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Values of certain L -series in positive characteristic*

Federico Pellarin^{†‡}

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Abstract. We introduce a family of L -series specialising to both L -series associated to certain Dirichlet characters over $\mathbb{F}_q[\theta]$ and to integral values of Carlitz-Goss zeta function associated to $\mathbb{F}_q[\theta]$. We prove, with the use of the theory of deformations of vectorial modular forms, a formula for their value at 1, as well as some arithmetic properties of other “even” values.

1 Introduction, results

let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let \mathbb{F}_q be the finite field with q elements. We consider the polynomial ring $A = \mathbb{F}_q[\theta]$ and its fraction field $K = \mathbb{F}_q(\theta)$, with θ an indeterminate over \mathbb{F}_q . On K , we will consider the absolute value $|\cdot|$ defined by $|a| = q^{\deg_\theta a}$, a being in K , so that $|\theta| = q$. Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let $K_\infty^{\text{alg.}}$ be an algebraic closure of K_∞ , let \mathbb{C}_∞ be the completion of $K_\infty^{\text{alg.}}$ for the unique extension of $|\cdot|$ to $K_\infty^{\text{alg.}}$, and let $K^{\text{alg.}}$ be the algebraic closure of K in $K_\infty^{\text{alg.}}$.

We consider an element t of \mathbb{C}_∞ . We have the “evaluating at t ” ring homomorphism

$$\chi_t : A \rightarrow \mathbb{F}_q[t]$$

defined by $\chi_t(a) = a(t)$. In other words, $\chi_t(a)$ is the image of the polynomial map $a(t)$ obtained by substituting, in $a(\theta)$, θ by t . For example, $\chi_t(1) = 1$ and $\chi_t(\theta) = t$. If we choose $t \in \mathbb{F}_q^{\text{alg.}}$ then χ_t factors through a Dirichlet character modulo the ideal generated by the minimal polynomial of t in A .

We can also consider t as an indeterminate; for $\alpha > 0$ an integer, we then have a well defined formal series

$$L(\chi_t, \alpha) = \sum_{a \in A^+} \chi_t(a) a^{-\alpha} = \prod_{\mathfrak{p}} (1 - \chi_t(\mathfrak{p}) \mathfrak{p}^{-\alpha})^{-1} \in K_\infty[[t]],$$

where A^+ denotes the set of monic polynomials of A , and where the eulerian product runs over the monic irreducible polynomials of A . This formal series converges for $\log_q |t| < \alpha$, \log_q being the logarithm in base q . In this paper, we are interested in the function $L(\chi_t, \alpha)$ of the variable $t \in \mathbb{C}_\infty$, for fixed α .

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We give some relevant examples of values of these series. If $t = \theta$ and $\alpha > 1$, then $L(\chi_\theta, \alpha)$ converges to the value of Carlitz-Goss zeta value at α :

$$L(\chi_\theta, \alpha) = \zeta(\alpha) = \sum_{a \in A^+} a^{-\alpha}.$$

If on the other side we consider $t \in \mathbb{F}_q^{\text{alg}}$, then, for $\alpha > 0$, $L(\chi_t, \alpha)$ converges to the value at α of the L -series associated to a Dirichlet character, see Goss' book [10].

We will need some classical functions related to Carlitz's module. First of all, its *exponential function* e_{Car} defined, for all $\zeta \in \mathbb{C}_\infty$, by the sum of the convergent series:

$$e_{\text{Car}}(\zeta) = \sum_{n \geq 0} \frac{\zeta^{q^n}}{d_n}, \quad (1)$$

where $d_0 := 1$ and $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$, with $[i] = \theta^{q^i} - \theta$ if $i > 0$.

We choose once and for all a fundamental period $\tilde{\pi}$ of e_{Car} . It is possible to show that $\tilde{\pi}$ is equal, up to the choice of a $(q-1)$ -th root of $-\theta$, to the (value of the) convergent product:

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_\infty((-\theta)^{\frac{1}{q-1}}) \setminus K_\infty.$$

Next, we need the following series of $K^{\text{sep}}[[t]]$:

$$s_{\text{Car}}(t) := \sum_{i=0}^{\infty} e_{\text{Car}}\left(\frac{\tilde{\pi}}{\theta^{i+1}}\right) t^i = \sum_{n=0}^{\infty} \frac{\tilde{\pi}^{q^n}}{d_n(\theta^{q^n} - t)}, \quad (2)$$

converging for $|t| < q$. This is the *canonical rigid analytic trivialisation* of the so-called *Carlitz's motive*. This leads to the property that s_{Car} generates the one-dimensional $\mathbb{F}_q((t))$ -vector space of solutions of the τ -difference equation

$$\tau s_{\text{Car}} = (t - \theta) s_{\text{Car}}, \quad (3)$$

in a suitable difference field, where $\tau : \mathbb{C}_\infty((t)) \rightarrow \mathbb{C}_\infty((t))$ is the operator defined by $\tau \sum c_i t^i = \sum c_i^q t^i$. We refer to [15] for a description of the main properties of this function s_{Car} , or to the papers [1, 4, 14], where it was originally introduced and appears under different notations.

We shall prove:

Theorem 1 *The following identity holds:*

$$L(\chi_t, 1) = -\frac{\tilde{\pi}}{(t - \theta) s_{\text{Car}}}.$$

The inverse of $(t - \theta) s_{\text{Car}}$, denoted by Ω in [4], is entire. This implies that $L(\chi_t, 1)$ allows, beyond its domain of convergence, entire analytic continuation in terms of the variable t .

It is interesting to notice that Theorem 1 directly implies the classical formulas for the values

$$\zeta(q^k - 1) = \sum_{a \in A^+} a^{1-q^k} = (-1)^k \frac{\tilde{\pi}^{q^k - 1}}{[k][k-1] \cdots [1]}$$

of Carlitz-Goss' zeta value at $q^k - 1$ for $k > 0$. This follows easily from the computation of the limit $t \rightarrow \theta$ in the formula $\tau^k L(\chi_t, 1) = -\tilde{\pi}^{q^k} / (\tau^k (t - \theta) s_{\text{Car}})$, observing that $\tau^k ((t - \theta) s_{\text{Car}}) = (t - \theta^{q^k}) \cdots (t - \theta^q)(t - \theta) s_{\text{Car}}$ by (3).

Also, if $t = \xi \in \mathbb{F}_q^{\text{alg}}$, Theorem 1 directly implies that $L(\chi_\xi, 1)$, the value of an L -function associated to a Dirichlet character, is a multiple of $\tilde{\pi}$ by an algebraic element of \mathbb{C}_∞ , solution of an algebraic equation

$$X^{q^r-1} = (\xi - \theta^{q^{r-1}}) \cdots (\xi - \theta).$$

Some of these consequences are also covered by the so-called Anderson *log-algebraic power series identities for twisted harmonic sums*, see [2, 3, 12], see also [5].

