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► **To cite this version:**

Zhivko Dimitrov Georgiev. Analysis and synthesis of oscillator systems described by perturbed single well Duffing equations. *Nonlinear Dynamics*, 2010, 62 (4), pp.883-893. 10.1007/s11071-010-9771-x . hal-00611663

HAL Id: hal-00611663

<https://hal.science/hal-00611663>

Submitted on 27 Jul 2011

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Analysis and Synthesis of Oscillator Systems Described by Perturbed Single Well Duffing Equations

Zhivko Dimitrov Georgiev

Department of Theoretical Electrical Engineering,
Technical University of Sofia,
8 Kl. Ohridski Blvd., Sofia 1000, Bulgaria
E-mail: zhdgeorg@tu-sofia.bg,

Abstract:

The paper presents an analysis as well as a synthesis of oscillator systems described by single well Duffing equations under polynomial perturbations of fourth degree. It is proved that such a system can have a unique hyperbolic limit cycle. There has been obtained an analytical condition for the arising of a limit cycle and an equation giving the parameters of this limit cycle. There has been proposed a method for the synthesis of oscillator systems of the considered type, having preliminarily assigned properties. The synthesis consists of an appropriate choice of the perturbation coefficients in such a way, that the oscillator equation should have a preliminary assigned limit cycle. Both the analysis and the synthesis are performed with the help of the Melnikov function.

Keywords: Duffing equation, Duffing oscillators, limit cycles, Melnikov function, elliptic functions, synthesis of oscillator systems.

1. Introduction

The paper presents an analysis as well as a synthesis of oscillator systems allowing self-sustained oscillations. The results obtained are based on the qualitative investigation of dynamic systems, whereas the self-sustained oscillations are regarded as limit cycles on the phase plane.

The Melnikov theory allows us to establish the number, position, shape and stability of the limit cycles arising in autonomous perturbed planar systems. According to this theory the problem of finding the limit cycles is replaced by an equivalent problem of finding the zeros of a given function – the Melnikov function for autonomous systems [1-3].

Using the Melnikov theory, there has been performed an analysis of the limit cycles in oscillator systems described by single well Duffing equations under polynomial perturbations of 4th degree. It is proved that such a system can have a unique hyperbolic (simple) limit cycle. Conditions are derived for the arising of a limit cycle (or periodic oscillations) and the establishing whether this limit cycle is stable or unstable. Moreover, a method is suggested for the synthesis of oscillator systems having preliminarily assigned properties.

In general, the synthesis of oscillators systems includes the following two stages: first, finding a differential equation to satisfy the preliminary assigned requirements and second, obtaining an electronic circuit corresponding to this differential equation. In this paper, it should be emphasized, that we are only interested in the first stage. The second stage is a separate problem and we are not examining it here. It is worth mentioning that detailed information concerning the circuit simulation of a given differential equation is presented in [4] (see also the references therein cited). In this expose, the synthesis of an oscillator system consists of choosing the perturbation coefficients in such a way, that the perturbed Duffing

equation should have preliminary assigned limit cycles, respectively self-sustained oscillations.

The Melnikov theory is a successful approach for the synthesis of oscillator systems with preliminary assigned solutions and desired characteristics. Some applications of the Melnikov function for synthesizing generalized Van der Pol systems (perturbed harmonic oscillators) and some types of perturbed Duffing systems are given in References [5-6].

The analysis of the Melnikov function is a separate mathematical problem and there exist various mathematical techniques for its solving [7-12]. In the case of a polynomial perturbed Hamiltonian system the Melnikov function is generally an Abelian integral. Finding the zeros of an Abelian integral is known as the weakened Hilbert's 16th problem (which still remains open [13-14]).

In this paper expressing the unperturbed solution through the Jacobi elliptic functions allows us to compute the Melnikov function and the result is a function involving complete elliptic integrals. There exist some advantages in the study of the Melnikov function involving complete elliptic integrals compared to the study of the Melnikov function involving Abelian integrals. First of all, the complete elliptic integrals have been extensively studied - their properties and series expansions are well known. Furthermore, there exist a lot of handbooks, tables and computer software giving the values of the complete elliptic integrals [15-20]. All this facilitates the engineering applications of the obtained results. In addition, the analytical expression of the interdependence between both the Hamiltonian level h and the modulus of the complete elliptic integrals k is an essential element of the synthesis of oscillator systems.

In order to analyze the Melnikov function, it is necessary to prove that a given function involving complete elliptic integrals is a monotonic function. The monotonicity of this function is proved with the help of appropriate Picard-Fuchs and Riccati equations. Then this monotonic function is used to establish that the Melnikov function has a single simple zero. Equations of Picard-Fuchs and Riccati are widely used to prove that a given function defined as a ratio of two Abelian integrals is monotonic [7-8, 21-23]. In this paper these equations are obtained for the functions involving complete elliptic integrals.

Brief information concerning the Melnikov function and the limit cycles bifurcations in perturbed planar Hamiltonian system is given below. It should be mentioned, that only the case of first order perturbations, respectively first order bifurcations of limit cycles from a "centre" and a first order Melnikov function are examined.

