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# Values of certain $L$ -series in positive characteristic\*

Federico Pellarin<sup>†‡</sup>

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**Abstract.** We introduce a family of  $L$ -series specialising to both  $L$ -series associated to certain Dirichlet characters over  $\mathbb{F}_q[\theta]$  and to integral values of Carlitz-Goss zeta function associated to  $\mathbb{F}_q[\theta]$ . We prove, with the use of the theory of deformations of vectorial modular forms, a formula for their value at 1, as well as some arithmetic properties of other “even” values.

## 1 Introduction, results

let  $q = p^e$  be a power of a prime number  $p$  with  $e > 0$  an integer, let  $\mathbb{F}_q$  be the finite field with  $q$  elements. We consider the polynomial ring  $A = \mathbb{F}_q[\theta]$  and its fraction field  $K = \mathbb{F}_q(\theta)$ , with  $\theta$  an indeterminate over  $\mathbb{F}_q$ . On  $K$ , we will consider the absolute value  $|\cdot|$  defined by  $|a| = q^{\deg_\theta a}$ ,  $a$  being in  $K$ , so that  $|\theta| = q$ . Let  $K_\infty := \mathbb{F}_q((1/\theta))$  be the completion of  $K$  for this absolute value, let  $K_\infty^{\text{alg}}$  be an algebraic closure of  $K_\infty$ , let  $\mathbb{C}_\infty$  be the completion of  $K_\infty^{\text{alg}}$  for the unique extension of  $|\cdot|$  to  $K_\infty^{\text{alg}}$ , and let  $K^{\text{alg}}$  be the algebraic closure of  $K$  in  $K_\infty^{\text{alg}}$ .

We consider an element  $t$  of  $\mathbb{C}_\infty$ . We have the “evaluating at  $t$ ” ring homomorphism

$$\chi_t : A \rightarrow \mathbb{F}_q[t]$$

defined by  $\chi_t(a) = a(t)$ . In other words,  $\chi_t(a)$  is the image of the polynomial map  $a(t)$  obtained by substituting, in  $a(\theta)$ ,  $\theta$  by  $t$ . For example,  $\chi_t(1) = 1$  and  $\chi_t(\theta) = t$ . If we choose  $t \in \mathbb{F}_q^{\text{alg}}$  then  $\chi_t$  factors through a Dirichlet character modulo the ideal generated by the minimal polynomial of  $t$  in  $A$ .

We can also consider  $t$  as an indeterminate; for  $\alpha > 0$  an integer, we then have a well defined formal series

$$L(\chi_t, \alpha) = \sum_{a \in A^+} \chi_t(a) a^{-\alpha} = \prod_{\mathfrak{p}} (1 - \chi_t(\mathfrak{p}) \mathfrak{p}^{-\alpha})^{-1} \in K_\infty[[t]],$$

where  $A^+$  denotes the set of monic polynomials of  $A$ , and where the eulerian product runs over the monic irreducible polynomials of  $A$ . This formal series converges for  $\log_q |t| < \alpha$ ,  $\log_q$  being the logarithm in base  $q$ . In this paper, we are interested in the function  $L(\chi_t, \alpha)$  of the variable  $t \in \mathbb{C}_\infty$ , for fixed  $\alpha$ .

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We give some relevant examples of values of these series. If  $t = \theta$  and  $\alpha > 1$ , then  $L(\chi_\theta, \alpha)$  converges to the value of Carlitz-Goss zeta value at  $\alpha$ :

$$L(\chi_\theta, \alpha) = \zeta(\alpha) = \sum_{a \in A^+} a^{-\alpha}.$$

If on the other side we consider  $t \in \mathbb{F}_q^{\text{alg}}$ , then, for  $\alpha > 0$ ,  $L(\chi_t, \alpha)$  converges to the value at  $\alpha$  of the  $L$ -series associated to a Dirichlet character, see Goss' book [10].

We will need some classical functions related to Carlitz's module. First of all, its *exponential function*  $e_{\text{Car}}$  defined, for all  $\zeta \in \mathbb{C}_\infty$ , by the sum of the convergent series:

$$e_{\text{Car}}(\zeta) = \sum_{n \geq 0} \frac{\zeta^{q^n}}{d_n}, \quad (1)$$

where  $d_0 := 1$  and  $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$ , with  $[i] = \theta^{q^i} - \theta$  if  $i > 0$ .

We choose once and for all a fundamental period  $\tilde{\pi}$  of  $e_{\text{Car}}$ . It is possible to show that  $\tilde{\pi}$  is equal, up to the choice of a  $(q-1)$ -th root of  $-\theta$ , to the (value of the) convergent product:

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_\infty((-\theta)^{\frac{1}{q-1}}) \setminus K_\infty.$$

Next, we need the following series of  $K^{\text{sep}}[[t]]$ :

$$s_{\text{Car}}(t) := \sum_{i=0}^{\infty} e_{\text{Car}}\left(\frac{\tilde{\pi}}{\theta^{i+1}}\right) t^i = \sum_{n=0}^{\infty} \frac{\tilde{\pi}^{q^n}}{d_n(\theta^{q^n} - t)}, \quad (2)$$

converging for  $|t| < q$ . This is the *canonical rigid analytic trivialisation* of the so-called *Carlitz's motive*. This leads to the property that  $s_{\text{Car}}$  generates the one-dimensional  $\mathbb{F}_q((t))$ -vector space of solutions of the  $\tau$ -difference equation

$$\tau s_{\text{Car}} = (t - \theta) s_{\text{Car}}, \quad (3)$$

in a suitable difference field, where  $\tau : \mathbb{C}_\infty((t)) \rightarrow \mathbb{C}_\infty((t))$  is the operator defined by  $\tau \sum c_i t^i = \sum c_i^q t^i$ . We refer to [15] for a description of the main properties of this function  $s_{\text{Car}}$ , or to the papers [1, 4, 14], where it was originally introduced and appears under different notations.

We shall prove:

**Theorem 1** *The following identity holds:*

$$L(\chi_t, 1) = -\frac{\tilde{\pi}}{(t - \theta) s_{\text{Car}}}.$$

The inverse of  $(t - \theta) s_{\text{Car}}$ , denoted by  $\Omega$  in [4], is entire. This implies that  $L(\chi_t, 1)$  allows, beyond its domain of convergence, entire analytic continuation in terms of the variable  $t$ .

It is interesting to notice that Theorem 1 directly implies the classical formulas for the values

$$\zeta(q^k - 1) = \sum_{a \in A^+} a^{1-q^k} = (-1)^k \frac{\tilde{\pi}^{q^k - 1}}{[k][k-1] \cdots [1]}$$

of Carlitz-Goss' zeta value at  $q^k - 1$  for  $k > 0$ . This follows easily from the computation of the limit  $t \rightarrow \theta$  in the formula  $\tau^k L(\chi_t, 1) = -\tilde{\pi}^{q^k} / (\tau^k (t - \theta) s_{\text{Car}})$ , observing that  $\tau^k ((t - \theta) s_{\text{Car}}) = (t - \theta^{q^k}) \cdots (t - \theta^q)(t - \theta) s_{\text{Car}}$  by (3).

