



HAL
open science

A spatial rank test and corresponding estimators for several samples

Jaakko Nevalainen, Jyrki Möttönen, Hannu Oja

► **To cite this version:**

Jaakko Nevalainen, Jyrki Möttönen, Hannu Oja. A spatial rank test and corresponding estimators for several samples. *Statistics and Probability Letters*, 2010, 78 (6), pp.661. 10.1016/j.spl.2007.09.028 . hal-00612013

HAL Id: hal-00612013

<https://hal.science/hal-00612013>

Submitted on 28 Jul 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Author's Accepted Manuscript

A spatial rank test and corresponding estimators for several samples

Jaakko Nevalainen, Jyrki Möttönen, Hannu Oja

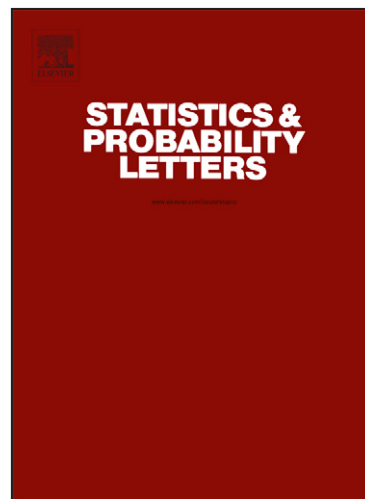
PII: S0167-7152(07)00309-4
DOI: doi:10.1016/j.spl.2007.09.028
Reference: STAPRO 4761

To appear in: *Statistics & Probability Letters*

Received date: 31 October 2006
Accepted date: 7 September 2007

Cite this article as: Jaakko Nevalainen, Jyrki Möttönen and Hannu Oja, A spatial rank test and corresponding estimators for several samples, *Statistics & Probability Letters* (2007), doi:10.1016/j.spl.2007.09.028

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



www.elsevier.com/locate/stapro

A Spatial Rank Test and Corresponding Estimators for Several Samples

by Jaakko Nevalainen¹, Jyrki Möttönen² and Hannu Oja³

Version: June 12, 2007

¹*Department of Mathematics, Statistics and Philosophy*

33014 University of Tampere, Finland

e-mail: jaakko.nevalainen@uta.fi

²*Department of Mathematics and Statistics, University of Kuopio, Finland*

³*Tampere School of Public Health, University of Tampere, Finland*

Abstract

In the several samples location problem, it is usually of interest to present estimates of treatment effects along with the test. The spatial Hodges-Lehmann estimators $\widehat{\Delta}_{ij}$ of the differences between treatments i and j are apparent companions to a multivariate Kruskal-Wallis test. However, these estimators generally fail to satisfy the property $\widehat{\Delta}_{ij} = \widehat{\Delta}_{ik} + \widehat{\Delta}_{kj}$, making them incompatible with each other. In this paper we consider adjusted estimators possessing this property. A simulation study is carried out in order to study their finite sample efficiencies. Limiting distributions and efficiencies are presented as well.

Key words: Kruskal-Wallis test; Multivariate several samples rank test; Spatial Hodges-Lehmann estimator; Spatial rank.

1 Introduction

Let $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_c)$ be a $p \times N$ data matrix consisting of c independent random samples

$$\begin{aligned}\mathbf{X}_1 &= (\mathbf{x}_{11}, \dots, \mathbf{x}_{1n_1}), \\ \mathbf{X}_2 &= (\mathbf{x}_{21}, \dots, \mathbf{x}_{2n_2}), \\ &\vdots \\ \mathbf{X}_c &= (\mathbf{x}_{c1}, \dots, \mathbf{x}_{cn_c}),\end{aligned}$$

from p -variate continuous distributions $F(\mathbf{x} - \boldsymbol{\mu}_1)$, $F(\mathbf{x} - \boldsymbol{\mu}_2)$, \dots , $F(\mathbf{x} - \boldsymbol{\mu}_c)$, respectively, where $N = n_1 + \dots + n_c$. Knowing little of the underlying distribution, we are interested in finding out whether there are differences in location between the samples, and if so, estimating those differences.