Let's consider the following system

$$\begin{cases} \dot{x} = \partial H / \partial y + \varepsilon p(x, y, \varepsilon) \\ \dot{y} = -\partial H / \partial x + \varepsilon q(x, y, \varepsilon) \end{cases} \quad (1)$$

where the Hamiltonian $H = H(x, y)$ is an analytic function in \mathbf{R}^2 , the perturbation functions $p(x, y, \varepsilon)$ and $q(x, y, \varepsilon)$ are analytic in $\mathbf{R}^2 \times \mathbf{R}$, ε is a small parameter, $|\varepsilon| \ll 1$, $(\dot{}) \equiv d/dt$ and t is time. Let's denote the solution of system (1) by $(x, y) = (\varphi_\varepsilon(t), \psi_\varepsilon(t))$. In this case the solution of the unperturbed system (at $\varepsilon = 0$) is $(x, y) = (\varphi_0(t), \psi_0(t))$.

Further on, let's assume that the unperturbed system (at $\varepsilon = 0$) has at least one equilibrium point, which is a "centre", surrounded by closed concentric trajectories parameterized by the set of constant Hamiltonian levels. Then the equation on the phase plane of a given closed trajectory $\Gamma_0(h)$, having a parameter h , is expressed by

$$\Gamma_0(h): H(x, y) = h, \quad h \in \Omega \subset \mathbf{R}, \quad (2)$$

where Ω is an open set of values h , for which the trajectories are not degenerated in a point or a separatrix loop.

Let's denote the period of the unperturbed solution $(x, y) = (\varphi_0(t), \psi_0(t))$, corresponding to the closed trajectory $\Gamma_0(h)$, by $T_0(h)$. Using these notions the following function

$$M(h) = \int_0^{T_0(h)} [q(\varphi_0, \psi_0, 0)\dot{\varphi}_0 - p(\varphi_0, \psi_0, 0)\dot{\psi}_0] dt, \quad (3)$$

is called First Order Melnikov function (briefly, Melnikov function). It plays a basic role in the study of limit cycles. The zeros of the Melnikov function determine the limit cycles in the system (1) emerging from the periodic orbits of the unperturbed Hamiltonian system. The stability of the bifurcating limit cycle is determined by the derivative of the function $M(h)$. More details on this subject can be found in [24-26].

In all computations further on the complete elliptic integrals of the first and second kind \mathbf{K} and \mathbf{E} are regarded as functions of k^2 (k is the modulus), i.e. $\mathbf{K} = \mathbf{K}(k^2)$ and $\mathbf{E} = \mathbf{E}(k^2)$. Some properties and identities concerning the Jacobi elliptic function sd and the complete elliptic integrals \mathbf{K} and \mathbf{E} , that have been used in the presentation, are given in an Appendix at the end of the paper for quick reference. The equations in the Appendix are marked by the letter A.

2. Oscillator circuits and oscillator equations

Examples of AC equivalent circuits of oscillator systems governed by perturbed Duffing equations are shown in Fig. 1. The parallel circuit, shown in Fig. 1a, consists of a resistor with conductance G , an inductive element with inductance L , a nonlinear resistor NR of the type N , having a voltage-current characteristic $i = f_1(u)$ and a nonlinear capacitive element having a charge-voltage characteristic $u = \alpha q + \beta q^3$, where q is the electric charge associated with the element, and α, β are constants, $\alpha > 0, \beta > 0$. The series circuit, shown in Fig. 1b, consists of a resistor with resistance R , a capacitive element with capacitance C , a nonlinear resistor NR of the type S having a current-voltage characteristic $u = f_1(i)$ and a nonlinear inductive element having a flux-current characteristic $i = \alpha \psi + \beta \psi^3$, where ψ is the magnetic flux associated with the element, and here, too, α, β are constants, $\alpha > 0, \beta > 0$.

Let's consider the parallel oscillator circuit. The characteristic of the nonlinear resistor is transformed in the following way:

$$i = f_1(u) = f_1(\alpha q + \beta q^3) = f_2(q). \quad (4)$$

Using Kirchoff's current law we obtain

$$Gu + \frac{1}{L} \int u du + \frac{dq}{dt} + f_2(q) = 0. \quad (5)$$

After differentiating the equation above, there follows

$$\frac{d^2 q}{dt^2} + \frac{d}{dt} [Gu + f_2(q)] + \frac{1}{L}u = 0 .$$

Taking into account the expression for u , we find

$$\frac{d^2 q}{dt^2} + \frac{d}{dq} [G\alpha q + G\beta q^3 + f_2(q)] \frac{dq}{dt} + \frac{\alpha}{L}q + \frac{\beta}{L}q^3 = 0 . \quad (6)$$

If we perform the substitutions

$$G\alpha q + G\beta q^3 + f_2(q) = -\varepsilon F(q) , \quad q = x , \quad \frac{\alpha}{L} = a , \quad \frac{\beta}{L} = b ,$$

Equation (6) takes its final form

$$\frac{d^2 x}{dt^2} - \varepsilon \frac{dF(x)}{dx} \frac{dx}{dt} + ax + bx^3 = 0 , \quad a > 0 , \quad b > 0 . \quad (7)$$

Equation (7) is known as a perturbed single well Duffing equation. The small parameter ε depends on the circuit parameters.

Starting from the series circuit (Fig. 1b) and using the Kirchhoff's voltage law, we get the same equation as in (7).

3. Melnikov function for perturbed single well Duffing equation

In our work we assume that the function $F(x)$ is represented by the polynomial

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 . \quad (8)$$

In this case Equation (7) can be rewritten as the following perturbed Hamiltonian system

$$\begin{cases} \dot{x} = y + \varepsilon(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \\ \dot{y} = -ax - bx^3 \end{cases} , \quad (9)$$

where $a > 0$ and $b > 0$. Further on, we will investigate the system (9) from a point of view of the existence of limit cycles. As in section 1 let $(x, y) = (\varphi_\varepsilon(t), \psi_\varepsilon(t))$ be the solution of the perturbed system (9).

