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ALGEBRAIC TORI AS NISNEVICH SHEAVES WITH TRANSFERS

BRUNO KAHN

ABSTRACT. We relate R -equivalence on tori with Voevodsky's theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

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1. MAIN RESULTS

Let k be a field and let T be a k -torus. The R -equivalence classes on T have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [3] and [4]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky's triangulated category of motives sheds a new light on this question: see Corollaries 1, 3 and 4 below.

More generally, let G be a semi-abelian variety over k , which is an extension of an abelian variety A by a torus T . Denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over k in the sense of Voevodsky [19]. Then G has a natural structure of an object of HI ([17, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let L be the group of cocharacters of T .

Proposition 1. *There is a natural isomorphism $G_{-1} \xrightarrow{\sim} L$ in HI .*

Here $_{-1}$ is the contraction operation of [18, p. 96], whose definition is recalled in the proof below.

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Proof. Recall that if \mathcal{F} is a presheaf [with transfers] on smooth k -schemes, the presheaf [with transfers] \mathcal{F}_{-1}^p is defined by

$$U \mapsto \text{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \rightarrow \mathcal{F}(U \times \mathbb{G}_m)).$$

If \mathcal{F} is homotopy invariant, we may replace $U \times \mathbf{A}^1$ by U and the rational point $1 \in \mathbb{G}_m$ realises $\mathcal{F}_{-1}^p(U)$ as a functorial direct summand of $\mathcal{F}(U \times \mathbb{G}_m)$.

If \mathcal{F} is a Nisnevich sheaf [with transfers], \mathcal{F}_{-1} is defined as the sheaf associated to \mathcal{F}_{-1}^p .

Now $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$ since A is an abelian variety, hence $A_{-1}^p = 0$. We therefore have an isomorphism of presheaves $T_{-1}^p \xrightarrow{\sim} G_{-1}^p$, and *a fortiori* an isomorphism of Nisnevich sheaves $T_{-1} \xrightarrow{\sim} G_{-1}$.

Let $p : \mathbb{G}_m \rightarrow \text{Spec } k$ be the structural map. One easily checks that the *étale* sheaf $\text{Coker}(T \xrightarrow{i} p_*p^*T)$ is canonically isomorphic to L . Since i is split, its cokernel is still L if we view it as a morphism of presheaves, hence of Nisnevich sheaves. \square

From now on, we assume k perfect. Let $\text{DM}_{-}^{\text{eff}}$ be the triangulated category of effective motivic complexes introduced in [19]: it has a t -structure with heart HI . It also has a tensor structure and a (partially defined) internal Hom . We then have an isomorphism

$$L[0] = G_{-1}[0] \simeq \underline{\text{Hom}}_{\text{DM}_{-}^{\text{eff}}}(\mathbb{G}_m[0], G[0])$$

[10, Rk. 4.4], hence by adjunction a morphism in $\text{DM}_{-}^{\text{eff}}$

$$(1) \quad L[0] \otimes \mathbb{G}_m[0] \rightarrow G.$$

Let $\nu_{\leq 0}G[0]$ denote the cone of (1): by [11, Lemma 6.3] or [8, §2], $\nu_{\leq 0}G[0]$ is the *birational motivic complex* associated to G . We want to compute its homology sheaves.

For this, consider a coflasque resolution

$$(2) \quad 0 \rightarrow Q \rightarrow L_0 \rightarrow L \rightarrow 0$$

of L in the sense of [3, p. 179]. Taking a coflasque resolution of Q and iterating, we get a resolution of L by invertible lattices¹:

$$(3) \quad \cdots \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow L \rightarrow 0.$$

We set

$$Q_n = \begin{cases} Q & \text{for } n = 1 \\ \text{Ker}(L_{n-1} \rightarrow L_{n-2}) & \text{for } n > 1. \end{cases}$$

¹Recall that a *lattice* is a free finitely generated Galois module; a lattice is *invertible* if it is a direct summand of a permutation lattice.

Theorem 1. a) Let T_n denote the torus with cocharacter group L_n . Then $\nu_{\leq 0}G[0]$ is isomorphic to the complex

$$\cdots \rightarrow T_n \rightarrow \cdots \rightarrow T_0 \rightarrow G \rightarrow 0.$$

b) Let S_n be the torus with cocharacter group Q_n . For any connected smooth k -scheme X with function field K , we have

$$H_n(\nu_{\leq 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

The proof is given in Section 2.