First recall the notions of spatial sign and rank. A multivariate extension of univariate sign, the spatial sign of vector \mathbf{x}_{ij} , is defined as

$$\mathbf{S}(\mathbf{x}_{ij}) = \begin{cases} \|\mathbf{x}_{ij}\|^{-1} \mathbf{x}_{ij}, & \text{if } \mathbf{x}_{ij} \neq \mathbf{0}; \\ \mathbf{0}, & \text{if } \mathbf{x}_{ij} = \mathbf{0}, \end{cases}$$

where $\|\cdot\|$ denotes the Euclidian length. Thus, the spatial sign is a p -variate unit vector.

The empirical spatial centered rank of \mathbf{x}_{ij} among the data set \mathbf{X} is defined as

$$\mathbf{R}_N(\mathbf{x}_{ij}) = \frac{1}{N} \sum_{k=1}^c \sum_{l=1}^{n_k} \mathbf{S}(\mathbf{x}_{ij} - \mathbf{x}_{kl}).$$

This gives a vector inside the unit sphere pointing from the center of the data cloud \mathbf{X} approximately to the direction of \mathbf{x}_{ij} . Spatial ranks are data dependent, but they converge uniformly in probability to their theoretical values $\mathbf{R}(\mathbf{x}_{ij}) = E[\mathbf{S}(\mathbf{x}_{ij} - \mathbf{x})]$, $\mathbf{x} \sim F$.

2 A Multivariate Kruskal-Wallis Test

The hypotheses of interest are

$$H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_c \quad \text{versus} \quad H_1 : \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_c \text{ not all equal.}$$

A classical univariate nonparametric test for the problem is the Kruskal-Wallis test. Hettmansperger et al. (1998) gave a multivariate extension of the Kruskal-Wallis test, identical to the classical test in the univariate case, based on affine invariant ranks. Similar approach can be taken with spatial ranks as outlined next.

Write $\bar{\mathbf{R}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{R}_N(\mathbf{x}_{ij})$ for the group average of empirical spatial ranks with respect to the entire sample. Under the null hypothesis,

$$Q^2 = \sum_{i=1}^c n_i \bar{\mathbf{R}}_i^T \hat{\mathbf{B}}^{-1} \bar{\mathbf{R}}_i \xrightarrow{D} \chi_{p(c-1)}^2$$

(Möttönen and Oja, 1995; Choi and Marden, 1997), where

$$\hat{\mathbf{B}} = \text{ave} \{ \mathbf{R}_N(\mathbf{x}_{ij}) \mathbf{R}_N(\mathbf{x}_{ij})^T \}$$

is a consistent estimator (under H_0) of the spatial rank covariance matrix

$$\mathbf{B} = E_0 [\mathbf{R}(\mathbf{x}_{ij}) \mathbf{R}(\mathbf{x}_{ij})^T].$$

3 Estimation of Treatment Effects

Denote the difference between treatments i and j by $\boldsymbol{\Delta}_{ij} = \boldsymbol{\mu}_j - \boldsymbol{\mu}_i$. Natural companion estimator $\hat{\boldsymbol{\Delta}}_{ij}$ to the test is the multivariate two-sample spatial Hodges-Lehmann estimator, that is, the sample spatial median of the $n_i n_j$ pairwise differences $\mathbf{x}_{jk} - \mathbf{x}_{il}$ ($k = 1, \dots, n_j$; $l = 1, \dots, n_i$). Note that $\hat{\boldsymbol{\Delta}}_{ij} = -\hat{\boldsymbol{\Delta}}_{ji}$ and $\hat{\boldsymbol{\Delta}}_{ii} = \mathbf{0}$. Möttönen and Oja (1995) state but do not prove the following asymptotic result concerning this estimator:

Theorem 1 Assume that $\min\{n_i, n_j\} \rightarrow \infty$ and that $\frac{n_i}{N} \rightarrow \lambda_i$ and $\frac{n_j}{N} \rightarrow \lambda_j$, $0 < \lambda_i, \lambda_j <$

1. Under general assumptions,

$$\sqrt{N}(\widehat{\Delta}_{ij} - \Delta_{ij}) \xrightarrow{D} N_p \left(\mathbf{0}, \frac{\lambda_i + \lambda_j}{\lambda_i \lambda_j} \mathbf{A}^{-1} \mathbf{B} \mathbf{A}^{-1} \right).$$