The unperturbed system (at $\varepsilon = 0$) has a Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4 \quad (10)$$

and a single equilibrium point $(0, 0)$ which is a ‘‘centre’’ surrounded by a continuous one-parameter family of closed trajectories, which can be expressed as follows:

$$\Gamma_0(h): H(x, y) = \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4 = h, \quad h \in (0, \infty). \quad (11)$$

The solution of the unperturbed system $(x, y) = (\varphi_0(t), \psi_0(t))$ in the time domain and its period are expressed as follows [27]

$$x = \varphi_0(t) = \frac{\sqrt{2}\sqrt{ak}\sqrt{1-k^2}}{\sqrt{b}\sqrt{1-2k^2}} \operatorname{sd}\left(\frac{\sqrt{a}}{\sqrt{1-2k^2}}t, k\right), \quad (12a)$$

$$y = \psi_0(t) = \frac{\sqrt{2}ak\sqrt{1-k^2}}{\sqrt{b}(1-2k^2)} \operatorname{cd}\left(\frac{\sqrt{a}}{\sqrt{1-2k^2}}t, k\right) \operatorname{nd}\left(\frac{\sqrt{a}}{\sqrt{1-2k^2}}t, k\right), \quad (12b)$$

$$T_0 = T_0(h) = 4\mathbf{K}(k^2) \frac{\sqrt{1-2k^2}}{\sqrt{a}}, \quad (13)$$

where sd , cd and nd are Jacobi elliptic functions.

The modulus k and the constant h are related in the following way:

$$k^2 = \frac{1}{2} \left(1 - \frac{a}{\sqrt{a^2 + 4bh}}\right), \quad h = \frac{a^2 k^2 (1-k^2)}{b(1-2k^2)^2}, \quad (14a)$$

$$0 < k^2 < 1/2, \quad 0 < h < \infty. \quad (14b)$$

It is easy to prove that $dh/dk^2 > 0$, and $dk^2/dh > 0$, i.e. the function $h = h(k^2)$ and $k^2 = k^2(h)$ are monotone increasing.

According to Equation (3), the Melnikov function for the system (9) is

$$\begin{aligned} M(h) &= -\int_0^{T_0} \left(\sum_{n=1}^4 (a_n \varphi_0^n) \right) \dot{\psi}_0 dt = \int_0^{T_0} \left[\left(\sum_{n=1}^4 a_n \varphi_0^n \right) (a \varphi_0 + b \varphi_0^3) \right] dt = \\ &= \int_0^{T_0} \left[\sum_{n=2}^7 c_n \left(\frac{\sqrt{2}\sqrt{ak}\sqrt{1-k^2}}{\sqrt{b}\sqrt{1-2k^2}} \right)^n \operatorname{sd}^n\left(\frac{\sqrt{a}}{\sqrt{1-2k^2}}t, k\right) \right] dt, \end{aligned}$$

where $c_2 = aa_1$, $c_3 = aa_2$, $c_4 = aa_3 + ba_1$, $c_5 = aa_4 + ba_2$, $c_6 = ba_3$, $c_7 = ba_4$. Introducing a new variable $z = \sqrt{at}/\sqrt{1-2k^2}$ and taking into account Equation (13) and the properties of the function sd given in the Appendix (Equations (A1) – (A3)), we see that the integrals containing $\operatorname{sd}^{2n+1}(z, k)$, $n = 1, 2, 3$ vanish and the Melnikov function becomes

$$M(h) = \frac{4\sqrt{1-2k^2}}{\sqrt{a}} \int_0^{\mathbf{K}} \left[\sum_{n=1}^3 c_{2n} \left(\frac{\sqrt{2}\sqrt{ak}\sqrt{1-k^2}}{\sqrt{b}\sqrt{1-2k^2}} \right)^{2n} \operatorname{sd}^{2n}(z, k) \right] dz. \quad (15)$$

As it is seen from the last result, the Melnikov function and the arising of limit cycles do not depend on the terms with coefficients having odd indices - c_3, c_5, c_7 . This means that the terms with even degrees in the polynomial perturbation in System (9) exert no influence on the arising of limit cycles and further we will assume that $a_2 = a_4 = 0$. This fact is important for the circuit realization of the oscillator equation. It is worth mentioning that similar assertions are valid for the generalized Van der Pol oscillators [5]. Finally, System (9), which we investigate, is reduced to the following system with respect to the limit cycles:

$$\begin{cases} \dot{x} = y + \varepsilon(a_1x + a_3x^3) \\ \dot{y} = -ax - bx^3 \end{cases} . \quad (16)$$

Under these conditions, after quite long but straightforward calculations, using Equations (A7)–(A9) and introducing the notation $m = k^2$, we obtain the final expression for the Melnikov function

$$M(h) = M(h(m)) = \mathbf{M}(m) = \frac{8\lambda\sqrt{a}\sqrt{1-2m}}{(1-2m)^3} \mathbf{B}(m; a_1, a_3) , \quad (17)$$

where $\lambda = a/b$ and

$$\mathbf{B}(m; a_1, a_3) = a_1 I_1(m) + \frac{2}{5} \lambda a_3 I_2(m) , \quad (18)$$

$$I_1(m) = \frac{1}{3} (1-2m) [(1-m)\mathbf{K} - (1-2m)\mathbf{E}] , \quad (19)$$

$$I_2(m) = 2(1-m+m^2)\mathbf{E} - (2-m)(1-m)\mathbf{K} . \quad (20)$$

Let's remind that here and in the equations below, the complete elliptic integrals \mathbf{K} and \mathbf{E} are regarded as functions of m , where $m = k^2$ and k is a modulus, i.e. $\mathbf{K} = \mathbf{K}(m)$ and $\mathbf{E} = \mathbf{E}(m)$ [15].