Corollary 1. The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$ provides G/R with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $\varphi : Y \rightarrow X$ of smooth connected k -schemes induces a morphism $\varphi^* : G(k(X))/R \rightarrow G(k(Y))/R$. \square

This functoriality is essential to formulate Theorem 2 below. For φ a closed immersion of codimension 1, it recovers a specialisation map on R -equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, e.g. [4, Th. 3.1 and Cor. 4.2] or [7]. (I am indebted to Colliot-Thélène for pointing out these references.)

Corollary 2. a) If k is finitely generated, the n -th homology sheaf of $\nu_{\leq 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or G is a torus.

b) If G is a torus, then $\nu_{\leq 0}G[0] = 0$ if G is split by a Galois extension E/k whose Galois group has cyclic Sylow subgroups. This condition is automatic if k is (quasi-)finite.

The proof is also given in Section 2.

Given two semi-abelian varieties G, G' , we would now like to understand the maps

$$\mathrm{Hom}_k(G, G') \rightarrow \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0]) \rightarrow \mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R).$$

In Section 3, we succeed in elucidating the nature of their composition to a large extent, at least if G is a torus. Our main result, in the spirit of Yoneda's lemma, is

Theorem 2. Let G, G' be two semi-abelian varieties, with G a torus. Suppose given, for every function field K/k , a homomorphism $f_K : G(K)/R \rightarrow G'(K)/R$ such that f_K is natural with respect to the functoriality of Corollary 1. Then

a) There exists an extension \tilde{G} of G by a permutation torus, and a homomorphism $f : \tilde{G} \rightarrow G'$ inducing (f_K) .

b) f_K is surjective for all K if and only if there exist extensions \tilde{G}, \tilde{G}' of G and G' by permutation tori such that f_K is induced by a split surjective homomorphism $\tilde{G} \rightarrow \tilde{G}'$.

The proof is given in §3.3. See Proposition 2, Corollary 5, Remark 4 and Proposition 3 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [3] and [4], providing alternate proofs and strengthening of some of their results (at least over a perfect field). More precisely:

Corollary 3. a) Let G' be a semi-abelian k -variety such that $G'(K)/R = 0$ for any function field K/k . Then G' is an invertible torus.

b) In Theorem 2 b), assume that f_K is bijective for all K/k . Then there exist extensions \tilde{G}, \tilde{G}' of G and G' by invertible tori such that f_K is induced by an isomorphism $\tilde{G} \xrightarrow{\sim} \tilde{G}'$.

Proof. a) This is the special case $G = 0$ of Theorem 2 b).

b) By Theorem 2 b), we may replace G and G' by extensions by permutation tori such that f_K is induced by a split surjection $f : G \rightarrow G'$. Let $T = \text{Ker } f$. Then $T/R = 0$ universally. By a), T is invertible. \square

Corollary 3 a) is a version of [4, Prop. 7.4] (taking [3, p. 199, Th. 2] into account). Theorem 2 was inspired by the desire to understand this result from a different viewpoint.

Corollary 4. Let $f : G \dashrightarrow G'$ be a rational map of semi-abelian varieties, with G a torus. Then the following conditions are equivalent:

- (i) $f_* : \nu_{\leq 0} G[0] \rightarrow \nu_{\leq 0} G'[0]$ is an isomorphism (see Proposition 2).
- (ii) $f_* : G(K)/R \rightarrow G'(K)/R$ is bijective for any function field K/k .
- (iii) f is an isomorphism, up to extensions of G and G' by invertible tori and up to a translation. (See Lemma 6.) \square

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2. PROOFS OF THEOREM 1 AND COROLLARY 2

Lemma 1. *The exact sequence*

$$0 \rightarrow T(k) \rightarrow G(k) \rightarrow A(k)$$

induces an exact sequence

$$0 \rightarrow T(k)/R \xrightarrow{i} G(k)/R \rightarrow A(k).$$

Proof. Let $f : \mathbf{P}^1 \dashrightarrow G$ be a k -rational map defined at 0 and 1. Its composition with the projection $G \rightarrow A$ is constant: thus the image of f lies in a T -coset of G defined by a rational point. This implies the injectivity of i , and the rest is clear. \square

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_-^{\mathrm{eff}}$ may be viewed as a localisation of $D^-(\mathrm{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [19, Prop. 3.2.3].

