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Values of certain L -series in positive characteristic*

Federico Pellarin^{†‡}

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Abstract. We introduce a family of L -series specialising to both L -series associated to certain Dirichlet characters over $\mathbb{F}_q[\theta]$ and to integral values of Carlitz-Goss zeta function associated to $\mathbb{F}_q[\theta]$. We prove, with the use of the theory of deformations of vectorial modular forms, a formula for their value at 1, as well as some arithmetic properties of values at other positive integers.

1 Introduction, results

let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let \mathbb{F}_q be the finite field with q elements. We consider, for an indeterminate θ , the polynomial ring $A = \mathbb{F}_q[\theta]$ and its fraction field $K = \mathbb{F}_q(\theta)$. On K , we consider the absolute value $|\cdot|$ defined by $|a| = q^{\deg_\theta a}$, a being in K , so that $|\theta| = q$. Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let K_∞^{alg} be an algebraic closure of K_∞ , let \mathbb{C}_∞ be the completion of K_∞^{alg} for the unique extension of $|\cdot|$ to K_∞^{alg} , and let K^{sep} be the separable closure of K in K_∞^{alg} .

We consider an element t of \mathbb{C}_∞ . We have the “evaluating at t ” ring homomorphism

$$\chi_t : A \rightarrow \mathbb{F}_q[t]$$

defined by $\chi_t(a) = a(t)$. In other words, $\chi_t(a)$ is the image of the polynomial map $a(t)$ obtained by substituting, in $a(\theta)$, θ by t . For example, $\chi_t(1) = 1$ and $\chi_t(\theta) = t$. If we choose $t \in \mathbb{F}_q^{\text{alg}}$ then χ_t factors through a Dirichlet character modulo the ideal generated by the minimal polynomial of t in A .

We can also consider t as an indeterminate; for $\alpha > 0$ an integer, we then have a well defined formal series

$$L(\chi_t, \alpha) = \sum_{a \in A^+} \chi_t(a) a^{-\alpha} = \prod_{\mathfrak{p}} (1 - \chi_t(\mathfrak{p}) \mathfrak{p}^{-\alpha})^{-1} \in K_\infty[[t]],$$

where A^+ denotes the set of monic polynomials of A , and where the eulerian product runs over the monic irreducible polynomials of A . This formal series converges for $\log_q |t| < \alpha$, \log_q being the logarithm in base q . In this paper, we are interested in the function $L(\chi_t, \alpha)$ of the variable $t \in \mathbb{C}_\infty$, for a fixed positive integer α .

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We give some relevant examples of values of these series. If $t = \theta$ and $\alpha > 1$, then $L(\chi_\theta, \alpha)$ converges to the value of Carlitz-Goss' zeta series at $\alpha - 1$:

$$L(\chi_\theta, \alpha) = \zeta(\alpha - 1) = \sum_{a \in A^+} a^{1-\alpha}.$$

If on the other side we consider $t \in \mathbb{F}_q^{\text{alg}}$, then, for $\alpha > 0$, $L(\chi_t, \alpha)$ converges to the value at α of the L -series associated to a Dirichlet character, see Goss' book [10].

We will need some classical functions related to Carlitz's module. First of all, its *exponential function* e_{Car} defined, for all $\eta \in \mathbb{C}_\infty$, by the sum of the convergent series:

$$e_{\text{Car}}(\eta) = \sum_{n \geq 0} \frac{\eta^{q^n}}{d_n}, \quad (1)$$

where $d_0 := 1$ and $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$, with $[i] = \theta^{q^i} - \theta$ if $i > 0$.

We choose once and for all a fundamental period $\tilde{\pi}$ of e_{Car} . It is possible to show that $\tilde{\pi}$ is equal, up to the choice of a $(q-1)$ -th root of $-\theta$, to the (value of the) convergent infinite product:

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty.$$

Next, we need the following series of $K^{\text{sep}}[[t]]$:

$$s_{\text{Car}}(t) := \sum_{i=0}^{\infty} e_{\text{Car}}\left(\frac{\tilde{\pi}}{\theta^{i+1}}\right) t^i = \sum_{n=0}^{\infty} \frac{\tilde{\pi}^{q^n}}{d_n(\theta^{q^n} - t)}, \quad (2)$$

converging for $|t| < q$. We shall prove:

Theorem 1 *The following identity holds:*

$$L(\chi_t, 1) = -\frac{\tilde{\pi}}{(t - \theta)s_{\text{Car}}}.$$

The function s_{Car} is, after Anderson in [1], the *canonical rigid analytic trivialisation* of the so-called *Carlitz's motive*, property leading to three important facts that we recall now. First of all, s_{Car} generates the one-dimensional $\mathbb{F}_q(t)$ -vector space of solutions of the τ -difference equation

$$\tau X = (t - \theta)X, \quad (3)$$

in the fraction field of the ring of formal power series in t converging for $|t| \leq 1$, where $\tau : \mathbb{C}_\infty((t)) \rightarrow \mathbb{C}_\infty((t))$ is the operator defined by $\tau \sum c_i t^i = \sum c_i^q t^i$. Second, $\Omega := (\tau s_{\text{Car}})^{-1}$ is entire holomorphic on \mathbb{C}_∞ with the only zeros at $t = \theta^q, \theta^{q^2}, \dots$ and third, we have the limit $\lim_{t \rightarrow \theta} \Omega(t) = -\frac{1}{\tilde{\pi}}$. We refer to [16] for a description of the main properties of the functions s_{Car}, Ω , or to the papers [1, 4, 15], where Ω was originally introduced and studied.

The function Ω being entire, the function $L(\chi_t, 1) = -\tilde{\pi}\Omega(t)$ allows, beyond its domain of convergence, entire analytic continuation in terms of the variable t with the only zeros at the points $t = \theta^q, \theta^{q^2}, \dots$. Also, it is well known (see [4]) that Ω has the following infinite product expansion:

$$\Omega(t) = -\theta^{-1}(-\theta)^{-\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - t\theta^{-q^i}) \in (-\theta)^{\frac{1}{q-1}} K_\infty[[t]].$$

We deduce that the function $L(\chi_t, 1)$, apart from its eulerian product, also has the following product expansion converging for all $t \in \mathbb{C}_\infty$:

$$L(\chi_t, 1) = \prod_{i=1}^{\infty} \left(1 - \frac{t - \theta}{[i]} \right).$$

It is interesting to notice that Theorem 1 directly implies the classical formulas for the values

$$\zeta(q^k - 1) = \sum_{a \in A^+} a^{1-q^k} = (-1)^k \frac{\tilde{\pi}^{q^k - 1}}{[k][k-1] \cdots [1]}$$

of Carlitz-Goss' zeta value at $q^k - 1$ for $k > 0$. This follows easily from the computation of the limit $t \rightarrow \theta$ in the formula $\tau^k L(\chi_t, 1) = -\tilde{\pi}^{q^k} / (\tau^k(t - \theta)_{S_{\text{Car}}})$, observing that $\tau^k((t - \theta)_{S_{\text{Car}}}) = (t - \theta^{q^k}) \cdots (t - \theta^q)(t - \theta)_{S_{\text{Car}}}$ by (3).

