the above mentioned problem.

**Theorem 1.** *The process  $M(t) = X_1(t)X_2(t)Y$ ,  $t \geq 0$ , is a continuous martingale such that  $M(t) =_d B(t)$  for  $t \geq 0$ , where  $=_d$  denotes equality in distribution, but which is not Brownian motion.*

The problem to construct martingales with prescribed univariate marginal distributions has received attention recently, see e.g, Bibby et al. (2005), Campi (2004), Carr and Madan (2005), Hamza and Klebaner (2006a, 2006b), and Madan and Yor (2002). In part, this is due to the significance the problem has in modelling in mathematical finance. In particular, martingales with Brownian motion marginals are solutions to the Bachelier model (see e.g, Schachermayer and Teichmann, 2006, on this model).

Based on the fundamental work of Kellerer (1972), who gave necessary and sufficient conditions for the existence of Markovian martingales with prescribed marginal distributions, Madan and Yor (2002) gave three constructive solutions to that problem. Hamza and Klebaner (2006b) noted that, in the case of Brownian motion marginals, two of these solutions reduced to Brownian motion itself, while the third resulted in a non-continuous process. Hamza and Klebaner proceeded to construct a whole family of non-continuous martingales with Brownian motion marginal distributions. They also noted

that it remains an open problem whether there exists a non-Brownian motion continuous martingale with Brownian motion marginals.

## 2 Proof of Theorem 1

As the diffusion coefficient  $\sigma(y) = 1/(2y)$  of (1) is non-zero with

$$\int_{-\varepsilon}^{\varepsilon} \frac{dy}{\sigma(x+y)^2} = \frac{4((x+\varepsilon)^3 - (x-\varepsilon)^3)}{3} < \infty \quad \text{for } x \in \mathbb{R} \text{ and } \varepsilon > 0,$$

the Engelbert-Schmidt theory shows that (1) has a unique weak solution  $X$  (see e.g, Karatzas and Shreve, 1991, Theorem 5.5.7). To find the transition density

$$p_t(x, y) = \frac{\mathbf{P}\{X(t+s) \in (y, y+dy) \mid X(s) = x\}}{dy}$$

of  $X$ , we consider the transformed process  $Y = X^4$ , which by Itô's formula satisfies

$$dY(t) = 2\sqrt{Y(t)} dB(t) + \frac{3}{2} dt, \quad Y(0) = 0. \quad (3)$$

The process  $Y$  is usually called the CIR process, after Cox et al. (1985), but was considered already by Feller (1951). According to these authors,  $Y$  has transition density

$$\frac{\mathbf{P}\{Y(t+s) \in (y, y+dy) \mid Y(s) = x\}}{dy} = \begin{cases} \frac{x^{1/8}}{2t y^{1/8}} \exp\left\{-\frac{x+y}{2t}\right\} I_{-1/4}\left(\frac{\sqrt{xy}}{t}\right) & \text{for } x > 0, \\ \frac{1}{(2t)^{3/4} y^{1/4} \Gamma(\frac{3}{4})} \exp\left\{-\frac{y}{2t}\right\} & \text{for } x = 0, \end{cases}$$

where  $I$  is the modified Bessel function of the first kind [recall that  $I_{-1/4}(x) \sim (2/x)^{1/4}/\Gamma(\frac{3}{4})$  as  $x \downarrow 0$ , see e.g, Erdélyi et al., 1953b, Equation 7.2.2.12]. By the lower of these formulas, together with symmetry of the solution to (1), we get

$$p_t(0, y) = \frac{2^{1/4} y^2}{\Gamma(\frac{3}{4}) t^{3/4}} \exp\left\{-\frac{y^4}{2t}\right\} \quad \text{for } y \in \mathbb{R} \text{ and } t > 0. \quad (4)$$

The solution  $X$  to (1) is a continuous local martingale (see e.g, Karatzas and Shreve, 1991, Proposition 3.2.24). Therefore  $X$  is a continuous martingale if

$$\mathbf{E}\left\{\sup_{t \in [0, T]} |X(t)|^n\right\} = \mathbf{E}\left\{\sup_{t \in [0, T]} Y(t)^{n/4}\right\} < \infty \quad \text{for } T > 0, \quad (5)$$

for  $n = 1$  (see e.g, Karatzas and Shreve, 1991, Theorem 3.3.28). Since  $I_{-1/4}(x) \sim e^x/\sqrt{2\pi x}$  as  $x \rightarrow \infty$  (see e.g, Erdélyi et al., 1953b, Equation 7.13.1.6), Equation 3.1 of Albin (1993) holds. Further, Albin (1993), Proposition (v), shows that Equation 3.2-3.5 of Albin (1993) hold. And so we may apply Albin (1993), Theorem 1, to see that

$$\mathbf{P}\left\{\sup_{t \in [0, T]} Y(t)^{n/4} > u\right\} \sim 2\mathbf{P}\{Y(T) > u^{4/n}\} = 2 \int_{u^{4/n}}^{\infty} \frac{1}{(2T)^{3/4} y^{1/4} \Gamma(\frac{3}{4})} \exp\left\{-\frac{y}{2T}\right\} dy$$

as  $u \rightarrow \infty$ . From this it is obvious that (5) holds for  $n \geq 0$ , so that  $X$  is a martingale.

