# Hyperbolic Eisenstein series for geometrically finite hyperbolic surfaces of infinite volume.

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## Introduction.

The spectrum of the Laplace–Beltrami operator for a compact Riemann surface is discrete, but this is no longer the case when $M$ is non-compact. For example, when we remove one point from $M$, there arises a continuous part in the spectrum whose spectral measure is described by an Eisenstein series. The study of the limiting behaviour of the spectrum of the Laplace–Beltrami operator for a degenerating family of Riemann surfaces with finite area hyperbolic metric has been used to explain this phenomenon (see for example [27], [17], [14]). The present paper has one of its motivations in the general study of the approximation of Eisenstein series (see for example the question of Ji in [17], p. 308, line 15, concerning the approximation of Eisenstein series by suitable eigenfunctions of a degenerating family of hyperbolic Riemann surfaces). We hope to surround it via hyperbolic Eisenstein series (for results on degenerating Eisenstein series, see, for example [22], [23], [8], [9]). This article is organized as follows. In section 1 we provide the necessary background on geometrically finite hyperbolic surfaces of infinite volume. In section 2 we recall the definition and verify the convergence of hyperbolic Eisenstein series in the infinite volume case, like suggested in [20]. In section 3 we review the spectral decomposition for a geometrically finite hyperbolic surface of infinite volume. We obtain the analytic continuation of hyperbolic Eisenstein series and then the fact that this permits realizing a harmonic dual form to a simple closed geodesic on a geometrically finite hyperbolic surface of infinite volume (Theorem 3.1). In a similar way, in section 4, we realize a harmonic dual form to an infinite geodesic joining a pair of punctures (Theorem 4.2). Moreover in Section 5 we generalize the definition of hyperbolic Eisenstein series to the case of $q-$forms (see Section 5 formula (18)). After we study the limiting behavior of these $q-$forms on a degenerating family of geometrically finite hyperbolic surfaces of infinite volume (see Theorem 5.2 and Theorem 5.3). In particular we obtain a degeneration of hyperbolic Eisenstein series to horocyclic ones (Theorem 5.3(2)). Since this new result is at the heart of our motivation, we will be a little more precise.

**Main Theorem**

Let $(S_l)_l$ be a degenerating family of Riemann surfaces with infinite area hyperbolic metrics, $S_l$ having a funnel $F_l$ whose boundary geodesic is denoted $c_l$. We denote by $\Omega_{c_l} = \Omega_l$ the hyperbolic Eisenstein series of the Kudla–Millson theory associated to the pinching geodesic $c_l$ (see Section 2), $S_0 = \Gamma\backslash H$ the component of the limiting surface containing the cusp $\infty$ of stabilizer $\Gamma_\infty$ (see Section 5.1) and $\mathcal{E}_\infty$ the horocyclic Eisenstein series associated to the limiting cusp defined, for $\operatorname{Re} s > 0$ by $\mathcal{E}_\infty(s+1, z) = \sum_{\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\sigma z)^s \; d(\sigma z)$ (see (12)). For $s \in \{\operatorname{Re} s > -1/2\}$, except

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possibly a finite number of points in $(-1/2, 0)$, the family of $C^\infty$ 1-forms on $S_l$ depending meromorphically on $s$, $\frac{1}{l^{s+1}}\Omega_l(s,.)$ converges uniformly on compact subsets of $S_0$ to $\frac{\Gamma(1+\frac{s}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{s}{2})}\operatorname{Im}\mathcal{E}_\infty(s+1,.)$; where $\operatorname{Im}\mathcal{E}_\infty(s+1,.)$ is to be understood as $\operatorname{Im}\mathcal{E}_\infty(s+1,z) = \frac{1}{2i}(\sum_{\Gamma_\infty\backslash\Gamma}\operatorname{Im}(\sigma z)^s\ d(\sigma z) - \sum_{\Gamma_\infty\backslash\Gamma}\operatorname{Im}(\sigma z)^s\ d\overline{(\sigma z)})$.

## 1. Preliminary definitions

Let us recall the standard analytic and geometric notations which will be used. In this paper, a surface is a connected orientable two-dimensional manifold, without boundary unless otherwise specified. We denote by $H$ the hyperbolic upper half-plane endowed with its standard metric of constant Gaussian curvature $-1$. A topologically finite (i.e. finite Euler characteristic) surface is a surface homeomorphic to a compact surface with finitely many points excised and a geometrically finite hyperbolic surface $M$ is a topologically finite, complete Riemann surface of constant curvature -1. We will require that $M$ be of infinite volume, i.e. there exists a finitely generated, torsion free, discrete subgroup $\Gamma$ of $PSL(2,\mathbb{R})$, unique up to conjugation, such that $M$ is the quotient of $H$ by $\Gamma$ acting as Möbius transformations, $\Gamma$ is a Fuchsian group of the second kind, and $\Gamma$ has no elliptic elements. The group $\Gamma$ admits a finite sided polygonal fundamental domain in $H$. We recall now the description of the fundamental domain of $M = \Gamma\backslash H$ (see [1]). Let $L(\Gamma)$ be the limit set of $\Gamma$, that is the set of limit points (in the Riemann sphere topology) of all orbits $\Gamma z$ for $z \in H$ and $O(\Gamma) = \mathbb{R}\cup\{\infty\} - L(\Gamma)$. As $L(\Gamma)$ is closed in $\mathbb{R}\cup\{\infty\}$, $O(\Gamma)$ is open and so can be written as a countable union $O(\Gamma) = \cup_{\alpha\in A}O_\alpha$ where the $O_\alpha$ are disjoint open intervals in $\mathbb{R}\cup\{\infty\}$. Now let $\Gamma_\alpha = \{\gamma\in\Gamma, \gamma(O_\alpha) = O_\alpha\}$. This is an elementary hyperbolic subgroup of $\Gamma$. The fixed points of $\Gamma_\alpha$ are exactly the end-points of $O_\alpha$. There is a finite subset $\{\alpha(1),\alpha(2),...,\alpha(n_f)\}\subset A$ such that for $\alpha\in A$, $O_\alpha$ is conjugate to precisely one $O_{\alpha(j)}$ $(1\le j\le n_f)$. Let $\lambda_\alpha$ be the half-circle, lying in $H$, joining the end-points of $O_\alpha$. Let $\Delta_\alpha$ be the region in $H$ bounded by $O_\alpha$ and $\lambda_\alpha$. The $\Delta_\alpha$ $(\alpha\in A)$ are mutually disjoint.
Let $P$ be the set of parabolic vertices of $\Gamma$, and for $p\in P$, let $\Gamma_p$ be the parabolic subgroup of $\Gamma$ fixing $p$. There is a finite subset $\{p(1),p(2),...,p(n_c)\}\subset P$ such that $\Gamma_p$ is conjugate to precisely one $\Gamma_{p(j)}$ $(1\le j\le n_c)$. A circle lying in $H$ and tangent to $\partial H$ at $p$ is called a horocycle at $p$. We can construct an open disc $C_p$ determined by a horocycle at $p\in P$ such that

$$\begin{aligned}
&(i) \quad \text{if } p,\, q\in P, p\neq q, \text{then } C_p\cap C_q = \emptyset,\\
&(ii) \qquad \gamma(C_p) = C_{\gamma(p)}\ (\gamma\in\Gamma),\\
&(iii) \qquad C_p\cap\Delta_\alpha = \emptyset\ \ (p\in P,\alpha\in A).
\end{aligned}$$

If we consider the set $H - (\bigcup_{p\in P}C_p\cup\bigcup_{\alpha\in A}\Delta_\alpha)$, we see that it is invariant under $\Gamma$. We can find a finite-sided fundamental domain $D$ for the action of $\Gamma$ on this set; $D$ is relatively compact in $H$.

**Proposition 1.1.** *There is a fundamental domain $D$ for $\Gamma$ of the form*

$$D = K^*\cup\cup_{j=1}^{n_f}D_{\alpha(j)}\cup_{k=1}^{n_c}D^*_{p(k)}$$

*where*
*1)* $K^*$ *is relatively compact in* $H$*,*
*2)* $D_{\alpha(j)}$ *is a standard fundamental domain of* $\Gamma_{\alpha(j)}$ *on* $\Delta_{\alpha(j)}$*,*
*3)* $D^*_{p(k)}$ *is a standard fundamental domain for* $\Gamma_{p(k)}$ *on* $C_{p(k)}$*.*

We should note that $n_f \neq 0$ if and only if $\Gamma$ is of the second kind.

The Nielsen region of the group $\Gamma$ is the set $\tilde{N} = H - (\cup_{\alpha\in A}\Delta_\alpha)$, the truncated Nielsen region of $\Gamma$ is $\tilde{K} = \tilde{N} - (\cup_{p\in P}C_p)$, $K = \Gamma\backslash\tilde{K}$ is called the compact core of $M$. So the surface $M = \Gamma\backslash H$ can be decomposed into a finite area surface with geodesic boundary $N$, called the Nielsen region, on which infinite area ends $F_i$ are glued: the funnels. The Nielsen region $N$ is itself decomposed into a compact surface $K$ with geodesic and horocyclic boundary on which non compact, finite area ends $C_i$ are glued: the cusps. We have $M = K \cup C \cup F$ , where $C = C_1 \cup ... \cup C_{n_c}$ and $F = F_1 \cup ... \cup F_{n_f}$.

A hyperbolic transformation $T \in PSL(2,\mathbb{R})$ generates a cyclic hyperbolic group $\langle T\rangle$. The quotient $C_l = \langle T\rangle\backslash H$ is a hyperbolic cylinder of diameter $l = l(T)$. By conjugation, we can identify the generator $T$ with the map $z \mapsto e^l z$, and we define $\Gamma_l$ to be the corresponding cyclic group. A natural fundamental domain for $\Gamma_l$ would be the region $\mathcal{F}_l = \{z \in H, 1 \leq |z| \leq e^l\}$. The $y-$axis is the lift of the only simple closed geodesic on $C_l$, whose length is $l$. The standard funnel of diameter $l > 0$, $F_l$, is the half hyperbolic cylinder $\Gamma_l\backslash H$, $F_l = (\mathbb{R}^+)_r \times (\mathbb{R}/\mathbb{Z})_x$ with the metric $ds^2 = dr^2 + l^2\cosh^2(r)dx^2$.
We can always conjugate a parabolic cyclic group $\langle T\rangle$ to the group $\Gamma_\infty$ generated by $z \mapsto z + 1$, so the parabolic cylinder is unique up to isometry. A natural fundamental domain for $\Gamma_\infty$ is $\mathcal{F}_\infty = \{0 \leq \mathrm{Re}\, z \leq 1\} \subset H$. The standard cusp $C_\infty$ is the half parabolic cylinder $\Gamma_\infty\backslash H$, $C_\infty = ([0,\infty[)_r \times (\mathbb{R}/\mathbb{Z})_x$ with the metric $ds^2 = dr^2 + e^{-2r}dx^2$. The funnels $F_i$ and the cusps $C_i$ are isometric to the preceding standard models. We define the function $r$ as the distance to the compact core $K$ and the function $\rho$ by

$$\rho(r) = \begin{cases} 2e^{-r} & \text{in } F \\ e^{-r} & \text{in } C \end{cases}, \tag{1}$$

with $\rho$ extended to a smooth non vanishing function inside $K$ in some arbitrary way. We will adopt $(\rho, t) \in (0, 2] \times \mathbb{R}/l_j\mathbb{Z}$ as the standard coordinates for the funnel $F_j$, where $t$ is the arc length around the central geodesic at $\rho = 2$.
For the cusp, our standard coordinates $(\rho, t) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$ are based on the model defined by the cyclic group $\Gamma_\infty$. The cusp boundary is $y = 1$, so that $y = e^r$ and $\rho = 1/y$. We set $t \equiv x \pmod{\mathbb{Z}}$.