It is well known that $\lim_{t \rightarrow \theta} (t - \theta)_{s_{\text{Car}}} = -\tilde{\pi}$. It follows that

$$\lim_{t \rightarrow \theta} L(\chi_t, 1) = 1. \quad (4)$$

Surprisingly, this value coincides with $\zeta(0)$. Again, the well known fact that $(t - \theta)_{s_{\text{Car}}}$ has poles at $\theta^q, \theta^{q^2}, \dots$ implies that $\lim_{t \rightarrow \theta} L(\chi_t, q^k - 1) = 0$ for $k > 0$, and we know that $\zeta(s(q-1)) = 0$ for negative “even” integral values of s . It would be interesting to interpret these coincidences throughout the study of a generalisation of both the L -series considered in this paper, and *Carlitz-Goss’ zeta function*, which is an analytic function defined on the space $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ interpolating the above-mentioned zeta values.

For other positive values of $\alpha \equiv 1 \pmod{q-1}$, we have a result on $L(\chi_t, \alpha)$ at once more general and less precise.

Theorem 2 *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. There exists a non-zero element $\lambda_\alpha \in \mathbb{F}_q(t, \theta)$ such that*

$$L(\chi_t, \alpha) = \lambda_\alpha \frac{\tilde{\pi}^\alpha}{(t - \theta)_{s_{\text{Car}}}}.$$

We have seen that $\lambda_1 = -1$. On the other hand, again by (3), we have the formula

$$\tau^k \lambda_\alpha = (t - \theta^{q^k}) \cdots (t - \theta^q) \lambda_{q^k \alpha}.$$

Apart from this, the explicit computation of the λ_α ’s is difficult and very little is known on these coefficients which could encode, we hope, an interesting generalisation in $\mathbb{F}_q(t, \theta)$ of the theory of Bernoulli-Carlitz numbers.

The proofs of Theorems 1 and 2 that we propose rely on certain properties of *deformations of vectorial modular forms* (see Section 2). In fact, Theorem 1 is a corollary of an identity involving such functions that we describe now (Theorem 3 below) and Theorem 2 will be obtained from a simple modification of the techniques introduced to prove Theorem 3.

A fundamental identity for deformations of vectorial modular forms. To present Theorem 3, we need to introduce more tools. Let Ω be the rigid analytic space $\mathbb{C}_\infty \setminus K_\infty$. For $z \in \Omega$, we denote by Λ_z the A -module $A + zA$, free of rank 2. The evaluation at $\zeta \in \mathbb{C}_\infty$ of the *exponential function* e_{Λ_z} associated to the lattice Λ_z is given by the series

$$e_{\Lambda_z}(\zeta) = \sum_{i=0}^{\infty} \alpha_i(z) \zeta^{q^i}, \quad (5)$$

for functions $\alpha_i : \Omega \rightarrow \mathbb{C}_\infty$ with $\alpha_0 = 1$. We recall that for $i > 0$, α_i is a *Drinfeld modular form* of weight $q^i - 1$ and type 0 in the sense of Gekeler, [6].

We also recall from [16] the series:

$$\begin{aligned} \mathbf{s}_1(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t}, \\ \mathbf{s}_2(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}, \end{aligned}$$

which converge for $|t| < q$ and define two functions $\Omega \rightarrow \mathbb{C}_{\infty}[[t]]$ with the series in the image converging for $|t| < q$. We point out that for a fixed choice of $z \in \Omega$, the matrix function ${}^t(\mathbf{s}_1(z, t), \mathbf{s}_2(z, t))$ is the *canonical rigid analytic trivialisation* of the t -motive associated to the lattice Λ_z discussed in [15]. We set, for $i = 1, 2$:

$$\mathbf{d}_i(z, t) := \tilde{\pi}_{s_{\text{Car}}}(t)^{-1} \mathbf{s}_i(z, t),$$

remembering that in the notations of [16], we have $\mathbf{d}_2 = \mathbf{d}$. The advantage of using these functions instead of the \mathbf{s}_i 's, is that evaluation at $t = \theta$ makes sense, and we can check:

$$\mathbf{d}_1(z, \theta) = z, \quad \mathbf{d}_2(z, \theta) = 1. \quad (6)$$

On the other hand, both the series

$$\mathbf{e}_1(z, t) = \sum'_{c, d \in A} \frac{\chi_t(c)}{cz + d}, \quad \mathbf{e}_2(z, t) = \sum'_{c, d \in A} \frac{\chi_t(d)}{cz + d}$$

converge for $(z, t) \in \Omega \times B_q$, where the dash ' denotes a sum avoiding the couple $(c, d) = (0, 0)$, B_r denotes the open disk of center 0 and radius $r > 0$. The series $\mathbf{e}_1, \mathbf{e}_2$ define functions $\Omega \rightarrow \mathbb{C}_{\infty}[[t]]$ such that all the series in the images converge over B_q .

Let $\mathbf{Hol}(\Omega)$ be the ring of holomorphic functions $\Omega \rightarrow \mathbb{C}_{\infty}$, over which the Frobenius \mathbb{F}_q -linear map τ is well defined: if $f \in \mathbf{Hol}(\Omega)$, then $\tau f = f^q$. We consider the unique $\mathbb{F}_q((t))$ -linear extension of τ :

$$\mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)) \rightarrow \mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)),$$

again denoted by τ .

We shall prove the fundamental theorem:

Theorem 3 *The following identities hold in the domain $\Omega \times B_q$:*

$$\mathbf{e}_1 = -\tau(s_{\text{Car}} \mathbf{d}_2) h \quad \mathbf{e}_2 = \tau(s_{\text{Car}} \mathbf{d}_1) h. \quad (7)$$

In the statement of the theorem, h is the opposite of the unique normalised Drinfeld cusp form of weight $q + 1$ and type 1 for $\Gamma = \mathbf{GL}_2(A)$ as in [6].

Let us write $\mathcal{E} = L(\chi_t, 1)^{-1}(\mathbf{e}_1, \mathbf{e}_2)$ ⁽¹⁾ and $\mathcal{F} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$, and let us consider the representation $\rho_t : \mathbf{GL}_2(A) \rightarrow \mathbf{GL}_2(\mathbb{F}_q[[t]])$ defined by

$$\rho_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix},$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(A)$. Then, for any such a choice of γ we have the functional equations:

$$\begin{aligned} \mathcal{F}(\gamma(z)) &= (cz + d)^{-1} \rho_t(\gamma) \cdot \mathcal{F}(z), \\ {}^t \mathcal{E}(\gamma(z)) &= (cz + d) {}^t \rho_t^{-1}(\gamma) \cdot {}^t \mathcal{E}(z). \end{aligned}$$

¹We will also adopt the notation $\mathcal{E}_{1,0} = \mathcal{E}$.