According to the Melnikov theory the zeros of $M(h)$ determine the limit cycles emerging from the periodic orbits of the unperturbed Hamiltonian systems. The stability of the limit cycle is determined by the sign of the quantity $\varepsilon(dM/dh)$. Since $h = h(m)$ and $m = m(h)$ are single-valued functions the zeros of $M(h)$ and $\mathbf{M}(m)$ coincide. In other words, if m_0 is a zero of $\mathbf{M}(m)$, i.e. $\mathbf{M}(m_0) = 0$, then $h_0 = h(m_0)$ is a zero of $M(h)$, i.e. $M(h_0) = 0$. Moreover, from the relations

$$\frac{dM}{dh} = \frac{d\mathbf{M}(m)}{dh} = \frac{d\mathbf{M}(m)}{dm} \frac{dm}{dh} , \quad \frac{dm}{dh} > 0 ,$$

it follows that the signs of the quantities dM/dh and $d\mathbf{M}/dm$ also coincide. Having this in mind, we will look for the zeros of the function $\mathbf{M}(m)$ and the sign of the quantity $d\mathbf{M}/dm$. On the other hand, with $m \in (0, 1/2)$, the zeros of $\mathbf{M}(m)$ and $\mathbf{B}(m; a_1, a_3)$ coincide. Besides,

if m_0 is a zero of $\mathbf{B}(m; a_1, a_3)$, i.e. $\mathbf{B}(m_0; a_1, a_3) = 0$, then the quantities $M'(m_0)$ and $\mathbf{B}'(m_0; a_1, a_3)$ will have the same signs. Without any loss of generality, we may assume $a_1 = 1$ and then

$$B(m) := \mathbf{B}(m; 1, a_3) = I_1(m) + \frac{2}{5} \lambda a_3 I_2(m) . \quad (21)$$

Therefore we will look for the zeros of the function $B(m)$, or the roots of the equation $B(m) = 0$, and the signs of the derivative $B'(m)$ evaluated at these roots.

For further analysis it is convenient to rewrite the function $B(m)$ in the following way

$$B(m) = I_2(m) \left[Y(m) + \frac{2}{5} \lambda a_3 \right] , \quad (22)$$

where

$$Y(m) = \frac{I_1(m)}{I_2(m)} = \frac{\frac{1}{3}(1-2m)[(1-m)\mathbf{K} - (1-2m)\mathbf{E}]}{2(1-m+m^2)\mathbf{E} - (2-m)(1-m)\mathbf{K}} . \quad (23)$$

The analysis of the Melnikov function (respectively the analysis of function $B(m)$), requires some preliminary results that are given in the following section.

4. Functions involving complete elliptic integrals and equations of Picard-Fuchs and Riccati

In order to find the zeros of the Melnikov function we need to analyze the functions $I_1(m)$, $I_2(m)$ and their ratio $Y(m)$. The equations of Picard-Fuchs and Riccati play a crucial role in this analysis. At the beginning we will investigate the functions $I_1(m)$ and $I_2(m)$. The properties of these functions, which will be needed later, are summed up in the following lemma:

Lemma 1: The following assertions hold:

(a) $I_1(0) = 0$, $I_1(1/2) = 0$, $I_1'(0) = \pi/4$ and the function $I_1(m)$ is positive on $(0, 1/2)$, i.e.

$$I_1(m) > 0 \quad \text{for } m \in (0, 1/2) . \quad (24)$$

(b) $I_2(0) = 0$, $I_2(1/2) > 0$, $I_2'(0) = 0$ and the function $I_2(m)$ is positive and strictly monotone increasing on $(0, 1/2)$, i.e.

$$I_2(m) > 0 \quad , \quad I_2'(m) > 0 \quad \text{for } m \in (0, 1/2) . \quad (25)$$

(c) The functions $I_1(m)$ and $I_2(m)$ satisfy the following system

$$6m(1-m)(1-2m) \frac{d}{dm} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 6(1-6m+6m^2) & (1-2m)^2 \\ 45m(1-m) & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad (26)$$

which is called the Picard-Fuchs equation.

Proof: (a), (b). In the first place, using Equations (A12), we calculate the derivatives

$$I_1'(m) = \frac{1}{6} [10(1-2m)\mathbf{E} - (7-10m)\mathbf{K}], \quad (27)$$

$$I_2'(m) = \frac{5}{2} [(1-m)\mathbf{K} - (1-2m)\mathbf{E}] = \frac{15}{2} \frac{I_1(m)}{(1-2m)}. \quad (28)$$

Then the values $I_1(0)$, $I_1(1/2)$, $I_1'(0)$, $I_2(0)$, $I_2'(0)$ are obtained by direct calculations. Moreover, using elementary properties of the complete elliptic integrals, we find

$$I_1(m) = \frac{1}{3}(1-2m)[(1-m)\mathbf{K} - (1-2m)\mathbf{E}] = \frac{1}{3}(1-2m)[(1-m)(\mathbf{K} - \mathbf{E}) + m\mathbf{E}] > 0,$$

From here it follows that $I_2'(m) > 0$, i.e. the function $I_2(m)$ is monotone increasing. Further, in view of Equation (A11) we get

$$I_2(1/2) = \frac{3}{2} \left[\mathbf{E}(1/2) - \frac{1}{2} \mathbf{K}(1/2) \right] > 0.$$

From the relations $I_2(0) = 0$, $I_2(1/2) > 0$ and taking into account that $I_2(m)$ is a monotone increasing function, it follows the inequality $I_2(m) > 0$.