Here

$$\mathbf{A} = E \left[\frac{1}{\|\mathbf{x}_{jk} - \mathbf{x}_{il} - \Delta_{ij}\|} (\mathbf{I}_p - \mathbf{S}(\mathbf{x}_{jk} - \mathbf{x}_{il} - \Delta_{ij}) \mathbf{S}(\mathbf{x}_{jk} - \mathbf{x}_{il} - \Delta_{ij})^T) \right].$$

To find a covariance matrix estimate for $\widehat{\Delta}_{ij}$, the \mathbf{A} matrix can be estimated by

$$\widehat{\mathbf{A}} = \text{ave} \left\{ \frac{1}{\|\mathbf{x}_{jk} - \mathbf{x}_{il} - \widehat{\Delta}_{ij}\|} (\mathbf{I}_p - \mathbf{S}(\mathbf{x}_{jk} - \mathbf{x}_{il} - \widehat{\Delta}_{ij}) \mathbf{S}(\mathbf{x}_{jk} - \mathbf{x}_{il} - \widehat{\Delta}_{ij})^T) \right\},$$

where the average is taken over all possible pairs $(\mathbf{x}_{ij}, \mathbf{x}_{kl})$ from all the samples. An estimate of the \mathbf{B} matrix can be obtained via

$$\widehat{\mathbf{B}} = \text{ave} \left\{ \mathbf{R}_N(\mathbf{x}_{ij} - \widehat{\Delta}_{1i}) \mathbf{R}_N(\mathbf{x}_{ij} - \widehat{\Delta}_{1i})^T \right\}.$$

For completeness of this paper, a heuristic proof of the limiting normality is presented in the Appendix. Chaudhuri (1992) considers the spatial one-sample Hodges-Lehmann estimator and Hodges and Lehmann (1963) the univariate two-sample estimation problem.

The inconvenience with the above estimators—just like in the univariate case, or when using multivariate marginal ranks—is that the obtained estimates are not generally compatible in the sense that $\widehat{\Delta}_{ij} = \widehat{\Delta}_{ik} + \widehat{\Delta}_{kj}$.

To overcome this problem, consider competing estimators of treatment effects. An estimator of the difference between the i th and the j th treatment using the k th treatment as a reference is

$$\widetilde{\Delta}_{ij \cdot k} = \widehat{\Delta}_{ik} + \widehat{\Delta}_{kj}. \quad (1)$$

This type of estimator can be useful in a situation where the treatment effect of interest cannot be estimated directly, but only via a third treatment. Note that $\widetilde{\Delta}_{ij \cdot i} = \widetilde{\Delta}_{ij \cdot j} =$

$\widehat{\Delta}_{ij}$. Taking the average over the treatments groups

$$\widetilde{\Delta}_{ij} = \frac{1}{c} \sum_{k=1}^c \widetilde{\Delta}_{ij \cdot k} \quad (2)$$

yields a generalization of the univariate estimator proposed by Lehmann (1963). The adjusted estimators (1) and (2) are consistent only if $\frac{n_i}{N} \rightarrow \lambda_i$, $0 < \lambda_i < 1$, for all $i = 1, \dots, c$. Estimators that are consistent under a weaker condition that $\frac{n_i}{N} \rightarrow \lambda_i$ and $\frac{n_j}{N} \rightarrow \lambda_j$ are obtained by weighting the estimators by the relative group size of the reference sample (Spjøtvoll, 1968):

$$\bar{\Delta}_{ij} = \frac{1}{N} \sum_{k=1}^c n_k \widetilde{\Delta}_{ij \cdot k} \quad (3)$$

When $n_1 = \dots = n_c$ the spatial Spjøtvoll's estimators (3) reduce to the spatial Lehmann's estimators (2).

Theorem 2 *Assume that $\min \{n_i, n_j, n_k\} \rightarrow \infty$ and that $\frac{n_i}{N} \rightarrow \lambda_i$, $\frac{n_j}{N} \rightarrow \lambda_j$ and $\frac{n_k}{N} \rightarrow \lambda_k$, $0 < \lambda_i, \lambda_j, \lambda_k < 1$. Then, under general assumptions,*

$$\sqrt{N} \left(\widetilde{\Delta}_{ij \cdot k} - \widehat{\Delta}_{ij} \right) \xrightarrow{P} \mathbf{0}$$

for all i, j, k .