Also, if  $t = \xi \in \mathbb{F}_q^{\text{alg}}$ , Theorem 1 directly implies that  $L(\chi_\xi, 1)$ , the value of an  $L$ -function associated to a Dirichlet character, is a multiple of  $\tilde{\pi}$  by an algebraic element of  $\mathbb{C}_\infty$ , solution of an algebraic equation

$$X^{q^r-1} = (\xi - \theta^{q^{r-1}}) \cdots (\xi - \theta).$$

Some of these consequences are also covered by the so-called Anderson *log-algebraic power series* identities for *twisted harmonic sums*, see [2, 3, 12], see also [5].

It is well known that  $\lim_{t \rightarrow \theta} (t - \theta)_{s_{\text{Car}}} = -\tilde{\pi}$ . It follows that

$$\lim_{t \rightarrow \theta} L(\chi_t, 1) = 1. \quad (4)$$

Surprisingly, this value coincides with  $\zeta(0)$ . Again, the well known fact that  $(t - \theta)_{s_{\text{Car}}}$  has poles at  $\theta^q, \theta^{q^2}, \dots$  implies that  $\lim_{t \rightarrow \theta} L(\chi_t, q^k - 1) = 0$  for  $k > 0$ , and we know that  $\zeta(s(q-1)) = 0$  for negative “even” integral values of  $s$ . It would be interesting to interpret these coincidences throughout the study of a generalisation of both the  $L$ -series considered in this paper, and *Carlitz-Goss’ zeta function*, which is an analytic function defined on the space  $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$  interpolating the above-mentioned zeta values.

For other positive values of  $\alpha \equiv 1 \pmod{q-1}$ , we have a result on  $L(\chi_t, \alpha)$  at once more general and less precise.

**Theorem 2** *Let  $\alpha$  be a positive integer such that  $\alpha \equiv 1 \pmod{q-1}$ . There exists a non-zero element  $\lambda_\alpha \in \mathbb{F}_q(t, \theta)$  such that*

$$L(\chi_t, \alpha) = \lambda_\alpha \frac{\tilde{\pi}^\alpha}{(t - \theta)_{s_{\text{Car}}}}.$$

We have seen that  $\lambda_1 = -1$ . On the other hand, again by (3), we have the formula

$$\tau^k \lambda_\alpha = (t - \theta^{q^k}) \cdots (t - \theta^q) \lambda_{q^k \alpha}.$$

Apart from this, the explicit computation of the  $\lambda_\alpha$ ’s is difficult and very little is known on these coefficients which could encode, we hope, an interesting generalisation in  $\mathbb{F}_q(t, \theta)$  of the theory of Bernoulli-Carlitz numbers.

The proofs of Theorems 1 and 2 that we propose rely on certain properties of *deformations of vectorial modular forms* (see Section 2). In fact, Theorem 1 is a corollary of an identity involving such functions that we describe now (Theorem 3 below) and Theorem 2 will be obtained from a simple modification of the techniques introduced to prove Theorem 3.

*A fundamental identity for deformations of vectorial modular forms.* To present Theorem 3, we need to introduce more tools. Let  $\Omega$  be the rigid analytic space  $\mathbb{C}_\infty \setminus K_\infty$ . For  $z \in \Omega$ , we denote by  $\Lambda_z$  the  $A$ -module  $A + zA$ , free of rank 2. The evaluation at  $\zeta \in \mathbb{C}_\infty$  of the *exponential function*  $e_{\Lambda_z}$  associated to the lattice  $\Lambda_z$  is given by the series

$$e_{\Lambda_z}(\zeta) = \sum_{i=0}^{\infty} \alpha_i(z) \zeta^{q^i}, \quad (5)$$

for functions  $\alpha_i : \Omega \rightarrow \mathbb{C}_\infty$  with  $\alpha_0 = 1$ . We recall that for  $i > 0$ ,  $\alpha_i$  is a *Drinfeld modular form* of weight  $q^i - 1$  and type 0 in the sense of Gekeler, [6].

We also recall from [16] the series:

$$\begin{aligned} \mathbf{s}_1(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t}, \\ \mathbf{s}_2(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}, \end{aligned}$$

which converge for  $|t| < q$  and define two functions  $\Omega \rightarrow \mathbb{C}_{\infty}[[t]]$  with the series in the image converging for  $|t| < q$ . We point out that for a fixed choice of  $z \in \Omega$ , the matrix function  ${}^t(\mathbf{s}_1(z, t), \mathbf{s}_2(z, t))$  is the *canonical rigid analytic trivialisation* of the  $t$ -motive associated to the lattice  $\Lambda_z$  discussed in [15]. We set, for  $i = 1, 2$ :

$$\mathbf{d}_i(z, t) := \tilde{\pi}_{s_{\text{Car}}}(t)^{-1} \mathbf{s}_i(z, t),$$

remembering that in the notations of [16], we have  $\mathbf{d}_2 = \mathbf{d}$ . The advantage of using these functions instead of the  $\mathbf{s}_i$ 's, is that evaluation at  $t = \theta$  makes sense, and we can check:

$$\mathbf{d}_1(z, \theta) = z, \quad \mathbf{d}_2(z, \theta) = 1. \quad (6)$$

On the other hand, both the series

$$\mathbf{e}_1(z, t) = \sum'_{c, d \in A} \frac{\chi_t(c)}{cz + d}, \quad \mathbf{e}_2(z, t) = \sum'_{c, d \in A} \frac{\chi_t(d)}{cz + d}$$

converge for  $(z, t) \in \Omega \times B_q$ , where the dash ' denotes a sum avoiding the couple  $(c, d) = (0, 0)$ ,  $B_r$  denotes the open disk of center 0 and radius  $r > 0$ . The series  $\mathbf{e}_1, \mathbf{e}_2$  define functions  $\Omega \rightarrow \mathbb{C}_{\infty}[[t]]$  such that all the series in the images converge over  $B_q$ .

Let  $\mathbf{Hol}(\Omega)$  be the ring of holomorphic functions  $\Omega \rightarrow \mathbb{C}_{\infty}$ , over which the Frobenius  $\mathbb{F}_q$ -linear map  $\tau$  is well defined: if  $f \in \mathbf{Hol}(\Omega)$ , then  $\tau f = f^q$ . We consider the unique  $\mathbb{F}_q((t))$ -linear extension of  $\tau$ :

$$\mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)) \rightarrow \mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)),$$

again denoted by  $\tau$ .

We shall prove the fundamental theorem:

**Theorem 3** *The following identities hold in the domain  $\Omega \times B_q$ :*

$$\mathbf{e}_1 = -\tau(s_{\text{Car}} \mathbf{d}_2) h \quad \mathbf{e}_2 = \tau(s_{\text{Car}} \mathbf{d}_1) h. \quad (7)$$

In the statement of the theorem,  $h$  is the opposite of the unique normalised Drinfeld cusp form of weight  $q + 1$  and type 1 for  $\Gamma = \mathbf{GL}_2(A)$  as in [6].