(c) In order to obtain the system (26), we use the method of undetermined coefficients.

Let's consider the system

$$\begin{cases} I_1' = AI_1 + BI_2 \\ I_2' = CI_1 + DI_2 \end{cases},$$

where A , B , C and D are unknown expressions. Replacing the expressions I_1 , I_2 , I_1' , I_2' and using the fact that \mathbf{K}/\mathbf{E} is not a rational function, we make equal the expressions in front of \mathbf{K} , respectively \mathbf{E} , on the left and right sides of the system equations. In this way we obtain the coefficients A , B , C , D and this yields the system (26). ■

The following lemma gives the properties of the function $Y(m)$.

Lemma 2: The following assertion is valid:

(a) The function $Y(m)$ satisfies the following Riccati equation

$$6m(1-m)(1-2m)Y' = -45m(1-m)Y^2 + 6(1-6m+6m^2)Y + (1-2m)^2. \quad (29)$$

(b) The function $Y(m)$ satisfies the following second order differential equation

$$6m(1-m)(1-2m)Y'' = -45(1-2m)Y^2 - 36(1-2m)Y - 4(1-2m) - 90m(1-m)YY' . \quad (30)$$

(c) $Y(0) = \lim_{m \rightarrow 0^+} Y(m) = +\infty$, $Y(1/2) = 0$ and

$$Y(m) > 0 \quad \text{for } m \in (0, 1/2) . \quad (31)$$

(d) The function $Y(m)$ is strictly monotone decreasing on the interval $(0, 1/2)$, i.e.

$$Y'(m) < 0 \quad \text{for } m \in (0, 1/2) . \quad (32)$$

Proof: (a) Differentiating (23) and introducing $c = 6m(1-m)(1-2m)$ we obtain

$$cY' = c \frac{I_1' I_2 - I_1 I_2'}{I_2^2} = \frac{cI_1'}{I_2} - Y \frac{cI_2'}{I_2} .$$

Replacing in this equation cI_1' and cI_2' from (26), and making some transformations we get (29).

(b) Equation (30) is obtained after differentiation of (29).

(c) The values of $Y(m)$ with $m \rightarrow 0^+$, and $Y(1/2)$ are obtained by using l'Hôpital rule and direct calculations using the results of Lemma 1. We obtain

$$Y(0) = \lim_{m \rightarrow 0^+} \frac{I_1(m)}{I_2(m)} = \lim_{m \rightarrow 0^+} \frac{I_1'(m)}{I_2'(m)} = +\infty ,$$

$$Y(1/2) = \frac{I_1(1/2)}{I_2(1/2)} = \frac{0}{(3/2)[\mathbf{E}(1/2) - (1/2)\mathbf{K}(1/2)]} = 0 .$$

Moreover, $Y(m)$ is a positive function on the interval $(0, 1/2)$, since $I_1(m)$ and $I_2(m)$ are positive functions on the same interval.

(d) We have $Y(0) = +\infty$, $Y(1/2) = 0$, and $Y(m) > 0$ for $m \in (0, 1/2)$. Suppose now, that function $Y(m)$ is not monotone decreasing i.e. it has stationary points. Let \bar{m} be the first stationary point, which is a point of minimum. In this case the following relations hold

$$Y'(\bar{m}) = 0 , \quad Y''(\bar{m}) > 0 \quad \text{for } \bar{m} \in (0, 1/2) .$$

Taking into account the last relations, Equation (30) yields

$$6\bar{m}(1-\bar{m})(1-2\bar{m})Y''(\bar{m}) = -45(1-2\bar{m})Y^2(\bar{m}) - 36(1-2\bar{m})Y(\bar{m}) - 4(1-2\bar{m}) < 0 .$$

The last result gives the inequality $Y''(\bar{m}) < 0$, which means that the stationary point \bar{m} is a point of maximum (it could be stated that all stationary points are points of maximum). This fact is in contradiction to our supposition and this proves part (d) of the lemma. ■

The graph of the function $Y(m)$ is shown in Figure 2.

5. Analysis of the Melnikov Function and limit cycles

We are now able to formulate the following theorem which gives the main results concerning the limit cycles analysis in perturbed single well Duffing oscillators.

Theorem: The system (16) (with $a_1 = 1$) can have a unique hyperbolic limit cycle. The condition for the arising of a limit cycle is the fulfillment of the following inequality

$$a_3 < 0 . \quad (33)$$

The parameters of the limit cycle are determined by the equation

$$Y(m) + \frac{2}{5} \lambda a_3 = 0 . \quad (34)$$

In addition, the limit cycle is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$.