Also, if $t = \xi \in \mathbb{F}_q^{\text{alg}}$, Theorem 1 directly implies that $L(\chi_\xi, 1)$, the value of an L -function associated to a Dirichlet character, is a multiple of $\tilde{\pi}$ by $-(\tau_{S_{\text{Car}}})(\xi)^{-1}$ which is this time an element of \mathbb{C}_∞ algebraic over K , solution of an algebraic equation of the type:

$$X^{q^r - 1} = (\xi - \theta^{q^{r-1}}) \cdots (\xi - \theta).$$

Some of these consequences are also covered by the so-called Anderson's *log-algebraic power series* identities for *twisted harmonic sums*, see [2, 3, 13], see also [5].

We briefly mention on the way that the series $\zeta(\alpha)$, $L(\chi_t, \alpha)$ can be viewed as a special case of the (often divergent) multivariable formal series:

$$\mathcal{L}(\chi_{t_1}, \dots, \chi_{t_r}; \alpha_1, \dots, \alpha_r) = \sum_{a \in A^+} \chi_{t_1}^{-\alpha_1}(a) \cdots \chi_{t_r}^{-\alpha_r}(a).$$

For $r = 1$ we get $\mathcal{L}(\chi_\theta, \alpha) = \zeta(\alpha)$ and for $r = 2$ we get $\mathcal{L}(\chi_t, \chi_\theta; -1, \alpha) = L(\chi_t, \alpha)$. These series may also be of some interest in the arithmetic theory of function fields of positive characteristic.

By $\lim_{t \rightarrow \theta} \Omega(t) = -\tilde{\pi}$ it follows that

$$\lim_{t \rightarrow \theta} L(\chi_t, 1) = 1. \tag{4}$$

This value coincides with $\zeta(0)$. Again, the location of the zeros of Ω implies that $\lim_{t \rightarrow \theta} L(\chi_t, q^k - 1) = 0$ for $k > 0$, and we know that $\zeta(s(q - 1)) = 0$ for negative "even" integral values of s . These observations agree with the results of Goss in [11], where he considers, for β an integer, an analytic continuation of the function $\alpha \mapsto L(\chi_t^\beta, \alpha) = \mathcal{L}(\chi_t, \chi_\theta; -\beta, \alpha)$ to the space $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ via the map $\alpha \mapsto ((1/\theta)^{-\alpha}, \alpha) \in \mathbb{S}_\infty$.

For other positive values of $\alpha \equiv 1 \pmod{q - 1}$, we have a result on $L(\chi_t, \alpha)$ at once more general and less precise.

Theorem 2 *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q - 1}$. There exists a non-zero element $\lambda_\alpha \in \mathbb{F}_q(t, \theta)$ such that*

$$L(\chi_t, \alpha) = \lambda_\alpha \frac{\tilde{\pi}^\alpha}{(t - \theta)_{S_{\text{Car}}}}.$$

Theorem 1 implies that $\lambda_1 = -1$. On the other hand, again by (3), we have the formula

$$\tau^k \lambda_\alpha = (t - \theta^{q^k}) \cdots (t - \theta^q) \lambda_{q^k \alpha}.$$

Apart from this, the explicit computation of the λ_α 's is difficult and very little is known on these coefficients which could encode, we hope, an interesting generalisation in $\mathbb{F}_q(t, \theta)$ of the theory of Bernoulli-Carlitz numbers.

The proofs of Theorems 1 and 2 that we propose rely on certain properties of *deformations of vectorial modular forms* (see Section 2). In fact, Theorem 1 is a corollary of an identity involving such functions that we describe now (Theorem 3 below) and Theorem 2 will be obtained from a simple modification of the techniques introduced to prove Theorem 3.

A fundamental identity for deformations of vectorial modular forms. To present Theorem 3, we need to introduce more tools. Let Ω be the rigid analytic space $\mathbb{C}_\infty \setminus K_\infty$. For $z \in \Omega$, we denote by Λ_z the A -module $A + zA$, free of rank 2. The evaluation at $\zeta \in \mathbb{C}_\infty$ of the *exponential function* e_{Λ_z} associated to the lattice Λ_z is given by the series

$$e_{\Lambda_z}(\zeta) = \sum_{i=0}^{\infty} \alpha_i(z) \zeta^{q^i}, \quad (5)$$

for functions $\alpha_i : \Omega \rightarrow \mathbb{C}_\infty$ with $\alpha_0 = 1$. We recall that for $i > 0$, α_i is a *Drinfeld modular form* of weight $q^i - 1$ and type 0 in the sense of Gekeler, [6].

We also recall from [17] the series:

$$\begin{aligned} \mathbf{s}_1(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t}, \\ \mathbf{s}_2(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}, \end{aligned}$$

which converge for $|t| < q$ and define two functions $\Omega \rightarrow \mathbb{C}_\infty[[t]]$ with the series in the image converging for $|t| < q$. We point out that for a fixed choice of $z \in \Omega$, the matrix function ${}^t(\mathbf{s}_1(z, t), \mathbf{s}_2(z, t))$ is the *canonical rigid analytic trivialisation* of the t -*motive* associated to the lattice Λ_z discussed in [16]. We set, for $i = 1, 2$:

$$\mathbf{d}_i(z, t) := \tilde{\pi}_{S_{\text{Car}}}(t)^{-1} \mathbf{s}_i(z, t),$$

remembering that in the notations of [17], we have $\mathbf{d}_2 = \mathbf{d}$. The advantage of using these functions instead of the \mathbf{s}_i 's, is that \mathbf{d}_2 has a u -*expansion* defined over $\mathbb{F}_q[t, \theta]$ (see Proposition 7); it can be proved that for all $z \in \Omega$, $\mathbf{d}_1(z, t)$ and $\mathbf{d}_2(z, t)$ are entire functions of the variable $t \in \mathbb{C}_\infty$.

On the other hand, both the series

$$\mathbf{e}_1(z, t) = \sum'_{c, d \in A} \frac{\chi_t(c)}{cz + d}, \quad \mathbf{e}_2(z, t) = \sum'_{c, d \in A} \frac{\chi_t(d)}{cz + d}$$

converge for $(z, t) \in \Omega \times \mathbb{C}_\infty$ with $|t| \leq 1$, where the dash ' denotes a sum avoiding the couple $(c, d) = (0, 0)$. The series $\mathbf{e}_1, \mathbf{e}_2$ then define functions $\Omega \rightarrow \mathbb{C}_\infty[[t]]$ such that all the series in the images converge for $|t| \leq 1$.

Let $\mathbf{Hol}(\Omega)$ be the ring of holomorphic functions $\Omega \rightarrow \mathbb{C}_\infty$, over which the Frobenius \mathbb{F}_q -linear map τ is well defined: if $f \in \mathbf{Hol}(\Omega)$, then $\tau f = f^q$. We consider the unique $\mathbb{F}_q((t))$ -linear extension of τ :

$$\mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)) \rightarrow \mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)),$$

again denoted by τ .