Let us write  $\mathcal{E} = L(\chi_t, 1)^{-1}(\mathbf{e}_1, \mathbf{e}_2)$  <sup>(1)</sup> and  $\mathcal{F} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$ , and let us consider the representation  $\rho_t : \mathbf{GL}_2(A) \rightarrow \mathbf{GL}_2(\mathbb{F}_q[[t]])$  defined by

$$\rho_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix},$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(A)$ . Then, for any such a choice of  $\gamma$  we have the functional equations:

$$\begin{aligned} \mathcal{F}(\gamma(z)) &= (cz + d)^{-1} \rho_t(\gamma) \cdot \mathcal{F}(z), \\ {}^t \mathcal{E}(\gamma(z)) &= (cz + d) {}^t \rho_t^{-1}(\gamma) \cdot {}^t \mathcal{E}(z). \end{aligned}$$

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<sup>1</sup>We will also adopt the notation  $\mathcal{E}_{1,0} = \mathcal{E}$ .

This puts right away the functions  ${}^t\mathcal{E}$  and  $\mathcal{F}$  in the framework of *deformations of vectorial modular forms*, topic that will be developed in Section 2.

We will make use of a remarkable sequence  $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$  of functions  $\Omega \times B_q \rightarrow \mathbb{C}_\infty$  defined by the scalar product (with component-wise action of  $\tau$ ):

$$\mathcal{G}_k = \mathcal{G}_{1,0,k} = (\tau^k \mathcal{E}) \cdot \mathcal{F},$$

such that, for  $k \geq 0$ ,  $\mathcal{G}_k$  belongs to  $M_{q^{k-1},0} \otimes \mathbb{F}_q[t, \theta]$  where  $M_{w,m}$  denotes the  $\mathbb{C}_\infty$ -vector space of Drinfeld modular forms of weight  $w$  and type  $m$ . In fact, only the terms  $\mathcal{G}_0, \mathcal{G}_1$  are needed to prove our Theorem. Once their explicit computation is accomplished (see Proposition 13), the proof of Theorem 3 is only a matter of solving a non-homogeneous system in two equations and two indeterminates  $e_1, e_2$ . Furthermore, Theorem 1 will be deduced with the computation of a limit in the identity  $e_1 = -\tau(s_{\text{Car}} \mathbf{d}_2)h$  and a similar path will be followed for Theorem 2, by using this time the sequences  $\mathcal{G}_{\alpha,0,k} = (\tau^k \mathcal{E}_{\alpha,0}) \cdot \mathcal{F}$  defined later.

We end this introduction by pointing out that the sequence  $\mathcal{G}$  has itself several interesting features. For example, the functions  $\mathcal{G}_k$  already appeared in [16] (they are denoted by  $g_k^*$  there) as the coefficients of the ‘‘cocycle terms  $\mathbf{L}_\gamma$ ’’ of the functional equations of the *deformations of Drinfeld quasi-modular forms*  $\tau^k \mathbf{E}$  introduced there.

It is also interesting to notice that the *deformation of Legendre’s identity* (10) that we quote here (proved in [16]):

$$h^{-1}(\tau s_{\text{Car}})^{-1} = \mathbf{d}_1(\tau \mathbf{d}_2) - \mathbf{d}_2(\tau \mathbf{d}_1)$$

can be deduced from Theorem 3 by using the fact that  $\mathcal{G}_0 = -1$  obtained in Proposition 13.

Moreover, it can be proved, again with the help of the theory of  $\tau$ -linear recurrent sequences and  $\tau$ -linearised recurrent sequences (they will not be described here), that for  $k \geq 0$ , the function  $\mathcal{G}_k(z, \theta)$ , well defined, is equal to the *ortho-Eisenstein* series  $g_k(z)$ , and that  $\mathcal{G}_k(z, \theta^{q^k})$ , also well defined, is equal to the *para-Eisenstein* series  $m_k(z)$ , in the notations and the terminology of [8]. Hence, the sequence  $\mathcal{G}$  provides an interesting tool also in the study of both these kinds of functions. This program will be however pursued in another paper.

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## 2 Vectorial modular forms and their deformations

In this paper,  $t$  will be an indeterminate independent on  $\theta$  or a parameter varying in  $\mathbb{C}_\infty$ , and we will freely switch from formal series to functions.

For a positive real number  $r$ , we denote by  $\mathbb{T}_{<r}$  the sub- $\mathbb{C}_\infty$ -algebra of  $\mathbb{C}_\infty[[t]]$  whose elements are formal series  $\sum_{i \geq 0} c_i t^i$  that converge for any  $t \in \mathbb{C}_\infty$  with  $|t| < r$ . We also denote by  $\mathbb{T}_\infty$  the sub- $\mathbb{C}_\infty$ -algebra of series that converge everywhere in  $\mathbb{C}_\infty$ . If  $r_1 > r_2 > 0$ , we have

$$\mathbb{T}_{<r_2} \supset \mathbb{T}_{<r_1} \supset \mathbb{T}_\infty.$$

The *Tate algebra* of formal series of  $\mathbb{C}_\infty[[t]]$  converging for all  $t$  such that  $|t| \leq 1$  will be denoted by  $\mathbb{T}_1$  or  $\mathbb{T}$ ; it is contained in  $\mathbb{T}_{<1}$  and contains  $\mathbb{T}_{<1+\epsilon}$  for all  $\epsilon > 0$ ; clearly,  $\mathbb{C}_\infty[[t]] \supset \mathbb{T}_1 \supset \mathbb{T}_\infty$ .

The ring  $\mathbb{C}_\infty[[t]]$  is endowed with the  $\mathbb{F}_q[[t]]$ -linear automorphism  $\tau$  acting on formal series as follows:

$$\tau \sum_i c_i t^i = \sum_i c_i^q t^i.$$

This automorphism induces automorphisms of  $\mathbb{T}_1, \mathbb{T}_\infty$ .

We will work with certain functions  $f : \Omega \times B_r \rightarrow \mathbb{C}_\infty$  with the property that for all  $z \in \Omega$ ,  $f(z, t)$  can be identified with an element of  $\mathbb{T}_{<r}$  in the obvious way. For such functions we will then also write  $f(z)$  to stress the dependence on  $z \in \Omega$  when we want to consider them as functions  $\Omega \rightarrow \mathbb{T}_{<r}$  for some  $r$ . Sometimes, we will not specify the variables  $z, t$  and just write  $f$  instead of  $f(z, t)$  or  $f(z)$  to lighten our formulas just as we did in some places of the introduction. Moreover,  $z$  will denote a variable in  $\Omega$  all along the paper.

In all the following,  $\mathbf{Hol}(\Omega)$  denotes the ring of holomorphic functions on  $\Omega$ . For  $r$  a positive real number, let us denote by  $\mathcal{R}_{<r}$  (resp.  $\mathcal{R}$  or  $\mathcal{R}_1$ ) the (integral) ring whose elements are the formal series  $f = \sum_{i \geq 0} f_i t^i$ , such that

1. For all  $i$ ,  $f_i$  is a map  $\Omega \rightarrow \mathbb{C}_\infty$  belonging to  $\mathbf{Hol}(\Omega)$ .
2. For all  $z \in \Omega$ ,  $\sum_{i \geq 0} f_i(z) t^i$  is an element of  $\mathbb{T}_{<r}$  (resp.  $\mathbb{T}$ ).