## 2. Hyperbolic Eisenstein series on a geometrically finite hyperbolic surface of infinite volume.

2.1. **Return to the definition of Kudla and Millson of a hyperbolic Eisenstein series.** In the following, $M$ will denote an arbitrary Riemann surface and $L^2(M)$, the Hilbert space of square integrable 1-forms with inner product

$$(w_1, w_2) = \frac{1}{2}\int_M w_1 \wedge *\overline{w_2}\,,$$

and corresponding norm $||.||_{L^2}$. The pointwise norm of a 1-form $w$ is defined by $w \wedge *\overline{w} = ||w||^2 * 1$ where $*1$ is the volume form.

Let $c$ be a simple closed curve on $M$. We may associate with $c$ a real smooth closed differential $n_c$ with compact support such that

$$\int_c \omega = \int_M \omega \wedge n_c \,, \tag{2}$$

for all closed differentials $\omega$. Since every cycle $c$ on $M$ is a finite sum of cycles corresponding to simple closed curves, we conclude that to each such $c$, we can associate a real closed differential $n_c$ with compact support such that (2) holds. Let $a$ and $b$ be two cycles on the Riemann surface $M$. We define the intersection number of $a$ and $b$ by

$$a.b = \int_M n_a \wedge n_b \,.$$

In [20], Kudla and Millson construct the harmonic 1-form dual to a simple closed geodesic on a hyperbolic surface of finite volume $M$ in terms of Eisenstein series. Let us recall the definition:

**Definition 2.1.** *Let $\eta$ be a simple closed geodesic or an infinite geodesic joining $p$ and $q$. A 1-form $\alpha$ is dual to $\eta$ if for any closed 1-form $\omega$ with compact support,*

$$\int_M \omega \wedge \alpha = \int_\eta \omega \ .$$

*Or, equivalently, for any closed oriented cycle $c'$,*

$$\int_{c'} \alpha = \eta \ . \ c'. \tag{3}$$

Kudla and Millson construct a meromorphic family of forms on $M$, called the hyperbolic Eisenstein series associated to an oriented simple closed geodesic $c$. Let $\tilde{c}$ be a component of the inverse image of $c$ in the covering $H \to M$ and $\Gamma_1$ the stabilizer of $\tilde{c}$ in $\Gamma$. The hyperbolic Eisenstein series are expressed in Fermi coordinates in the following way for $\operatorname{Re} s > 0$:

$$\Omega_c(s,z) = \Omega(s,z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \frac{dx_2}{(\cosh x_2)^{s+1}}, \ \ k(s) := \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s}{2}\right)}. \tag{4}$$

By applying an element of $SL(2,\mathbb{R})$, we may assume that $\tilde{c}$ is the $y-$axis in $H$ and that $\Gamma_1$ is generated by $\gamma_1 : z \mapsto e^l z$. The Fermi coordinates $(x_1, x_2)$ associated to $\tilde{c}$ are related to Euclidean polar coordinates by

$$\begin{aligned} r &= e^{x_1}, \\ \sin\theta &= \frac{1}{\cosh x_2} \,. \end{aligned}$$

We obtain

$$\Omega(s,z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* (\sin\theta)^s \, d\,\theta. \tag{5}$$

Using the following equalities

$$\begin{aligned} \operatorname{Im} \gamma z &= |\gamma z| \sin\theta(\gamma z) = |\gamma' z| \operatorname{Im} z, \\ d \log \frac{\gamma z}{\overline{\gamma z}} &= 2id\theta(\gamma z) \,, \end{aligned}$$

we find

$$\Omega(s,z)=\frac{1}{2ik(s)}(\operatorname{Im} z)^s\left(\sum_{\Gamma_1\backslash\Gamma}\left(\frac{\gamma' z}{|\gamma z|}\right)^s\frac{\gamma' z}{\gamma z}dz-\left(\frac{\gamma' z}{|\gamma z|}\right)^s\frac{\overline{\gamma' z}}{\overline{\gamma z}}d\bar{z}\right). \tag{6}$$

We denote, with a little misuse of notation, to simplify $\Omega(s,z)=\operatorname{Im}(\Theta(s,z))$ with

$$\Theta(s,z)=\frac{1}{k(s)}\sum_{\Gamma_1\backslash\Gamma}\gamma^*\left(\frac{y^s}{z|z|^s}dz\right)=i\Omega(s,z)-*\Omega(s,z), \tag{7}$$

keeping in mind that the complex conjugation is apply on z and "not on s".

At the end of their paper, they make the remark that "it is also interesting to consider the infinite volume case."

2.2. **The infinite volume case.** We are going to verify that this definition is still meaningful in the case of a geometrically finite hyperbolic surface of infinite volume, $M=\Gamma\backslash H$, $\Gamma$ being a Fuchsian group of the second kind without elliptic elements. We identify $M$ locally with its universal cover $H$. By $d(z,w)$ we denote the hyperbolic distance from $z\in H$ to $w\in H$. For $z_0\in H$ and $\epsilon>0$, by $B(z_0,\epsilon)\subset H$ we denote the hyperbolic metric ball centered at $z_0$ with radius $\epsilon$.

**Proposition 2.1.** *The hyperbolic Eisenstein series $\Omega(s,z)$ converges for $\operatorname{Re} s>0$, uniformly on compact subsets of $H$, is bounded on $M$, and represents a $C^\infty$ closed form which is dual to c. Moreover, it is an analytic function of $s$ in $\operatorname{Re} s>0$.*

The proof in the infinite volume case is as straightforward as in the finite volume case ([20], [10]), but for the convenience of the reader we give some details.

*Proof.* Recall first that if $K$ is a compact subset of the fundamental domain $D$ of $\Gamma$, then there exists $\eta>0$ such that for all $z_0\in K$, the balls $(B(\gamma z_0,\eta))_{\gamma\in\Gamma_1\backslash\Gamma}$ are disjoint.

For a fundamental domain of $\Gamma_1$, let us choose $\mathcal{D}_1=\{z\in H: 1\le|z|\le\beta\}$. After passing to ordinary Euclidean polar coordinates $(r,\theta)$, with $\sigma=\operatorname{Re} s$, where $||\Omega||$ denotes the pointwise norm of $\Omega$, we obtain

$$\begin{aligned}||\Omega(s,z)|| &\le \frac{1}{|k(s)|}\sum_{\Gamma_1\backslash\Gamma}\frac{1}{(\operatorname{ch} x_2(\gamma z))^{\sigma+1}}\\ &\le \frac{1}{|k(s)|}\sum_{\Gamma_1\backslash\Gamma}\left(\frac{y}{r}\right)^{\sigma+1}(\gamma z)\le\frac{1}{|k(s)|}\sum_{[\gamma]\in\Gamma_1\backslash\Gamma,\gamma z\in\mathcal{D}_1}y^{\sigma+1}(\gamma z).\end{aligned}$$

Now $y^s$ is an eigenfunction of all the invariant integral operators on $H$. Let $k(z,z')$ be the point-pair invariant defined by $k(z,z')=1$ or $0$ according as the distance between $z$ and $z'$ is less or no less than $\eta$. Then there exists a $\Lambda_\eta$ independent of $z_0$ such that

$$\int_{B(z_0,\eta)}y^\sigma\frac{dxdy}{y^2}=\int_H k(z_0,z)y^\sigma\frac{dxdy}{y^2}$$

and

$$\int_{B(z_0,\eta)}y^\sigma\frac{dxdy}{y^2}=\Lambda_\eta y(z_0)^\sigma\,.$$

This is a particular case of a more general result we will need further on (see Proposition 5.2).
We write, as in [20], $R(T_1,T_2)=\{P\in\mathcal{D}_1 : T_1<x_2(P)<T_2\}$. So for $T>2\eta$,

$$\frac{1}{|k(s)|}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}\frac{1}{(\mathrm{ch}(x_2(\gamma z)))^{\sigma+1}}\leq\frac{1}{|k(s)|}\sum_{[\gamma]\in\Gamma_1\backslash\Gamma,\gamma z\in\mathcal{D}_1\backslash R(-T,T)}y^{\sigma+1}(\gamma z).$$

We need the following.

**Lemma 2.1.** *Let $\gamma\in\Gamma_1\backslash\Gamma$, $z$, $\zeta\in H$ such that for $\gamma z\notin R(-T,T)$, $\gamma\zeta\in B(\gamma z,\eta)$. Then $\gamma\zeta\notin R(-T+2\eta,T-2\eta)$.*

*Proof.* Let $\pi : H\to\tilde{c}$ be the orthogonal projection on $\tilde{c}$. As $\pi$ is 1-Lipschitzian, we have for the hyperbolic distance $d(\pi\gamma z,\pi\gamma\zeta)\leq d(\gamma z,\gamma\zeta)\leq\eta$. If $x_2(\gamma z)\geq T$,

$$T\leq d(\gamma z,\pi\gamma z)\leq d(\gamma z,\gamma\zeta)+d(\gamma\zeta,\pi\gamma\zeta)+d(\pi\gamma z,\pi\gamma\zeta)\,.$$

Then

$$T-2\eta\leq d(\gamma\zeta,\pi\gamma\zeta)\,.$$

□

Then

$$\begin{aligned}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}y^{\sigma+1}(\gamma z) &= \frac{1}{\Lambda_\eta}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}\int_{B(\gamma z,\eta)}y^{\sigma+1}\,\frac{dxdy}{y^2}\\ &\leq \frac{1}{\Lambda_\eta}\int_{R^c(-T+2\eta,T-2\eta)}y^{\sigma+1}\,\frac{dxdy}{y^2}\end{aligned}$$

where $R^c(-T+2\eta,T-2\eta)$ is the complement in $\mathcal{D}_1$ of $R(-T+2\eta,T-2\eta)$. Note that if $\gamma z\notin R(-T,T)$, then $y(\gamma z)\leq\dfrac{\beta}{\mathrm{ch}\,T}$, so

$$\begin{aligned}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}y^{\sigma+1}(\gamma z) &\leq \frac{\beta}{\Lambda_\eta}\int_0^{\frac{\beta}{\mathrm{ch}(T-2\eta)}}y^{\sigma-1}\,dy\\ &\leq \frac{\beta}{\Lambda_\eta\sigma}\left(\frac{\beta}{\mathrm{ch}(T-2\eta)}\right)^{\sigma}\,.\end{aligned}$$

From this, there follows the uniform convergence of $\Omega(s,z)$ on compact subsets of $H$, uniformly on compact subsets of the half-plane $\mathrm{Re}\,s>0$.
We next show that $\Omega(s,z)$ is bounded on $D$. For this we use a very useful fundamental lemma (see [15], p. 178, [12], pp. 27, 214 note 30):

**Proposition 2.2.** *Suppose that $q>1$. For any Fuchsian group $\Gamma$, there exists a $\mathcal{C}(q,\Gamma)$ such that for all $z\in H$,*

$$\sum_{\gamma\in\Gamma}\frac{y(\gamma z)^q}{[1+|\gamma z|]^{2q}}\leq\mathcal{C}(q,\Gamma).$$

*The constant $\mathcal{C}(q,\Gamma)$ depends only on $q$ and $\Gamma$.*

*Proof.* Let $z \in H$. Then there exists a system of representatives $S$ of $\Gamma_1\backslash\Gamma$ (depending on $z$) such that for all $\gamma \in S$, $|\gamma z| \leq \beta$. In effect, let $(\delta_1, ..., \delta_n, ...)$ be a system of representatives of $\Gamma_1\backslash\Gamma$, then for all $n$ there exists $p_n \in \mathbb{Z}$ such that $\gamma_1^{p_n}\delta_n(z) \in \mathcal{D}_1$.