We shall prove the fundamental theorem:

Theorem 3 *The following identities hold for $z \in \Omega, t \in \mathbb{C}_\infty$ such that $|t| \leq 1$:*

$$\mathbf{e}_1(z, t) = -(t - \theta) s_{Car}(t) (\tau \mathbf{d}_2)(z, t) h(z) \quad \mathbf{e}_2(z, t) = (t - \theta) s_{Car}(t) (\tau \mathbf{d}_1)(z, t) h(z). \quad (6)$$

In the statement of the theorem, h is the opposite of the unique normalised Drinfeld cusp form of weight $q + 1$ and type 1 for $\Gamma = \mathbf{GL}_2(A)$ as in [6]. It can be observed that both right-hand sides in (6) are well defined for $t \in \mathbb{C}_\infty$ with $|t| < q^q$ so that these equalities provide analytic continuations of the functions $\mathbf{e}_1, \mathbf{e}_2$ in terms of the variable t .

Let us write $\mathcal{E} = L(\chi_t, 1)^{-1}(\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{F} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$, and let us consider the representation $\rho_t : \mathbf{GL}_2(A) \rightarrow \mathbf{GL}_2(\mathbb{F}_q[t])$ defined by

$$\rho_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix},$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(A)$. Then, for any such a choice of γ we have the functional equations:

$$\begin{aligned} \mathcal{F}(\gamma(z)) &= (cz + d)^{-1} \rho_t(\gamma) \cdot \mathcal{F}(z), \\ {}^t \mathcal{E}(\gamma(z)) &= (cz + d) {}^t \rho_t^{-1}(\gamma) \cdot {}^t \mathcal{E}(z). \end{aligned}$$

This puts right away the functions ${}^t \mathcal{E}$ and \mathcal{F} in the framework of *deformations of vectorial modular forms*, topic that will be developed in Section 2.

Let B_1 be the set of $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$. We will make use of a remarkable sequence $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$ of functions $\Omega \times B_1 \rightarrow \mathbb{C}_\infty$ defined by the scalar product (with component-wise action of τ):

$$\mathcal{G}_k = \mathcal{G}_{1,0,k} = (\tau^k \mathcal{E}) \cdot \mathcal{F},$$

such that, for $k \geq 0$, \mathcal{G}_k ⁽¹⁾ turns out to be an element of $M_{q^{k-1},0} \otimes \mathbb{F}_q[t, \theta]$ where $M_{w,m}$ denotes the \mathbb{C}_∞ -vector space of Drinfeld modular forms of weight w and type m . In fact, only the terms $\mathcal{G}_0, \mathcal{G}_1$ are needed to be examined to prove our Theorem. Once their explicit computation is accomplished (see Proposition 13), the proof of Theorem 3 is only a matter of solving a non-homogeneous system in two equations and two indeterminates $\mathbf{e}_1, \mathbf{e}_2$. Furthermore, Theorem 1 will be deduced with the computation of a limit in the identity $\mathbf{e}_1 = -\tau(s_{Car} \mathbf{d}_2) h$ and a similar path will be followed for Theorem 2, by using this time the more general sequences $\mathcal{G}_{\alpha,0,k} = (\tau^k \mathcal{E}_{\alpha,0}) \cdot \mathcal{F}$ defined later.

We end this introduction by pointing out that the sequence \mathcal{G} has itself several interesting features. For example, the functions \mathcal{G}_k already appeared in [17] (they are denoted by g_k^* there) as the coefficients of the ‘‘cocycle terms \mathbf{L}_γ ’’ of the functional equations of the *deformations of Drinfeld quasi-modular forms $\tau^k \mathbf{E}$* introduced there.

It is also interesting to notice that the *deformation of Legendre’s identity* (9) that we quote here (proved in [17]):

$$h^{-1}(\tau s_{Car})^{-1} = \mathbf{d}_1(\tau \mathbf{d}_2) - \mathbf{d}_2(\tau \mathbf{d}_1)$$

¹We will also adopt the notation $\mathcal{E}_{1,0} = \mathcal{E}$ and $\mathcal{G}_k = \mathcal{G}_{1,0,k}$.

can be deduced from Theorem 3 by using the fact that $\mathcal{G}_0 = -1$ obtained in Proposition 13.

Moreover, it can be proved, again with the help of the theory of τ -linear recurrent sequences and τ -linearised recurrent sequences (they will not be described here), that for $k \geq 0$, the function $\mathcal{G}_k(z, \theta)$, well defined, is equal to the *ortho-Eisenstein* series $g_k(z)$, and that $\mathcal{G}_k(z, \theta^{q^k})$, also well defined, is equal to the *para-Eisenstein* series $m_k(z)$, in the notations and the terminology of [8]. Hence, the sequence \mathcal{G} provides an interesting tool also in the study of both these kinds of functions. This program will be however pursued in another paper.

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2 Vectorial modular forms and their deformations

In this paper, t will be an indeterminate independent on θ or a parameter varying in \mathbb{C}_∞ , and we will freely switch from formal series to functions.

The ring of formal series in $\mathbb{C}_\infty[[t]]$ converging for all t such that $|t| \leq 1$ will be denoted by \mathbb{T} . It is endowed with the $\mathbb{F}_q[t]$ -linear automorphism τ acting on formal series as follows:

$$\tau \sum_i c_i t^i = \sum_i c_i^q t^i.$$

We will work with certain functions $f : \Omega \times B_1 \rightarrow \mathbb{C}_\infty$ with the property that for all $z \in \Omega$, $f(z, t)$ can be identified with a series of \mathbb{T} converging for all $t_0 \in B_1$ to $f(z, t_0)$. For such functions we will then also write $f(z)$ to stress the dependence on $z \in \Omega$ when we want to consider them as functions $\Omega \rightarrow \mathbb{T}$. Sometimes, we will not specify the variables z, t and just write f instead of $f(z, t)$ or $f(z)$ to lighten our formulas just as we did in some parts of the introduction. Moreover, z will denote a variable in Ω all along the paper.

In all the following, $\mathbf{Hol}(\Omega)$ denotes the ring of rigid holomorphic functions on Ω . Let us denote by \mathcal{R} the integral ring whose elements are the formal series $f = \sum_{i \geq 0} f_i t^i$, such that

1. For all i , f_i is a map $\Omega \rightarrow \mathbb{C}_\infty$ belonging to $\mathbf{Hol}(\Omega)$.
2. For all $z \in \Omega$, $\sum_{i \geq 0} f_i(z) t^i$ is an element of \mathbb{T} .

The ring \mathcal{R} is endowed with the injective endomorphism τ acting on formal series as follows:

$$\tau \sum_{i \geq 0} f_i(z) t^i = \sum_{i \geq 0} f_i(z)^q t^i.$$

2.1 Deformations of vectorial modular forms.

In this subsection we develop some tools in the theory of *deformations of vectorial modular forms*. This part is inspired by works about vectorial modular forms for $\mathbf{SL}_2(\mathbb{Z})$ by Knopp, Mason such as [12, 14].