We shall write

$$\mathcal{R}_\infty = \bigcap_{r > 0} \mathcal{R}_{<r}$$

and allow  $r$  to vary in  $\mathbb{R}_{>0} \cup \{\infty\}$ . The rings  $\mathcal{R}$  and  $\mathcal{R}_\infty$  are endowed with injective endomorphisms  $\tau$  acting on formal series as follows:

$$\tau \sum_{i \geq 0} f_i(z) t^i = \sum_{i \geq 0} f_i(z) q^i t^i.$$

We end this preparatory section with some conventions on  $u$ -expansions. We will say that a series  $\sum_{i \geq i_0} c_i u^i$  (with the coefficients  $c_i$  in some ring) is *normalised*, if  $c_{i_0} = 1$ . We will also say that the series is *of type*  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$  if  $i \not\equiv m \pmod{q-1}$  implies  $c_i = 0$ . This definition is obviously compatible with the notion of type of a Drinfeld modular form already mentioned in the introduction.

## 2.1 Deformations of vectorial modular forms.

In this subsection we develop some tools in the theory of *deformations of vectorial modular forms*. This part is inspired by works about vectorial modular forms for  $\mathbf{SL}_2(\mathbb{Z})$  by Knopp, Mason [11, 13].

Let us consider a representation

$$\rho : \Gamma \rightarrow \mathbf{GL}_s(\mathbb{F}_q((t))). \tag{8}$$

We assume that the determinant representation  $\det(\rho)$  is the  $\mu$ -th power of the determinant character, for some  $\mu \in \mathbb{Z}/(q-1)\mathbb{Z}$ . In all the following, given  $\gamma \in \Gamma$ , we denote by  $J_\gamma$  the associated factor of automorphy  $(\gamma, z) \mapsto cz + d$ , if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

**Definition 4** A *deformation of vectorial modular form* of weight  $w$ , dimension  $s$ , type  $m$  and radius  $r \in \mathbb{R}_{>0} \cup \{\infty\}$  associated with a representation  $\rho$  as in (8) is a column matrix  $\mathcal{F} \in \mathbf{Mat}_{s \times 1}(\mathcal{R}_{<r})$  such that, considering  $\mathcal{F}$  as a map  $\Omega \rightarrow \mathbf{Mat}_{s \times 1}(\mathbb{T}_{<r})$  we have, for all  $\gamma \in \Gamma$ ,

$$\mathcal{F}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} \rho(\gamma) \cdot \mathcal{F}(z).$$

The definition means that if the radius is  $\infty$ , then the entries of  $\mathcal{F}$  are in  $\mathcal{R}_\infty$ .

The set of deformations of vectorial modular forms of weight  $w$ , dimension  $s$ , type  $m$  and radius  $r$  associated to a representation  $\rho$  is a  $\mathbb{T}_{<r}$ -module (or  $\mathbb{T}_\infty$ -module if  $r = \infty$ ) that we will denote by  $\mathcal{M}_{w,m}^s(\rho, r)$  or  $\mathcal{M}_{w,m}^s(\rho)$  when the reference to a particular radius is clear.

Let us denote by  $M_{w,m}^1$  the  $\mathbb{C}_\infty$ -vector space (of infinite dimension) generated by quotients  $f/g$  with  $f \in M_{w',m'}$ ,  $g \in M_{w'',m''} \setminus \{0\}$  such that  $w' - w'' = w$ ,  $m' - m'' = m$ , and such that  $g$  does not vanish on  $\Omega$  (so that it is a power of  $h$ ).

If  $s = 1$  and if  $\rho = \mathbf{1}$  is the constant map, then  $\mathcal{M}_{w,m}^1(\mathbf{1}, r) = M_{w,m}^1 \otimes \mathbb{T}_{<r}$ . Similarly, for general  $s$ , we have a graded  $M_{w,m}^1 \otimes \mathbb{T}_{<r}$ -module

$$\mathcal{M}^s(\rho, r) = \bigoplus_{w,m} \mathcal{M}_{w,m}^s(\rho, r).$$

**Lemma 5** *Let  $k$  be a non-negative integer and  $r \geq 1$ . If  $\mathcal{F}$  is in  $\mathcal{M}_{w,m}^s(\rho, r)$ , then  $\tau^k \mathcal{F} \in \mathcal{M}_{wq^k, m}^s(\rho, r^{q^k})$ . Therefore, if we choose nonnegative integers  $k_1, \dots, k_s$ , then*

$$\det(\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}) \in M_{w(q^{k_1} + \dots + q^{k_s}), sm + \mu}^1 \otimes \mathbb{T}_{<r}.$$

In particular,

$$W_\tau(\mathcal{F}) = \det(\tau^0 \mathcal{F}, \dots, \tau^{s-1} \mathcal{F}) \in M_{w(1+q+q^2+\dots+q^{s-1}), sm + \mu}^1 \otimes \mathbb{T}_{<r}.$$

*Proof.* From the definition,

$$(\tau^k \mathcal{F})(\gamma(z)) = J_\gamma^{wq^k} \det(\gamma)^{-m} \rho(\gamma) (\tau^k \mathcal{F})$$

because  $\tau(\rho(\gamma)) = \rho(\gamma)$  and the first part of the lemma holds. Now define the matrix function:

$$\mathbf{M}_{k_1, \dots, k_s} = (\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}).$$

After the first part of the lemma we have, for  $\gamma \in \mathbf{GL}_2(A)$ :

$$\mathbf{M}_{k_1, \dots, k_s}(\gamma(z)) = \det(\gamma)^{-m} \rho(\gamma) \cdot \mathbf{M}_{k_1, \dots, k_s}(z) \cdot \mathbf{Diag}(J_\gamma^{wq^{k_1}}, \dots, J_\gamma^{wq^{k_s}}),$$

from which we can conclude the proof taking determinants of both sides.  $\square$

**Lemma 6** *Assume that  $r \geq 1$ . Let us consider  $\mathcal{F}$  in  $\mathcal{M}_{w,m}^s(\rho, r)$  and let  $\mathcal{E}$  be such that  ${}^t \mathcal{E}$  is in  $\mathcal{M}_{w',m'}^s({}^t \rho^{-1}, r)$ . For nonnegative  $k$ , if  $\mathcal{G}_k$  denotes the scalar product  $(\tau^k \mathcal{E}) \cdot \mathcal{F}$ , then,*

$$\mathcal{G}_k \in M_{w+w'q^k, m+m'}^1 \otimes \mathbb{T}_{<r}.$$

*Proof.* By Lemma 5,  $\tau^k({}^t \mathcal{E})$  is in  $\mathcal{M}_{wq^k, m'}^s({}^t \rho^{-1}, r)$ . Let  $\gamma$  be in  $\mathbf{GL}_2(A)$ . We thus have, after transposition, that

$$(\tau^k \mathcal{E})(\gamma(z)) = J_\gamma^{wq^k} \det(\gamma)^{-m} \mathcal{E}(z) \cdot \rho^{-1}(\gamma),$$

and

$$\mathcal{F}(\gamma(z)) = J_\gamma^{w'} \det(\gamma)^{-m'} \rho(\gamma) \cdot \mathcal{F}(z).$$

Hence,

$$\mathcal{G}_k(\gamma(z)) = J_\gamma^{wq^k + w'} \det(\gamma)^{-m-m'} \mathcal{G}_k(z),$$

from which we deduce that  $\mathcal{G}_k \in M_{wq^k + w', m+m'}^1 \otimes \mathbb{T}_{<r}$ .  $\square$



## 2.2 Main examples of deformations of vectorial modular forms.

From now on, we will use the representation  $\rho = \rho_t$  and the transposed of its inverse.