Lemma 2. *If G is an invertible torus, there is a canonical isomorphism in $D^-(\mathrm{NST})$*

$$L[0] \otimes \mathbb{G}_m \xrightarrow{\sim} G[0].$$

In particular, $\nu_{\leq 0}G[0] = 0$.

Proof. We reduce to the case $T = R_{E/k}\mathbb{G}_m$, where E is a finite extension of k . Let us write more precisely $\mathrm{NST}(k)$ and $\mathrm{NST}(E)$. There is a pair of adjoint functors

$$\mathrm{NST}(k) \xrightarrow{f^*} \mathrm{NST}(E), \quad \mathrm{NST}(E) \xrightarrow{f_*} \mathrm{HI}(k)$$

where $f : \mathrm{Spec} E \rightarrow \mathrm{Spec} k$ is the projection. Clearly,

$$f_*\mathbf{Z} = \mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec} E), \quad f_*\mathbb{G}_m = T$$

where $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec} E)$ is the Nisnevich sheaf with transfers represented by $\mathrm{Spec} E$. Since $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec} E) = L$, this proves the claim. \square

Proof of Theorem 1. a) Recall that L_0 is an invertible lattice chosen so that $L_0(E) \rightarrow L(E)$ is surjective for any extension E/k . In particular, (2) and (3) are exact as sequences of Nisnevich sheaves; hence $L[0]$ is isomorphic in $D^-(\mathrm{NST})$ to the complex

$$L = \cdots \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow 0.$$

(We may view (3) as a version of Voevodsky's "canonical resolutions" as in [19, §3.2 p. 206].)

By Lemma 2, $L_n[0] \otimes \mathbb{G}_m[0] \simeq T_n[0]$ is homologically concentrated in degree 0 for all n . It follows that the complex

$$T = \cdots \rightarrow T_n \rightarrow \cdots \rightarrow T_0 \rightarrow 0$$

is isomorphic to $L[0] \otimes \mathbb{G}_m[0]$ in $D^-(\text{NST})$, hence *a fortiori* in DM_-^{eff} .

b) For any nonempty open subscheme $U \subseteq X$ we have isomorphisms

$$(4) \quad H_n(\nu_{\leq 0}G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(K)$$

(e.g. [8, p. 912]). By a), the right hand term is the n -th homology group of the complex

$$\cdots \rightarrow T_n(K) \rightarrow \cdots \rightarrow T_0(K) \rightarrow G(K) \rightarrow 0$$

with $G(K)$ in degree 0. By [3, p. 199, Th. 2], the sequences

$$0 \rightarrow S_1(K) \rightarrow T_0(K) \rightarrow T(K) \rightarrow T(K)/R \rightarrow 0$$

$$0 \rightarrow S_{n+1}(K) \rightarrow T_n(K) \rightarrow S_n(K) \rightarrow S_n(K)/R \rightarrow 0$$

are all exact. Using Lemma 1 for H_0 , the conclusion follows from an easy diagram chase. \square

Remark 1. As a corollary to Theorem 1, $S_n(K)/R$ only depends on G . This can be seen without mentioning DM_-^{eff} : in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (3), which easily follows from the definition of coflasque modules.

Proof of Corollary 2. a) This follows via Theorem 1 and Lemma 1 from [3, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the L_n , hence the S_n split by E/k . The conclusion now follows from Theorem 1 and [3, p. 200, Cor. 3]. The last claim is clear. \square

Remark 2. In characteristic $p > 0$, all finitely generated perfect fields are finite. To give some contents to Corollary 2 a) in this characteristic, one may pass to the perfect [one should say radical] closure k of a finitely generated field k_0 . If G is a semi-abelian k -variety, it is defined over some finite extension k_1 of k_0 . If k_2/k_1 is a finite (purely inseparable) subextension of k/k_1 , then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \rightarrow G(k_2)$$

equals multiplication by $[k_2 : k_1]$. Hence Corollary 2 a) remains true at least after inverting p .

3. STABLE BIRATIONALITY

If X is a smooth variety over a field k , we write $\text{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [16]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \rightarrow \text{Alb}(X)$ which is universal for morphisms from X to semi-abelian varieties sending x_0 to 0.