Then

$$\begin{aligned}\sum_{\Gamma_1\backslash\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1+\beta)^{2(\sigma+1)}} &\leq \sum_{\Gamma_1\backslash\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1+|\gamma z|)^{2(\sigma+1)}} \\ &\leq \sum_{\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1+|\gamma z|)^{2(\sigma+1)}} \\ &\leq \mathcal{C}(\sigma+1, \Gamma)\end{aligned}$$

and the result follows.

□

**Remark 2.1.** *The constant $\mathcal{C}(q, \Gamma)$ depends solely on $q$ and $\mathcal{D}$ where $\mathcal{D}$ is a tiny circular disk such that:*

*$\mathcal{D} \cup \partial\mathcal{D} \subset H$ and $T(\mathcal{D}) \cap \mathcal{D} = \emptyset$ for all $T \in \Gamma - \{I\}$.*

The fact that $\Omega(s, z)$ is dual to $c$ follows straightforwardly from the construction of Kudla and Millson. □

## 3. Spectral decomposition and analytic continuation.

The aim is to realize the injection $H_c^1 \to \mathcal{H}^1$, where $H_c^1$ is the first de Rham cohomology group with compact support of $M$ and $\mathcal{H}^1$ is the space of $L^2$ harmonic 1-forms of $M$. Recall that in our context, $\dim \mathcal{H}^1 = \infty$ (see [2], p. 27).
We are going to prove, as in [20], that the hyperbolic Eisenstein series have an analytic continuation. The essential difference from the finite volume case is the spectral decomposition of $L^2(M)$.

3.1. **Spectral theory.** For any non-compact geometrically finite hyperbolic surface $M$, the essential spectrum of the (positive) Laplacian $\Delta_M$ defined by the hyperbolic metric on $M$ (the Laplacian on functions) is $[1/4, \infty)$ and this is absolutely continuous. The discrete spectrum consists of finitely many eigenvalues in the range $(0, 1/4)$. In the finite-volume case, one may also have embedded eigenvalues in the continuous spectrum, but these do not occur for infinite volume surfaces. Then if $M$ has infinite volume, the discrete spectrum of $\Delta_M$ is finite (possibly empty). The exponent of convergence $\delta$ of a Fuchsian group $\Gamma$ is defined to be the abscissa of convergence of the Dirichlet series

$$\delta = \inf\{s > 0, \sum_{T\in\Gamma} e^{-sd(z,Tw)} < \infty\}$$

for some $z, w \in H$, where $d(z, w)$ again denotes the hyperbolic distance from $z \in H$ to $w \in H$.

Let $\Gamma$ be a Fuchsian group of the second kind and $L(\Gamma)$ be its limit set. Then $0 < \delta < 1$ with $\delta > 1/2$ if $\Gamma$ has parabolic elements. Patterson and Sullivan showed that $\delta$ is the Hausdorff dimension of the limit set when $\Gamma$ is geometrically finite. Furthermore, if $\delta > 1/2$, then $\delta(1-\delta)$ is the lowest eigenvalue of the Laplacian $\Delta_M$. The connection with spectral theory was later extended to the case $\delta \leq 1/2$

by Patterson. In this case, the discrete spectrum of $\Delta_M$ is empty and $\delta$ is the location of the first resonance. For a detailed account of the spectral theory of infinite area surfaces, we refer the reader to [1].

3.2. **Tensors and automorphic forms.** This section introduces the notations used in the following subsection 3.3 and Section 5. Let $M$ be a geometrically finite hyperbolic surface. Let $z$ be a local conformal coordinate and $ds^2 = \rho|dz|^2$ the Poincaré metric. Let $\omega_M = T^*M$ be the holomorphic cotangent bundle of $M$ and for any integers $n$ and $m$, let $\mathcal{E}^{r,s}(M, \omega_M^n \otimes \overline{\omega}_M^m)$ be the vector space of smooth differential forms of type $(r,s)$ on $M$ with values in the line bundle $\omega_M^n \otimes \overline{\omega}_M^m$. For an integer $q$, a $q-$form (or $q-$differential) is an element of $T^q = \mathcal{E}^{0,0}(M, \omega_M^q)$, the space of tensors of type $q$ on $M$, written locally as $f(z)(dz)^q$. $M$ may be realized as $\Gamma\backslash H$, where $H$ is the upper half-plane and $\Gamma$ a discrete subgroup of $PSL(2,\mathbb{R})$. The hyperbolic metric on $M$ induces the natural scalar product on $T^q$:

$$(\varphi, \psi) = \int_{\Gamma\backslash H} \varphi(z)\overline{\psi(z)} y^{2q-2}\, dxdy\,.$$

Let $\mathfrak{H}^q$ be the $L^2-$closure of $T^q$ with respect to this scalar product.

We recall now the link between $q-$forms and automorphic forms of weight[2] $2q$, also called $q-$automorphic forms for the following reason. As before, we make use of the uniformization theorem.

Using the notations of [11] and [7], set

$$j_\gamma(z) = \frac{(cz+d)^2}{|cz+d|^2} = \frac{cz+d}{c\bar{z}+d} = \left(\frac{\gamma' z}{|\gamma' z|}\right)^{-1} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let $\mathcal{F}_q$ be the space of all functions $f : H \to \mathbb{C}$ with

$$f(\gamma z) = j_\gamma(z)^q f(z), \quad \gamma \in \Gamma\,,$$

and if $\mathcal{D} = \Gamma\backslash H$ is a fundamental domain for $\Gamma$, define the Hilbert space $\mathcal{H}_q = \{f \in \mathcal{F}_q, \langle f, f\rangle_{\mathcal{D}} = \int_{\mathcal{D}} |f(z)|^2\, d\mu(z) < \infty\}$ with areal measure $d\mu(z) = \frac{dxdy}{y^2}$ and the inner product

$$\langle f, g\rangle = \int_{\mathcal{D}} f(z)\overline{g(z)} d\mu(z). \tag{8}$$

An element in $\mathcal{H}_q$ is called an automorphic form of weight $2q$. $\mathcal{H}_q$ is isometric to $\mathfrak{H}^q$ through the correspondence

$$I : \mathfrak{H}^q \ni f(dz)^q \mapsto y^q f \in \mathcal{H}_q. \tag{9}$$

Maass introduced the differential operators

$$\begin{array}{rll} L_q = & (\bar{z}-z)\frac{\partial}{\partial \bar{z}} - q & : \mathcal{F}_q \to \mathcal{F}_{q-1} \\ K_q = & (z-\bar{z})\frac{\partial}{\partial z} + q & : \mathcal{F}_q \to \mathcal{F}_{q+1} \end{array}$$

We also have

$$\begin{aligned} L_q &= -2iy^{1+q}\frac{\partial}{\partial \bar{z}} y^{-q} = \overline{K_{-q}}, \\ K_q &= 2iy^{1-q}\frac{\partial}{\partial z} y^{q} = \overline{L_{-q}}. \end{aligned}$$

[2]Some authors refer to them as weight $q$ or $-2q$

We write

$$\begin{aligned} -L_{q+1}K_q &= -\Delta_{2q} + q(q+1), \\ -K_{q-1}L_q &= -\Delta_{2q} + q(q-1) \end{aligned}$$

with

$$\Delta_{2q} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 2iqy\frac{\partial}{\partial x}\,.$$

These second order differential operators are self-adjoint on $\mathcal{H}_q$.

Now, the metric and complex structure determine a covariant derivative

$$\nabla = \nabla^q \oplus \nabla_q : \mathcal{E}^{0,0}(M, \omega_M^q) \longrightarrow \mathcal{E}^{1,0}(M, \omega_M^q) \oplus \mathcal{E}^{0,1}(M, \omega_M^q)$$

on the line bundle $\omega_M^q$. With the identifications $\mathcal{E}^{1,0}(M, \omega_M^q) \cong T^{q+1}$ and $\mathcal{E}^{0,1}(M, \omega_M^q) \cong T^{q-1}$, we have

$\nabla^q : T^q \to T^{q+1}$, $\nabla_q : T^q \to T^{q-1}$.

Under the correspondence $I_k$, $I_k : \mathfrak{H}^k \ni f(dz)^k \mapsto y^k f \in \mathcal{H}_k$, $k = q-1, q, q+1$ the operators $\nabla_q$, $\nabla^q$ go over to the Maass operators according to the commutative diagram

$$\begin{array}{ccccc} \mathfrak{H}^{q-1} & \overset{\nabla_q}{\leftarrow} & \mathfrak{H}^q & \overset{\nabla^q}{\rightarrow} & \mathfrak{H}^{q+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}_{q-1} & \overset{L_q}{\leftarrow} & \mathcal{H}_q & \overset{K_q}{\rightarrow} & \mathcal{H}_{q+1} \end{array}$$

and so are given locally by

$$\nabla^q = I_{q+1}^{-1} K_q I_q = 2i\rho^q \partial \rho^{-q} \text{ and } \nabla_q = I_{q-1}^{-1} L_q I_q = -2i\rho^{-1}\bar{\partial}\,,$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$.

The Laplacians $\Delta_q^+$ and $\Delta_q^-$ on $T^q$ are defined by $\Delta_q^+ = -\nabla_{q+1}\nabla^q = -I_q^{-1}L_{q+1}I_{q+1}I_{q+1}^{-1}K_qI_q = I_q^{-1}(-L_{q+1}K_q)I_q$, $\Delta_q^- = -\nabla^{q-1}\nabla_q = I_q^{-1}(-K_{q-1}L_q)I_q$ and then the isometry $I$ conjugates $\Delta_q^+$ with $-\Delta_{2q}+q(q+1)$ and $\Delta_q^-$ with $-\Delta_{2q}+q(q-1)$. Thus $\Delta_M = \Delta_0^{\pm}$ is the Laplacian on functions. The $\Delta_q^{\pm}$ are non-negative self-adjoint operators.

We are first interested in the case $q = 1$. Let $\Delta_{\text{Diff}}$ be the (positive) Laplacian on 1-forms on a geometrically finite hyperbolic surface, $\Delta_{\text{Diff}} = d\delta + \delta d$, $\delta = -*d*$ with $*$ the Hodge operator. In the following, we write $\Delta_{\text{Diff}} = \Delta$. If $\omega$ is a 1-form in the holomorphic cotangent bundle, $\omega = f(z)\,dz$, then we define its image by the isometry $I$ to be $I(w) = I(f\,dz) = yf(z) = \tilde{f}(z)$. We have $y\Delta(f\,dz) = -(\Delta_2\tilde{f})dz$, in other words, in the preceding notation, $\Delta = \Delta_1^-$.

3.3. **Generalized eigenfunctions.** We are going to give the spectral expansion in eigenforms of $\Delta$; we use [7], [24], [1]. With the notation of Section 1, for a finitely generated group of the second kind, for each cusp and for each funnel of the quotient there is a corresponding Eisenstein series, which is what we are going to develop now.

We will denote $R_{s,q}$ the resolvent operator defined for $\operatorname{Re} s > 1/2$, $s \notin [1/2, 1]$ by $R_{s,q} = (\Delta_{2q} + s(1-s))^{-1}$.

**Proposition 3.1.** *For an integer $q$, for $\operatorname{Re} s > \delta$, $G_s(z, w, q)$, the kernel of the resolvent $R_{s,q}$, for the self-adjoint operator $\Delta_{2q}$, acting on the Hilbert space $\mathcal{H}_q$ of*

*automorphic forms of weight 2q, is given by the convergent series*

$$G_s(z,w,q)=\sum_{\gamma\in\Gamma} j_\gamma(w)^q g_s(z,\gamma w,q)$$

*with*

$$g_s(z,w,q)=-\left(\frac{w-\bar z}{z-\bar w}\right)^q \frac{\Gamma(s+q)\Gamma(s-q)}{4\pi\Gamma(2s)}\cosh^{-2s}(d(z,w)/2)F(s+q,s-q,2s;\cosh^{-2}(d(z,w)/2))$$

*and F being Gauss's hypergeometric function.*

We collect some properties we can find for example in [24], vol I, p. 100, with [1], Chapters 6 and 7. The resolvent kernel $G_s(z,w,q)$ has a meromorphic continuation to all of $\mathbb{C}$. Following these references, we define generalized eigenfunctions.