Corollary 1 *Under the assumptions of Theorem 2,*

$$\sqrt{N} \left(\widetilde{\Delta}_{ij} - \widehat{\Delta}_{ij} \right) \xrightarrow{P} \mathbf{0} \text{ and}$$

$$\sqrt{N} \left(\bar{\Delta}_{ij} - \widehat{\Delta}_{ij} \right) \xrightarrow{P} \mathbf{0}$$

for all i, j, k .

Theorem 2 and Corollary 1 imply that the alignment of the estimates (with respect to the other $c - 2$ treatments) does not alter their limiting distributions. However, it is unclear what happens to the efficiency of the adjusted estimators in finite samples, particularly if the sample sizes are widely disparate. This question will be addressed in the next section.

4 Efficiencies

Recall that the spatial Hodges-Lehmann estimator is much more efficient than the mean difference vector for heavy-tailed distributions, and nearly as efficient at the normal model (Table 1). As the adjusted estimators share the limiting distribution of the spatial Hodges-Lehmann estimator, their asymptotic relative efficiencies are identical as well.

The finite sample efficiencies of the adjusted estimators $\tilde{\Delta}_{12,3}$, $\tilde{\Delta}_{12}$ and $\bar{\Delta}_{12}$ relative to the unadjusted estimator $\hat{\Delta}_{12}$ shown in Figures 1, 2 and 3, respectively, are based on simulations from a univariate normal distribution (10000 repetitions) and a bivariate spherical normal distribution (1000 repetitions) for three groups. Efficiencies for spherical distributions in general are likely to be approximately the same. The simulations were conducted in R (R Development Core Team, 2004).

The efficiency of the estimator $\tilde{\Delta}_{12,3}$, based on a reference sample of size 1 ($n_3 = 1$), is approximately 0.7 ($p = 1$) and 0.8 ($p = 2$). At this point, the value is merely the observed relative efficiency of the the difference of the (spatial) medians related to the (spatial) Hodges-Lehmann estimator. Adding a few observations to the third group quickly improves the performance of the estimator $\tilde{\Delta}_{12,3}$. At $n_1 = n_2 = n_3$, the observed relative efficiency is close to unity (Figure 1). Our further simulation studies suggest that, as $n_1 = n_2$ remain fixed and n_3 increases, the finite sample efficiency increases even beyond 1, but only very slightly.

The behavior of the spatial Lehmann's estimator $\tilde{\Delta}_{12}$, being the average of the $\tilde{\Delta}_{12,k}$ estimators, is very similar (Figure 2). It is superior to the estimator $\tilde{\Delta}_{12,3}$, because it is never worse than 90% efficient, and because it reaches the efficiency of the spatial Hodges-Lehmann estimator much faster. The reasons are easy to see: as $\tilde{\Delta}_{12} = \frac{1}{3}(2\hat{\Delta}_{12} + \tilde{\Delta}_{12,3})$, the (spatial) Hodges-Lehmann estimator $\hat{\Delta}_{12}$ receives the most weight in the computation.

Finally, the spatial Spjøtvoll's estimator $\bar{\Delta}_{12}$ seems to have the same efficiency as the spatial Hodges-Lehmann estimator $\hat{\Delta}_{12}$ when n_3 is small (Figure 3). As n_3 increases and n_1, n_2 remain fixed, the estimator tends to $\tilde{\Delta}_{12.3}$. Thus, for large n_3 it can be slightly better than $\hat{\Delta}_{12}$. The weighting procedure enables the spatial Spjøtvoll's estimator $\bar{\Delta}_{12}$ to protect itself against efficiency losses due to extreme group allocations in both directions, thus making it a superior estimator. If the sample sizes are approximately the same, it does not make a difference which estimator is used.