We also write $\text{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence. This group is finitely generated if k is algebraically closed [9, Th. 3].

3.1. Well-known lemmas. I include proofs for lack of reference.

Lemma 3. *a) Let G, G' be two semi-abelian k -varieties. Then any k -morphism $f : G \rightarrow G'$ can be written uniquely $f = f(0) + f'$, where f' is a homomorphism.*

b) For any semi-abelian k -variety G , the canonical map $G \rightarrow \text{Alb}(G)$ sending 0 to 0 is an isomorphism.

Proof. a) amounts to showing that if $f(0) = 0$, then f is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

Proof of a). We may assume k to be a universal domain. The statement is classical for abelian varieties [15, p. 41, Cor. 1] and an easy computation for tori. In the general case, let T, T' be the toric parts of G and G' and A, A' be their abelian parts. Let $g \in G(k)$. As any morphism from T to A' is constant, the k -morphism

$$\varphi_g : T \ni t \mapsto f(g + t) - f(g) \in G'$$

(which sends 0 to 0) lands in T' , hence is a homomorphism. Therefore it only depends on the image of g in $A(k)$. This defines a morphism $\varphi : A \rightarrow \underline{\text{Hom}}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g + h) - f(g) - f(h)$$

induces a morphism $A \times A \rightarrow T'$. Such a morphism is constant, of value 0.

Proof of b). This is true if G is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from G to an abelian variety is trivial on T . This shows that the abelian part of $\text{Alb}(G)$ is A . Let $T' = \text{Ker}(\text{Alb}(G) \rightarrow A)$. We also have the counit morphism $\text{Alb}(G) \rightarrow G$, and the composition $G \rightarrow \text{Alb}(G) \rightarrow G$ is the identity. Thus T is a direct summand of T' . It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$.

Then consider the line bundle completion $\bar{G} \rightarrow A$ of the \mathbb{G}_m -bundle $G \rightarrow A$. It is sufficient to show that the kernel of

$$\text{Alb}(G) \rightarrow \text{Alb}(\bar{G}) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. \square

Lemma 4. *Suppose k algebraically closed, and let G be a semi-abelian k -variety. Let A be the abelian quotient of G . Then the map*

$$(5) \quad \mathrm{NS}(A) \rightarrow \mathrm{NS}(G)$$

is an isomorphism.

Proof. Let $T = \mathrm{Ker}(G \rightarrow A)$ and $X(T)$ be its character group. Choosing a basis (e_i) of $X(T)$, we may complete the \mathbb{G}_m^n -torsor G into a product of line bundles $\bar{G} \rightarrow A$. The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\bar{G}) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (5). Its kernel is generated by the classes of the irreducible components D_i of the divisor with normal crossings $\bar{G} - G$. These components correspond to the basis elements e_i . Since the corresponding \mathbb{G}_m -bundle is a group extension of A by \mathbb{G}_m , the class of the 0 section of its line bundle completion lies in $\mathrm{Pic}^0(A)$, hence goes to 0 in $\mathrm{NS}(\bar{G})$. \square

Lemma 5. *Let X be a smooth k -variety, and let $U \subseteq X$ be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \rightarrow T \rightarrow \mathrm{Alb}(U) \rightarrow \mathrm{Alb}(X) \rightarrow 0$$

with T a torus. If $\mathrm{NS}(\bar{U}) = 0$ (this happens if U is small enough), there is an exact sequence of character groups

$$0 \rightarrow X(T) \rightarrow \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \rightarrow \mathrm{NS}(\bar{X}) \rightarrow 0.$$

Proof. This follows for example from [1, Cor. 10.5.1]. \square

Lemma 6. *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian k -varieties, with G a torus. Then there exists an extension \tilde{G} of G by a permutation torus and a homomorphism $\tilde{f} : \tilde{G} \rightarrow G'$ which extends f up to translation in the following sense: there exists a rational section $s : G \dashrightarrow \tilde{G}$ of the projection $\pi : \tilde{G} \rightarrow G$ and a rational point $g' \in G'(k)$ such that $f = \tilde{f}s + g'$. If f is defined at 0_G and sends it to $0_{G'}$, then $g' = 0$.*

Proof. Let U be an open subset of G where f is defined. We define $\tilde{G} = \mathrm{Alb}(U)$. Applying Lemmas 5 and 3 b) and using $\mathrm{NS}(\bar{G}) = 0$, we get an extension

$$0 \rightarrow P \rightarrow \tilde{G} \rightarrow G \rightarrow 0$$

where P is a permutation torus, as well as a morphism $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \rightarrow G'$.