For the funnel case, we identify $z'$ with the standard coordinates $(\rho',t')$ (see (1)) in the funnel $F_j$, and, with $x'=e^{t'}$, we denote

$$x'^s E^f_{j,q}(s,z,x')=(1-2s)\lim_{\rho'\to 0}\rho'^{-s}G_s(z,z',q)\,, \tag{10}$$

for $j=1,...,n_f$. In the cusp $C_j$, with standard coordinates $z'=(\rho',t')$, we set

$$E^c_{j,q}(s,z)=(1-2s)\lim_{\rho'\to 0}\rho'^{1-s}G_s(z,z',q)\,, \tag{11}$$

for $j=1,...,n_c$.

We will call them respectively (standard) funnel Eisenstein series and cuspidal Eisenstein series. The Poisson kernel is

$$P(z,\zeta)=\mathrm{Im}(z)/|z-\zeta|^2$$

where $z\in H$ and $\zeta\in\mathbb{R}$. For $b\in O(\Gamma)=\mathbb{R}\cup\{\infty\}-L(\Gamma)$, define the Eisenstein series ([24] [3])

$$E_b(z,s,k)=\sum_{\gamma\in\Gamma} j(\gamma,z)^k P(\gamma(z),b)^s(\gamma(z),b)^k\,,$$

where $j(\gamma,z)=\gamma'(z)/|\gamma'(z)|$ and $(z,b)=(\bar z-b)/(z-b)$. It converges uniformly on compact subsets of $H$ if $\mathrm{Re}\,s>\delta$.

**Proposition 3.2.** *The series $E_b(z,s,k)$ can be continued to the whole complex plane as a meromorphic function in s.*

One verifies that $-\Delta_{2k}E_b(z,s,k)=s(1-s)E_b(z,s,k)$.

**Remark 3.1.** *Thus, if $\delta<1/2$, then $E_b(z,s,k)$ is analytic in a neighbourhood of* $\mathrm{Re}\,s=1/2$.

For the standard funnel $F_l$ which corresponds to the region $\mathrm{Re}\,z\geq 0$ in the model $C_l=\Gamma_l\backslash H$, we have (see [7], p. 200)

$$\begin{aligned} E^f_{l,q}(s,z,x') &= (1-2s)\lim_{z'\to x'}(\mathrm{Im}\,z')^{-s}G_s(z,z',q)\\ &= -(1-2s)\frac{4^s}{4\pi}\frac{\Gamma(s+q)\Gamma(s-q)}{\Gamma(2s)}E_{x'}(z,s,q).\end{aligned}$$

In the case of 1-forms we write,

$$\mathcal{E}_{f_l}(s,z,x')=\frac{E^f_{l,1}(s,z,x')}{y}\,dz.$$

Now we point out the link between cuspidal and classical Eisenstein series.

**Remark 3.2.** *Recall the definition of the classical Eisenstein series. We will assimilate the standard cusp of a parabolic point with the latter. The stabilizer of a cusp* $\mathfrak{a}$ *is an infinite cyclic group generated by a parabolic motion,*

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\, \mathfrak{a} = \mathfrak{a}\} = \langle \gamma_{\mathfrak{a}} \rangle \ ,$$

*say. There exists a* $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$*, called a scaling matrix of the cusp* $\mathfrak{a}$*, such that* $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$*,* $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$*.* $\sigma_{\mathfrak{a}}$ *is determined up to composition with a translation from the right. The Eisenstein series for the cusp* $\mathfrak{a}$ *is then defined by*

$$E_{\mathfrak{a}}(s,z) = \sum_{\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \ ,$$

*where* $s$ *is a complex variable with* $\operatorname{Re} s > \delta$*.*

**Definition 3.1.** *In a similar way we define the Eisenstein series of weight* $2q$ *associated to a cusp* $\mathfrak{a}$ *as the automorphic form of weight* $2q$*, for* $\operatorname{Re} s > \delta$*:*

$$E_{\mathfrak{a},q}(s,z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-q} = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \left(\frac{(\sigma_{\mathfrak{a}}^{-1}\gamma)' z}{|(\sigma_{\mathfrak{a}}^{-1}\gamma)' z|}\right)^q \ .$$

*We call a horocyclic Eisenstein series the 1-form corresponding to the Eisenstein series of weight 2 associated to a cusp* $\mathfrak{a}$*,* $E_{\mathfrak{a},1}$ *and defined for* $\operatorname{Re} s > 1$ *by*

$$\mathcal{E}_{\mathfrak{a}}(s,z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\ d(\sigma_{\mathfrak{a}}^{-1}\gamma z) = \frac{E_{\mathfrak{a},1}(s,z)}{y}\ dz \ . \tag{12}$$

We now verify that this corresponds to the defining formula (11), that is $E^c_{\mathfrak{a},q} = E_{\mathfrak{a},q}$.

For the standard cusp, we write for $\operatorname{Re} s > \delta$,

$$G_s(z,z',q) = \sum_{\Gamma_\infty\backslash\Gamma} \left(\frac{c\bar{z}+d}{cz+d}\right)^q G_s^{\Gamma_\infty}(\gamma z, z', q)\,,$$

where $G_s^{\Gamma_\infty}(z,z',q)$ is the resolvent kernel of the standard cusp for automorphic forms of weight 2q. We use then [7], p. 155 (38), p. 177 for $\operatorname{Im} z' > \operatorname{Im}\gamma z$, p. 172 (see also [1], pp. 72, 102) to conclude that

$$\lim_{y'\to\infty} {y'}^{s-1} G_s(z,z',q) = \sum_{\Gamma_\infty\backslash\Gamma} \left(\frac{c\bar{z}+d}{cz+d}\right)^q \frac{(\operatorname{Im}\gamma z)^s}{1-2s} = \frac{1}{1-2s} E_{\infty,q}(s,z)\,.$$

In particular from the foregoing and the references cited, we have

**Lemma 3.1.** *The funnel and cuspidal Eisenstein series,* $E^f_{j,q}(s,z,t')$ *and* $E_{\mathfrak{a},q}(s,z)$*, can be continued to the whole complex plane as a meromorphic function in* $s$*.*

We recall the property (see for example [7], p. 196, [24], [1], p.94)

**Remark 3.3.** *The possible poles of the resolvent* $G_s(z,z',q)$ *in* $\operatorname{Re} s > 1/2$ *are simple and at* $s_{1,q}, ..., s_{N,q} \in (\frac{1}{2}, 1)$ *corresponding to the eigenvalues* $\lambda_{k,q}$ $1 \le k \le N$ *of the Laplacian* $-\Delta_{2q}$ *on* $S_l$*.*

Recall the decomposition (see Section 1) $M = K \cup_{j=1}^{n_c} C_j \cup_{j=1}^{n_f} F_j$, with the preceding notation and formula(8), we then have (see for example [7], [24])

**Proposition 3.3.** *For $f \in \mathcal{H}_q$,*

$$\begin{aligned} f(z) &= \sum_{k=1}^{N}(f)_{\lambda_{k,q}}(z) + \frac{1}{4\pi}\sum_{j=1}^{n_c}\int_{-\infty}^{+\infty}\langle f, E^c_{j,q}(1/2+it,.)\rangle E^c_{j,q}(1/2+it,z)\,dt \;+ \\ &\quad \frac{1}{4\pi}\sum_{j=1}^{n_f}\int_{-\infty}^{+\infty}\left[\int_1^{e^{l_j}}\langle f, E^f_{j,q}(1/2+it,.,b)\rangle E^f_{j,q}(1/2+it,z,b)\,db\right]dt; \end{aligned}$$

*where the first sum in the right member is the projection of $f$ on the discrete spectrum.*

Let's formulate the spectral decomposition we needed in the case of 1-forms.

**Proposition 3.4.** *For $w = f(z)\,dz$ square integrable,*

$$\begin{aligned} w(z) &= \sum_{i=1}^{m}(w)_{\lambda_i}(z) + \frac{1}{4\pi}\sum_{j=1}^{n_c}\int_{-\infty}^{+\infty}\left(w, \mathcal{E}_{c_j}(1/2+it,.)\right)\mathcal{E}_{c_j}(1/2+it,z)\,dt \;+ \\ &\quad \frac{1}{4\pi}\sum_{j=1}^{n_f}\int_{-\infty}^{+\infty}\left[\int_1^{e^{l_j}}\left(w, \mathcal{E}_{f_j}(1/2+it,.,b)\right)\mathcal{E}_{f_j}(1/2+it,z,b)\,db\right]dt; \end{aligned}$$

*where the first sum in the right member is the projection of $w$ on the discrete spectrum, $\mathcal{E}_{c_j}$ and $\mathcal{E}_{f_j}$ are the Eisenstein series associated to the cusp $C_j$ and the funnel $F_j$, respectively.*

**Remark 3.4.** *One can easily deduce the formula for an arbitrary square integrable 1-form, with the following notations,*

$$\begin{aligned} \mathcal{E}_{c_j}(s,z)_+ &= \mathcal{E}_{c_j}(s,z) & \mathcal{E}_{f_k}(s,z,b)_+ &= \mathcal{E}_{f_k}(s,z,b) \\ \mathcal{E}_{c_j}(s,z)_- &= \overline{\mathcal{E}_{c_j}(\bar{s},z)} & \mathcal{E}_{f_k}(s,z,b)_- &= \overline{\mathcal{E}_{f_k}(\bar{s},z,b)} \end{aligned}$$

$$\begin{aligned} &\Omega(z) = f\,dz + g\,d\bar{z} = \sum_{i=1}^{m}(\Omega)_{\lambda_i}(z) + \\ &\frac{1}{4\pi}\sum_{j=1}^{n_c}\int_{-\infty}^{+\infty}\left(\Omega, \mathcal{E}_{c_j}(1/2+it,.)_+\right)\mathcal{E}_{c_j}(1/2+it,z)_+ + \left(\Omega, \mathcal{E}_{c_j}(1/2+it,.)_-\right)\mathcal{E}_{c_j}(1/2+it,z)_-\,dt \\ &+\frac{1}{4\pi}\sum_{j=1}^{n_f}\int_{-\infty}^{+\infty}\int_1^{e^{l_j}}\left(\Omega, \mathcal{E}_{f_j}(1/2+it,.,b)_+\right)\mathcal{E}_{f_j}(1/2+it,z,b)_+\,db\,dt \\ &+\frac{1}{4\pi}\sum_{j=1}^{n_f}\int_{-\infty}^{+\infty}\int_1^{e^{l_j}}\left(\Omega, \mathcal{E}_{f_j}(1/2+it,.,b)_-\right)\mathcal{E}_{f_j}(1/2+it,z,b)_-\,db\,dt \end{aligned}$$

*For simplicity, we will write*

$$\begin{aligned} \Omega(z) &= (\Omega)_{\lambda_i}(z) + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\Omega, \mathcal{E}_{c_j}(1/2+it,.)_\pm\right)\mathcal{E}_{c_j}(1/2+it,z)_\pm\,dt + \\ &\quad \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left[\int_1^{e^{l_k}}\left(\Omega, \mathcal{E}_{f_k}(1/2+it,.,b)_\pm\right)\mathcal{E}_{f_k}(1/2+it,z,b)_\pm\,db\right]dt\,. \end{aligned} \tag{13}$$

3.4. **Harmonic dual form.** We are now going to see

**Proposition 3.5.** *If $c$ is an oriented simple closed geodesic on $M$, then, for $\operatorname{Re} s > 0$, the hyperbolic Eisenstein series $\Omega_c(s,.)$ are square integrable.*