As  $X_1$  and  $X_2$  are martingales it is an elementary fact that they are martingales with respect to the filtrations generated by themselves  $\mathcal{F}_t^1 = \sigma(X_1(s) : 0 \leq s \leq t)$  and  $\mathcal{F}_t^2 = \sigma(X_2(s) : 0 \leq s \leq t)$ , respectively. Using the independence of  $X_1$ ,  $X_2$  and  $Y$ , it is an elementary exercise to see that  $X_1 X_2$  and  $M(t) = X_1(t) X_2(t) Y$  are martingales with respect to the filtrations  $\{\mathcal{F}_t^1 \vee \mathcal{F}_t^2\}_{t \geq 0}$  and  $\{\mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \sigma(Y)\}_{t \geq 0}$ , respectively [where (5) together with elementary arguments ensure sufficient integrability properties]. Now, if  $M$  is Brownian motion, then  $M$  has quadratic variation process  $\langle M \rangle_t = t$ , so that  $\langle X_1 X_2 \rangle_t = t/Y^2$ , giving  $\mathbf{E}\{\langle X_1 X_2 \rangle_t\} = t \mathbf{E}\{1/Y^2\} = \infty$ , by (2). This is impossible as

$$\mathbf{E}\{\langle X_1 X_2 \rangle_t\} \leq K \mathbf{E}\left\{\sup_{s \in [0, t]} |X_1(s)|^2 |X_2(s)|^2\right\} \leq K \mathbf{E}\left\{\sup_{s \in [0, t]} |X_1(s)|^4\right\} < \infty$$

for some constant  $K > 0$ , by the Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities (see e.g. Karatzas and Shreve, 1991, Proposition 3.3.28), together with (5).

It remains to show that  $M(t) =_d B(t)$  for  $t \geq 0$ : As  $p_t(0, y) = p_1(0, y/t^{1/4})/t^{1/4}$  by (4), we have  $X_1(t) =_d t^{1/4} X_1(1)$ . As  $B(t) =_d t^{1/2} B(1)$ , it follows that it is enough to prove that  $X_1(1) X_2(1) Y =_d B(1)$ . By symmetry, this follows if the Mellin transform  $\mathbf{E}\{|X_1(1)| |X_2(1)| Y^{s-1}\} = (\mathbf{E}\{|X_1(1)|^{s-1}\})^2 \mathbf{E}\{Y^{s-1}\}$  of  $|X_1(1)| |X_2(1)| Y$  agrees with

$$\mathbf{E}\{|B(1)|^{s-1}\} = \int_0^\infty y^{s-1} \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy = \frac{2^{s/2} \Gamma(s/2)}{\sqrt{2\pi}} \quad \text{for } s \geq 1. \quad (6)$$

By (4),  $|X_1(1)|$  and  $|X_2(1)|$  have common Mellin transform

$$\int_0^\infty y^{s-1} \frac{2 \cdot 2^{1/4} y^2}{\Gamma(\frac{3}{4})} \exp\left\{-\frac{y^4}{2}\right\} dy = \frac{2^{s/4} \Gamma((s+2)/4)}{2^{1/4} \Gamma(\frac{3}{4})} \quad \text{for } s \geq 1.$$

By (6) it is therefore enough to prove that  $Y$  has Mellin transform

$$\frac{2^{s/2} \Gamma(s/2)}{\sqrt{2\pi}} \Big/ \left( \frac{2^{s/4} \Gamma((s+2)/4)}{2^{1/4} \Gamma(\frac{3}{4})} \right)^2 = \frac{(\Gamma(\frac{3}{4}))^2 \Gamma(s/2)}{\sqrt{\pi} (\Gamma((s+2)/4))^2} = \frac{\Gamma(\frac{3}{4})^2 2^{s/2} \Gamma(s/4)}{2\pi \Gamma((s+2)/4)}, \quad (7)$$

where the second equality follows from the Gauss-Legendre multiplication formula

$$\Gamma(s/4) \Gamma((s+2)/4) = 2\sqrt{\pi} 2^{-s/2} \Gamma(s/2),$$

see e.g. Erdélyi et al. (1953a), Equation 1.2.11. However, the right-hand side of (7) is the Mellin transform of  $Y$ , by Erdélyi et al. (1954), Equations 6.1.2 and 6.2.31.

### 3 Further remarks

A non-Brownian motion martingale  $M$  with Brownian motion marginal distributions does not have independent increments, because if  $M(t+s) - M(s)$  and  $M(s) =_d B(s)$  are independent with sum  $M(t+s) =_d B(t+s)$ , then  $M(t+s) - M(s)$  must be Gaussian by Cramer's theorem (see e.g, Bondesson, 1995), which implies that  $M(t+s) - M(s) =_d B(t+s) - B(s) =_d B(t)$ , so that  $M$  is a Gaussian process with stationary and independent increments, thereby contradicting the assumption that  $M$  is not Brownian motion.