*The function  $\mathcal{F}$ .* This is the vector valued function  $\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$  so we recall now some properties of the functions  $\mathbf{d}_1, \mathbf{d}_2$ . At first sight, we only have  $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_{<q}$ . However, one sees easily that  $s_{\text{Car}}^{-1} \in \mathbb{T}_\infty$  from which it follows that  $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_\infty$ . The next proposition follows immediately from the results of [16] where some of them are stated in slightly different, although equivalent forms. We write  $g$  for the unique normalised Drinfeld modular form of weight  $q-1$  and type 0 for  $\Gamma$  (proportional to an Eisenstein series), and  $\Delta$  for the cusp form  $-h^{q-1}$ .

**Proposition 7** *We have five properties for the  $\mathbf{d}_i$ 's.*

1.  $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_\infty$ .
2. We have  $\mathcal{F} \in \mathcal{M}_{-1,0}^2(\rho_t, \infty)$ .
3. The functions  $\mathbf{d}_1, \mathbf{d}_2$  span the  $\mathbb{F}_q(t)$ -vector space of dimension 2 of solutions of the following  $\tau$ -linear difference equation:

$$X = (t - \theta^q)\Delta\tau^2 X + g\tau X, \quad (9)$$

*in a suitable existentially closed inversive field containing  $\mathcal{R}_\infty$ .*

4. Let us consider the matrix function:

$$\Psi(z, t) := \begin{pmatrix} \mathbf{d}_1(z, t) & \mathbf{d}_2(z, t) \\ \mathbf{d}_1^{(1)}(z, t) & \mathbf{d}_2^{(1)}(z, t) \end{pmatrix}.$$

*For all  $z \in \Omega$  and  $t$  with  $|t| < q$ :*

$$\det(\Psi) = (t - \theta)^{-1}h(z)^{-1}s_{\text{Car}}(t)^{-1}. \quad (10)$$

5. We have the series expansion

$$\mathbf{d}_2 = \sum_{i \geq 0} c_i(t)u^{(q-1)i} \in 1 + u^{q-1}\mathbb{F}_q[t, \theta][[u^{q-1}]], \quad (11)$$

*convergent for  $t, u$  sufficiently close to  $(0, 0)$ .*

*Deformations of vectorial Poincaré series.* Following [6], we consider the subgroup

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

of  $\Gamma = \mathbf{GL}_2(A)$  and its left action on  $\Gamma$ . For  $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , the map  $\delta \mapsto (c, d)$  induces a bijection between the orbit set  $H \backslash \Gamma$  and the set of  $(c, d) \in A^2$  with  $c, d$  relatively prime.

We consider the factor of automorphy

$$\mu_{\alpha, m}(\delta, z) = \det(\delta)^{-m} J_\gamma^\alpha,$$

where  $m$  and  $\alpha$  are positive integers (later,  $m$  will also determine a type, that is, a class modulo  $q-1$ ).

Let  $V_1(\delta)$  be the row matrix  $(\chi_t(c), \chi_t(d))$ . It is easy to show that the row matrix

$$\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta)$$

only depends on the class of  $\delta \in H \backslash \Gamma$ , so that we can consider the following expression:

$$\mathcal{E}_{\alpha,m}(z) = \sum_{\delta \in H \backslash \Gamma} \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta),$$

which is a row matrix whose two entries are formal series.

Let  $\mathcal{V}$  be the set of functions  $\Omega \rightarrow \mathbf{Mat}_{1 \times 2}(\mathbb{C}_\infty[[t]])$ . We introduce, for  $\alpha, m$  integers,  $f \in \mathcal{V}$  and  $\gamma \in \Gamma$ , the Petersson slash operator:

$$f|_{\alpha,m}\gamma = \det(\gamma)^m (cz + d)^{-\alpha} f(\gamma(z)) \cdot \rho_t(\gamma).$$

This will be used in the next proposition, where we recall that  $\log_q^+(x)$  denotes the maximum between 0 and  $\log_q(x)$ , the logarithm in base  $q$  of  $x > 0$ . We point out that we will not apply this proposition in full generality.

**Proposition 8** *Let  $\alpha, m$  be non-negative integers with  $\alpha \geq 2m + 1$ , and write  $r(\alpha, m) = \alpha - 2m - 1$ . We have the following properties.*

1. For  $\gamma \in \Gamma$ , the map  $f \mapsto f|_{\alpha,m}\gamma$  induces a permutation of the subset of  $\mathcal{V}$ :

$$\mathcal{S} = \{\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta); \delta \in H \backslash \Gamma\}.$$

2. If  $t \in \mathbb{C}_\infty$  and  $\alpha, m$  are chosen so that  $r(\alpha, m) > \log_q^+ |t|$ , then the components of  $\mathcal{E}_{\alpha,m}(z, t)$  are series of functions of  $z \in \Omega$  which converge absolutely and uniformly on every compact subset of  $\Omega$  to holomorphic functions.
3. If  $|t| < 1$ , then the components of  $\mathcal{E}_{\alpha,m}(z, t)$  converge absolutely and uniformly on every compact subset of  $\Omega$  also if  $\alpha - 2m > 0$ .
4. For any choice of  $\alpha, m, t$  submitted to the convergence conditions above, the function  ${}^t\mathcal{E}_{\alpha,m}(z, t)$  belongs to the space  $\mathcal{M}_{\alpha,m}^2({}^t\rho_t^{-1}, r(\alpha, m))$ .
5. If  $\alpha - 1 \not\equiv 2m \pmod{(q-1)}$ , the matrix function  $\mathcal{E}_{\alpha,m}(z, t)$  is identically zero.
6. If  $\alpha - 1 \equiv 2m \pmod{(q-1)}$ ,  $\alpha \geq (q+1)m + 1$  so that  $\mathcal{E}_{\alpha,m}$  converges, then  $\mathcal{E}_{\alpha,m}$  is not identically zero in its domain of convergence.

*Proof.* 1. We choose  $\delta \in H \backslash \Gamma$  corresponding to a couple  $(c, d) \in A^2$  with  $c, d$  relatively prime, and set  $f_\delta = \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta) \in \mathcal{S}$ . We have

$$\begin{aligned} f_\delta(\gamma(z)) &= \mu_{\alpha,m}(\delta, \gamma(z))^{-1} u^m(\delta(\gamma(z))) V_1(\delta) \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta), \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta\gamma) \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta') \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha,m}(\gamma, z) f_{\delta'} \cdot \rho_t(\gamma)^{-1}, \end{aligned}$$

with  $\delta' = \delta\gamma$  and  $f_{\delta'} = \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta')$ , from which part 1 of the proposition follows.