Let us consider a representation

$$\rho : \Gamma \rightarrow \mathbf{GL}_s(\mathbb{F}_q((t))). \tag{7}$$

We assume that the determinant representation $\det(\rho)$ is the μ -th power of the determinant character, for some $\mu \in \mathbb{Z}/(q-1)\mathbb{Z}$. In all the following, given $\gamma \in \Gamma$, we denote by J_γ the associated factor of automorphy $(\gamma, z) \mapsto cz + d$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 4 A deformation of vectorial modular form of weight w , dimension s and type m associated with a representation ρ as in (7) is a column matrix $\mathcal{F} \in \mathbf{Mat}_{s \times 1}(\mathcal{R})$ such that, considering \mathcal{F} as a map $\Omega \rightarrow \mathbf{Mat}_{s \times 1}(\mathbb{T})$ we have, for all $\gamma \in \Gamma$,

$$\mathcal{F}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} \rho(\gamma) \cdot \mathcal{F}(z).$$

The set of deformations of vectorial modular forms of weight w , dimension s and type m associated to a representation ρ is a \mathbb{T} -module that we will denote by $\mathcal{M}_{w,m}^s(\rho)$.

Let us denote by $M_{w,m}^!$ the \mathbb{C}_∞ -vector space (of infinite dimension) generated by the rigid holomorphic functions $f : \Omega \rightarrow \mathbb{C}$ satisfying, for all $z \in \Omega$ and $\gamma \in \Gamma$, $f(\gamma(z)) = \det(\gamma)^{-m} J_\gamma^w f(z)$ (they behave as Drinfeld modular forms except that they do not necessarily are regular at the cusp at infinity).

If $s = 1$ and if $\rho = \mathbf{1}$ is the constant map, then $\mathcal{M}_{w,m}^1(\mathbf{1})$ is the space $M_{w,m}^! \otimes \mathbb{T}$. Similarly, for general s , we have a graded $M^! \otimes \mathbb{T}$ -module

$$\mathcal{M}^s(\rho) = \bigoplus_{w,m} \mathcal{M}_{w,m}^s(\rho),$$

where $M^! = \bigoplus_{w,m} M_{w,m}^!$.

Lemma 5 Let k be a non-negative integer. If \mathcal{F} is in $\mathcal{M}_{w,m}^s(\rho)$, then $\tau^k \mathcal{F} \in \mathcal{M}_{wq^k,m}^s(\rho)$. If we choose nonnegative integers k_1, \dots, k_s , then

$$\det(\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}) \in M_{w(q^{k_1} + \dots + q^{k_s}), sm + \mu}^! \otimes \mathbb{T}.$$

In particular,

$$W_\tau(\mathcal{F}) = \det(\tau^0 \mathcal{F}, \dots, \tau^{s-1} \mathcal{F}) \in M_{w(1+q+q^2+\dots+q^{s-1}), sm + \mu}^! \otimes \mathbb{T}.$$

Proof. From the definition, and for all $k' \in \mathbb{Z}$,

$$(\tau^{k'} \mathcal{F})(\gamma(z)) = J_\gamma^{wq^{k'}} \det(\gamma)^{-m} \rho(\gamma) (\tau^{k'} \mathcal{F})$$

because $\tau(\rho(\gamma)) = \rho(\gamma)$. Moreover, τ is an endomorphism of \mathcal{R} and we get the first part of the lemma.

Now define the matrix function:

$$\mathbf{M}_{k_1, \dots, k_s} = (\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}).$$

After the first part of the lemma we have, for $\gamma \in \mathbf{GL}_2(A)$:

$$\mathbf{M}_{k_1, \dots, k_s}(\gamma(z)) = \det(\gamma)^{-m} \rho(\gamma) \cdot \mathbf{M}_{k_1, \dots, k_s}(z) \cdot \mathbf{Diag}(J_\gamma^{wq^{k_1}}, \dots, J_\gamma^{wq^{k_s}}),$$

and we conclude the proof taking determinants of both sides. \square

Lemma 6 Let us consider \mathcal{F} in $\mathcal{M}_{w,m}^s(\rho)$ and let \mathcal{E} be such that ${}^t \mathcal{E}$ is in $\mathcal{M}_{w',m'}^s({}^t \rho^{-1})$. Let us denote by \mathcal{G}_k the scalar product $(\tau^k \mathcal{E}) \cdot \mathcal{F}$, then, for nonnegative k ,

$$\mathcal{G}_k \in M_{w+w'q^k, m+m'}^! \otimes \mathbb{T}.$$

Proof. By Lemma 5, $\tau^k({}^t\mathcal{E})$ is in $\mathcal{M}_{wq^k, m'}^s({}^t\rho^{-1})$. Let γ be in $\mathbf{GL}_2(A)$. We thus have, after transposition, and for all $k' \in \mathbb{Z}$, that

$$({}^{\tau^{k'}}\mathcal{E})(\gamma(z)) = J_\gamma^{wq^{k'}} \det(\gamma)^{-m} \mathcal{E}(z) \cdot \rho^{-1}(\gamma),$$

and

$$\mathcal{F}(\gamma(z)) = J_\gamma^{w'} \det(\gamma)^{-m'} \rho(\gamma) \cdot \mathcal{F}(z).$$

Hence,

$$\mathcal{G}_{k'}(\gamma(z)) = J_\gamma^{wq^{k'} + w'} \det(\gamma)^{-m-m'} \mathcal{G}_{k'}(z),$$

from which we can conclude. \square

2.2 Main examples of deformations of vectorial modular forms.

From now on, we will use the representation $\rho = \rho_t$ and the transposed of its inverse.

We will write, for $z \in \Omega$, $u = u(z) = 1/e_{\text{Car}}(\tilde{\pi}z)$; this is the local parameter at infinity of $\Gamma \backslash \Omega$. For all w, m , we have an embedding of $M_{w, m}$ in $\mathbb{C}_\infty[[u]]$, see [6]. We will say that a series $\sum_{i \geq i_0} c_i u^i$ (with the coefficients c_i in some field extension of $\mathbb{F}_q(t, \theta)$) is *normalised*, if $c_{i_0} = 1$. We will also say that the series is of *type* $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ if $i \not\equiv m \pmod{q-1}$ implies $c_i = 0$. This definition is obviously compatible with the notion of *type* of a Drinfeld modular form already mentioned in the introduction, see [6].

The function \mathcal{F} . This is the vector valued function $(\mathbf{d}_1, \mathbf{d}_2)$ so we recall now some properties of the functions $\mathbf{d}_1, \mathbf{d}_2$. The next proposition follows immediately from the results of [17] where some of them are stated in slightly different, although equivalent forms. We write g for the unique normalised Drinfeld modular form of weight $q-1$ and type 0 for Γ (proportional to an Eisenstein series), and Δ for the cusp form $-h^{q-1}$.

Proposition 7 *We have four properties for the \mathbf{d}_i 's.*

1. We have $\mathcal{F} \in \mathcal{M}_{-1, 0}^2(\rho_t)$.
2. The functions $\mathbf{d}_1, \mathbf{d}_2$ span the $\mathbb{F}_q(t)$ -vector space of dimension 2 of solutions of the following τ -linear difference equation:

$$X = (t - \theta^q)\Delta\tau^2 X + g\tau X, \quad (8)$$

in a suitable existentially closed inversive field containing \mathcal{R} .