This puts right away the functions ${}^t\mathcal{E}$ and \mathcal{F} in the framework of *deformations of vectorial modular forms*, topic that will be developed in Section 2.

We will make use of a remarkable sequence $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$ of functions $\Omega \times B_q \rightarrow \mathbb{C}_\infty$ defined by the scalar product (with component-wise action of τ):

$$\mathcal{G}_k = \mathcal{G}_{1,0,k} = (\tau^k \mathcal{E}) \cdot \mathcal{F},$$

such that, for $k \geq 0$, \mathcal{G}_k belongs to $M_{q^{k-1},0} \otimes \mathbb{F}_q[t, \theta]$ where $M_{w,m}$ denotes the \mathbb{C}_∞ -vector space of Drinfeld modular forms of weight w and type m . In fact, only the terms $\mathcal{G}_0, \mathcal{G}_1$ are needed to prove our Theorem. Once their explicit computation is accomplished (see Proposition 13), the proof of Theorem 3 is only a matter of solving a non-homogeneous system in two equations and two indeterminates e_1, e_2 . Furthermore, Theorem 1 will be deduced with the computation of a limit in the identity $e_1 = -\tau(s_{\text{Car}} \mathbf{d}_2)h$ and a similar path will be followed for Theorem 2, by using this time the sequences $\mathcal{G}_{\alpha,0,k} = (\tau^k \mathcal{E}_{\alpha,0}) \cdot \mathcal{F}$ defined later.

We end this introduction by pointing out that the sequence \mathcal{G} has itself several interesting features. For example, the functions \mathcal{G}_k already appeared in [16] (they are denoted by g_k^* there) as the coefficients of the ‘‘cocycle terms \mathbf{L}_γ ’’ of the functional equations of the *deformations of Drinfeld quasi-modular forms* $\tau^k \mathbf{E}$ introduced there.

It is also interesting to notice that the *deformation of Legendre’s identity* (10) that we quote here (proved in [16]):

$$h^{-1}(\tau s_{\text{Car}})^{-1} = \mathbf{d}_1(\tau \mathbf{d}_2) - \mathbf{d}_2(\tau \mathbf{d}_1)$$

can be deduced from Theorem 3 by using the fact that $\mathcal{G}_0 = -1$ obtained in Proposition 13.

Moreover, it can be proved, again with the help of the theory of τ -linear recurrent sequences and τ -linearised recurrent sequences (they will not be described here), that for $k \geq 0$, the function $\mathcal{G}_k(z, \theta)$, well defined, is equal to the *ortho-Eisenstein* series $g_k(z)$, and that $\mathcal{G}_k(z, \theta^{q^k})$, also well defined, is equal to the *para-Eisenstein* series $m_k(z)$, in the notations and the terminology of [8]. Hence, the sequence \mathcal{G} provides an interesting tool also in the study of both these kinds of functions. This program will be however pursued in another paper.

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2 Vectorial modular forms and their deformations

In this paper, t will be an indeterminate independent on θ or a parameter varying in \mathbb{C}_∞ , and we will freely switch from formal series to functions.

For a positive real number r , we denote by $\mathbb{T}_{<r}$ the sub- \mathbb{C}_∞ -algebra of $\mathbb{C}_\infty[[t]]$ whose elements are formal series $\sum_{i \geq 0} c_i t^i$ that converge for any $t \in \mathbb{C}_\infty$ with $|t| < r$. We also denote by \mathbb{T}_∞ the sub- \mathbb{C}_∞ -algebra of series that converge everywhere in \mathbb{C}_∞ . If $r_1 > r_2 > 0$, we have

$$\mathbb{T}_{<r_2} \supset \mathbb{T}_{<r_1} \supset \mathbb{T}_\infty.$$

The *Tate algebra* of formal series of $\mathbb{C}_\infty[[t]]$ converging for all t such that $|t| \leq 1$ will be denoted by \mathbb{T}_1 or \mathbb{T} ; it is contained in $\mathbb{T}_{<1}$ and contains $\mathbb{T}_{<1+\epsilon}$ for all $\epsilon > 0$; clearly, $\mathbb{C}_\infty[[t]] \supset \mathbb{T}_1 \supset \mathbb{T}_\infty$.

The ring $\mathbb{C}_\infty[[t]]$ is endowed with the $\mathbb{F}_q[[t]]$ -linear automorphism τ acting on formal series as follows:

$$\tau \sum_i c_i t^i = \sum_i c_i^q t^i.$$

This automorphism induces automorphisms of $\mathbb{T}_1, \mathbb{T}_\infty$.

We will work with certain functions $f : \Omega \times B_r \rightarrow \mathbb{C}_\infty$ with the property that for all $z \in \Omega$, $f(z, t)$ can be identified with an element of $\mathbb{T}_{<r}$ in the obvious way. For such functions we will then also write $f(z)$ to stress the dependence on $z \in \Omega$ when we want to consider them as functions $\Omega \rightarrow \mathbb{T}_{<r}$ for some r . Sometimes, we will not specify the variables z, t and just write f instead of $f(z, t)$ or $f(z)$ to lighten our formulas just as we did in some places of the introduction. Moreover, z will denote a variable in Ω all along the paper.

In all the following, $\mathbf{Hol}(\Omega)$ denotes the ring of holomorphic functions on Ω . For r a positive real number, let us denote by $\mathcal{R}_{<r}$ (resp. \mathcal{R} or \mathcal{R}_1) the (integral) ring whose elements are the formal series $f = \sum_{i \geq 0} f_i t^i$, such that

1. For all i , f_i is a map $\Omega \rightarrow \mathbb{C}_\infty$ belonging to $\mathbf{Hol}(\Omega)$.
2. For all $z \in \Omega$, $\sum_{i \geq 0} f_i(z) t^i$ is an element of $\mathbb{T}_{<r}$ (resp. \mathbb{T}).

We shall write

$$\mathcal{R}_\infty = \bigcap_{r > 0} \mathcal{R}_{<r}$$

and allow r to vary in $\mathbb{R}_{>0} \cup \{\infty\}$. The rings \mathcal{R} and \mathcal{R}_∞ are endowed with injective endomorphisms τ acting on formal series as follows:

$$\tau \sum_{i \geq 0} f_i(z) t^i = \sum_{i \geq 0} f_i(z) q^i t^i.$$

We end this preparatory section with some conventions on u -expansions. We will say that a series $\sum_{i \geq i_0} c_i u^i$ (with the coefficients c_i in some ring) is *normalised*, if $c_{i_0} = 1$. We will also say that the series is *of type* $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ if $i \not\equiv m \pmod{q-1}$ implies $c_i = 0$. This definition is obviously compatible with the notion of type of a Drinfeld modular form already mentioned in the introduction.