*Proof.* We consider a fundamental domain $D$ contained in $\{z, 1 \leq |z| \leq e^l\}$ in which the segment $(i, ie^l)$ represents the geodesic $c$. We write $C_\lambda = \{z \in D, d(z,c) = \lambda\}$ and $F_\lambda = \{z \in D, d(z,c) \geq \lambda\}$. Without loss of generality we may assume that there is only one funnel on $M$ and no cusps. Let $V_\lambda$ be the volume of $F_\lambda - F_{\lambda+1}$. Then there exists a constant $c_1$ such that $V_\lambda \geq c_1(\sinh(\lambda+1) - \sinh(\lambda))$. For $\operatorname{Re} s = \sigma > 0$, $||\Omega_c(s,z)|| = ||\Omega(s,z)|| \leq \frac{1}{|k(s)|} \sum_{\Gamma_1\backslash\Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$. Let $\eta(z) = \sum_{\Gamma_1\backslash\Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$. We know from Section 2 that there exists a constant $K > 0$ such that $\forall z \in H$, $\eta(z) \leq K$.
We have

$$\begin{aligned}
\int_D ||\Omega_c(s,z)||^2 d\mu(z) &\leq \frac{1}{|k(s)|^2} \int_D \eta^2(z) d\mu(z) \\
&\leq \frac{1}{|k(s)|^2} \int_{1\leq x_1 \leq e^l, -\infty < x_2 < +\infty} \eta(z) \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 \, dx_1 \, dx_2 \\
&\leq \frac{K}{|k(s)|^2} \int_{1\leq x_1 \leq e^l, -\infty < x_2 < +\infty} \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 \, dx_1 \, dx_2 \, .
\end{aligned}$$

The last integral is $\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\sigma}{2})}{\Gamma(\frac{1}{2}+\frac{\sigma}{2})}(e^l - 1)$ and the result follows. □

As in [20], we verify that

$$\Delta(\Omega(s,z)) + s(s+1)\Omega(s,z) = s(s+1)\Omega(s+2,z) \, .$$

This formula has the consequence that for fixed $s$ with $\operatorname{Re} s > 0$, the function $\Delta^k(\Omega(s,z))$ is again square integrable for any $k > 0$.
Set $\operatorname{Re} s > 0$. Then with the notation of Remark 3.4, in particular in the last equality (13), we decompose the first sum of the right member as $\Omega_0(z)$ the harmonic part of $\Omega(s,z)$ plus the projection on the non zero discrete spectrum. More precisely

$$\Omega(s,z) = \Omega_0(z) + a_i(s)\varphi_i(z) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h_\pm^{c_j}(s,t) \mathcal{E}_c(1/2+it, z)_\pm \, dt$$

$$+ \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left[ \int_1^{e^{l_k}} H_\pm^{f_k}(s,t,b) \mathcal{E}_f(1/2+it, z, b)_\pm \, db \right] dt \, , \tag{14}$$

where, $\{\varphi_i\}$ is a complete orthonormal basis of eigenforms of $\Delta$ with corresponding positive eigenvalues, $a_i(s) = (\Omega(s,.), \varphi_i)$, $h_\pm^{c_j}(s,t) = \left(\Omega(s,.), \mathcal{E}_{c_j}(1/2+it,.)_\pm\right)$ and $H_\pm^{f_k}(s,t,b) = (\Omega(s,.), \mathcal{E}_{f_k}(1/2+it,.,b)_\pm)$.

We obtain with $H$ corresponding to any $H_\pm^{f_k}$

$$\left(1/4 + t^2 + s(s+1)\right) H(s,t,b) = s(s+1)H(s+2,t,b) \, .$$

From this we get a continuation of $H$ to the region $\operatorname{Re} s > -1/2$ and we note that for all $t$ and all $b$ we have $H(0,t,b) = 0$.
Moreover, for $\operatorname{Re} s > -1/2$, $\operatorname{Re}(s+2) > 0$ and we may substitute in (14) to obtain a continuation of $\Omega(s,z)$ to $\operatorname{Re} s > -1/2$. In particular $s = 0$ is a regular value. Substituting $s = 0$ into $\Delta\Omega(s,z) + s(s+1)\Omega(s,z) = s(s+1)\Omega(s+2,z)$ we obtain $\Delta\Omega(0,z) = 0$. Moreover for any closed oriented cycle $c'$, for $\operatorname{Re} s > 0$,

$\int_{c'} \Omega(s,.) = c.c'$ and by analytic continuation, for $\operatorname{Re} s > -1/2$, in particular for $s = 0$. Thus we have proved the following theorem:

**Theorem 3.1.** *$\Omega(s,z)$ has a meromorphic continuation to $\operatorname{Re} s > -1/2$ with $s = 0$ a regular point and $\Omega(0,z)$ is a harmonic form which is dual to $c$.*

**Remark 3.5.** *1) Another way to see this is:*
*write $\Omega(s,z) = (\Delta + s(s+1))^{-1}(s(s+1)\Omega(s+2,z))$ and use the meromorphic continuation of the resolvent (see for example* [1], [25]*).*
*2) Using methods analogous to those of* [20] *(see also* [18]*), we can obtain a complete description of the singularities of the hyperbolic Eisenstein series.*

## 4. The case of an infinite geodesic joining two points.

Without loss of generality, we suppose that the two cusps $p$ and $q$ are 0 and $\infty$ and, as the lift of the geodesic, we take the imaginary axis. Let $\eta$ be the infinite geodesic $]p,q[$. Can we carry out the same construction as Kudla and Millson? As in the finite volume case, the problem reduces to studying the following series for $\operatorname{Re} s > 1$:

$$\hat{\eta}^s(z) = \frac{1}{k(s-1)} \sum_{\gamma\in\Gamma} \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1}dz) \right] = \operatorname{Im}(\theta^s(z)) \ , \tag{15}$$

where

$$\theta^s(z) = \frac{1}{k(s-1)} \sum_{\gamma\in\Gamma} \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right] \ ,$$

$k(s-1) = \dfrac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2+s/2)}$, and the notation

$$\operatorname{Im}(\theta^s(z)) = \frac{1}{2ik(s-1)} \left( \sum_{\gamma\in\Gamma} \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right] - \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \frac{d\bar{z}}{\bar{z}} \right] \right) .$$

4.1. **Some useful estimates.** As usual, we can suppose $\Gamma_\infty = \langle z \mapsto z+1 \rangle$ to be the stabilizer of $\infty$ in $\Gamma$ and the stabilizer of 0, $\Gamma_0$, is then generated by $z \mapsto \frac{z}{-c_0^2 z+1}$ (for some non-zero constant $c_0$).
First of all we note that, unlike in the finite volume case (see Proposition 3.1 and (11)):

**Lemma 4.1.** *The series $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)$ is convergent.*

Another way to see this is 'by hand': We know that for $\operatorname{Re} s > \delta$, $\sum_{T\in\Gamma} e^{-sd(i,Tz)}$ converges; moreover there exists a constant $C > 0$ such that $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}^{\operatorname{Re} s}(\gamma z) \leq C \sum_{T\in\Gamma} e^{-\operatorname{Re} s\, d(i,Tz)}$, as in our case $\delta < 1$, we have the result.
As $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left| \operatorname{Im}^s(\gamma z) \frac{c\bar{z}+d}{cz+d} \right| = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}^{\operatorname{Re} s}(\gamma z)$, we also deduce the convergence of the series representing $E_{\infty,1}(1,z)$ (3.2).

**Lemma 4.2.** *We have the following asymptotic behaviour for $\operatorname{Re} s > \delta$:*
*1) in a funnel, for all cusps $\mathfrak{a}$, $E_\mathfrak{a}(s,z)$ is square integrable;*
*2) at $\mathfrak{a} = \infty$, $E_\infty(s,z) - y^s = O(y^{1-s})$ and $E_0(s,z) = O(y^{1-s})$;*

*3) near* $\mathfrak{a} = 0$, $E_0(s,z) - y^s/(c_0^2|z|^2)^s = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$ *and* $E_\infty(s,z) = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$.

*Proof.* With the notations of 3.1 and taking $\rho$ to be the standard coordinate for the cusp $\mathfrak{a}$, we re-write the results of [1](p. 110) in the following way:

$$E_\mathfrak{a}(s,.) = \rho^{-s}(1-\chi_0(\rho)) + 0(\rho_f^s \rho_c^{s-1}), \tag{16}$$

where we define $\chi_0 \in \mathcal{C}_0^\infty(M)$ such that, $r$ being the distance to the the compact core for a point in cusp $\mathfrak{a}$,

$$\chi_0(\rho) = \chi_0(e^{-r}) = \begin{cases} 1, & r \leq 0 \\ 0, & r \geq 1 \end{cases} ;$$

and $\rho_f$ (resp. $\rho_c$) is the standard coordinate in the funnels (resp. cusps). Now $\Gamma$ contains parabolic elements then $\delta > 1/2$. The area form in standard funnel coordinates is $dA = \rho_f^{-2} d\rho_f dt + O(1)$, so $\rho_f^{1/2}$ is the threshold for $L^2$ asymptotic behavior in a funnel. Then for $\operatorname{Re} s > \delta > 1/2$ we have 1). The cusp area form is $dA = d\rho_c dt$, so borderline $L^2$ behavior is $\rho_c^{-1/2}$. For 2) and 3) we can use (16), see also [5] proposition 3.1. □

4.2. **Convergence of the hyperbolic Eisenstein series and its analytic continuation.** The computations to prove the convergence of (15) are easily adapted from the finite volume case. For the convenience of the reader, we recall the essential points.

We have $|| \sum_{\gamma\in\Gamma} \gamma * \left(\frac{y}{|z|}\right)^{s-1} \operatorname{Im}(z^{-1}dz)|| \leq \sum_{\gamma\in\Gamma} \left(\frac{y}{|z|}\right)^{\sigma} (\gamma z)$ , where $\sigma = \operatorname{Re} s > 1$ and if we put $\mathcal{S} = \displaystyle\sum_{\gamma\in\Gamma} \left(\frac{y}{|z|}\right)^{\sigma} (\gamma z)$, we have

$$\mathcal{S} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \sum_{n\in\mathbb{Z}} \frac{1}{|\gamma z + n|^\sigma} = S_1 + S_2 \ ,$$

with

$$\mathcal{S}_1 = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \qquad \text{and} \qquad \mathcal{S}_2 = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \sum_{n\in\mathbb{Z}^*} \frac{1}{|\gamma z + n|^\sigma} \, .$$

Let $S_z$ be a system of representatives of $\Gamma_\infty\backslash\Gamma$ such that $|\operatorname{Re}\gamma z| \leq 1/2$. Then

$$||\mathcal{S}|| \leq \mathcal{S}_1 + 2 \sum_{\gamma\in S_z} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma} \, .$$

We have

$$\begin{aligned} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &= \sum_{\Gamma_0\backslash(\Gamma_\infty\backslash\Gamma)} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n\in\mathbb{Z}} \frac{1}{|-nc_0^2\gamma z + 1|^\sigma} \\ &= \sum_{\gamma\in\Gamma_0\backslash(\Gamma_\infty\backslash\Gamma)} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + \sum_{\gamma\in\Gamma_0\backslash(\Gamma_\infty\backslash\Gamma)} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n\in\mathbb{Z}^*} \frac{1}{|nc_0^2|^\sigma [(x(\gamma z) - 1/nc_0^2)^2 + y^2(\gamma z)]^{\sigma/2}} . \end{aligned}$$

For $K$ a compact set in $H$ there exists $m$ in $H$ such that

$$\forall z \in K,\ \forall \gamma \in \Gamma_0\backslash\Gamma_\infty\backslash\Gamma,\ |\gamma z| \geq |m| \text{ and } \operatorname{Im}\gamma z \geq \operatorname{Im} m \, .$$

So

$$\begin{aligned}\mathcal{S}_1 &\leq \sum_{\gamma\in\Gamma_0\backslash(\Gamma_\infty\backslash\Gamma)} \frac{y^\sigma(\gamma z)}{|m|^\sigma} + \sum_{\gamma\in\Gamma_0\backslash(\Gamma_\infty\backslash\Gamma)} \frac{y^\sigma(\gamma z)}{|m|^\sigma} \sum_{n\in\mathbb{Z}^*} \frac{1}{|nc_0^2|^\sigma(\operatorname{Im} m)^\sigma} \\ &\leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) + 2\sum_{n\in\mathbb{N}^*} \frac{1}{(nc_0^2)^\sigma} \frac{1}{|m|^\sigma(\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z)\ ;\end{aligned}$$

and lastly, for all $z$ in $K$

$$\begin{aligned}||\mathcal{S}|| \leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) + 2\sum_{n\in\mathbb{N}^*} \frac{1}{(nc_0^2)^\sigma} \frac{1}{|m|^\sigma(\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) + \\ 2\sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma},\end{aligned}$$

with uniform convergence on all compact subsets of $H$ and all compact subsets of $\operatorname{Re} s > 1$.