Let us first assume k infinite. Then $U(k) \neq \emptyset$ because G is unirational. A rational point $g \in U$ defines an Albanese map $s : U \rightarrow \tilde{G}$ sending g to $0_{\tilde{G}}$. Since P is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace s by a morphism sending g to \tilde{g} . Then s is a rational section of π . Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows.

If k is finite, then U has at least a zero-cycle g of degree 1, which is enough to define the Albanese map s . We then proceed as above (lift every closed point involved in g to a closed point of \tilde{G} with the same residue field). \square

Lemma 7. *Let G be a finite group, and let A be a finitely generated G -module. Then*

- a) *There exists a short exact sequence of G -modules $0 \rightarrow P \rightarrow F \rightarrow A \rightarrow 0$, with F torsion-free and flasque, and P permutation.*
- b) *Let B be another finitely generated G -module, and let $0 \rightarrow P' \rightarrow E \rightarrow B \rightarrow 0$ be an exact sequence with P' an invertible module. Then any G -morphism $f : A \rightarrow B$ lifts to $\tilde{f} : F \rightarrow E$.*

Proof. a) is the contents of [4, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting f lies in $\text{Ext}_G^1(F, P') = 0$ [3, p. 182, Lemme 9]. \square

3.2. Functoriality of $\nu_{\leq 0}G$. We now assume k perfect.

Lemma 8. *Let*

$$(6) \quad 0 \rightarrow P \rightarrow G \rightarrow H \rightarrow 0$$

be an exact sequence of semi-abelian varieties, with P an invertible torus. Then $\nu_{\leq 0}G[0] \xrightarrow{\sim} \nu_{\leq 0}H[0]$.

Proof. As P is invertible, (6) is exact in NST hence defines an exact triangle

$$P[0] \rightarrow G[0] \rightarrow H[0] \xrightarrow{+1}$$

in DM_-^{eff} . The conclusion then follows from Lemma 2. \square

Proposition 2. *Let G, G' be two semi-abelian k -varieties, with G a torus. Then a rational map $f : G \dashrightarrow G'$ induces a morphism $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$, hence a homomorphism $f_* : G(K)/R \rightarrow G'(K)/R$ for any extension K/k . If K is infinite, f_* agrees up to translation with the morphism induced by f via the isomorphism $U(K)/R \xrightarrow{\sim} G(K)/R$ from [3, p. 196 Prop. 11], where U is an open subset of definition of f .*

Proof. By Lemma 6, f induces a homomorphism $\tilde{G} \rightarrow G'$ where \tilde{G} is an extension of G by a permutation torus. By Lemma 8, the induced

morphism

$$\nu_{\leq 0}\tilde{G}[0] \rightarrow \nu_{\leq 0}G'[0]$$

factors through a morphism $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$.

The claims about R -equivalence classes follow from Theorem 1 b) and Lemma 6. \square

Remark 3. The proof shows that $f'_* = f_*$ if f' differs from f by a translation by an element of $G(k)$ or $G'(k)$.

Corollary 5. *If T and T' are birationally equivalent k -tori, then $\nu_{\leq 0}T[0] \simeq \nu_{\leq 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.*

Proof. The proof of Proposition 2 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_*f_* = 1_{\nu_{\leq 0}T[0]}$ and $f_*g_* = 1_{\nu_{\leq 0}T'[0]}$. The last claim follows from Theorem 1. \square

Remark 4. It is proven in [3] that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$ (p. 197, Cor. to Prop. 11) and that the group $T(k)/R$ is abstractly a birational invariant of T (p. 200, Cor. 4). The proof above shows that f_* is an isomorphism of groups if f respects the origins of T and T' . This solves the question raised in [3, mid. p. 397]. The proofs of Lemma 6 and Proposition 2 may be seen as dual to the proof of [3, p. 189, Prop. 5], and are directly inspired from it.

3.3. Faithfulness and fullness.

Proposition 3. *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with G a torus. Assume that the map $f_* : G(K)/R \rightarrow G'(K)/R$ from Proposition 2 is identically 0 when K runs through the finitely generated extensions of k . Then there exists a permutation torus P and a factorisation of f as*

$$G \dashrightarrow P \xrightarrow{g} G'$$

where \tilde{f} is a rational map and g is a homomorphism. If f is a morphism, we may choose \tilde{f} as a homomorphism.