3. Let us consider the matrix function:

$$\Psi(z, t) := \begin{pmatrix} \mathbf{d}_1(z, t) & \mathbf{d}_2(z, t) \\ \mathbf{d}_1^{(1)}(z, t) & \mathbf{d}_2^{(1)}(z, t) \end{pmatrix}.$$

For all $z \in \Omega$ and t with $|t| < q$:

$$\det(\Psi) = (t - \theta)^{-1} h(z)^{-1} s_{\text{Car}}(t)^{-1}. \quad (9)$$

4. We have the series expansion

$$\mathbf{d}_2 = \sum_{i \geq 0} c_i(t) u^{(q-1)i} \in 1 + u^{q-1} \mathbb{F}_q[t, \theta][[u^{q-1}]], \quad (10)$$

convergent for t, u sufficiently close to $(0, 0)$.

Deformations of vectorial Eisenstein and Poincaré series. Following Gekeler [6, Section 3], we recall that for all $\alpha > 0$ there exists a polynomial $G_\alpha(u) \in \mathbb{C}_\infty[[u]]$, called the α -th Goss polynomial, such that, for all $z \in \Omega$, $G_\alpha(u(z))$ equals the sum of the convergent series

$$\tilde{\pi}^{-\alpha} \sum_{a \in A} \frac{1}{(z+a)^\alpha}.$$

Several properties of these polynomials are collected in [6, Proposition (3.4)]. Here, we will need that for all α , G_α is of type α as a formal series of $\mathbb{C}_\infty[[u]]$. Namely:

$$G_\alpha(\lambda u) = \lambda^\alpha G_\alpha(u), \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We also recall, for $a \in A$, the function

$$u_a(z) := u(az) = e_{\text{Car}}(\tilde{\pi}az)^{-1} = u^{|a|} f_a(u) = u^{|a|} + \dots \in A[[u]],$$

where $f_a \in A[[u]]$ is the a -th inverse cyclotomic polynomial defined in [6, (4.6)]. Obviously, we have

$$u_{\lambda a} = \lambda^{-1} u_a \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We will use the following lemma.

Lemma 8 *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. We have, for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$ and $z \in \Omega$, convergence of the series below, and equality:*

$$\sum'_{c,d \in A} \frac{\chi_t(c)}{(cz+d)^\alpha} = -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c(z)),$$

from which it follows that the series is not identically zero.

Proof. Convergence is ensured by Lemma 10 (or Proposition 9) and the elementary properties of Goss' polynomials. By the way, the series on the right-hand side converges for all $t \in \mathbb{C}_\infty$. We then compute:

$$\begin{aligned} \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} &= \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(cz+d)^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(c\tilde{\pi}z + d\tilde{\pi})^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) G_\alpha(u_c) \\ &= \tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{1-\alpha} \\ &= -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c). \end{aligned}$$

The non-vanishing of the series comes from the non-vanishing contribution of the term $G_\alpha(u)$ in the last series. \square

Following [6], we consider the subgroup

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

of $\Gamma = \mathbf{GL}_2(A)$ and its left action on Γ . For $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the map $\delta \mapsto (c, d)$ induces a bijection between the orbit set $H \backslash \Gamma$ and the set of $(c, d) \in A^2$ with c, d relatively prime.

We consider the factor of automorphy

$$\mu_{\alpha, m}(\delta, z) = \det(\delta)^{-m} J_{\gamma}^{\alpha},$$

where m and α are positive integers (later, m will also determine a type, that is, a class modulo $q - 1$).

Let $V_1(\delta)$ be the row matrix $(\chi_t(c), \chi_t(d))$. It is easy to show that the row matrix

$$\mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta)$$

only depends on the class of $\delta \in H \backslash \Gamma$, so that we can consider the following expression:

$$\mathcal{E}_{\alpha, m}(z) = \sum_{\delta \in H \backslash \Gamma} \mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta),$$

which is a row matrix whose two entries are formal series.

Let \mathcal{V} be the set of functions $\Omega \rightarrow \mathbf{Mat}_{1 \times 2}(\mathbb{C}_{\infty}[[t]])$. We introduce, for α, m integers, $f \in \mathcal{V}$ and $\gamma \in \Gamma$, the Petersson slash operator:

$$f|_{\alpha, m} \gamma = \det(\gamma)^m (cz + d)^{-\alpha} f(\gamma(z)) \cdot \rho_t(\gamma).$$

This will be used in the next proposition, where we recall that $\log_q^+(x)$ denotes the maximum between 0 and $\log_q(x)$, the logarithm in base q of $x > 0$. We point out that we will not apply this proposition in full generality.

Proposition 9 *Let α, m be non-negative integers with $\alpha \geq 2m + 1$, and write $r(\alpha, m) = \alpha - 2m - 1$. We have the following properties.*

1. For $\gamma \in \Gamma$, the map $f \mapsto f|_{\alpha, m} \gamma$ induces a permutation of the subset of \mathcal{V} :

$$\mathcal{S} = \{\mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta); \delta \in H \backslash \Gamma\}.$$

2. If $t \in \mathbb{C}_{\infty}$ and α, m are chosen so that $r(\alpha, m) > \log_q^+ |t|$, then the components of $\mathcal{E}_{\alpha, m}(z, t)$ are series of functions of $z \in \Omega$ which converge absolutely and uniformly on every compact subset of Ω to holomorphic functions.

3. If $|t| \leq 1$, then the components of $\mathcal{E}_{\alpha, m}(z, t)$ converge absolutely and uniformly on every compact subset of Ω also if $\alpha - 2m > 0$.

4. For any choice of α, m, t submitted to the convergence conditions above, the function ${}^t \mathcal{E}_{\alpha, m}(z, t)$ belongs to the space $\mathcal{M}_{\alpha, m}^2({}^t \rho_t^{-1})$.

5. If $\alpha - 1 \not\equiv 2m \pmod{q - 1}$, the matrix function $\mathcal{E}_{\alpha, m}(z, t)$ is identically zero.

6. If $\alpha - 1 \equiv 2m \pmod{q - 1}$, $\alpha \geq (q + 1)m + 1$ so that $\mathcal{E}_{\alpha, m}$ converges, then $\mathcal{E}_{\alpha, m}$ is not identically zero in its domain of convergence.

Proof. 1. We choose $\delta \in H \setminus \Gamma$ corresponding to a couple $(c, d) \in A^2$ with c, d relatively prime, and set $f_\delta = \mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta) \in \mathcal{S}$. We have

$$\begin{aligned} f_\delta(\gamma(z)) &= \mu_{\alpha, m}(\delta, \gamma(z))^{-1} u^m(\delta(\gamma(z))) V_1(\delta) \\ &= \mu_{\alpha, m}(\gamma, z) \mu_{\alpha, m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta), \\ &= \mu_{\alpha, m}(\gamma, z) \mu_{\alpha, m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta\gamma) \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha, m}(\gamma, z) \mu_{\alpha, m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta') \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha, m}(\gamma, z) f_{\delta'} \cdot \rho_t(\gamma)^{-1}, \end{aligned}$$

with $\delta' = \delta\gamma$ and $f_{\delta'} = \mu_{\alpha, m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta')$, from which part 1 of the proposition follows.