2.1 Deformations of vectorial modular forms.

In this subsection we develop some tools in the theory of *deformations of vectorial modular forms*. This part is inspired by works about vectorial modular forms for $\mathbf{SL}_2(\mathbb{Z})$ by Knopp, Mason [11, 13].

Let us consider a representation

$$\rho : \Gamma \rightarrow \mathbf{GL}_s(\mathbb{F}_q((t))). \tag{8}$$

We assume that the determinant representation $\det(\rho)$ is the μ -th power of the determinant character, for some $\mu \in \mathbb{Z}/(q-1)\mathbb{Z}$. In all the following, given $\gamma \in \Gamma$, we denote by J_γ the associated factor of automorphy $(\gamma, z) \mapsto cz + d$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 4 A *deformation of vectorial modular form* of weight w , dimension s , type m and radius $r \in \mathbb{R}_{>0} \cup \{\infty\}$ associated with a representation ρ as in (8) is a column matrix $\mathcal{F} \in \mathbf{Mat}_{s \times 1}(\mathcal{R}_{<r})$ such that, considering \mathcal{F} as a map $\Omega \rightarrow \mathbf{Mat}_{s \times 1}(\mathbb{T}_{<r})$ we have, for all $\gamma \in \Gamma$,

$$\mathcal{F}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} \rho(\gamma) \cdot \mathcal{F}(z).$$

The definition means that if the radius is ∞ , then the entries of \mathcal{F} are in \mathcal{R}_∞ .

The set of deformations of vectorial modular forms of weight w , dimension s , type m and radius r associated to a representation ρ is a $\mathbb{T}_{<r}$ -module (or \mathbb{T}_∞ -module if $r = \infty$) that we will denote by $\mathcal{M}_{w,m}^s(\rho, r)$ or $\mathcal{M}_{w,m}^s(\rho)$ when the reference to a particular radius is clear.

Let us denote by $M_{w,m}^1$ the \mathbb{C}_∞ -vector space (of infinite dimension) generated by quotients f/g with $f \in M_{w',m'}$, $g \in M_{w'',m''} \setminus \{0\}$ such that $w' - w'' = w$, $m' - m'' = m$, and such that g does not vanish on Ω (so that it is a power of h).

If $s = 1$ and if $\rho = \mathbf{1}$ is the constant map, then $\mathcal{M}_{w,m}^1(\mathbf{1}, r) = M_{w,m}^1 \otimes \mathbb{T}_{<r}$. Similarly, for general s , we have a graded $M_{w,m}^1 \otimes \mathbb{T}_{<r}$ -module

$$\mathcal{M}^s(\rho, r) = \bigoplus_{w,m} \mathcal{M}_{w,m}^s(\rho, r).$$

Lemma 5 *Let k be a non-negative integer and $r \geq 1$. If \mathcal{F} is in $\mathcal{M}_{w,m}^s(\rho, r)$, then $\tau^k \mathcal{F} \in \mathcal{M}_{wq^k, m}^s(\rho, r^{q^k})$. Therefore, if we choose nonnegative integers k_1, \dots, k_s , then*

$$\det(\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}) \in M_{w(q^{k_1} + \dots + q^{k_s}), sm + \mu}^1 \otimes \mathbb{T}_{<r}.$$

In particular,

$$W_\tau(\mathcal{F}) = \det(\tau^0 \mathcal{F}, \dots, \tau^{s-1} \mathcal{F}) \in M_{w(1+q+q^2+\dots+q^{s-1}), sm + \mu}^1 \otimes \mathbb{T}_{<r}.$$

Proof. From the definition,

$$(\tau^k \mathcal{F})(\gamma(z)) = J_\gamma^{wq^k} \det(\gamma)^{-m} \rho(\gamma) (\tau^k \mathcal{F})$$

because $\tau(\rho(\gamma)) = \rho(\gamma)$ and the first part of the lemma holds. Now define the matrix function:

$$\mathbf{M}_{k_1, \dots, k_s} = (\tau^{k_1} \mathcal{F}, \dots, \tau^{k_s} \mathcal{F}).$$

After the first part of the lemma we have, for $\gamma \in \mathbf{GL}_2(A)$:

$$\mathbf{M}_{k_1, \dots, k_s}(\gamma(z)) = \det(\gamma)^{-m} \rho(\gamma) \cdot \mathbf{M}_{k_1, \dots, k_s}(z) \cdot \mathbf{Diag}(J_\gamma^{wq^{k_1}}, \dots, J_\gamma^{wq^{k_s}}),$$

from which we can conclude the proof taking determinants of both sides. \square

Lemma 6 *Assume that $r \geq 1$. Let us consider \mathcal{F} in $\mathcal{M}_{w,m}^s(\rho, r)$ and let \mathcal{E} be such that ${}^t \mathcal{E}$ is in $\mathcal{M}_{w',m'}^s({}^t \rho^{-1}, r)$. For nonnegative k , if \mathcal{G}_k denotes the scalar product $(\tau^k \mathcal{E}) \cdot \mathcal{F}$, then,*

$$\mathcal{G}_k \in M_{w+w'q^k, m+m'}^1 \otimes \mathbb{T}_{<r}.$$

Proof. By Lemma 5, $\tau^k({}^t \mathcal{E})$ is in $\mathcal{M}_{wq^k, m'}^s({}^t \rho^{-1}, r)$. Let γ be in $\mathbf{GL}_2(A)$. We thus have, after transposition, that

$$(\tau^k \mathcal{E})(\gamma(z)) = J_\gamma^{wq^k} \det(\gamma)^{-m} \mathcal{E}(z) \cdot \rho^{-1}(\gamma),$$

and

$$\mathcal{F}(\gamma(z)) = J_\gamma^{w'} \det(\gamma)^{-m'} \rho(\gamma) \cdot \mathcal{F}(z).$$

Hence,

$$\mathcal{G}_k(\gamma(z)) = J_\gamma^{wq^k + w'} \det(\gamma)^{-m-m'} \mathcal{G}_k(z),$$

from which we deduce that $\mathcal{G}_k \in M_{wq^k + w', m+m'}^1 \otimes \mathbb{T}_{<r}$. \square

2.2 Main examples of deformations of vectorial modular forms.

From now on, we will use the representation $\rho = \rho_t$ and the transposed of its inverse.

The function \mathcal{F} . This is the vector valued function $\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$ so we recall now some properties of the functions $\mathbf{d}_1, \mathbf{d}_2$. At first sight, we only have $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_{<q}$. However, one sees easily that $s_{\text{Car}}^{-1} \in \mathbb{T}_\infty$ from which it follows that $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_\infty$. The next proposition follows immediately from the results of [16] where some of them are stated in slightly different, although equivalent forms. We write g for the unique normalised Drinfeld modular form of weight $q-1$ and type 0 for Γ (proportional to an Eisenstein series), and Δ for the cusp form $-h^{q-1}$.