Proof: As it has been emphasized above, the zeros of the Melnikov function are determined by the solution of the following equation

$$B(m) = I_2(m) \left[Y(m) + \frac{2}{5} \lambda a_3 \right] = 0 . \quad (35)$$

We have reached the result $I_2(m) > 0$ and $Y(m) > 0$ with $m \in (0, 1/2)$. Since $\lambda = a/b > 0$, Equation (35) has a solution if and only if the inequality (33) is satisfied. Moreover the zeros of $B(m)$ are determined by Equation (34). Equation (34) has a unique solution with a given negative constant a_3 , since $Y(m)$ is a monotonic function. Then the function $B(m)$ has also a unique zero. Denote the solution of Equation (34) by m_0 . In this case the following relations are valid

$$B(m_0) = 0 \quad , \quad B'(m_0) = I_2(m_0) Y'(m_0) < 0 . \quad (36)$$

This, according to the Melnikov theory, means that for a sufficiently small $\varepsilon \neq 0$, there exists such an h_ε in an $O(\varepsilon)$ neighbourhood of $h_0 = h(m_0)$ (see Equations (14a)), that the perturbed Hamiltonian system (16) has a hyperbolic (simple) limit cycle $\Gamma_\varepsilon(h_\varepsilon)$. The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ is localized in a $O(\varepsilon)$ neighbourhood of the curve $\Gamma_0(h_0)$ and tends to $\Gamma_0(h_0)$ with $\varepsilon \rightarrow 0$. The stability of the limit cycle is determined by the sign of the quantity $\varepsilon B'(m_0)$ and it follows from Equation (36) that the limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ is stable when $\varepsilon > 0$, and unstable when $\varepsilon < 0$. The theorem is proved. ■

The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ in practice coincides with the curve $\Gamma_0(h_0)$ and its first order approximation in the time domain is expressed by the functions (12).

6. Synthesis of perturbed single well Duffing oscillators

The presented theory can be used to perform a synthesis of the considered oscillators. The relations between the Hamiltonian level h and the elliptic modulus k and $m = k^2$ given in Equation (14a) allow us to compute the value of m_0 corresponding to the preliminarily assigned Hamiltonian level h_0 . Then Equation (34), for a given m_0 , yields the coefficient a_3 in such a way that the system (16) is to have a limit cycle localized in a $O(\varepsilon)$ neighbourhood of the assigned Hamiltonian level. The **synthesis procedure** includes the following steps:

1) Formulation of the synthesis problem:

Find an oscillator system of the type (16) (with $a_1 = 1$), having a limit cycle localized in an $O(\varepsilon)$ neighborhood of the curve

$$\Gamma_0(h_0): \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4 = h_0, \quad (37)$$

where the values of a , b , h_0 are given. We note that the preliminarily assigned values of a , b ($\lambda = a/b$) and h_0 form the desired characteristics of the oscillator. In other words, these values determine the shape, position and period of the limit cycle which will arise in system (16).

2) Computation of the values of m_0 and modulus k_0 of the complete elliptic integrals:

$$m_0 = \frac{1}{2} \left(1 - \frac{a}{\sqrt{a^2 + 4bh_0}} \right) \cdot \quad k_0 = \sqrt{m_0}; \quad (38)$$

2) Computation of the value of $Y(m_0)$:

$$Y(m_0) = \frac{\frac{1}{3}(1-2m_0)[(1-m_0)\mathbf{K} - (1-2m_0)\mathbf{E}]}{2(1-m_0+m_0^2)\mathbf{E} - (2-m_0)(1-m_0)\mathbf{K}}. \quad (39)$$

4) Computation of the perturbation coefficient a_3 :

$$a_3 = -\frac{Y(m_0)}{(2/5)\lambda}. \quad (40)$$

The perturbation coefficient a_3 obtained in this way ensures that the system (16) is to have a simple limit cycle $\Gamma_\varepsilon(h_\varepsilon)$, which is localized in a $O(\varepsilon)$ neighborhood of the curve (37). The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ is stable with $\varepsilon > 0$, and unstable with $\varepsilon < 0$.

Finally, as an application of the proposed method, we shall briefly consider an example of a synthesis of an oscillator system. The computation of the numerical data is given after the synthesis procedure.

Example: Find an oscillator system of the type (16) (with $a_1 = 1$), having a simple limit cycle which is localized in a $O(\varepsilon)$ neighbourhood of the curve (37) with $a = 1$, $b = 2$ and $h_0 = 3$.

Solution:

- 1) $a = 1$, $b = 2$, $h_0 = 3$, $\lambda = a/b = 0.5$;
- 2) $m_0 = 0.4$, $k_0 = 0.632456$;
- 3) $\mathbf{K}(0.4) = 1.777519$, $\mathbf{E}(0.4) = 1.399392$, $Y(0.4) = 0.124667$;
- 4) $a_3 = -0.623335 \approx -0.623$.

In this way we find the following system

$$\begin{cases} \dot{x} = y + \varepsilon(x - 0.623x^3) \\ \dot{y} = -x - 2x^3 \end{cases} \quad (41)$$

The system (41) has a hyperbolic limit cycle $\Gamma_\varepsilon(h_\varepsilon)$, which in an $O(\varepsilon)$ approximation is presented by the curve $\Gamma_0(3): (1/2)y^2 + (1/2)x^2 + (1/2)x^4 = 3$. The self-sustained oscillations generated by system (41) are expressed practically in the time domain as follows

$$x(t) = 1.095 \operatorname{sd}\left(\frac{1}{\sqrt{0.2}}t, 0.632\right). \quad (42)$$

The phase portrait of the system (41) for $\varepsilon = 0.25$ and initial data $(0, 0.5)$ and $(0, 4)$ obtained by a numerical integration is shown in Fig. 3. The numerical perfectly computations confirm the analytical results.