A non-Brownian motion martingale  $M$  with Brownian motion marginal distributions is not a Gaussian process, because  $M$  has uncorrelated increments (being a martingale), that are independent if  $M$  is Gaussian, which is impossible by the previous paragraph.

Our construction of a martingale  $M(t) = X_1(t)X_2(t)Y$  with Brownian motion marginal distributions crucially relied on the self-similarity  $p_t(0, y) = p_1(0, y/t^H)/t^H$  for some  $H \in (0, \frac{1}{2}]$ , which made it enough to prove that  $X_1(1)X_2(1)Y =_d B(1)$ . Using this self-similarity together with the Kolmogorov forward equation

$$\frac{\partial^2}{\partial y^2} \left( \frac{\sigma(y)^2}{2} p_t(x, y) \right) - \frac{\partial}{\partial t} p_t(x, y) = \frac{\partial^2}{\partial y^2} \left( \frac{1}{8y^2} p_t(x, y) \right) - \frac{\partial}{\partial t} p_t(x, y) = 0,$$

elementary calculations and the change of variable  $z = y/t^H$  give

$$\frac{\partial^2}{\partial z^2} \left( \frac{\sigma(t^H z)^2}{2t^{2H-1}} p_1(0, z) \right) + \frac{\partial}{\partial z} (H z p_1(0, z)) = 0. \quad (8)$$

This makes necessary that  $\sigma(z)^2/z^{2-1/H}$  is a constant (where in our case  $H = \frac{1}{4}$ ). With such a choice  $\sigma(z)^2 = \frac{1}{4}z^{2-1/H}$  of  $\sigma$ , (8) is the equation for the stationary density of a diffusion with this diffusion coefficient and drift coefficient  $\mu(z) = -H z$ , so that

$$p_1(0, z) = \frac{2C}{\sigma(z)^2} \exp \left\{ \int_0^z \frac{2\mu(y)}{\sigma(y)^2} dy \right\} = 8C z^{1/H-2} \exp \left\{ -8H^2 z^{1/H} \right\} \quad \text{for } z \in \mathbb{R},$$

for a suitable normalizing constant  $C > 0$  (see e.g, Karatzas and Shreve, 1991, Section 5.5), which gives (4) using that  $p_t(0, y) = p_1(0, y/t^{1/4})/t^{1/4}$ .

As the self-similarity imposed in the previous paragraph severely limits the possible diffusion coefficients  $\sigma$  of diffusions  $X$  that can feature in our construction of  $M$ , one might look for non-diffusion type self-similar martingales. In order for such a martingale  $X$  to feature in our construction it has to have tail probabilities at least as light as Gaussian tails, as  $X(1)Z$  cannot have lighter tails than  $X(1)$ , except for a scaling factor, when  $Z$  is a non-degenerate random variable independent of  $X$ . Examples of

non-diffusion type light-tailed self-similar martingales includes, for example,

$$X(t) = \int_0^t g(B(s)/\sqrt{s}) dB(s) \quad \text{for a bounded function } g.$$

However, for this  $X$  it seems near impossible to find the distribution of  $X(1)$ , to check whether there exists a  $Z$  such that  $X(1)Z \stackrel{d}{=} B(1)$ .

With the above notation,  $X(1)Z \stackrel{d}{=} B(1)$  holds if and only if  $\ln(|B(1)|) \stackrel{d}{=} \ln(|X(1)|) + \ln(|Z|)$ . This is the much researched factorization problem for probability distributions (see e.g. Bondesson, 1995), for which it is well-known that non-trivial factorizations do not always exist, especially not when one of the factors  $X(1)$  is given.

The family of non-continuous martingales with Brownian motion marginal distributions constructed by Hamza and Klebaner (2006b) is rather non-explicit. However, Olle Häggström has communicated a simple explicit example of such a martingale to us: Let  $M(t) = \sqrt{t} B(1) (-1)^{N(t)}$  with the filtration  $\mathcal{F}_t = \sigma(B(1)) \vee \sigma(N(s) : s \leq t)$ , where  $\{N(t)\}_{t>0}$  is a non-homogeneous Poisson process independent of  $B$  with intensity  $\lambda(r) = 1/(4r)$ . As  $M$  is not continuous, but has the univariate marginal distributions of Brownian motion, we only have to show that  $M$  is a martingale. As

$$\mathbf{E}\{M(t) | \mathcal{F}_s\} = \sqrt{t} \xi (-1)^{N(s)} \mathbf{E}\{(-1)^{N(t)-N(s)}\} = M(s) \sqrt{\frac{t}{s}} (1 - 2 \mathbf{P}\{N(t)-N(s) \text{ odd}\}),$$

this follows from the fact that (by basic properties of Poisson processes)

$$\mathbf{P}\{N(t) - N(s) \text{ odd}\} = \frac{1}{2} \left( 1 - \exp \left\{ -2 \int_s^t \lambda(r) dr \right\} \right) = \frac{1}{2} \left( 1 - \sqrt{\frac{s}{t}} \right).$$

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