2. Convergence and holomorphy are ensured by simple modifications of [6, (5.5)], or by the arguments in [9, Chapter 10]. More precisely, let us choose  $0 \leq s \leq 1$  and look at the component at the place  $s + 1$

$$\mathcal{E}_s(z, t) = \sum_{\delta \in H \setminus \Gamma} \mu_{\alpha, m}(\delta, z)^{-1} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

of the vector series  $\mathcal{E}_{\alpha, m}$ . Writing  $\alpha = n(q-1) + 2m + l'$  with  $n$  non-negative integer and  $l' \geq 1$  we see, following Gerritzen and van der Put, [9, pp. 304-305] and taking into account the inequality  $|u(\delta(z))| \leq |cz + d|^2 / |z|_i$  ( $|z|_i$  denotes, for  $z \in \mathbb{C}_\infty$ , the infimum  $\inf_{a \in K_\infty} \{|z - a|\}$ ), that the term of the series  $\mathcal{E}_s$ :

$$\mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}) = (cz + d)^{-n(q-1) - l' - 2m} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

(where  $\delta$  corresponds to  $(c, d)$ ) has absolute value bounded from above by

$$|z|_i^{-m} \left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right|.$$

Applying the first part of the proposition, to check convergence, we can freely substitute  $z$  with  $z + a$  with  $a \in A$  and we may assume, without loss of generality, that  $|z| = |z|_i$ . We verify that, either  $\lambda = \deg_\theta z = \max(\mathcal{I}) \in \mathbb{Q} \setminus \mathbb{Z}$ , or  $\lambda \in \mathbb{Z}$  case in which  $c_\lambda \notin \mathbb{F}_q$ . In both cases, for all  $c, d, |cz + d| = \max\{|cz|, |d|\}$ . Then, the series defining  $\mathcal{E}_s$  can be decomposed as follows:

$$\mathcal{E}_s = \sum_{f_\delta \in H \setminus \Gamma} f_\delta = \left( \sum'_{|cz| < |d|} + \sum'_{|cz| \geq |d|} \right) \mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}).$$

We now look for upper bounds for the absolute values of the terms of the series above separating the two cases in a way similar to that of Gerritzen and van der Put in loc. cit.

Assume first that  $|cz| < |d|$ , that is,  $\deg_\theta c + \lambda < \deg_\theta d$ . Then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa \max\{1, |t|\}^{\deg_\theta d} |d|^{-n(q-1) - l'} \leq \kappa q^{\deg_\theta d (\log_q^+ |t| - n(q-1) - l')},$$

where  $\kappa$  is a constant depending on  $\lambda$ , and the corresponding sub-series converges with the imposed conditions on the parameters, because  $\log_q^+ |t| - n(q-1) - l' < 0$ .

If on the other side  $|cz| \geq |d|$ , that is,  $\deg_\theta c + \lambda \geq \deg_\theta d$ , then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa' \max\{1, |t|\}^{\deg_\theta d} |c|^{-n(q-1) - l'} \leq \kappa' q^{\deg_\theta c (\log_q^+ |t| - n(q-1) - l')},$$

with a constant  $\kappa'$  depending on  $\lambda$ , again because  $\log_q^+ |t| - n(q-1) - l' < 0$ . This completes the proof of the second part of the Proposition.

3. This property can be deduced from the proof of the second part because if  $\log_q |t| < 0$ , then  $|\chi_t(c^s d^{1-s})| \leq 1$ .

4. The property is obvious by the first part of the proposition, because  $\mathcal{E}_{\alpha, m} = \sum_{f \in \mathcal{S}} f$ .

5. We consider  $\gamma = \mathbf{Diag}(1, \lambda)$  with  $\lambda \in \mathbb{F}_q^\times$ ; the corresponding homography, multiplication by  $\lambda^{-1}$ , is equal to that defined by  $\mathbf{Diag}(\lambda^{-1}, 1)$ . Hence, we have:

$$\begin{aligned} \mathcal{E}_{\alpha, m}(\gamma(z)) &= \lambda^{\alpha - m} \mathcal{E}_{\alpha, m}(z) \cdot \mathbf{Diag}(1, \lambda^{-1}) \\ &= \lambda^m \mathcal{E}_{\alpha, m}(z) \cdot \mathbf{Diag}(\lambda, 1), \end{aligned}$$

from which it follows that  $\mathcal{E}_{\alpha,m}$  is identically zero if  $\alpha - 1 \not\equiv 2m \pmod{q-1}$ .

6. If  $m = 0$ , we postpone the proof to Lemma 10. Assuming now that  $m > 0$ , the series  $z\mathcal{E}_1(z, \theta) + \mathcal{E}_0(z, \theta)$  converges to the Poincaré series of weight  $\alpha - 1$  for  $\Gamma$  so that [9, Proposition 10.5.2] suffices for our purposes.  $\square$

Let  $\alpha, m$  be non-negative integers such that  $\alpha - 2m > 1$  and  $\alpha - 1 \equiv 2m \pmod{q-1}$ . We have functions:

$$\begin{aligned}\mathcal{E}_{\alpha,m} : \Omega &\rightarrow \mathbf{Mat}_{1 \times 2}(\mathcal{R}_{<r}), \\ \mathcal{F} : \Omega &\rightarrow \mathbf{Mat}_{2 \times 1}(\mathcal{R}_{\infty}),\end{aligned}$$

with  $r = r(\alpha, m)$  as in Proposition 8, and  ${}^t\mathcal{E}_{\alpha,m} \in \mathcal{M}_{\alpha,m}^2({}^t\rho_t^{-1}, r)$ ,  $\mathcal{F} \in \mathcal{M}_{-1,0}^2(\rho_t, \infty)$ . Therefore, after Lemma 6, the functions

$$\mathcal{G}_{\alpha,m,k} = (\tau^k \mathcal{E}_{\alpha,m}) \cdot \mathcal{F} = \mathcal{E}_{q^k \alpha, m} \cdot \mathcal{F} : \Omega \rightarrow \mathbb{T}_{<r}$$

satisfy  $\mathcal{G}_{\alpha,m,k} \in M_{q^k \alpha - 1, m}^1 \otimes \mathbb{T}_r$ .

*A special case.* After Proposition 8, if  $\alpha > 0$  and  $\alpha \equiv 1 \pmod{q-1}$ , then  $\mathcal{E}_{\alpha,0} \neq 0$ . We call these series *deformations of vectorial Eisenstein series*.

**Lemma 9** *With  $\alpha > 0$  such that  $\alpha \equiv 1 \pmod{q-1}$ , the following identity holds:*

$$\mathcal{E}_{\alpha,0}(z, t) = L(\chi_t, \alpha)^{-1} \sum'_{c,d} (cz + d)^{-\alpha} V_1(c, d),$$

and  $\mathcal{E}_{\alpha,0}$  is not identically zero.

*Proof.* We recall the notation

$$V_1(c, d) = (\chi_t(c), \chi_t(d)) \in \mathbf{Mat}_{1 \times 2}(\mathbb{F}_q[t]).$$

We have

$$\begin{aligned}\sum'_{c,d} (cz + d)^{-\alpha} V_1(c, d) &= \sum_{(c',d')=1} \sum_{a \in A^+} a^{-\alpha} (c'z + d')^{-\alpha} V_1(ac', ad') \\ &= L(\chi_t, \alpha) \mathcal{E}_{\alpha,0}(z, t),\end{aligned}$$

where the first sum is over couples of  $A^2$  distinct from  $(0,0)$ , while the second sum is over the couples  $(c', d')$  of relatively prime elements of  $A^2$ . Non vanishing of the function follows from Proposition 8.  $\square$

### 3 Proof of the Theorems

Following Gekeler [6, Section 3], we recall that for all  $\alpha > 0$  there exists a polynomial  $G_\alpha(u) \in \mathbb{C}_\infty[u]$ , called the  $\alpha$ -th *Goss polynomial*, such that, for all  $z \in \Omega$ ,  $G_\alpha(u(z))$  equals the sum of the convergent series

$$\tilde{\pi}^{-\alpha} \sum_{a \in A} \frac{1}{(z+a)^\alpha}.$$

Several properties of these polynomials are collected in [6, Proposition (3.4)]. Here, we will need that for all  $\alpha$ ,  $G_\alpha$  is of type  $\alpha$  as a formal series of  $\mathbb{C}_\infty[[u]]$ . Namely:

$$G_\alpha(\lambda u) = \lambda^\alpha G_\alpha(u), \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We also recall, for  $a \in A$ , the function

$$u_a(z) := u(az) = e_{\text{Car}}(\tilde{\pi}az)^{-1} = u^{|\alpha|} f_a(u) = u^{|\alpha|} + \cdots \in A[[u]],$$

where  $f_a \in A[[u]]$  is the  $a$ -th inverse cyclotomic polynomial defined in [6, (4.6)]. Obviously, we have

$$u_{\lambda a} = \lambda^{-1} u_a \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

To continue, we will state and prove three auxiliary lemmas.