From this last inequality we conclude that $\theta^s$ is square integrable in the funnels as the Eisenstein series $\mathcal{E}_\infty$. Moreover using the notation in formula (4 ) we can write $\eta^s(z) = \sum_{\gamma\in\Gamma} \gamma^*\varphi_s$, where $\varphi_s = \frac{1}{k(s-1)} \frac{dx_2}{(\cosh x_2)^s}$ satisfies $\Delta\varphi_s + s(s-1)\varphi_s = s(s-1)\varphi_{s+2}$. To conclude, we have the following theorem:

**Theorem 4.1.** *For $\operatorname{Re} s > 1$, the Eisenstein series associated to the geodesic $\eta = (p, q)$ converges uniformly on all compact sets. It represents a $\mathcal{C}^\infty$ closed form which is dual to $\eta$. For $\operatorname{Re} s > 1$ it satisfies the differential functional equation:*

$$\Delta\hat{\eta}^s = s(1-s)[\hat{\eta}^s - \hat{\eta}^{s+2}].$$

Now we want to prove that $\hat{\eta}^s$ has an analytic continuation to $\operatorname{Re} s > 1/2$ and $s = 1$ is a regular value. For this, first of all, we are going to show that $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ is square integrable. As we have shown that $\theta^s$ is square integrable in the funnels, what we have to do is to investigate the Fourier expansion of $\theta^s$ at each inequivalent cusp, i.e. at 0 and $\infty$, and to show that $||\theta^s||$ is bounded at the cusps. As in the finite volume case, we have ([5]):

**Proposition 4.1.** *At $\infty$*

$$\theta^s(z) = (\frac{1}{i} + O(1/y))\ dz\ ,$$

*and at 0*

$$\theta^s(z) = (-\frac{1}{ic_0^2 z^2} + O(1/y))\ dz\ .$$

By Proposition 4.1 and Lemma 4.2, we conclude:

**Proposition 4.2.** *The 1-forms $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ and $\hat{\eta}^s(z) + \operatorname{Re}(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ are square integrable.*

*Proof.* Let's, for example, treat the case at $\infty$. Recall that

$$\eta^s(z) = \frac{1}{2ik(s-1)} \left( \sum_{\gamma\in\Gamma} \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right] - \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \frac{d\bar{z}}{\bar{z}} \right] \right) .$$

The asymptotic behavior of $\frac{1}{k(s-1)}\sum_{\gamma\in\Gamma}\gamma*\left[\left(\frac{y}{|z|}\right)^{s-1}\frac{d\bar{z}}{\bar{z}}\right]$ can be deduced from the preceding study. For example at $\infty$ it behaves like $-(\frac{1}{i}+O(1/y))\,dz$. We conclude that $\eta^s(z)=-\frac{1}{2}(dz+d\bar{z})+O(1/y\,dz)$ and the result.

□

Lastly, as in [5]:

**Theorem 4.2.** *The 1-form $\hat{\eta}^s$ has a meromorphic continuation to $\operatorname{Re} s > 1/2$, with $s=1$ a regular point and $\hat{\eta}$ is a harmonic form which is dual to $\eta$.*

## 5. Degenerations.

5.1. **Background material and the main results.** A family of degenerating geometrically finite hyperbolic surfaces consists of a surface $M$ and a smooth family $(g_l)_{l>0}$ of Riemannian metrics that meet the following assumptions:

(1) The Riemannian manifold $M_l=(M,g_l)$ is a geometrically finite hyperbolic surface for each $l$.
(2) There are finitely many disjoint open subsets $\mathcal{C}_{l,i}\subset M$ that are diffeomorphic to cylinders $\mathbb{R}/\mathbb{Z}\times J_i$ where $J_i\subset\mathbb{R}$ is a connected neighborhood of 0 with the metric $(x,a)\mapsto(l_i(l)^2+a^2)dx^2+((l_i(l)^2+a^2)^{-1}da^2$ and $l_i(l)\to 0$ as $l\to 0$. The curve $c_i=\mathbb{R}/\mathbb{Z}\times\{0\}$ is a closed geodesic of length $l_i(l)$.
(3) The complement of $(C_1\cup\ldots\cup C_{n_c})\cup(F_1\cup\ldots\cup F_{n_f})\cup_i\mathcal{C}_{l,i}$ where we may have some $F_j\subset\mathcal{C}_{l,i}$ is relatively compact.
(4) On $M_0:=M\setminus\cup_i c_i$, the metrics $g_l$ converge smoothly to a hyperbolic metric $g_0$ as $l\to 0$. $M_0$ is a possibly non connected hyperbolic surface that contains a pair of cusps for each $i$.

First of all, we recall some material and results. The following lemma can be found, for example, in [1], p. 252-253. The neighbourhood of points within a distance $a$ of a geodesic $\gamma$, where $d(z,\gamma)$ is the hyperbolic distance from $z$ to $\gamma$,

$$G_a=\{z\in K, d(z,\gamma)\le a\},$$

is isometric for small $a$ to a half-collar $[0,a]\times S^1$, $ds^2=dr^2+l^2\cosh^2 r\,d\theta^2$.

**Lemma 5.1.** *Suppose that $\gamma$ is a simple closed geodesic of length $l(\gamma)$ on a geometrically finite hyperbolic surface $M$. Then $\gamma$ has a collar neighbourhood of half-width $d$, such that*

$$\sinh(d)=\frac{1}{\sinh(l(\gamma)/2)}.$$

*As a consequence, if $\eta$ is any other closed geodesic intersecting $\gamma$ transversally (still assuming $\gamma$ is simple), then the lengths of the two geodesics satisfy the inequality*

$$\sinh(l(\eta)/2)\ge\frac{1}{\sinh(l(\gamma)/2)}.$$

**Lemma 5.2.** *Let $\gamma$ be a simple closed geodesic of length $l$ on a complete hyperbolic surface $M$. If $\alpha$ is a simple closed geodesic that does not intersect $\gamma$, then, putting $d(\gamma,\alpha)$ to be the hyperbolic distance of $\gamma$ to $\alpha$,*

$$\cosh d(\gamma,\alpha)\ge\coth(l/2).$$

A standard collar for a geodesic of length $l$ is a cylinder isometric to $\langle z \mapsto e^l z\rangle \backslash \mathcal{C}$ with $\mathcal{C} = \{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi - l\} \subset H$ with the restriction of the hyperbolic metric, and $\langle z \mapsto e^l z\rangle$ the cyclic group generated by the transformation $z \mapsto e^l z$. There is a constant $k_0$ (the short geodesic constant) such that each closed geodesic on $M_l$ of length at most $k_0$ has a neighbourhood isometric to the standard collar and each cusp for $M_l$ has a neighbourhood isometric to the standard cusp; furthermore, the collars for short geodesics and the cusp regions are all mutually disjoint.

Here we consider a family of surfaces $S_l = \Gamma_l \backslash H$ degenerating to the surface $S$ with only one geodesic $c_l$ being pinched, $\Gamma_l$ containing the transformation $\sigma_l(z) = e^l z$ corresponding to $c_l$. Let $K_l$ be $S_l$ minus $C_l$ the standard collar for $c_l$. There exist homeomorphisms $f_l$ from $S_l \backslash c_l$ to $S$, with $f_l$ tending to isometries $C^2$-uniformly on the compact core $K_l \subset S_l$; define $\pi_l = f_l^{-1}$. Suppose that $p$ is one of the two cusps of $S$ arising from pinching $c_l$. Let $S_0 = \Gamma \backslash H$ be the component of $S$ containing $p$ and conjugate $\Gamma$ to represent the cusp by the translation $w \mapsto w + 1$. In the following, $p = \infty$.

Let, for $\operatorname{Re} s > 1$, $\alpha_l(s,z) = \sum_{\gamma \in \langle \sigma_l \rangle \backslash \Gamma_l} \gamma * \left[ \left( \frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \right]$ be such that the hyperbolic Eisenstein series $\Omega_{c_l} = \Omega_l$ is related by $\Omega_l(s,z) = \frac{1}{k(s)} \alpha_l(s+1, z)$. Without loss of generality we suppose $S_l$ has only one funnel $F_1$. With the notation of the beginning of the paper, $S_l = K \cup (C_1 \cup ... \cup C_{n_c}) \cup F_1$ and $c_l$ is the one geodesic of the boundary of the compact core $K$. We consider the specific case of $p$, the limit of the right side of the $c_l$-collar, contained in $S_l \backslash F_1$. With the misuse of notation already mentioned

**Theorem 5.1.** *Let* $\operatorname{Re} s > 1$. *Then the family of 1-forms* $\frac{1}{l^s} \alpha_l(s, \pi_l(.))$ *converges uniformly on compact subsets of* $S_0$ *to* $\operatorname{Im} \mathcal{E}_\infty(s, .)$.

This is a particular case of Theorem 5.2 below.

The sketch of the proof of this theorem follows those of the finite volume case ([4], [27], see also [8]). To study the right side of the $c_l$-collar, let $w = \frac{1}{l} \log z$, with the principal branch $z \in H$, and conjugate $\Gamma_l$ by the map $w$ to obtain $\tilde{\Gamma}_l$ acting on $\mathcal{S}_l = \{w, 0 < \operatorname{Im} w < \pi / l\}$. The hyperbolic metric on $\mathcal{S}_l$ is $ds_l^2 = \left( \frac{l|dw|}{\sin(l \operatorname{Im} w)} \right)^2$, which tends uniformly on compact subsets to $\left( \frac{|dw|}{\operatorname{Im} w} \right)^2$. $\tilde{\Gamma}_l$ is a (non-Möbius) group of deck transformations acting on $\mathcal{S}_l$; the quotient $\tilde{\Gamma}_l \backslash \mathcal{S}_l$ is $S_l$.

In the following if a group $G$ acts on a domain $D$, we will denote by $G_A$ the stabilizer in $G$ of a subset $A \subset D$.

Let $\hat{f}_l$ be the restriction of $f_l$ to the component $\mathcal{S}_l^{(r)}$ of $S_l \backslash c_l$ containing the right half-collar for $c_l$. Let $F_l$ be a lift of $\hat{f}_l$ to the universal covers $\mathcal{A}_l$ and $H$, where $\mathcal{A}_l$ is the simply connected component of $H \backslash \pi_l^{-1}(c_l)$ which contains the standard right collar $\{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi/2\}$. $\mathcal{S}_l^{(r)} = (\Gamma_l)_{\mathcal{A}_l} \backslash \mathcal{A}_l$. More precisely [4], p. 350, [27]:

**Lemma 5.3.** *The simply connected component* $\mathcal{A}_l$ *contains* $\{z = re^{i\theta}, 1 \leq r \leq e^l, lc(l) < \theta < \pi/2\}$ *where* $c(l) \to 0$, $l \to 0$.