2. Convergence and holomorphy are ensured by simple modifications of [6, (5.5)], or by the arguments in [9, Chapter 10]. More precisely, let us choose $0 \leq s \leq 1$ and look at the component at the place $s + 1$

$$\mathcal{E}_s(z, t) = \sum_{\delta \in H \setminus \Gamma} \mu_{\alpha, m}(\delta, z)^{-1} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

of the vector series $\mathcal{E}_{\alpha, m}$. Writing $\alpha = n(q-1) + 2m + l'$ with n non-negative integer and $l' \geq 1$ we see, following Gerritzen and van der Put, [9, pp. 304-305] and taking into account the inequality $|u(\delta(z))| \leq |cz + d|^2 / |z|_i$ ($|z|_i$ denotes, for $z \in \mathbb{C}_\infty$, the infimum $\inf_{a \in K_\infty} \{|z - a|\}$), that the term of the series \mathcal{E}_s :

$$\mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}) = (cz + d)^{-n(q-1) - l' - 2m} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

(where δ corresponds to (c, d)) has absolute value bounded from above by

$$|z|_i^{-m} \left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right|.$$

Applying the first part of the proposition, to check convergence, we can freely substitute z with $z + a$ with $a \in A$ and we may assume, without loss of generality, that $|z| = |z|_i$. We verify that, either $\lambda = \deg_\theta z = \max(\mathcal{I}) \in \mathbb{Q} \setminus \mathbb{Z}$, or $\lambda \in \mathbb{Z}$ case in which for all $\rho \in K_\infty$ with $|\rho| = |z|$, we have $|z - \rho| = |z|$. In both cases, for all c, d , $|cz + d| = \max\{|cz|, |d|\}$. Then, the series defining \mathcal{E}_s can be decomposed as follows:

$$\mathcal{E}_s = \left(\sum'_{|cz| < |d|} + \sum'_{|cz| \geq |d|} \right) \mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}).$$

We now look for upper bounds for the absolute values of the terms of the series above separating the two cases in a way similar to that of Gerritzen and van der Put in loc. cit.

Assume first that $|cz| < |d|$, that is, $\deg_\theta c + \lambda < \deg_\theta d$. Then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa \max\{1, |t|\}^{\deg_\theta d} |d|^{-n(q-1) - l'} \leq \kappa q^{\deg_\theta d (\log_q^+ |t| - n(q-1) - l')},$$

where κ is a constant depending on λ , and the corresponding sub-series converges with the imposed conditions on the parameters, because $\log_q^+ |t| - n(q-1) - l' < 0$.

If on the other side $|cz| \geq |d|$, that is, $\deg_\theta c + \lambda \geq \deg_\theta d$, then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa' \max\{1, |t|\}^{\deg_\theta d} |c|^{-n(q-1) - l'} \leq \kappa' q^{\deg_\theta c (\log_q^+ |t| - n(q-1) - l')},$$

with a constant κ' depending on λ , again because $\log_q^+ |t| - n(q-1) - l' < 0$. This completes the proof of the second part of the Proposition.

3. This property can be deduced from the proof of the second part because if $|t| \leq 1$, then $|\chi_t(c^s d^{1-s})| \leq 1$.

4. The property is obvious by the first part of the proposition, because $\mathcal{E}_{\alpha,m} = \sum_{f \in \mathcal{S}} f$.

5. We consider $\gamma = \mathbf{Diag}(1, \lambda)$ with $\lambda \in \mathbb{F}_q^\times$; the corresponding homography, multiplication by λ^{-1} , is equal to that defined by $\mathbf{Diag}(\lambda^{-1}, 1)$. Hence, we have:

$$\begin{aligned} \mathcal{E}_{\alpha,m}(\gamma(z)) &= \lambda^{\alpha-m} \mathcal{E}_{\alpha,m}(z) \cdot \mathbf{Diag}(1, \lambda^{-1}) \\ &= \lambda^m \mathcal{E}_{\alpha,m}(z) \cdot \mathbf{Diag}(\lambda, 1), \end{aligned}$$

from which it follows that $\mathcal{E}_{\alpha,m}$ is identically zero if $\alpha - 1 \not\equiv 2m \pmod{q-1}$.

6. If $m = 0$ and $\alpha = 1$, we simply appeal to Lemma 8. Assuming now that either $m > 0$ or $\alpha > 1$, we have that $\mathcal{E}_{\alpha,m}$ converges at $t = \theta$ and:

$$\begin{aligned} z\mathcal{E}_0(z, \theta) + \mathcal{E}_1(z, \theta) &= \sum_{\delta \in H \setminus \Gamma} \det(\delta)^m (cz + d)^{1-\alpha} u(\delta(z))^m \\ &= P_{\alpha-1,m}, \end{aligned}$$

where $P_{\alpha-1,m} \in M_{\alpha-1,m}$ is the Poincaré series of weight $\alpha-1$ and type m so that [9, Proposition 10.5.2] suffices for our purposes. \square

Let α, m be non-negative integers such that $\alpha - 2m > 1$ and $\alpha - 1 \equiv 2m \pmod{q-1}$. We have constructed functions:

$$\begin{aligned} \mathcal{E}_{\alpha,m} : \Omega &\rightarrow \mathbf{Mat}_{1 \times 2}(\mathcal{R}), \\ \mathcal{F} : \Omega &\rightarrow \mathbf{Mat}_{2 \times 1}(\mathcal{R}), \end{aligned}$$

and ${}^t\mathcal{E}_{\alpha,m} \in \mathcal{M}_{\alpha,m}^2({}^t\rho_t^{-1})$, $\mathcal{F} \in \mathcal{M}_{-1,0}^2(\rho_t)$. Therefore, after Lemma 6, the functions

$$\mathcal{G}_{\alpha,m,k} = (\tau^k \mathcal{E}_{\alpha,m}) \cdot \mathcal{F} = \mathcal{E}_{q^k \alpha, m} \cdot \mathcal{F} : \Omega \rightarrow \mathbb{T}$$

satisfy $\mathcal{G}_{\alpha,m,k} \in M_{q^k \alpha - 1, m}^1 \otimes \mathbb{T}$.

A special case. After Proposition 9, if $\alpha > 0$ and $\alpha \equiv 1 \pmod{q-1}$, then $\mathcal{E}_{\alpha,0} \neq 0$. We call these series *deformations of vectorial Eisenstein series*.

Lemma 10 *With $\alpha > 0$ such that $\alpha \equiv 1 \pmod{q-1}$, the following identity holds for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$:*

$$\mathcal{E}_{\alpha,0}(z, t) = L(\chi_t, \alpha)^{-1} \sum'_{c,d} (cz + d)^{-\alpha} V_1(c, d),$$

and $\mathcal{E}_{\alpha,0}$ is not identically zero.