Conversely, if there is such a factorisation, then $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$ is the 0 morphism.

Proof. By Lemma 6, we may reduce to the case where f is a morphism. Let $K = k(G)$. By hypothesis, the image of the generic point $\eta_G \in G(K)$ is R -equivalent to 0 on $G'(K)$. By a lemma of Gille [6, Lemme

II.1.1 b)], it is directly R -equivalent to 0: in other words, there exists a rational map $h : G \times \mathbf{A}^1 \dashrightarrow G'$, defined in the neighbourhood of 0 and 1, such that $h|_{G \times \{0\}} = 0$ and $h|_{G \times \{1\}} = f$.

Let $U \subseteq G \times \mathbf{A}^1$ be an open set of definition of h . The 0 and 1-sections of $G \times \mathbf{A}^1 \rightarrow G$ induce sections

$$s_0, s_1 : G \rightarrow \text{Alb}(U)$$

of the projection $\pi : \text{Alb}(U) \rightarrow \text{Alb}(G \times \mathbf{A}^1) = G$ such that $\text{Alb}(h) \circ s_0 = 0$ and $\text{Alb}(h) \circ s_1 = f$. If $P = \text{Ker } \pi$, then $s_0 - s_1$ induces a homomorphism $\tilde{f} : G \rightarrow P$ such that the composition

$$G \xrightarrow{\tilde{f}} P \rightarrow \text{Alb}(U) \xrightarrow{\text{Alb}(h)} G'$$

equals f . Finally, P is a permutation torus by Lemma 5.

The last claim follows from Lemma 2. \square

Proof of Theorem 2. a) Take $K = k(G)$. The image of the generic point η_G by f_K lifts to a (non unique) rational map $f : G \dashrightarrow G'$. Using Lemma 6, we may extend f to a homomorphism

$$\tilde{f} : \tilde{G} \rightarrow G'$$

where \tilde{G} is an extension of G by a permutation torus P . Since $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$, we reduce to $\tilde{G} = G$ and $\tilde{f} = f$.

Let L/k be a function field, and let $g \in G(L)$. Then g arises from a morphism $g : X \rightarrow G$ for a suitable smooth model X of L . By assumption on $K \mapsto f_K$, the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow{f_K} & G'(K)/R \\ g^* \downarrow & & g^* \downarrow \\ G(L)/R & \xrightarrow{f_L} & G'(L)/R \end{array}$$

commutes. Applying this to $\eta_K \in G(K)$, we find that $f_L([g]) = [g \circ f]$, which means that f_L is the map induced by f .

b) The hypothesis implies that $G'(E)/R = 0$ for any algebraically closed extension E/k , which in turn implies that G' is also a torus. Applying a), we may, and do, convert f into a true homomorphism by replacing G by a suitable extension by a permutation torus. Applying Lemma 7 a) to the cocharacter group of G , we get a resolution $0 \rightarrow P_1 \rightarrow Q \rightarrow G \rightarrow 0$ with Q coflasque and P_1 permutation. Hence we may (and do) further assume G coflasque.

Let $K = k(G')$ and choose some $g \in G(K)$ mapping modulo R -equivalence to the generic point of G' . Then g defines a rational map

$g : G' \dashrightarrow G$ such that fg is R -equivalent to $1_{G'}$. It follows that the induced map

$$(7) \quad 1 - fg : G'/R \rightarrow G'/R$$

is identically 0.

Reapplying Lemma 6, we may find an extension \tilde{G}' of G' by a suitable permutation torus which converts g into a true homomorphism. Since G is coflasque, Lemma 7 b) shows that $f : G \rightarrow G'$ lifts to $\tilde{f} : G \rightarrow \tilde{G}'$. Then (7) is still identically 0 when replacing (G', f) by (\tilde{G}', \tilde{f}) .