Start with the standard $\Gamma$ fundamental domain $\mathcal{F} = \{w \in H, 0 \leq \operatorname{Re} w < 1, \operatorname{Im} w \geq \operatorname{Im} A(w), \forall A \in \Gamma\}$. Set $D_l = F_l^{-1}(\mathcal{F})$, then $D_l$ is a fundamental domain of $\mathcal{S}_l^{(r)}$. Divide the cosets of $\langle\sigma_l\rangle\backslash(\Gamma_l - \langle\sigma_l\rangle)$ into two classes $D = \{[A], A \in \Gamma_l, \inf \operatorname{Re} A(D_l) > 0\}$ and $G = \{[A], A \in \Gamma_l, \sup \operatorname{Re} A(D_l) < 0\}$.

Then $\hat{f}_l$ has a lift $\tilde{f}_l$, a homeomorphism from a subdomain of $\mathcal{S}_l$ to $H$: $\tilde{f}_l = F_l \circ w^{-1} : w(\mathcal{A}_l) \to H$. The homeomorphism $\tilde{f}_l$ induces a group homomorphism $\rho_l : \Gamma \to \tilde{\Gamma}_l$, $A \mapsto \tilde{f}_l^{-1} A \tilde{f}_l$, $A \in \Gamma$. Now by our normalizations for $\tilde{\Gamma}_l$ and $\Gamma$, the translation $w \mapsto w + 1$ corresponds to itself and induces an isomorphism from $\Gamma$ to $(\tilde{\Gamma}_l)_{w(\mathcal{A}_l)}$. We call $\rho_l(A) \in \tilde{\Gamma}_l$ the element corresponding to $A \in \Gamma$. If we specify the further normalization $\tilde{f}_l(i) = i$, then the lifts $\tilde{f}_l$ are uniquely determined and then we have (see, e.g. [27], p. 107)

**Lemma 5.4.** *The $\tilde{f}_l$ tend uniformly on compact subsets to the identity, and thus for $A \in \Gamma$, the corresponding elements $\rho_l(A)$ tend uniformly on compact subsets to $A$.*

For $q \in \mathbb{N}$, we associate to the pinching geodesic $c_l$, the $q-$form defined for $\operatorname{Re} s > 1$ by

$$A_{l,q}(s, z) = \sum_{\langle\sigma_l\rangle\backslash\Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)}\right)^q \sin^{s-q} \theta(\gamma z)\, dz^q\,.$$

**Remark 5.1.** *As suggested by the referee, it would be interesting to have a geometric meaning of $A_{l,q}(s, z)$. As far as I know, in the case of functions, they have been study in particular in* [8], [18], [19]. *In the case of 1-forms, the analytic continuation of the hyperbolic Eisenstein series gives a harmonic dual form to the corresponding geodesic and the construction of an explicit geometric basis for the space of holomorphic 1-forms (*[20]*). In the case of 2-forms (quadratic differentials), they permit to define vector fields on Teichmüller space (*[27]*, Chapter 2). For $q-$forms ($q > 2$) I could maybe find some answer in* [21].

Divide the cosets $\langle z \mapsto z + 1\rangle\backslash(\tilde{\Gamma}_l - \langle z \mapsto z + 1\rangle)$ into two classes, the left and the right: for $\mathcal{F}_l = \tilde{f}_l^{-1}(\mathcal{F})$, $L = wGw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \inf \operatorname{Im} A(\mathcal{F}_l) > \pi/2l\}$ and $R = wDw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \sup \operatorname{Im} A(\mathcal{F}_l) < \pi/2l\}$ (the line $\{\operatorname{Im} w = \pi/2l\}$ is a lift of $c_l$, and we write $[A]$ for the $\langle z \mapsto z + 1\rangle$ coset of $A$). In particular the cosets $\langle z \mapsto z + 1\rangle\backslash(\Gamma - \langle z \mapsto z + 1\rangle)$ correspond to the right cosets of $\tilde{\Gamma}_l$: $\{[\rho_l(A)], A \in \Gamma, \langle w \mapsto w + 1\rangle\} \subset R$. Then we can write, where $\chi_+$ is the characteristic function of $\{\operatorname{Re} z > 0\}$ and $\chi_-$ that of $\{\operatorname{Re} z \leq 0\}$,

$$\begin{aligned} A_{l,q}(s, z) &= \sum_{\langle\sigma_l\rangle\backslash\Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)}\right)^q \sin^{s-q} \theta(\gamma z)\, dz^q \\ &= y(z)^{s-q} \left(\frac{1}{z^q |z|^{s-q}} \chi_+ + \sum_D \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}\right) dz^q + \\ &\quad y(z)^{s-q} \left(\frac{1}{z^q |z|^{s-q}} \chi_- + \sum_G \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}\right) dz^q \end{aligned}$$

and the $q$-form on $\mathcal{S}_l^{(r)}$:

$$A_{l,q}^R(s, z) = y(z)^{s-q} \left(\frac{1}{z^q |z|^{s-q}} \chi_+ + \sum_D \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}\right) dz^q$$

$$= l^q(\sin l \operatorname{Im} w)^{s-q}(\chi + \sum_R (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q}) dw^q, \tag{17}$$

with $w = \frac{1}{l}\log z$ and $\chi$ the characteristic function of $w(\{\operatorname{Re} z > 0\})$.

**Theorem 5.2.** *Let $S_l = \Gamma_l \backslash H$ be a family of geometrically finite hyperbolic surfaces degenerating to the surface $S$ with only one geodesic $c_l$ being pinched and with $S_l$ having only one funnel $F_1$; $\Gamma_l$ contains the transformation $\sigma_l(z) = e^l z$ corresponding to $c_l$ and the right half-collar for $c_l$ is in $S_l \backslash F_1$. Let $S_0 = \Gamma \backslash H$ be the component of $S$ containing $p$, the cusp arising from the right half-collar for $c_l$, $p = \infty$. Let $w = \frac{1}{l}\log z$ and write again with a little misuse of notation $A_{l,q}(s,w) = a_{l,q}(s,w)\, dw^q$ and the corresponding $q$-automorphic form (for $\tilde{\Gamma}_l$) $\hat{a}_{l,q}(s,w) = \operatorname{Im}(w)^q a_{l,q}(s,w)$. Denote by $\hat{a}^R_{l,q}$ the right half of $\hat{a}_{l,q}$. Then the family $(\frac{1}{l^s}\hat{a}^R_{l,q}(s, \pi_l(.)))_l$ converges uniformly on compact subsets of $S_0$ and on compact subsets of $\operatorname{Re} s > 1$ to the Eisenstein series for weight $2q$:*

$$E_{\infty,q}(s,w) = \sum_{\Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma w)^s \left(\frac{c\bar{w} + d}{cw + d}\right)^q .$$

**Remark 5.2.** *We have the analogous result: if we denote by $\hat{a}^L_{l,q}$ the left half of $\hat{a}_{l,q}$, then $\frac{1}{l^s}\hat{a}^L_{l,q}(s, \pi_l(.))$ converges to $(-1)^q E_{\mathfrak{a},q}(s,.)$, where $E_{\mathfrak{a},q}(s,.)$ is the Eisenstein series of weight $2q$ (see Definition 3.1) associated to $\mathfrak{a}$, the other cusp arising from the left half-collar for $c_l$.*

Before proving this theorem we give some complements and its corollaries.

For $\operatorname{Re} s > 1$ let $b_q(s) = e^{i\pi q/2}\int_0^\pi (\sin u)^{s-2} e^{-iqu}\, du$. Note that $b_1(s) = k(s-1)$. The function $b_q$ has the following properties (see e.g. [18]): $b_q(s+2) = \dfrac{s(s-1)}{s^2-q^2} b_q(s)$, $b_q$ admits a meromorphic continuation to all $s \in \mathbb{C}$, more precisely

$$b_q(s) = \pi 2^{-s+2} \frac{\Gamma(s-1)}{\Gamma(\frac{s+q}{2})\Gamma(\frac{s-q}{2})} .$$

In order to be consistent with the definition of Kudla and Millson's hyperbolic Eisenstein series, we may use the normalized $q-$automorphic forms

$$\Xi_{l,q}(s,z) = \frac{1}{b_q(s)} A_{l,q}(s,z). \tag{18}$$

We have indeed $\Xi_{l,1}(s,z) = \Theta(s-1,z)$ defined in (7) Section 2.1.
We recall that the series (18) converges absolutely and locally uniformly for any $z \in H$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, and that it is invariant with respect to $\Gamma$.

Let $\tilde{a}_{l,q}(s,z)$ (respectively, $\tilde{f}_{l,q}(s,z)$) be the $q-$automorphic form associated to $A_{l,q}(s,z)$ (respectively, $\Xi_{l,q}(s,z)$) via the correspondence (9). More precisely and with the notation of Theorem 5.2

$$\begin{aligned} A_{l,q}(s,z) &= \frac{\tilde{a}_{l,q}(s,z)}{y^q}\, dz^q \\ &= \frac{\hat{a}_{l,q}(s,w)}{w^q}\, dw^q, \\ \Xi_{l,q}(s,z) &= \frac{\tilde{f}_{l,q}(s,z)}{y^q}\, dz^q, \qquad \hat{f}_{l,q}(s,w) = \frac{1}{b_q(s)}\hat{a}_{l,q}(s,w). \end{aligned}$$

A straightforward computation shows that these series satisfy the functional differential equations

$$\Delta_{2q}\tilde{a}_{l,q}(s,z)+s(1-s)\tilde{a}_{l,q}(s,z)=(s+q)(q-s)\tilde{a}_{l,q}(s+2,z),$$

and

$$\Delta_{2q}\tilde{f}_{l,q}(s,z)+s(1-s)\tilde{f}_{l,q}(s,z)=s(1-s)\tilde{f}_{l,q}(s+2,z); \tag{19}$$

and so, the series $A_{l,q}(s,z)$ satisfies the functional differential equation

$$\Delta_q^{\pm}A_{l,q}(s,z)-(s\pm q)(1-s\pm q)A_{l,q}(s,z)=(s+q)(s-q)A_{l,q}(s+2,z),$$

and the series (18)

$$\Delta_q^{\pm}\Xi_{l,q}(s,z)-(s\pm q)(1-s\pm q)\Xi_{l,q}(s,z)=s(s-1)\Xi_{l,q}(s+2,z).$$

**Proposition 5.1.** *The series $A_{l,q}(s,z)$ (as well as $\Xi_{l,q}(s,z)$) admits a meromorphic continuation to all of $\mathbb{C}$.*

*Proof.* There are different ways to prove this: one is to use the functional differential equation (18) and to apply the method developed in [20] (see also [18]). More precisely, from (19) we have

$$\tilde{f}_{l,q}(s,z)=s(1-s)\int_{\mathcal{D}_l}G_s^l(z,z',q)\tilde{f}_{l,q}(s+2,z')\,d\mu(z')\,,$$

where $\mathcal{D}_l$ is a fundamental domain of $S_l$ and $G_s^l(z,z',q)$ has been defined in proposition 3.1. Another way is to use [7]. We make precise another calculation which we will develop further. We can rewrite $\tilde{a}_{l,q}(s,z)$ as

$$\tilde{a}_{l,q}(s,z)=\sum_{\langle\sigma_l\rangle\backslash\Gamma_l}\left(\frac{c\bar{z}+d}{cz+d}\right)^q\left(\frac{\overline{\gamma z}}{\gamma z}\right)^{q/2}\left(\frac{\operatorname{Im}\gamma z}{|\gamma z|}\right)^s$$

and using the Fourier development of $G_s(z,z',q)$ and the expansion of $\int_1^{e^l}G_s(z,iy',q)\,d\ln y'$ we obtain the result (see [7] corollary 4.2, p. 188). □

We recall from Section 3.3 that

**Remark 5.3.** *The possible poles of the resolvent $G_s^l(z,z',q)$ in $\operatorname{Re}s>1/2$ are simple, in finite number and at $s_{1,q}^l,...,s_{n_l,q}^l\in(\frac{1}{2},1)$ corresponding to the eigenvalues $\lambda_{k,q}^l$, $1\le k\le n_l$ of the Laplacian $-\Delta_{2q}$ on $S_l$.*

*In particular, $\tilde{f}_{l,q}(s,z)$ is holomorphic in $\Sigma_l=\{\operatorname{Re}s>1/2\}\backslash\{s_{1,q}^l,...,s_{n_l,q}^l\}$.*

We can then refine Theorem 5.2 to the following.