7. Conclusions

An analysis of oscillator systems described by single well Duffing equation under polynomial perturbations of 4th degree has been presented. It has been proved that the considered system can have a unique hyperbolic limit cycle. The obtained inequality (33) represents a condition for the arising of a limit cycle and the equation (34) determines the parameters of this limit cycle.

Moreover, a method for the synthesis of oscillator systems of the considered type is proposed. The synthesis consists of finding such appropriate perturbation coefficients that the prescribed properties are fulfilled. Both the analysis and the synthesis are performed with the help of the Melnikov function.

An example of the synthesis of oscillator system is given. Together with the numerical simulation, it illustrates and confirms the presented theory.

The results obtained can be used for more complete analysis and synthesis of oscillator systems.

Appendix

In this section we give some properties and identities concerning the Jacobi elliptic function $\operatorname{sd} = \operatorname{sd} z = \operatorname{sd}(z, k)$ and the complete elliptic integrals of the first and second kind, \mathbf{K} and \mathbf{E} , which are used in the presentation. Recall that the complete elliptic integrals \mathbf{K}

and \mathbf{E} are regarded as functions of k^2 , where k is a modulus, i.e. $\mathbf{K} = \mathbf{K}(k^2)$ and $\mathbf{E} = \mathbf{E}(k^2)$.

The function $\text{sd} = \text{sd } z = \text{sd}(z, k)$ is a periodic and odd function with a period $4\mathbf{K}$. Moreover the following identities are valid

$$\text{sd}(0, k) = \text{sd}(4\mathbf{K}, k) = 0 \quad , \quad \text{sd}(z, k) = -\text{sd}(z + 2\mathbf{K}, k) \quad , \quad (\text{A1})$$

$$\int_0^{4\mathbf{K}} \text{sd}^{2n+1}(z, k) dz = 0 \quad , \quad n = 0, 1, 2, \dots, \quad (\text{A2})$$

$$\int_0^{4\mathbf{K}} \text{sd}^{2n}(z, k) dz = 4 \int_0^{\mathbf{K}} \text{sd}^{2n}(z, k) dz \quad , \quad n = 1, 2, \dots, \quad (\text{A3})$$

The results concerning the integration of Jacobi elliptic function sd are given in two lemmas.

Lemma A1: The following recurrence formula is valid

$$\int_0^{\mathbf{K}} \text{sd}^n dz = \left(\frac{n-2}{n-1} \right) \left(\frac{2k^2-1}{k^2(1-k^2)} \right) \int_0^{\mathbf{K}} \text{sd}^{n-2} dz + \left(\frac{n-3}{n-1} \right) \left(\frac{1}{k^2(1-k^2)} \right) \int_0^{\mathbf{K}} \text{sd}^{n-4} dz \quad , \quad (\text{A4})$$

where n is an even number and $n \geq 4$, ($n = 4, 6, 8, \dots$).

Proof: In order to prove the recurrence formula (A4) we start from the identity

$$\frac{d}{dz} (\text{sd}^{n-3} \text{cd nd}) = -(n-1)k^2(1-k^2)\text{sd}^n + (n-2)(2k^2-1)\text{sd}^{n-2} + (n-3)\text{sd}^{n-4} \quad . \quad (\text{A5})$$

where cd and nd are Jacobi elliptic functions. This identity can be easily obtained using the usual dependencies between the Jacobi elliptic functions [15-20]. Integrating (A5) and taking into account the relations (A1), we obtain

$$\int_0^{4\mathbf{K}} \text{sd}^n dz = \frac{1}{(n-1)k^2(1-k^2)} \left[(n-2)(2k^2-1) \int_0^{4\mathbf{K}} \text{sd}^{n-2} dz + (n-3) \int_0^{4\mathbf{K}} \text{sd}^{n-4} dz \right] \quad . \quad (\text{A6})$$

Equation (A6) is transformed to (A4) with the aid of (A2) and (A3). ■

Lemma A2: The following identities hold

$$\int_0^{\mathbf{K}} \text{sd}^2 dz = \frac{1}{k^2(1-k^2)} [\mathbf{E} - (1-k^2)\mathbf{K}] \quad , \quad (\text{A7})$$

$$\int_0^{\mathbf{K}} \text{sd}^4 dz = \frac{1}{3k^4(1-k^2)^2} [2(2k^2-1)\mathbf{E} + (3k^4-5k^2+2)\mathbf{K}] \quad , \quad (\text{A8})$$

$$\int_0^{\mathbf{K}} \text{sd}^6 dz = \frac{1}{15k^6(1-k^2)^3} [(23k^4 - 23k^2 + 8)\mathbf{E} + (15k^6 - 34k^4 + 27k^2 - 8)\mathbf{K}]. \quad (\text{A9})$$

Proof: Equation (A7) is given in [16]. Equations (A8) and (A9) are obtained with the help of the recurrence formula (A4). ■