5 Affine Invariant/Equivariant Versions

The test and the estimators based on spatial ranks are orthogonally invariant and equivariant but not affine invariant and equivariant. However, if $\hat{\mathbf{V}}$ is an affine equivariant estimator of shape in the sense that $\hat{\mathbf{V}}(\mathbf{H}\mathbf{X} + \mathbf{g}\mathbf{1}_N^T) \propto \mathbf{H}\hat{\mathbf{V}}\mathbf{H}^T$, then

- any orthogonally invariant test computed on the transformed data set $\mathbf{Y} = \hat{\mathbf{V}}^{-1/2}\mathbf{X}$ is affine invariant (Randles, 2000; Möttönen et al., 2003), and
- any orthogonally equivariant estimator computed on the transformed data set, and retransformed, $\hat{\mathbf{V}}^{1/2}\hat{\Delta}(\mathbf{Y})$, is affine equivariant. This procedure is widely known as the transformation retransformation technique (Chakraborty and Chaudhuri, 1996; Chakraborty et al., 1998; Chakraborty and Chaudhuri, 1998).

Therefore, a solution is to perform the multivariate spatial rank test on a transformed data set \mathbf{Y} , and to retransform the treatment difference estimates obtained from \mathbf{Y} back to the original scale by $\hat{\mathbf{V}}^{1/2}$. Here the most natural approach is to use spatial ranks in the estimation of the shape matrix \mathbf{V} (Oja and Randles, 2004). One possibility is to apply

a shape matrix defined by the implicit equation

$$p \sum_{i=1}^c \left[\frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{R}_{n_i}(\mathbf{y}_{ij}) \mathbf{R}_{n_i}(\mathbf{y}_{ij})^T \right] = \sum_{i=1}^c \left[\frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{R}_{n_i}(\mathbf{y}_{ij})^T \mathbf{R}_{n_i}(\mathbf{y}_{ij}) \right] \mathbf{I}_p, \quad (4)$$

where $\mathbf{y}_{ij} = \widehat{\mathbf{V}}^{-1/2} \mathbf{x}_{ij}$ and $\mathbf{R}_{n_i}(\mathbf{y}_{ij})$ is the spatial rank of \mathbf{y}_{ij} among $\mathbf{y}_{i1}, \dots, \mathbf{y}_{in_i}$. After standardization by $\widehat{\mathbf{V}}^{-1/2}$ obtained from (4) the spatial ranks appear as if they were from a spherical distribution. For similar definitions of shape matrices based on spatial signs, see earlier work of Tyler (1987) and Dümbgen (1998). Oja and Randles (2004) also gave an algorithm for the computation of a shape matrix similar to (4). The algorithm always seems to converge, but the actual proof is missing.

Acknowledgements

The research work of Jaakko Nevalainen and Hannu Oja was partially supported by research grants from the Academy of Finland and by the doctoral program of School of Statistical Information, Inference, and Data Analysis (SIIDA).

Appendix

PROOF OF THEOREM 1. Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ and $\mathbf{y}_1, \dots, \mathbf{y}_n$ be two independent random samples from $F(\mathbf{x} - \boldsymbol{\mu}_x)$ and $F(\mathbf{y} - \boldsymbol{\mu}_y)$. It is not a restriction to assume that $\boldsymbol{\Delta}_{xy} = \boldsymbol{\mu}_y - \boldsymbol{\mu}_x = \mathbf{0}$. The estimator $\widehat{\boldsymbol{\Delta}} = \widehat{\boldsymbol{\Delta}}_{xy}$ satisfies

$$\frac{\sqrt{N}}{mn} \sum_{i=1}^m \sum_{j=1}^n \mathbf{S}(\mathbf{y}_j - \mathbf{x}_i - \widehat{\boldsymbol{\Delta}}) = \mathbf{0}.$$

Suppose that $\widehat{\boldsymbol{\Delta}}$ is \sqrt{N} -consistent for $\boldsymbol{\Delta}$, where $N = m + n$. This can be shown in the multivariate case as in Nevalainen et al. (2007). Write $\widehat{\boldsymbol{\Delta}}^* = \sqrt{N} \widehat{\boldsymbol{\Delta}}$. Then the Taylor

expansion around $\widehat{\Delta}^* = \mathbf{0}$ gives

$$\mathbf{0} = \frac{\sqrt{N}}{mn} \sum_{i=1}^m \sum_{j=1}^n \mathbf{S}(\mathbf{y}_j - \mathbf{x}_i) - \widehat{\mathbf{A}} \widehat{\Delta}^* + o_P(1),$$

where $\widehat{\mathbf{A}} \xrightarrow{P} \mathbf{A}$ and

$$\sqrt{N} \left[\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \mathbf{S}(\mathbf{y}_j - \mathbf{x}_i) - \frac{1}{n} \sum_{j=1}^n \mathbf{R}(\mathbf{y}_j) + \frac{1}{m} \sum_{i=1}^m \mathbf{R}(\mathbf{x}_i) \right] \xrightarrow{P} \mathbf{0}.$$