Proposition 7 *We have five properties for the \mathbf{d}_i 's.*

1. $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{R}_\infty$.
2. We have $\mathcal{F} \in \mathcal{M}_{-1,0}^2(\rho_t, \infty)$.
3. The functions $\mathbf{d}_1, \mathbf{d}_2$ span the $\mathbb{F}_q(t)$ -vector space of dimension 2 of solutions of the following τ -linear difference equation:

$$X = (t - \theta^q)\Delta\tau^2 X + g\tau X, \quad (9)$$

in a suitable existentially closed inversive field containing \mathcal{R}_∞ .

4. Let us consider the matrix function:

$$\Psi(z, t) := \begin{pmatrix} \mathbf{d}_1(z, t) & \mathbf{d}_2(z, t) \\ \mathbf{d}_1^{(1)}(z, t) & \mathbf{d}_2^{(1)}(z, t) \end{pmatrix}.$$

For all $z \in \Omega$ and t with $|t| < q$:

$$\det(\Psi) = (t - \theta)^{-1}h(z)^{-1}s_{\text{Car}}(t)^{-1}. \quad (10)$$

5. We have the series expansion

$$\mathbf{d}_2 = \sum_{i \geq 0} c_i(t)u^{(q-1)i} \in 1 + u^{q-1}\mathbb{F}_q[t, \theta][[u^{q-1}]], \quad (11)$$

convergent for t, u sufficiently close to $(0, 0)$.

Deformations of vectorial Poincaré series. Following [6], we consider the subgroup

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

of $\Gamma = \mathbf{GL}_2(A)$ and its left action on Γ . For $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the map $\delta \mapsto (c, d)$ induces a bijection between the orbit set $H \backslash \Gamma$ and the set of $(c, d) \in A^2$ with c, d relatively prime.

We consider the factor of automorphy

$$\mu_{\alpha, m}(\delta, z) = \det(\delta)^{-m} J_\gamma^\alpha,$$

where m and α are positive integers (later, m will also determine a type, that is, a class modulo $q-1$).

Let $V_1(\delta)$ be the row matrix $(\chi_t(c), \chi_t(d))$. It is easy to show that the row matrix

$$\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta)$$

only depends on the class of $\delta \in H \backslash \Gamma$, so that we can consider the following expression:

$$\mathcal{E}_{\alpha,m}(z) = \sum_{\delta \in H \backslash \Gamma} \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta),$$

which is a row matrix whose two entries are formal series.

Let \mathcal{V} be the set of functions $\Omega \rightarrow \mathbf{Mat}_{1 \times 2}(\mathbb{C}_\infty[[t]])$. We introduce, for α, m integers, $f \in \mathcal{V}$ and $\gamma \in \Gamma$, the Petersson slash operator:

$$f|_{\alpha,m}\gamma = \det(\gamma)^m (cz + d)^{-\alpha} f(\gamma(z)) \cdot \rho_t(\gamma).$$

This will be used in the next proposition, where we recall that $\log_q^+(x)$ denotes the maximum between 0 and $\log_q(x)$, the logarithm in base q of $x > 0$. We point out that we will not apply this proposition in full generality.

Proposition 8 *Let α, m be non-negative integers with $\alpha \geq 2m + 1$, and write $r(\alpha, m) = \alpha - 2m - 1$. We have the following properties.*

1. For $\gamma \in \Gamma$, the map $f \mapsto f|_{\alpha,m}\gamma$ induces a permutation of the subset of \mathcal{V} :

$$\mathcal{S} = \{\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta); \delta \in H \backslash \Gamma\}.$$

2. If $t \in \mathbb{C}_\infty$ and α, m are chosen so that $r(\alpha, m) > \log_q^+ |t|$, then the components of $\mathcal{E}_{\alpha,m}(z, t)$ are series of functions of $z \in \Omega$ which converge absolutely and uniformly on every compact subset of Ω to holomorphic functions.
3. If $|t| < 1$, then the components of $\mathcal{E}_{\alpha,m}(z, t)$ converge absolutely and uniformly on every compact subset of Ω also if $\alpha - 2m > 0$.
4. For any choice of α, m, t submitted to the convergence conditions above, the function ${}^t\mathcal{E}_{\alpha,m}(z, t)$ belongs to the space $\mathcal{M}_{\alpha,m}^2({}^t\rho_t^{-1}, r(\alpha, m))$.
5. If $\alpha - 1 \not\equiv 2m \pmod{(q-1)}$, the matrix function $\mathcal{E}_{\alpha,m}(z, t)$ is identically zero.
6. If $\alpha - 1 \equiv 2m \pmod{(q-1)}$, $\alpha \geq (q+1)m + 1$ so that $\mathcal{E}_{\alpha,m}$ converges, then $\mathcal{E}_{\alpha,m}$ is not identically zero in its domain of convergence.

Proof. 1. We choose $\delta \in H \backslash \Gamma$ corresponding to a couple $(c, d) \in A^2$ with c, d relatively prime, and set $f_\delta = \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta) \in \mathcal{S}$. We have

$$\begin{aligned} f_\delta(\gamma(z)) &= \mu_{\alpha,m}(\delta, \gamma(z))^{-1} u^m(\delta(\gamma(z))) V_1(\delta) \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta), \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta\gamma) \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta') \cdot \rho_t(\gamma)^{-1}, \\ &= \mu_{\alpha,m}(\gamma, z) f_{\delta'} \cdot \rho_t(\gamma)^{-1}, \end{aligned}$$

with $\delta' = \delta\gamma$ and $f_{\delta'} = \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta')$, from which part 1 of the proposition follows.

2. Convergence and holomorphy are ensured by simple modifications of [6, (5.5)], or by the arguments in [9, Chapter 10]. More precisely, let us choose $0 \leq s \leq 1$ and look at the component at the place $s + 1$

$$\mathcal{E}_s(z, t) = \sum_{\delta \in H \setminus \Gamma} \mu_{\alpha, m}(\delta, z)^{-1} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

of the vector series $\mathcal{E}_{\alpha, m}$. Writing $\alpha = n(q-1) + 2m + l'$ with n non-negative integer and $l' \geq 1$ we see, following Gerritzen and van der Put, [9, pp. 304-305] and taking into account the inequality $|u(\delta(z))| \leq |cz + d|^2 / |z|_i$ ($|z|_i$ denotes, for $z \in \mathbb{C}_\infty$, the infimum $\inf_{a \in K_\infty} \{|z - a|\}$), that the term of the series \mathcal{E}_s :

$$\mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}) = (cz + d)^{-n(q-1) - l' - 2m} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

(where δ corresponds to (c, d)) has absolute value bounded from above by

$$|z|_i^{-m} \left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right|.$$

Applying the first part of the proposition, to check convergence, we can freely substitute z with $z + a$ with $a \in A$ and we may assume, without loss of generality, that $|z| = |z|_i$. We verify that, either $\lambda = \deg_\theta z = \max(\mathcal{I}) \in \mathbb{Q} \setminus \mathbb{Z}$, or $\lambda \in \mathbb{Z}$ case in which $c_\lambda \notin \mathbb{F}_q$. In both cases, for all $c, d, |cz + d| = \max\{|cz|, |d|\}$. Then, the series defining \mathcal{E}_s can be decomposed as follows:

$$\mathcal{E}_s = \sum_{f_\delta \in H \setminus \Gamma} f_\delta = \left(\sum'_{|cz| < |d|} + \sum'_{|cz| \geq |d|} \right) \mu_{\alpha, m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}).$$

We now look for upper bounds for the absolute values of the terms of the series above separating the two cases in a way similar to that of Gerritzen and van der Put in loc. cit.