**Lemma 10** *Let  $\alpha$  be a positive integer such that  $\alpha \equiv 1 \pmod{q-1}$ . We have, for all  $t \in \mathbb{C}_\infty$  such that  $|t| < 1$  and  $z \in \Omega$ , convergence of the series below, and equality:*

$$\sum'_{c,d \in A} \frac{\chi_t(c)}{(cz+d)^\alpha} = -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c(z)),$$

from which it follows that  $\mathcal{E}_{\alpha,0} \neq 0$ .

*Proof.* Convergence features are easy to deduce from Proposition 8. Indeed, we have convergence if  $\log_q^+ |t| < r(\alpha, m) = \alpha - 1$ , that is,  $\max\{1, |t|\} \leq q^{\alpha-1}$  if  $\alpha > 1$  and we have convergence, for  $\alpha = 1$ , for  $|t| < 1$ . In all cases, convergence holds for  $|t| < 1$ .

We then compute:

$$\begin{aligned} \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} &= \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(cz+d)^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(c\tilde{\pi}z + d\tilde{\pi})^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) G_\alpha(u_c) \\ &= \tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{1-\alpha} \\ &= -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c). \end{aligned}$$

The non-vanishing of  $\mathcal{E}_{\alpha,0}$  comes from Lemma 9 and the non-vanishing contribution of the term  $G_\alpha(u)$  in the latter series.  $\square$

**Lemma 11** *Let  $\alpha > 0$  be an integer such that  $\alpha \equiv 1 \pmod{q-1}$ . For all  $t \in \mathbb{C}_\infty$  such that  $|t| < 1$ , we have*

$$\lim_{|z|=|z| \rightarrow \infty} \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} = 0.$$

*Proof.* We recall from [16] the series expansion

$$\mathbf{d}_1(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \mathbf{s}_2(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \sum_{n \geq 0} e_{\Lambda_z} \left( \frac{z}{\theta^{n+1}} \right) t^n,$$

converging for all  $t$  such that  $|t| < q$  and all  $z \in \Omega$ .

By a simple modification of the proof of [7, Lemma 5.9 p. 286], we have

$$\lim_{|z|=|z| \rightarrow \infty} u(z) t^n e_{\Lambda_z}(z/\theta^{n+1})^q = 0$$

uniformly in  $n > 0$ , for all  $t$  such that  $|t| \leq q$ .

Moreover, it is easy to show that

$$\lim_{|z|=|z| \rightarrow \infty} u(z) e_{\Lambda_z}(z/\theta)^q = \tilde{\pi}^{-q} \lim_{|z|=|z| \rightarrow \infty} e_{\text{Car}}^q(\tilde{\pi}z/\theta)/e_{\text{Car}}(\tilde{\pi}z) = 1.$$

This suffices to show that

$$\lim_{|z|=|z| \rightarrow \infty} \mathbf{d}_1(z) G_\alpha(u_c(z)) = 0$$

uniformly for  $c \in A^+$ , for all  $t$  such that  $|t| < q$ . The lemma then follows from an application of Lemma 10.  $\square$

**Lemma 12** *Let  $\alpha > 0$  be an integer such that  $\alpha \equiv 1 \pmod{q-1}$ . For all  $t \in \mathbb{C}_\infty$  such that  $|t| < 1$ , we have*

$$\lim_{|z|=|z| \rightarrow \infty} \sum'_{c,d} \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t, \alpha).$$

*Proof.* It suffices to show that

$$\lim_{|z|=|z| \rightarrow \infty} \sum_{c \neq 0} \sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = 0.$$

For all  $(c, d) \in A \setminus \{(0, 0)\}$ , we have  $|cz+d| = \max\{|cz|, |d|\}$ . We then have, for  $c \neq 0$ :

$$\sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = \left( \sum_{|d| > |cz|} + \sum_{|d| \leq |cz|} \right) \frac{\chi_t(d)}{(cz+d)^\alpha}.$$

Now, if  $|d| > |cz|$ , we have, for  $|t| < 1$ :

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{d^\alpha} \right| \leq |d|^{-\alpha} \leq |cz|^{-\alpha}.$$

If  $|d| \leq |cz|$ , we have, again for  $|t| < 1$ :

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{(cz)^\alpha} \right| \leq |cz|^{-\alpha}.$$

Therefore, for  $c \neq 0$ ,

$$\left| \sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq |cz|^{-\alpha}.$$

This implies that

$$\left| \sum_{c \neq 0} \sum_{d \in A} \frac{\chi_t(d)}{(cz + d)^\alpha} \right| \leq |z|^{-\alpha},$$

from which the Lemma follows.  $\square$

The next step is to prove the following Proposition.

**Proposition 13** *For all  $\alpha > 0$  with  $\alpha \equiv 1 \pmod{q-1}$ ,  $\mathcal{G}_{\alpha,0,0} \in M_{\alpha-1,0} \otimes \mathbb{T}_{<q}$  and we have the limit  $\lim_{|z|=|z|_i \rightarrow \infty} \mathcal{G}_{\alpha,0,0} = -1$ .*

*Moreover, if  $\alpha \leq q(q-1)$ , then:*

$$\mathcal{G}_{\alpha,0,0} = -E_{\alpha-1},$$

where  $E_{\alpha-1}$  is the normalised Eisenstein series of weight  $\alpha-1$  for  $\Gamma$ .

*Proof.* After Lemma 6, the sum of the series:

$$F_\alpha(z, t) := \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz + d)^\alpha} + \mathbf{d}_2(z) \sum'_{c,d} \frac{\chi_t(d)}{(cz + d)^\alpha}$$

converges on  $\Omega$  to an element of  $M_{\alpha-1,0}^! \otimes \mathbb{T}_{<q}$ .