Therefore

$$\sqrt{N} \left[\widehat{\Delta} + \mathbf{A}^{-1} \left(\frac{1}{n} \sum_{j=1}^n \mathbf{R}(\mathbf{y}_j) - \frac{1}{m} \sum_{i=1}^m \mathbf{R}(\mathbf{x}_i) \right) \right] \xrightarrow{P} \mathbf{0}$$

and the result follows.

PROOF OF THEOREM 2. Let $\mathbf{x}_1, \dots, \mathbf{x}_m$, $\mathbf{y}_1, \dots, \mathbf{y}_n$ and $\mathbf{z}_1, \dots, \mathbf{z}_l$ be three independent random samples from $F(\mathbf{x})$. Let $N = m + n + l$. By Theorem 1

$$\sqrt{N} \left[\widehat{\Delta}_{xz} + \mathbf{A}^{-1} \left(\frac{1}{l} \sum_{k=1}^l \mathbf{R}(\mathbf{z}_k) - \frac{1}{m} \sum_{i=1}^m \mathbf{R}(\mathbf{x}_i) \right) \right] \xrightarrow{P} \mathbf{0},$$

and

$$\sqrt{N} \left[\widehat{\Delta}_{zy} + \mathbf{A}^{-1} \left(\frac{1}{n} \sum_{j=1}^n \mathbf{R}(\mathbf{y}_j) - \frac{1}{l} \sum_{k=1}^l \mathbf{R}(\mathbf{z}_k) \right) \right] \xrightarrow{P} \mathbf{0}.$$

But then simply

$$\sqrt{N} \left[\widehat{\Delta}_{xy.z} + \mathbf{A}^{-1} \left(\frac{1}{n} \sum_{j=1}^n \mathbf{R}(\mathbf{y}_j) - \frac{1}{m} \sum_{i=1}^m \mathbf{R}(\mathbf{x}_i) \right) \right] \xrightarrow{P} \mathbf{0}$$

and therefore $\sqrt{N} \widehat{\Delta}_{xy.z}$ and $\sqrt{N} \widehat{\Delta}_{xy}$ have the same limiting distribution.

References

- B. Chakraborty and P. Chaudhuri. On a transformation and re-transformation technique for constructing affine equivariant multivariate median. In *Proceedings of the American Mathematical Society*, volume 124, pages 2539–2547, 1996.

- B. Chakraborty and P. Chaudhuri. On an adaptive transformation and retransformation estimate of multivariate location. *Journal of the Royal Statistical Society, Series B*, 60: 145–157, 1998.
- B. Chakraborty, P. Chaudhuri, and H. Oja. Operating transformation retransformation on spatial median and angle test. *Statistica Sinica*, 8:767–784, 1998.
- P. Chaudhuri. Multivariate location estimation using extension of R -estimates through U -statistics type approach. *The Annals of Statistics*, 20:897–916, 1992.
- K. Choi and J. Marden. An approach to multivariate rank tests in multivariate analysis of variance. *Journal of the American Statistical Association*, 92:1581–1590, 1997.
- L. Dümbgen. On Tyler’s M -functional of scatter in high dimension. *Annals of the Institute of Statistical Mathematics*, 50:471–491, 1998.
- T. P. Hettmansperger, J. Möttönen, and H. Oja. Affine invariant multivariate rank tests for several samples. *Statistica Sinica*, 8:785–800, 1998.
- J. L. Hodges and E. L. Lehmann. Estimates of location based on rank tests. *The Annals of Mathematical Statistics*, 34:598–611, 1963.
- E. L. Lehmann. Robust estimation in analysis of variance. *The Annals of Mathematical Statistics*, 34:957–966, 1963.
- J. Möttönen and H. Oja. Multivariate spatial sign and rank methods. *Nonparametric Statistics*, 5:201–213, 1995.
- J. Möttönen, H. Oja, and J. Tienari. On the efficiency of multivariate spatial sign and rank tests. *The Annals of Statistics*, 25:542–552, 1997.