Assume first that $|cz| < |d|$, that is, $\deg_\theta c + \lambda < \deg_\theta d$. Then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa \max\{1, |t|\}^{\deg_\theta d} |d|^{-n(q-1) - l'} \leq \kappa q^{\deg_\theta d (\log_q^+ |t| - n(q-1) - l')},$$

where κ is a constant depending on λ , and the corresponding sub-series converges with the imposed conditions on the parameters, because $\log_q^+ |t| - n(q-1) - l' < 0$.

If on the other side $|cz| \geq |d|$, that is, $\deg_\theta c + \lambda \geq \deg_\theta d$, then

$$\left| \frac{\chi_t(c^s d^{1-s})}{(cz + d)^{n(q-1) + l'}} \right| \leq \kappa' \max\{1, |t|\}^{\deg_\theta d} |c|^{-n(q-1) - l'} \leq \kappa' q^{\deg_\theta c (\log_q^+ |t| - n(q-1) - l')},$$

with a constant κ' depending on λ , again because $\log_q^+ |t| - n(q-1) - l' < 0$. This completes the proof of the second part of the Proposition.

3. This property can be deduced from the proof of the second part because if $\log_q |t| < 0$, then $|\chi_t(c^s d^{1-s})| \leq 1$.

4. The property is obvious by the first part of the proposition, because $\mathcal{E}_{\alpha, m} = \sum_{f \in \mathcal{S}} f$.

5. We consider $\gamma = \mathbf{Diag}(1, \lambda)$ with $\lambda \in \mathbb{F}_q^\times$; the corresponding homography, multiplication by λ^{-1} , is equal to that defined by $\mathbf{Diag}(\lambda^{-1}, 1)$. Hence, we have:

$$\begin{aligned} \mathcal{E}_{\alpha, m}(\gamma(z)) &= \lambda^{\alpha - m} \mathcal{E}_{\alpha, m}(z) \cdot \mathbf{Diag}(1, \lambda^{-1}) \\ &= \lambda^m \mathcal{E}_{\alpha, m}(z) \cdot \mathbf{Diag}(\lambda, 1), \end{aligned}$$

from which it follows that $\mathcal{E}_{\alpha,m}$ is identically zero if $\alpha - 1 \not\equiv 2m \pmod{q-1}$.

6. If $m = 0$, we postpone the proof to Lemma 10. Assuming now that $m > 0$, the series $z\mathcal{E}_1(z, \theta) + \mathcal{E}_0(z, \theta)$ converges to the Poincaré series of weight $\alpha - 1$ for Γ so that [9, Proposition 10.5.2] suffices for our purposes. \square

Let α, m be non-negative integers such that $\alpha - 2m > 1$ and $\alpha - 1 \equiv 2m \pmod{q-1}$. We have functions:

$$\begin{aligned}\mathcal{E}_{\alpha,m} : \Omega &\rightarrow \mathbf{Mat}_{1 \times 2}(\mathcal{R}_{<r}), \\ \mathcal{F} : \Omega &\rightarrow \mathbf{Mat}_{2 \times 1}(\mathcal{R}_{\infty}),\end{aligned}$$

with $r = r(\alpha, m)$ as in Proposition 8, and ${}^t\mathcal{E}_{\alpha,m} \in \mathcal{M}_{\alpha,m}^2({}^t\rho_t^{-1}, r)$, $\mathcal{F} \in \mathcal{M}_{-1,0}^2(\rho_t, \infty)$. Therefore, after Lemma 6, the functions

$$\mathcal{G}_{\alpha,m,k} = (\tau^k \mathcal{E}_{\alpha,m}) \cdot \mathcal{F} = \mathcal{E}_{q^k \alpha, m} \cdot \mathcal{F} : \Omega \rightarrow \mathbb{T}_{<r}$$

satisfy $\mathcal{G}_{\alpha,m,k} \in M_{q^k \alpha - 1, m}^1 \otimes \mathbb{T}_r$.

A special case. After Proposition 8, if $\alpha > 0$ and $\alpha \equiv 1 \pmod{q-1}$, then $\mathcal{E}_{\alpha,0} \neq 0$. We call these series *deformations of vectorial Eisenstein series*.

Lemma 9 *With $\alpha > 0$ such that $\alpha \equiv 1 \pmod{q-1}$, the following identity holds:*

$$\mathcal{E}_{\alpha,0}(z, t) = L(\chi_t, \alpha)^{-1} \sum'_{c,d} (cz + d)^{-\alpha} V_1(c, d),$$

and $\mathcal{E}_{\alpha,0}$ is not identically zero.

Proof. We recall the notation

$$V_1(c, d) = (\chi_t(c), \chi_t(d)) \in \mathbf{Mat}_{1 \times 2}(\mathbb{F}_q[t]).$$

We have

$$\begin{aligned}\sum'_{c,d} (cz + d)^{-\alpha} V_1(c, d) &= \sum_{(c',d')=1} \sum_{a \in A^+} a^{-\alpha} (c'z + d')^{-\alpha} V_1(ac', ad') \\ &= L(\chi_t, \alpha) \mathcal{E}_{\alpha,0}(z, t),\end{aligned}$$

where the first sum is over couples of A^2 distinct from $(0,0)$, while the second sum is over the couples (c', d') of relatively prime elements of A^2 . Non vanishing of the function follows from Proposition 8. \square

3 Proof of the Theorems

Following Gekeler [6, Section 3], we recall that for all $\alpha > 0$ there exists a polynomial $G_\alpha(u) \in \mathbb{C}_\infty[u]$, called the α -th *Goss polynomial*, such that, for all $z \in \Omega$, $G_\alpha(u(z))$ equals the sum of the convergent series

$$\tilde{\pi}^{-\alpha} \sum_{a \in A} \frac{1}{(z+a)^\alpha}.$$

Several properties of these polynomials are collected in [6, Proposition (3.4)]. Here, we will need that for all α , G_α is of type α as a formal series of $\mathbb{C}_\infty[[u]]$. Namely:

$$G_\alpha(\lambda u) = \lambda^\alpha G_\alpha(u), \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We also recall, for $a \in A$, the function

$$u_a(z) := u(az) = e_{\text{Car}}(\tilde{\pi}az)^{-1} = u^{|\alpha|} f_a(u) = u^{|\alpha|} + \cdots \in A[[u]],$$

where $f_a \in A[[u]]$ is the a -th inverse cyclotomic polynomial defined in [6, (4.6)]. Obviously, we have

$$u_{\lambda a} = \lambda^{-1} u_a \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

To continue, we will state and prove three auxiliary lemmas.