After (11), we have that for all  $t$  with  $|t| < 1$ ,  $\lim_{|z|_i=|z| \rightarrow \infty} \mathbf{d}_2(z) = 1$ . From Lemmas 11 and 12,

$$\lim_{|z|_i=|z| \rightarrow \infty} F_\alpha(z, t) = -L(\chi_t, \alpha).$$

Therefore, for all  $t$  such that  $|t| < q$ ,  $F_\alpha(z, t)$  converges to an holomorphic function on  $\Omega$  and is endowed with a  $u$ -expansion. In particular,  $F_\alpha(z, t)$  is a family of modular forms of  $M_{\alpha-1,0} \otimes \mathbb{T}_{<q}$  and we have the first part of the proposition. Since for the selected values of  $\alpha$ ,  $M_{\alpha-1,0} = \langle E_{\alpha-1} \rangle$ , we obtain that  $F_\alpha = -L(\chi_t, \alpha)E_{\alpha-1}$ . After Lemma 9, the proposition follows.  $\square$

*Proof of Theorem 3.* By Proposition 7, the matrix  $M = (\mathcal{F}, \tau^{-1}\mathcal{F})$  is invertible. From (10) we deduce that

$$\tau M^{-1} = (t - \theta)_{s_{\text{Car}}} h \begin{pmatrix} -\mathbf{d}_2 & \mathbf{d}_1 \\ \tau \mathbf{d}_2 & -\tau \mathbf{d}_1 \end{pmatrix}.$$

Therefore, we have the formulas:

$$\begin{aligned} \mathcal{E}_{\alpha,0} &= \\ &= (\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot M^{-1} \\ &= (t - \theta^{1/q}) h^{1/q} (\tau^{-1} s_{\text{Car}}) (\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot \begin{pmatrix} -\tau^{-1}\mathbf{d}_2 & \tau^{-1}\mathbf{d}_1 \\ \mathbf{d}_2 & -\mathbf{d}_1 \end{pmatrix} \\ &= (t - \theta^{1/q}) h^{1/q} (\tau^{-1} s_{\text{Car}}) (\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_2, -\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_1 + \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_1). \end{aligned} \quad (12)$$

Now, we have, for all  $k \in \mathbb{Z}$ ,

$$\mathcal{G}_{\alpha,0,k} = g\tau\mathcal{G}_{\alpha,0,k-1} + \Delta(t - \theta^q)\tau^2\mathcal{G}_{\alpha,0,k-2}. \quad (13)$$

Applying this formula for  $k = 1$  and by using Part 3 of Proposition 7, we obtain

$$\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_i = \Delta^{1/q}(t - \theta)((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_i)), \quad i = 1, 2.$$

Replacing in (12), and using  $\Delta^{1/q}h^{1/q} = -h$  and  $(t - \theta^{1/q})\tau^{-1}s_{\text{Car}} = s_{\text{Car}}$ , we get the formula:

$$\mathcal{E}_{\alpha,0} = (t - \theta)s_{\text{Car}}h(-(\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 + \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2), (\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_1 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_1)). \quad (14)$$

So far, we have not specified the value of  $\alpha$ . The general computation of  $\mathcal{G}_{\alpha,0,0}$  and  $\mathcal{G}_{\alpha,0,-1}$  is a difficult problem, but for  $\alpha = 1$  we can apply Proposition 13. We have  $\mathcal{G}_{1,0,0} = -1$  and  $\mathcal{G}_{1,0,1} = \mathcal{G}_{q,0,0} = -g = -E_{q-1}$ . Therefore,  $\mathcal{G}_{\alpha,0,-1} = 0$  and the Theorem 3 follows.  $\square$

*Proof of Theorem 1.* Lemma 10 and (14) imply that

$$L(\chi_t, \alpha) = \frac{\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c)}{(\tau s_{\text{Car}})h((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2))}.$$

For  $\alpha = 1$ , we get

$$L(\chi_t, \alpha) = -\frac{\tilde{\pi} \sum_{c \in A^+} \chi_t(c) u_c}{(\tau s_{\text{Car}}\mathbf{d}_2)h},$$

from which we deduce Theorem 1 and even some additional information, namely, the formula:

$$(\tau\mathbf{d}_2)h = -\sum_{c \in A^+} \chi_t(c) u_c.$$

$\square$

*Proof of Theorem 2.* For general  $\alpha$ , we know that there exists  $\lambda_\alpha \in \mathbb{T}_{<q}$  such that

$$\sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) = \lambda_\alpha h((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2)). \quad (15)$$

Let us write  $f$  for the series  $\sum_{c \in A^+} \chi_t(c) G_\alpha(u_c)$ ,  $\phi$  for  $\lambda_\alpha h \tau\mathcal{G}_{\alpha,0,-1}$  and  $\psi$  for  $-\lambda_\alpha h \mathcal{G}_{\alpha,0,0}$ , so that (15) becomes:

$$f = \phi\mathbf{d}_2 + \psi\tau\mathbf{d}_2.$$

We know, from the fact that  $h$  is in  $M_{q+1,1}$ , that  $\phi \in M_{\alpha+q,1} \otimes \mathbb{T}_{<q}$ . By (13), we also see that  $\psi \in M_{\alpha+1,1}^! \otimes \mathbb{T}_{<q}$ . Let  $L$  be an algebraically closed field containing  $\mathbb{T}_{<q}$ . As for any choice of  $w, m$ ,  $M_{w,m}^!$  embeds in  $\mathbb{C}_\infty((u))$  and there is a basis of this space with  $u$ -expansions defined over  $K$ ,  $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$  acts on  $M_{w,m}^! \otimes \mathbb{T}_{<q}$  through the coefficients of the  $u$ -expansions. Let  $\sigma$  be an element of  $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$  and, for  $\mu \in M_{w,m}^! \otimes \mathbb{T}_{<q}$ , let us denote by  $\mu^\sigma \in M_{w,m}^! \otimes \mathbb{T}_{<q}$  the form obtained applying  $\sigma$  on every coefficient of the  $u$ -expansion of  $\mu$ .

Since  $f, \mathbf{d}_2$  and  $\tau\mathbf{d}_2$  are defined over  $\mathbb{F}_q(t, \theta)$ , we get

$$f = \phi^\sigma \mathbf{d}_2 + \psi^\sigma \tau\mathbf{d}_2.$$

Assume that  $\phi^\sigma \neq \phi$  or  $\psi^\sigma \neq \psi$ . We can suppose in fact that  $\phi^\sigma \neq \phi$  and  $\psi^\sigma \neq \psi$ . Then  $\tau\mathbf{d}_2/\mathbf{d}_2 \in M_{q-1,0}^! \otimes \mathbb{T}_{<q}$  which is impossible after the results of [16] (there, it is even proven that the functions  $\mathbf{d}_2, \tau\mathbf{d}_2, g$  and  $h$  are algebraically independent over  $C((t))$ ). This means that the  $u$ -expansions of  $\phi, \psi$  are defined over  $\mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$  for some  $s \geq 0$ . By the fact that the constant coefficients of  $\mathcal{G}_{\alpha,0,0}$  and  $\tau\mathcal{G}_{\alpha,0,-1}$  are both equal to  $-1$  (this follows from the first part of Proposition 13), we get that  $\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$ .

We have proven that  $\tilde{\pi}^{-\alpha} L(\chi_t, \alpha)(t - \theta)s_{\text{Car}} \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$ . But we already know that  $L(\chi_t, \alpha) \in K_\infty[[t]]$ ,  $s_{\text{Car}} \in K^{\text{sep}}[[t]]$  (separable closure), and  $\tilde{\pi} \in K_\infty^{\text{sep}}$ . Therefore,

$$\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s}) \cap K_\infty^{\text{sep}}((t)) = \mathbb{F}_q(t, \theta).$$

$\square$



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