- J. Möttönen, J. Hüsler, and H. Oja. Multivariate nonparametric tests in a randomized complete block design. *Journal of Multivariate Analysis*, 85:106–129, 2003.
- J. Nevalainen, D. Larocque, and H. Oja. On the multivariate spatial median for clustered data. *The Canadian Journal of Statistics*, 35:215–231, 2007.
- H. Oja and R. H. Randles. Multivariate nonparametric tests. *Statistical Science*, 19:598–605, 2004.
- R Development Core Team. *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria, 2004. URL <http://www.R-project.org>. ISBN 3-900051-00-3.
- R. H. Randles. A simpler, affine-invariant, multivariate, distribution-free sign test. *Journal of the American Statistical Association*, 95:1263–1268, 2000.
- E. Spjøtvoll. A note on robust estimation in analysis of variance. *The Annals of Mathematical Statistics*, 39:1486–1492, 1968.
- D. E. Tyler. A distribution-free M -estimator of multivariate scatter. *The Annals of Statistics*, 15:234–251, 1987.

Table 1: Asymptotic relative efficiency of the spatial Hodges-Lehmann estimator relative to the mean difference vector under multivariate t -distributions (Möttönen et al., 1997).

| Dimension | Degrees of freedom | | | |
|-----------|--------------------|-------|-------|----------|
| | 3 | 6 | 10 | ∞ |
| 1 | 1.900 | 1.164 | 1.054 | 0.955 |
| 2 | 1.953 | 1.187 | 1.071 | 0.967 |
| 3 | 1.994 | 1.200 | 1.081 | 0.973 |
| 6 | 2.050 | 1.219 | 1.095 | 0.984 |
| 10 | 2.093 | 1.229 | 1.103 | 0.989 |

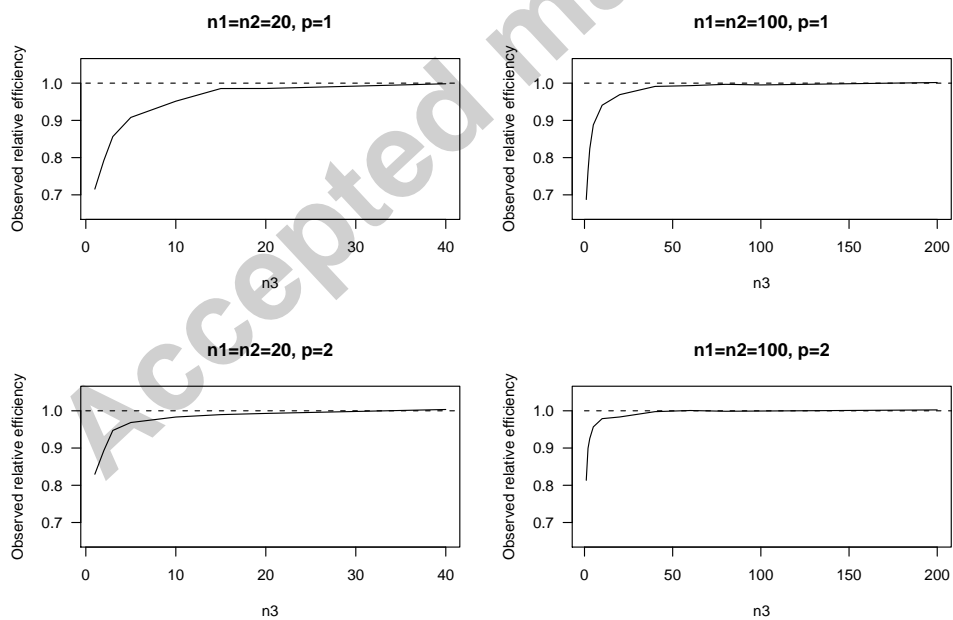


Figure 1: The observed relative efficiency of $\tilde{\Delta}_{12.3}$ relative to $\hat{\Delta}_{12}$.