Lemma 10 *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. We have, for all $t \in \mathbb{C}_\infty$ such that $|t| < 1$ and $z \in \Omega$, convergence of the series below, and equality:*

$$\sum'_{c,d \in A} \frac{\chi_t(c)}{(cz+d)^\alpha} = -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c(z)),$$

from which it follows that $\mathcal{E}_{\alpha,0} \neq 0$.

Proof. Convergence features are easy to deduce from Proposition 8. Indeed, we have convergence if $\log_q^+ |t| < r(\alpha, m) = \alpha - 1$, that is, $\max\{1, |t|\} \leq q^{\alpha-1}$ if $\alpha > 1$ and we have convergence, for $\alpha = 1$, for $|t| < 1$. In all cases, convergence holds for $|t| < 1$.

We then compute:

$$\begin{aligned} \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} &= \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(cz+d)^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(c\tilde{\pi}z + d\tilde{\pi})^\alpha} \\ &= \tilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) G_\alpha(u_c) \\ &= \tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{1-\alpha} \\ &= -\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c). \end{aligned}$$

The non-vanishing of $\mathcal{E}_{\alpha,0}$ comes from Lemma 9 and the non-vanishing contribution of the term $G_\alpha(u)$ in the latter series. \square

Lemma 11 *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| < 1$, we have*

$$\lim_{|z|=|z| \rightarrow \infty} \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} = 0.$$

Proof. We recall from [16] the series expansion

$$\mathbf{d}_1(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \mathbf{s}_2(z) = \frac{\tilde{\pi}}{s_{\text{Car}}(t)} \sum_{n \geq 0} e_{\Lambda_z} \left(\frac{z}{\theta^{n+1}} \right) t^n,$$

converging for all t such that $|t| < q$ and all $z \in \Omega$.

By a simple modification of the proof of [7, Lemma 5.9 p. 286], we have

$$\lim_{|z|=|z| \rightarrow \infty} u(z) t^n e_{\Lambda_z}(z/\theta^{n+1})^q = 0$$

uniformly in $n > 0$, for all t such that $|t| \leq q$.

Moreover, it is easy to show that

$$\lim_{|z|=|z| \rightarrow \infty} u(z) e_{\Lambda_z}(z/\theta)^q = \tilde{\pi}^{-q} \lim_{|z|=|z| \rightarrow \infty} e_{\text{Car}}^q(\tilde{\pi}z/\theta)/e_{\text{Car}}(\tilde{\pi}z) = 1.$$

This suffices to show that

$$\lim_{|z|=|z| \rightarrow \infty} \mathbf{d}_1(z) G_\alpha(u_c(z)) = 0$$

uniformly for $c \in A^+$, for all t such that $|t| < q$. The lemma then follows from an application of Lemma 10. \square

Lemma 12 *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| < 1$, we have*

$$\lim_{|z|=|z| \rightarrow \infty} \sum'_{c,d} \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t, \alpha).$$

Proof. It suffices to show that

$$\lim_{|z|=|z| \rightarrow \infty} \sum_{c \neq 0} \sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = 0.$$

For all $(c, d) \in A \setminus \{(0, 0)\}$, we have $|cz+d| = \max\{|cz|, |d|\}$. We then have, for $c \neq 0$:

$$\sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = \left(\sum_{|d| > |cz|} + \sum_{|d| \leq |cz|} \right) \frac{\chi_t(d)}{(cz+d)^\alpha}.$$

Now, if $|d| > |cz|$, we have, for $|t| < 1$:

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{d^\alpha} \right| \leq |d|^{-\alpha} \leq |cz|^{-\alpha}.$$

If $|d| \leq |cz|$, we have, again for $|t| < 1$:

$$\left| \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq \left| \frac{\chi_t(d)}{(cz)^\alpha} \right| \leq |cz|^{-\alpha}.$$

Therefore, for $c \neq 0$,

$$\left| \sum_{d \in A} \frac{\chi_t(d)}{(cz+d)^\alpha} \right| \leq |cz|^{-\alpha}.$$

This implies that

$$\left| \sum_{c \neq 0} \sum_{d \in A} \frac{\chi_t(d)}{(cz + d)^\alpha} \right| \leq |z|^{-\alpha},$$

from which the Lemma follows. \square

The next step is to prove the following Proposition.

Proposition 13 *For all $\alpha > 0$ with $\alpha \equiv 1 \pmod{q-1}$, $\mathcal{G}_{\alpha,0,0} \in M_{\alpha-1,0} \otimes \mathbb{T}_{<q}$ and we have the limit $\lim_{|z|=|z|_i \rightarrow \infty} \mathcal{G}_{\alpha,0,0} = -1$.*

Moreover, if $\alpha \leq q(q-1)$, then:

$$\mathcal{G}_{\alpha,0,0} = -E_{\alpha-1},$$

where $E_{\alpha-1}$ is the normalised Eisenstein series of weight $\alpha-1$ for Γ .

Proof. After Lemma 6, the sum of the series:

$$F_\alpha(z, t) := \mathbf{d}_1(z) \sum'_{c,d} \frac{\chi_t(c)}{(cz + d)^\alpha} + \mathbf{d}_2(z) \sum'_{c,d} \frac{\chi_t(d)}{(cz + d)^\alpha}$$

converges on Ω to an element of $M_{\alpha-1,0}^! \otimes \mathbb{T}_{<q}$.