Proof. We recall the notation

$$V_1(c, d) = (\chi_t(c), \chi_t(d)) \in \mathbf{Mat}_{1 \times 2}(\mathbb{F}_q[t]).$$

We have

$$\begin{aligned} \sum'_{c,d} (cz+d)^{-\alpha} V_1(c,d) &= \sum_{(c',d')=1} \sum_{a \in A^+} a^{-\alpha} (c'z+d')^{-\alpha} V_1(ac',ad') \\ &= L(\chi_t, \alpha) \mathcal{E}_{\alpha,0}(z,t), \end{aligned}$$

where the first sum is over couples of A^2 distinct from $(0,0)$, while the second sum is over the couples (c',d') of relatively prime elements of A^2 . Convergence features are easy to deduce from Proposition 9. Indeed, we have convergence if $\log_q^+ |t| < r(\alpha, m) = \alpha - 1$, that is, $\max\{1, |t|\} \leq q^{\alpha-1}$ if $\alpha > 1$ and we have convergence, for $\alpha = 1$, for $|t| \leq 1$. In all cases, convergence holds for $|t| \leq 1$. Non-vanishing of the function also follows from Proposition 9. \square

3 Proof of the Theorems

We will need two auxiliary lemmas.

Lemma 11 *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$, we have*

$$\lim_{|z|_i=|z| \rightarrow \infty} \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} = 0.$$

Proof. We recall from [17] the expansion

$$\mathbf{d}_1(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \mathbf{s}_2(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \sum_{n \geq 0} e_{\Lambda_z} \left(\frac{z}{\theta^{n+1}} \right) t^n,$$

converging for all t such that $|t| < q$ and for all $z \in \Omega$.

By a simple modification of the proof of [7, Lemma 5.9 p. 286], we have

$$\lim_{|z|_i=|z| \rightarrow \infty} u(z) t^n e_{\Lambda_z} (z/\theta^{n+1})^q = 0$$

uniformly in $n > 0$, for all t such that $|t| \leq 1$ (in fact, even if $|t| \leq q$).

Moreover, it is easy to show that

$$\lim_{|z|_i=|z| \rightarrow \infty} u(z) e_{\Lambda_z} (z/\theta)^q = \tilde{\pi}^{-q} \lim_{|z|_i=|z| \rightarrow \infty} e_{\text{Car}}^q(\tilde{\pi}z/\theta) / e_{\text{Car}}(\tilde{\pi}z) = 1.$$

This suffices to show that

$$\lim_{|z|_i=|z| \rightarrow \infty} \mathbf{d}_1(z) G_\alpha(u_c(z)) = 0$$

uniformly for $c \in A^+$, for all t such that $|t| \leq 1$. The lemma then follows from an application of Lemma 8. \square

Lemma 12 *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$, we have*

$$\lim_{|z|_i=|z| \rightarrow \infty} \sum'_{c,d} \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t, \alpha).$$

Proof. It suffices to show that

$$\lim_{|z|_i=|z|\rightarrow\infty} \sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = 0.$$

For all $(c, d) \in A \setminus \{(0, 0)\}$, we have $|cz+d| = \max\{|cz|, |d|\}$. We then have, for $c \neq 0$:

$$\sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = \left(\sum_{|d|>|cz|} + \sum_{|d|\leq|cz|} \right) \frac{\chi_t(d)}{(cz+d)^\alpha}.$$

Now, if $|d| > |cz|$, we have, for $|t| \leq 1$:

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{d^\alpha} \right| \leq |d|^{-\alpha} \leq |cz|^{-\alpha}.$$

If $|d| \leq |cz|$, we have, again for $|t| < 1$:

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{(cz)^\alpha} \right| \leq |cz|^{-\alpha}.$$

Therefore, for $c \neq 0$,

$$\left| \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq |cz|^{-\alpha}.$$

This implies that

$$\left| \sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq |z|^{-\alpha},$$

from which the Lemma follows. \square

The next step is to prove the following Proposition.

Proposition 13 *For all $\alpha > 0$ with $\alpha \equiv 1 \pmod{q-1}$, then $\mathcal{G}_{\alpha,0,0} \in M_{\alpha-1,0} \otimes \mathbb{T}$, and we have the limit $\lim_{|z|_i=|z|\rightarrow\infty} \mathcal{G}_{\alpha,0,0} = -1$.*

Moreover, if $\alpha \leq q(q-1)$, then:

$$\mathcal{G}_{\alpha,0,0} = -E_{\alpha-1},$$

where $E_{\alpha-1}$ is the normalised Eisenstein series of weight $\alpha-1$ for Γ .

Proof. Let us write:

$$F_\alpha(z, t) := \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} + \mathbf{d}_2(z) \sum'_{c,d} \frac{\chi_t(d)}{(cz+d)^\alpha},$$

series that converges for all $(z, t) \in \Omega \times \mathbb{C}_\infty$ with $|t| \leq 1$ to an element of $M_{\alpha-1,0} \otimes \mathbb{T}$. by Lemma 10, we have

$$F_\alpha(z, t) = L(\chi_t, \alpha) \mathcal{E}_{\alpha,0}(z, t) \cdot \mathcal{F}(z, t) = L(\chi_t, \alpha) \mathcal{G}_{\alpha,0,0}.$$

After (10), we verify that for all t with $|t| \leq 1$, $\lim_{|z|_i=|z|\rightarrow\infty} \mathbf{d}_2(z) = 1$. From Lemmas 11 and 12,

$$\lim_{|z|_i=|z|\rightarrow\infty} F_\alpha(z, t) = -L(\chi_t, \alpha).$$

Therefore, for all t such that $|t| \leq 1$, $F_\alpha(z, t)$ converges to an holomorphic function on Ω and is endowed with a u -expansion. In particular, $F_\alpha(z, t)$ is a family of modular forms of $M_{\alpha-1,0} \otimes \mathbb{T}$.

Since for the selected values of α , $M_{\alpha-1,0} = \langle E_{\alpha-1} \rangle$, we obtain that $F_\alpha = -L(\chi_t, \alpha)E_{\alpha-1}$. \square

Proof of Theorem 3. By Proposition 7, the matrix $M = (\mathcal{F}, \tau^{-1}\mathcal{F})$ is invertible. From (9) we deduce that

$$\tau M^{-1} = (t - \theta)_{s_{\text{Car}}} h \begin{pmatrix} -\mathbf{d}_2 & \mathbf{d}_1 \\ \tau \mathbf{d}_2 & -\tau \mathbf{d}_1 \end{pmatrix}.$$

Thanks to this expression, the identity:

$$\begin{pmatrix} \mathcal{E}_{\alpha,0} \\ \tau \mathcal{E}_{\alpha,0} \end{pmatrix} \cdot \mathcal{F} = \begin{pmatrix} \mathcal{G}_{\alpha,0,0} \\ \mathcal{G}_{\alpha,0,1} \end{pmatrix},$$

product of a 2×2 matrix with a column matrix, yields the formulas:

$$\begin{aligned} \mathcal{E}_{\alpha,0} &= \\ &= (\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot M^{-1} \\ &= (t - \theta^{1/q})h^{1/q}(\tau^{-1}s_{\text{Car}})(\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot \begin{pmatrix} -\tau^{-1}\mathbf{d}_2 & \tau^{-1}\mathbf{d}_1 \\ \mathbf{d}_2 & -\mathbf{d}_1 \end{pmatrix} \\ &= (t - \theta^{1/q})h^{1/q}(\tau^{-1}s_{\text{Car}})(\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_2, -\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_1 + \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_1). \end{aligned} \quad (11)$$