Summarising: we have replaced the initial G and G' by suitable extensions by permutation tori, such that f lifts to these extensions and there is a homomorphism $g : G' \rightarrow G$ such that (7) vanishes identically. Hence $1 - fg$ factors through a permutation torus P thanks to Proposition 3. Write $u : G' \rightarrow P$ and $v : P \rightarrow G'$ for homomorphisms such that $1 - fg = vu$. Let $G_1 = G \times P$ and consider the maps

$$f_1 = (f, v) : G_1 \rightarrow G', \quad g_1 = \begin{pmatrix} g \\ u \end{pmatrix} : G' \rightarrow G_1.$$

Then $f_1 g_1 = 1$ and G' is a direct summand of G_1 as requested. \square

4. SOME OPEN QUESTIONS

Question 1. Are lemma 6 and Proposition 2 still true when G is not a torus?

This is far from clear in general, starting with the case where G is an abelian variety and G' a torus. Let me give a positive answer in the case of an elliptic curve.

Proposition 4. *The answer to Question 1 is yes if the abelian part A of G is an elliptic curve.*

Proof. Arguing as in the proof of Proposition 2, we get for an open subset $U \subseteq G$ of definition for f an exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow P \rightarrow \text{Alb}(U) \rightarrow G \rightarrow 0$$

where P is a permutation torus. Here we used that $\text{NS}(\bar{G}) \simeq \mathbf{Z}$, which follows from Lemma 4.

The character group $X(P)$ has as a basis the geometric irreducible components of codimension 1 of $G - U$. Up to shrinking U , we may assume that $G - U$ contains the inverse image D of $0 \in A$. As the divisor class of 0 generates $\text{NS}(\bar{A})$, D provides a Galois-equivariant splitting of the map $\mathbb{G}_m \rightarrow P$. Thus its cokernel is still a permutation torus, and we conclude as before. \square

Question 2. Can one formulate a version of Theorem 2 and Corollary 3 providing a description of the groups $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0])$ and $\mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R)$ (at least when G and G' are tori)?

The proof of Theorem 2 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let G be a semi-abelian variety. Forgetting its group structure, it has a motive $M(G) \in \mathrm{DM}_-^{\mathrm{eff}}$. Recall the canonical morphism

$$M(G) \rightarrow G[0]$$

induced by the “sum” maps

$$(8) \quad c(X, G) \xrightarrow{\sigma} G(X)$$

for smooth varieties X ([17, (6), (7)], [1, §1.3]).

The morphism (8) has a canonical section

$$(9) \quad G(X) \xrightarrow{\gamma} c(X, G)$$

given by the graph of a morphism: this section is functorial in X but is not additive.

Consider now a smooth equivariant compactification \bar{G} of G . It exists in all characteristics. For tori, this is written up in [2]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if G is an extension of an abelian variety A by a torus T , take a smooth projective equivariant compactification Y of T . Then the bundle $G \times^T Y$ associated to the T -torsor $G \rightarrow A$ also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$(10) \quad \begin{array}{ccc} \nu_{\leq 0}M(G) & \xrightarrow{\sim} & \nu_{\leq 0}M(\bar{G}) \\ \nu_{\leq 0}\sigma \downarrow & & \\ \nu_{\leq 0}G[0] & & \end{array}$$

By [11], we have $H_0(\nu_{\leq 0}M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$ for any smooth connected X . Hence the above diagram induces a homomorphism

$$(11) \quad CH_0(\bar{G}_{k(X)}) \rightarrow G(k(X))/R$$

which is natural in X for the action of finite correspondences (compare Corollary 1). One can probably check that this is the homomorphism of [12, (17) p. 78], reformulating [3, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$(12) \quad G(k(X))/R \rightarrow CH_0(\bar{G}_{k(X)})$$

of [3, p. 197] can presumably be recovered as a birational version of (9), using perhaps the homotopy category of schemes of Morel and Voevodsky [14].

In [12], Merkurjev shows that (11) is an isomorphism for G a torus of dimension at most 3. This suggests:

Question 3. Is the map $\nu_{\leq 0}\sigma$ of Diagram (10) an isomorphism when G is a torus of dimension ≤ 3 ?

In [13], Merkurjev gives examples of tori G for which (12) is not a homomorphism; hence its (additive) left inverse (11) cannot be an isomorphism. Merkurjev's examples are of the form $G = R_{K/k}^1\mathbb{G}_m \times R_{L/k}^1\mathbb{G}_m$, where K and L are distinct biquadratic extensions of k . This suggests:

Question 4. Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\leq 0}\sigma$ of Diagram (10)?

We leave all these questions to the interested reader.

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