After (11), we have that for all t with $|t| < 1$, $\lim_{|z|=|z|_i \rightarrow \infty} \mathbf{d}_2(z) = 1$. From Lemmas 11 and 12,

$$\lim_{|z|=|z|_i \rightarrow \infty} F_\alpha(z, t) = -L(\chi_t, \alpha).$$

Therefore, for all t such that $|t| < q$, $F_\alpha(z, t)$ converges to an holomorphic function on Ω and is endowed with a u -expansion. In particular, $F_\alpha(z, t)$ is a family of modular forms of $M_{\alpha-1,0} \otimes \mathbb{T}_{<q}$ and we have the first part of the proposition. Since for the selected values of α , $M_{\alpha-1,0} = \langle E_{\alpha-1} \rangle$, we obtain that $F_\alpha = -L(\chi_t, \alpha)E_{\alpha-1}$. After Lemma 9, the proposition follows. \square

Proof of Theorem 3. By Proposition 7, the matrix $M = (\mathcal{F}, \tau^{-1}\mathcal{F})$ is invertible. From (10) we deduce that

$$\tau M^{-1} = (t - \theta)_{s_{\text{Car}}} h \begin{pmatrix} -\mathbf{d}_2 & \mathbf{d}_1 \\ \tau \mathbf{d}_2 & -\tau \mathbf{d}_1 \end{pmatrix}.$$

Therefore, we have the formulas:

$$\begin{aligned} \mathcal{E}_{\alpha,0} &= \\ &= (\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot M^{-1} \\ &= (t - \theta^{1/q}) h^{1/q} (\tau^{-1} s_{\text{Car}}) (\mathcal{G}_{\alpha,0,0}, \tau^{-1}\mathcal{G}_{\alpha,0,1}) \cdot \begin{pmatrix} -\tau^{-1}\mathbf{d}_2 & \tau^{-1}\mathbf{d}_1 \\ \mathbf{d}_2 & -\mathbf{d}_1 \end{pmatrix} \\ &= (t - \theta^{1/q}) h^{1/q} (\tau^{-1} s_{\text{Car}}) (\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_2, -\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_1 + \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_1). \end{aligned} \quad (12)$$

Now, we have, for all $k \in \mathbb{Z}$,

$$\mathcal{G}_{\alpha,0,k} = g\tau\mathcal{G}_{\alpha,0,k-1} + \Delta(t - \theta^q)\tau^2\mathcal{G}_{\alpha,0,k-2}. \quad (13)$$

Applying this formula for $k = 1$ and by using Part 3 of Proposition 7, we obtain

$$\tau^{-1}\mathcal{G}_{\alpha,0,1}\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}\tau^{-1}\mathbf{d}_i = \Delta^{1/q}(t - \theta)((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_i - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_i)), \quad i = 1, 2.$$

Replacing in (12), and using $\Delta^{1/q}h^{1/q} = -h$ and $(t - \theta^{1/q})\tau^{-1}s_{\text{Car}} = s_{\text{Car}}$, we get the formula:

$$\mathcal{E}_{\alpha,0} = (t - \theta)s_{\text{Car}}h(-(\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 + \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2), (\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_1 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_1)). \quad (14)$$

So far, we have not specified the value of α . The general computation of $\mathcal{G}_{\alpha,0,0}$ and $\mathcal{G}_{\alpha,0,-1}$ is a difficult problem, but for $\alpha = 1$ we can apply Proposition 13. We have $\mathcal{G}_{1,0,0} = -1$ and $\mathcal{G}_{1,0,1} = \mathcal{G}_{q,0,0} = -g = -E_{q-1}$. Therefore, $\mathcal{G}_{\alpha,0,-1} = 0$ and the Theorem 3 follows. \square

Proof of Theorem 1. Lemma 10 and (14) imply that

$$L(\chi_t, \alpha) = \frac{\tilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c)}{(\tau s_{\text{Car}})h((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2))}.$$

For $\alpha = 1$, we get

$$L(\chi_t, \alpha) = -\frac{\tilde{\pi} \sum_{c \in A^+} \chi_t(c) u_c}{(\tau s_{\text{Car}}\mathbf{d}_2)h},$$

from which we deduce Theorem 1 and even some additional information, namely, the formula:

$$(\tau\mathbf{d}_2)h = -\sum_{c \in A^+} \chi_t(c) u_c.$$

\square

Proof of Theorem 2. For general α , we know that there exists $\lambda_\alpha \in \mathbb{T}_{<q}$ such that

$$\sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) = \lambda_\alpha h((\tau\mathcal{G}_{\alpha,0,-1})\mathbf{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\mathbf{d}_2)). \quad (15)$$

Let us write f for the series $\sum_{c \in A^+} \chi_t(c) G_\alpha(u_c)$, ϕ for $\lambda_\alpha h \tau\mathcal{G}_{\alpha,0,-1}$ and ψ for $-\lambda_\alpha h \mathcal{G}_{\alpha,0,0}$, so that (15) becomes:

$$f = \phi\mathbf{d}_2 + \psi\tau\mathbf{d}_2.$$

We know, from the fact that h is in $M_{q+1,1}$, that $\phi \in M_{\alpha+q,1} \otimes \mathbb{T}_{<q}$. By (13), we also see that $\psi \in M_{\alpha+1,1}^! \otimes \mathbb{T}_{<q}$. Let L be an algebraically closed field containing $\mathbb{T}_{<q}$. As for any choice of w, m , $M_{w,m}^!$ embeds in $\mathbb{C}_\infty((u))$ and there is a basis of this space with u -expansions defined over K , $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$ acts on $M_{w,m}^! \otimes \mathbb{T}_{<q}$ through the coefficients of the u -expansions. Let σ be an element of $\mathbf{Aut}(L/\mathbb{F}_q(t, \theta))$ and, for $\mu \in M_{w,m}^! \otimes \mathbb{T}_{<q}$, let us denote by $\mu^\sigma \in M_{w,m}^! \otimes \mathbb{T}_{<q}$ the form obtained applying σ on every coefficient of the u -expansion of μ .

Since f, \mathbf{d}_2 and $\tau\mathbf{d}_2$ are defined over $\mathbb{F}_q(t, \theta)$, we get

$$f = \phi^\sigma \mathbf{d}_2 + \psi^\sigma \tau\mathbf{d}_2.$$

Assume that $\phi^\sigma \neq \phi$ or $\psi^\sigma \neq \psi$. We can suppose in fact that $\phi^\sigma \neq \phi$ and $\psi^\sigma \neq \psi$. Then $\tau\mathbf{d}_2/\mathbf{d}_2 \in M_{q-1,0}^! \otimes \mathbb{T}_{<q}$ which is impossible after the results of [16] (there, it is even proven that the functions $\mathbf{d}_2, \tau\mathbf{d}_2, g$ and h are algebraically independent over $C((t))$). This means that the u -expansions of ϕ, ψ are defined over $\mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$ for some $s \geq 0$. By the fact that the constant coefficients of $\mathcal{G}_{\alpha,0,0}$ and $\tau\mathcal{G}_{\alpha,0,-1}$ are both equal to -1 (this follows from the first part of Proposition 13), we get that $\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$.

We have proven that $\tilde{\pi}^{-\alpha} L(\chi_t, \alpha)(t - \theta)s_{\text{Car}} \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$. But we already know that $L(\chi_t, \alpha) \in K_\infty[[t]]$, $s_{\text{Car}} \in K^{\text{sep}}[[t]]$ (separable closure), and $\tilde{\pi} \in K_\infty^{\text{sep}}$. Therefore,

$$\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s}) \cap K_\infty^{\text{sep}}((t)) = \mathbb{F}_q(t, \theta).$$

\square

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