Now, we observe that we have, from the second part of Proposition 7 and for all $k \in \mathbb{Z}$,

$$\mathcal{G}_{\alpha,0,k} = g\tau\mathcal{G}_{\alpha,0,k-1} + \Delta(t - \theta^q)\tau^2\mathcal{G}_{\alpha,0,k-2}.$$

Applying this formula for $k = 1$ we obtain

$$\tau\mathcal{G}_{\alpha,0,-1} = \frac{\tau^{-1}\mathcal{G}_{\alpha,0,1} - g^{1/q}\mathcal{G}_{\alpha,0,0}}{(t - \theta)\Delta^{1/q}}, \quad (12)$$

and by using again Part two of Proposition 7, we eliminate $\tau^{-1}\mathcal{G}_{\alpha,0,1}$ and $\tau^{-1}\mathbf{d}_i$:

$$\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_i = \Delta^{1/q}(t - \theta)((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_i)), \quad i = 1, 2.$$

Replacing in (11), and using $\Delta^{1/q}h^{1/q} = -h$ and $(t - \theta^{1/q})\tau^{-1}s_{\text{Car}} = s_{\text{Car}}$, we get the formula:

$$\mathcal{E}_{\alpha,0} = (t - \theta)_{s_{\text{Car}}} h(-(\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 + \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2), (\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_1 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_1)). \quad (13)$$

So far, we have not specified the value of α . The general computation of $\mathcal{G}_{\alpha,0,0}$ and $\tau\mathcal{G}_{\alpha,0,-1}$ is difficult, but for $\alpha = 1$ we can apply Proposition 13. We have $\mathcal{G}_{1,0,0} = -1$ and $\mathcal{G}_{1,0,1} = \mathcal{G}_{q,0,0} = -g = -E_{q-1}$. Therefore, $\mathcal{G}_{\alpha,0,-1} = 0$ by (12) and Theorem 3 follows. \square

Proof of Theorem 1. Lemma 8 and the equality of the first components in (13) imply that

$$L(\chi_t, \alpha) = \frac{\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c)}{(\tau s_{\text{Car}}) h((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2))}, \quad (14)$$

so that in particular, the numerator and the denominator of the fraction are one proportional to the other. For $\alpha = 1$, we get, thanks to the fact that $h = -u + o(u)$ and $\sum_{c \in A^+} \chi_t(c) u_c = u + o(u)$,

$$L(\chi_t, \alpha) = \frac{\tilde{\pi} \sum_{c \in A^+} \chi_t(c) u_c}{(\tau s_{\text{Car}} \mathbf{d}_2) h} = -\frac{\tilde{\pi}}{\tau s_{\text{Car}}} + o(1),$$

from which we deduce Theorem 1 and even some additional information, namely, the formula:

$$(\tau \mathbf{d}_2)h = - \sum_{c \in A^+} \chi_t(c)u_c.$$

□

Proof of Theorem 2. For general α , we know by (14) that there exists $\lambda_\alpha \in \mathbb{L}$ (\mathbb{L} being the fraction field of \mathbb{T}) such that

$$\sum_{c \in A^+} \chi_t(c)G_\alpha(u_c) = \lambda_\alpha h((\tau \mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau \mathbf{d}_2)). \quad (15)$$

Since $L(\chi_t, \alpha) = \lambda_\alpha \tilde{\pi}^\alpha / (\tau s_{\text{Car}})$, it remains to show that λ_α belongs to $\mathbb{F}_q(t, \theta)$.

Let us write f for the series $\sum_{c \in A^+} \chi_t(c)G_\alpha(u_c)$, ϕ for $\lambda_\alpha h \tau \mathcal{G}_{\alpha,0,-1}$ and ψ for $-\lambda_\alpha h \mathcal{G}_{\alpha,0,0}$, so that (15) becomes:

$$f = \phi \mathbf{d}_2 + \psi \tau \mathbf{d}_2.$$

We recall that h is in $M_{q+1,1}$. Proposition 13) then tells that $\phi \in M_{\alpha+1,1} \otimes \mathbb{T}$. We also have $\psi = (f - \phi \mathbf{d}_2) / \tau \mathbf{d}_2$. Since $\mathbf{d}_2 = 1 + \dots$ has u -expansion in $\mathbb{F}_q[t, \theta][[u]]$ by the fourth part of Proposition 7, the same property holds for $\tau \mathbf{d}_2$ and ψ belongs to $M_{\alpha+q,1} \otimes \mathbb{L}$.

Let L be an algebraically closed field containing \mathbb{L} , hence containing also $\mathbb{F}_q(t, \theta)$. As for any choice of w, m , $M_{w,m}$ embeds in $\mathbb{C}_\infty[[u]]$ and since there is a basis of this space with u -expansions defined over K , we have that $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$ acts on $M_{w,m} \otimes L$ through the coefficients of the u -expansions. Let σ be an element of $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$ and, for $\mu \in M_{w,m} \otimes L$, let us denote by $\mu^\sigma \in M_{w,m} \otimes L$ the form obtained applying σ on every coefficient of the u -expansion of μ .

Since f, \mathbf{d}_2 and $\tau \mathbf{d}_2$ are defined over $\mathbb{F}_q[t, \theta]$, we get $f = \phi^\sigma \mathbf{d}_2 + \psi^\sigma \tau \mathbf{d}_2$, so that

$$(\phi - \phi^\sigma) \mathbf{d}_2 + (\psi - \psi^\sigma) \tau \mathbf{d}_2 = 0.$$

Assuming that $\phi^\sigma \neq \phi$ or $\psi^\sigma \neq \psi$, then $\phi^\sigma \neq \phi$ and $\psi^\sigma \neq \psi$, and $\tau \mathbf{d}_2 / \mathbf{d}_2 \in \mathbb{C}_\infty(g, h) \otimes \mathbb{L}$, which is impossible after the results of [17] (there, it is even proven that the functions $\mathbf{d}_2, \tau \mathbf{d}_2, g$ and h are algebraically independent over $\mathbb{C}_\infty((t))$). Hence, for all σ , $\phi^\sigma = \phi$ and $\psi^\sigma = \psi$. This means that the u -expansions of ϕ, ψ are both defined over $\mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$ for some $s \geq 0$. By the fact that the constant coefficient of $\mathcal{G}_{\alpha,0,0}$ is equal to -1 (this follows from the first part of Proposition 13), we get that $\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$ (2).

We have proven that $\lambda_\alpha = \tilde{\pi}^{-\alpha} L(\chi_t, \alpha)(t - \theta) s_{\text{Car}} \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$. But we already know that $L(\chi_t, \alpha) \in K_\infty[[t]]$, $s_{\text{Car}} \in K^{\text{sep}}[[t]]$, and $\tilde{\pi} \in K_\infty^{\text{sep}}$ (the separable closure of K_∞). Therefore,

$$\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s}) \cap K_\infty^{\text{sep}}((t)) = \mathbb{F}_q(t, \theta).$$

□

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²We observe the additional information that both $\mathcal{G}_{\alpha,0,0}$ and $\tau \mathcal{G}_{\alpha,0,-1}$ are defined over $\mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$; this will not be used here.

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