The following relation

$$\mathbf{E}(m)\mathbf{K}(1-m) + \mathbf{E}(1-m)\mathbf{K}(m) - \mathbf{K}(m)\mathbf{K}(1-m) = \pi/2. \quad (\text{A10})$$

is known as the Legendre's Relation [15-16]. At $m = k^2 = 1/2$ Equation (A10) becomes

$$\mathbf{K}(1/2)[2\mathbf{E}(1/2) - \mathbf{K}(1/2)] = \pi/2,$$

from where it follows

$$2\mathbf{E}(1/2) - \mathbf{K}(1/2) > 0. \quad (\text{A11})$$

The derivatives of the functions $\mathbf{K}(m)$ and $\mathbf{E}(m)$ with respect to $m = k^2$ are

$$\frac{d\mathbf{K}(m)}{dm} = \frac{\mathbf{E}(m) - (1-m)\mathbf{K}(m)}{2m(1-m)}, \quad \frac{d\mathbf{E}(m)}{dm} = \frac{\mathbf{E}(m) - \mathbf{K}(m)}{2m}. \quad (\text{A12})$$

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Captions for the figures

Figure 1. Oscillator circuits governed by perturbed Duffing equations

Figure 2. Graphical plot of the function $Y(m)$

Figure 3. Phase portrait of system (41)

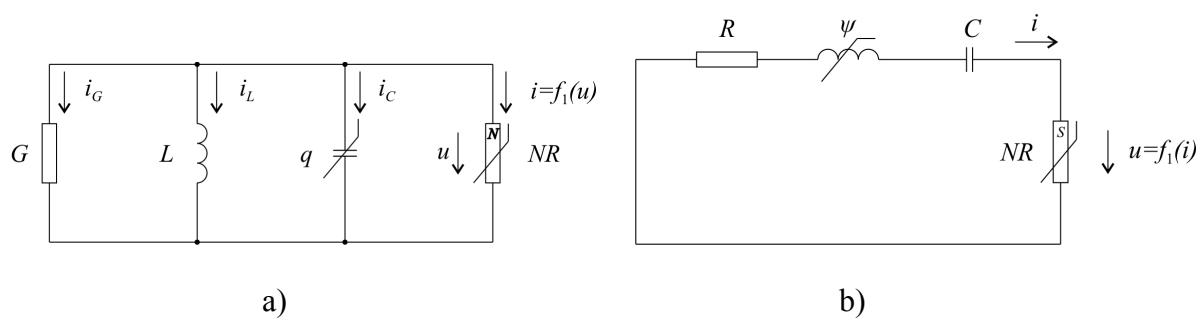


Figure 1.

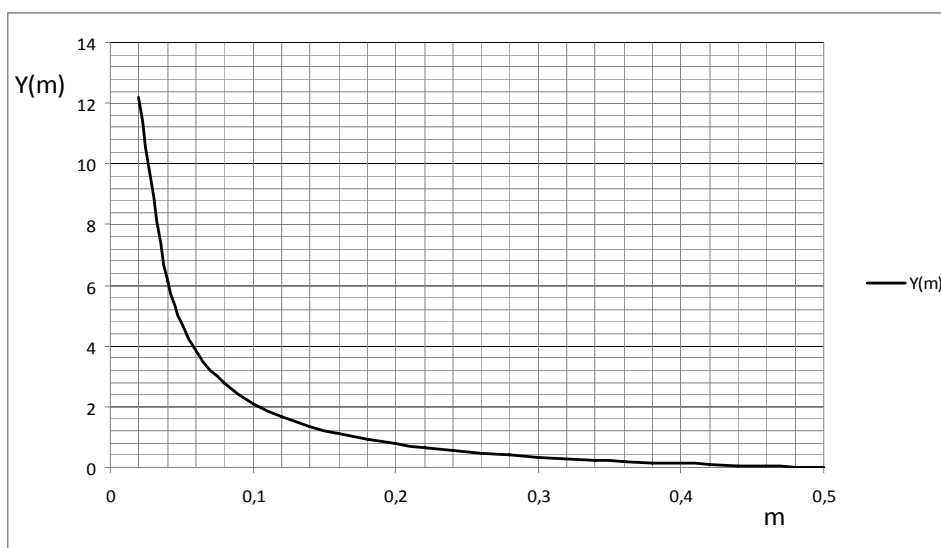


Figure 2.

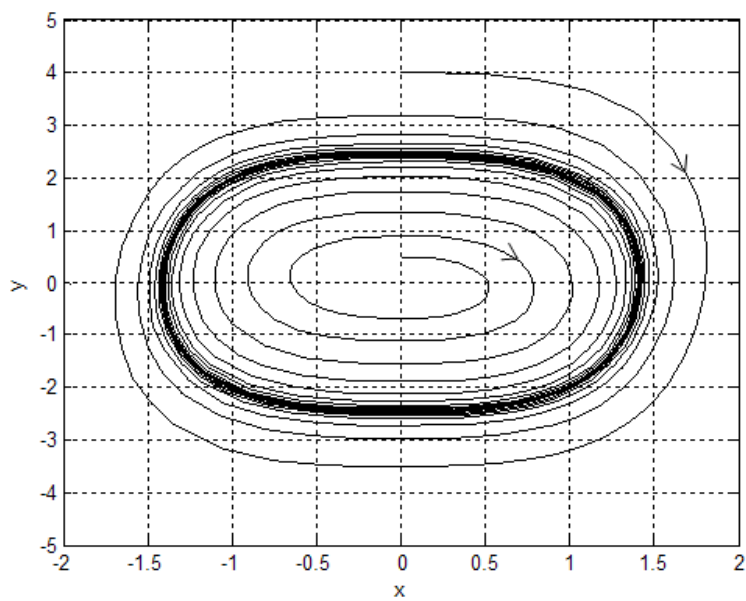


Figure 3.