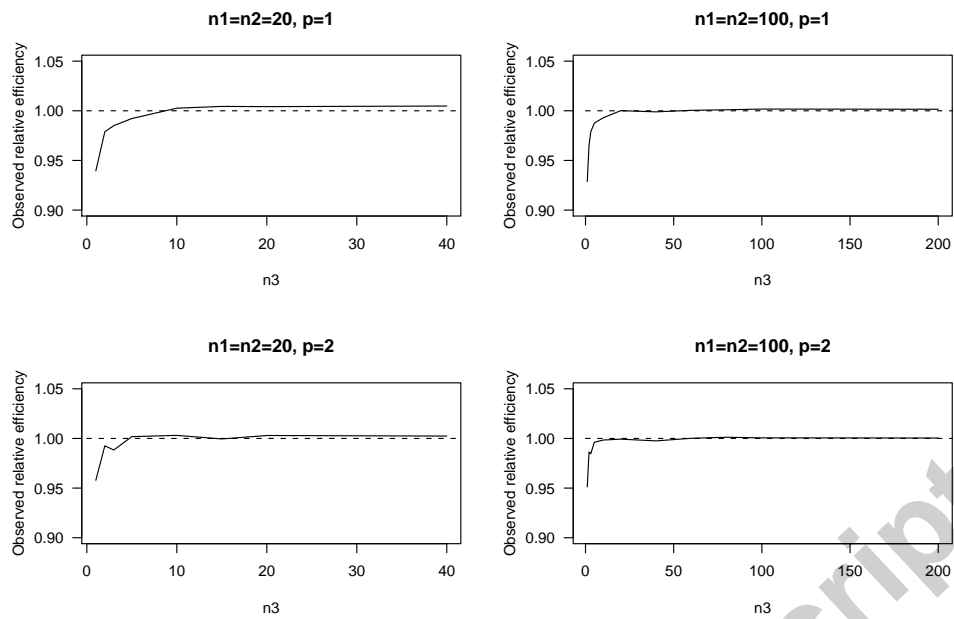


Figure 2: The observed relative efficiency of $\tilde{\Delta}_{12}$ relative to $\hat{\Delta}_{12}$.

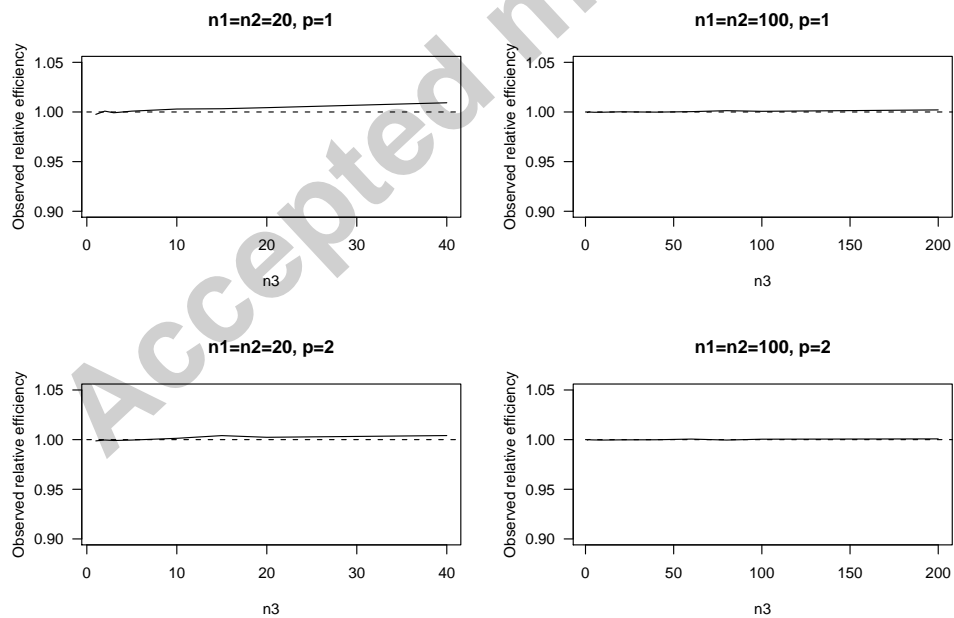


Figure 3: The observed relative efficiency of $\bar{\Delta}_{12}$ relative to $\hat{\Delta}_{12}$.