**Theorem 5.3.** *Let $\lambda_k^q$, $1\le k\le n$, be the eigenvalues of the Laplacian $-\Delta_{2q}$ on $S_0$ such that $0<\lambda_1^q<\lambda_2^q<...<\lambda_n^q<1/4\le\lambda_{n+1}^q$ and the corresponding $s_k^q=\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_k^q}$ with $\operatorname{Re}s_k^q\ge1/2$ and let $\Sigma=\{\operatorname{Re}s>1/2\}\backslash\{s_1^q,...,s_n^q\}$.*

*(1) In the case where the geodesic $c_l$ is not separating, without loss of generality we can suppose that the two limiting cusps are represented by $\infty$ and $0$, the family $(\frac{1}{l^s}\hat{f}_{l,q}(s,\pi_l(.)))_l$ converges uniformly on compact subsets of $S_0$ and on compact subsets of $\Sigma$ to $\dfrac{1}{b_q(s)}E_{\infty,q}(s,.)+\dfrac{(-1)^q}{b_q(s)}E_{0,q}(s,.)$.*

(2) *In the case where $c_l$ is the geodesic boundary of a funnel, the family $(\frac{1}{l^s}\hat{f}_{l,q}(s,\pi_l(.)))_l$ converges uniformly on compact subsets of $S_0$ and $\Sigma$ to $\dfrac{1}{b_q(s)}E_{\infty,q}(s,.)$.*

**Remark 5.4.** *In the case $c_l$ is separating and different from the geodesic boundary of a funnel, as in Theorem 5.2 and Remark 5.2, we can define the right and left half of $\frac{1}{l^s}\hat{f}_{l,q} = \frac{1}{l^s}\hat{a}^R_{l,q} + \frac{1}{l^s}\hat{f}^L_{l,q}$ each series in the right member converges to the Eisenstein series of the corresponding cusp.*

**Remark 5.5.** *We remark that as $\sigma_l$ is the geodesic in the funnel, it follows that $D = \langle\sigma_l\rangle\backslash(\Gamma_l - \langle\sigma_l\rangle)$ and $G = \emptyset$.*

We have seen that in the infinite volume case, $s = 1$ is a regular value for $E_{\infty,q}(s,z)$ for every $q \in \mathbb{N}$. In the finite volume case, we know that $s = 1$ is a simple pole for $E_{\infty,0}(s,z)$, which is no longer the case for $q \geq 1$:

**Remark 5.6.** *For $q \in \mathbb{N}^*$, $s = 1$ is a regular value of $E_{\infty,q}(s,z)$.*

*Proof.* This comes from the Fourier development of $E_{\infty,q}(s,z)$: we have ([7], p. 175)

$$E_{\infty,q}(s,z) = y^s + \varphi_q(s)y^{1-s} + \sum_{m\neq 0} a_m(y,s)_q e^{2i\pi mx},$$

with $\varphi_q(s) = \dfrac{\Gamma^2(s)}{\Gamma(s-q)\Gamma(s+q)}e^{-\pi iq}\varphi_0(s)$. The function $\varphi_0$ has a simple pole at $s = 1$ (with residue $\mathrm{Vol}(S_0)/\pi$) and $1/\Gamma(s-q)$ has a zero at $s = 1$. □

We now give the poles of $b_q(s)$:

**Lemma 5.5.** *For an even $q$, the poles of $b_q(s)$ are simple and at the numbers $s = 1 - 2k$, $k \in \mathbb{N}$.*
*For an odd $q$, the poles of $b_q(s)$ are simple and at the numbers $s = -2k$, $k \in \mathbb{N}$.*

From the preceding results and Theorem (5.3) we deduce

**Corollary 5.1.** *With the preceding notation,*

(1) *Suppose $q$ is odd. If the geodesic $c_l$ is not separating (respectively, the geodesic boundary of a funnel) the family $(\frac{1}{l^s}\hat{a}_{l,q}(s,\pi_l(.)))_l$ converges uniformly on compact subsets of $S_0$ and on compact subsets of $\Sigma$ to $E_{\infty,q}(s,.)+(-1)^q E_{0,q}(s,.)$ (respectively, $E_{\infty,q}(s,.)$).*
(2) *Suppose $q$ is even, we have the same results as the preceding, on replacing $\Sigma$ by $\Sigma\backslash\{1\}$.*

**Remark 5.7.** *For $q = 1$ we obtain the result announced in the Introduction.*

5.2. **Proofs of the main theorems and final remarks.**
*Proof of Theorem 5.2.*

It is enough to demonstrate the convergence for $\beta$ a relatively compact set in the fundamental domain $\mathcal{F}$. Given $\epsilon > 0$, denote by $G_\epsilon$ the set of cosets and representatives for $\langle z \mapsto z+1\rangle\backslash\Gamma$ such that $\sup \mathrm{Im}\, A(\mathcal{F}) < \epsilon$ for $[A] \notin G_\epsilon$ and let $R_l$ be the corresponding cosets of $\langle z \mapsto z+1\rangle\backslash\tilde{\Gamma}_l$ with the corresponding representatives.

**Lemma 5.6.** *The set $G_\epsilon$ is finite.*

*Proof.* We have to show that the number of representatives for $\langle z \mapsto z+1\rangle\backslash\Gamma$ such that $\sup \operatorname{Im} A(\mathcal{F}) \geq \epsilon$ is finite. Take a real $a$ such that the cusp for the parabolic point $\infty$, $\{0 \leq \operatorname{Re} z < 1, \operatorname{Im} z \geq a\} \subset \mathcal{F}$. As for all $g \in G_\epsilon$, different from the identity, $g(\mathcal{F})$ and $\mathcal{F}$ are disjoints, we have $\sup \operatorname{Im} g(\mathcal{F}) \geq \epsilon$ and $\sup \operatorname{Im} g(\mathcal{F}) \leq a$. Let $\mathcal{S}$ a system of representatives of $\langle z \mapsto z+1\rangle\backslash\Gamma$ such that the disjoints sets $\{[A](\mathcal{F}), [A] \in \mathcal{S}\}$ recover $B = \{0 \leq \operatorname{Re} z < 1\}$. The disjoints sets $\{g(\mathcal{F}), g \in G_\epsilon\}$ recovering the compact set $\{0 \leq \operatorname{Re} z \leq 1, \epsilon \leq \operatorname{Im} z \leq a\}$ of $B$, the cardinal of $G_\epsilon$ must be finite. □

The cosets $G_\epsilon$ of $\Gamma$ satisfy (modulo the action of $\langle z \mapsto z+1\rangle$) $\{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w > \epsilon\} \subset \cup_{A\in G_\epsilon} A(\mathcal{F})$; thus for $l$ sufficiently small the cosets $R_l$ satisfy (modulo the action of $\langle z \mapsto z+1\rangle$) $\{0 \leq \operatorname{Re} w < 1, 2\epsilon < \operatorname{Im} w < \pi/2l\} \subset \cup_{A\in R_l} A(\mathcal{F}_l)$ (this is a consequence of the convergence on compact subsets of the $\tilde{f}_l$ and of the fact that $\mathcal{F}_l$ contains the right half-collar for $c_l$, $\{0 \leq \operatorname{Re} w < 1, c(l) \leq \operatorname{Im} w < \pi/2l\}$, see Lemma 5.3. Now for a right coset $[A] \in R - R_l$, we have that $A(\mathcal{F}_l)$ lies below the $c_l$ geodesic $\{\operatorname{Im} w = \pi/2l\}$ and is disjoint modulo the action $\langle z \mapsto z+1\rangle$ from $\{0 \leq \operatorname{Re} w < 1, 2\epsilon < \operatorname{Im} w < \pi/2l\}$, since the latter is covered by the $R_l$ cosets. Thus for $[A] \in R - R_l$, modulo the action of $\langle z \mapsto z+1\rangle$, we have $A(\mathcal{F}_l) \subset \{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\}$. For $w \in \beta$, using (17), we write

$$\begin{aligned}
\frac{1}{l^s}\hat{a}^R_{l,q}(s,w) &= \frac{1}{l^{s-q}}(\operatorname{Im} w)^q(\sin l \operatorname{Im} w)^{s-q}(\chi + \sum_R (\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q}) \\
&= \frac{1}{l^{s-q}}(\operatorname{Im} w)^q(\sin l \operatorname{Im} w)^{s-q}(\chi + \sum_{R_l} (\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q}) \\
&+ \frac{1}{l^{s-q}}(\operatorname{Im} w)^q(\sin l \operatorname{Im} w)^{s-q}(\sum_{R-R_l} (\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q}).
\end{aligned}$$

Now on $\beta$ for $l$ sufficiently small, $\chi$ is identically unity and because of Lemma 5.4, the coset representatives for $R_l$ approximate the coset representatives for $G_\epsilon$. Thus for $\frac{1}{l^s}\hat{a}^R_{l,q}(s,.)$, in the last equality, the first sum is uniformly close to the corresponding terms for $E_{\infty,q}(s,.)$.

The principal problem lies in estimating the second sum: it remains to show that

$$\lim_{l\to 0} \sum_{R-R_l} |\tilde{\gamma}'w|^\sigma = 0\,,$$

where $\sigma = \operatorname{Re} s$.

$$\sum_{R-R_l} |\tilde{\gamma}'w|^\sigma = \sum_{w^{-1}(R-R_l)w} |z_l \frac{\gamma' z_l}{\gamma z_l}|^\sigma \leq |z_l|^\sigma \sum_{w^{-1}(R-R_l)w} |\gamma' z_l|^\sigma\,,$$

where $w = \frac{1}{l}\log z_l$ and $\gamma\mathcal{F}_l \subset w^{-1}(\{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\} \cap \beta)$. We deduce

$$\sum_{R-R_l} |\tilde{\gamma}'w|^\sigma \leq \frac{1}{\sin^\sigma(l \operatorname{Im} w)} \sum_{w^{-1}(R-R_l)w} \operatorname{Im}^\sigma(\gamma z_l)\,.$$

Let $\epsilon_0 \in ]0, \sinh^{-1} 1[$ such that for all $l$ sufficiently small and all $w \in \beta$, $B(z_l, \epsilon_0) \subset \mathcal{F}_l$, then $\cup_{\gamma\in w^{-1}(R-R_l)w} B(\gamma z_l, \epsilon_0) \subset \{z, 0 < \arg z < 2\epsilon l, 1 \leq |z| \leq e^l\}$. Moreover there exists a $\Lambda_{\epsilon_0}$ independent of $z_0 \in H$ such that

$$\int_{B(z_0,\epsilon_0)} y^\sigma \frac{dxdy}{y^2} = \Lambda_{\epsilon_0} y(z_0)^\sigma\,;$$

this is a particular case of the following Proposition 5.2, needed further in a more general case. Hence

$$\sum_{w^{-1}(R-R_l)w} (\operatorname{Im}\gamma z_l)^\sigma = \frac{1}{\Lambda_{\epsilon_0}} \sum_{w^{-1}(R-R_l)w} \int_{B(\gamma z_l,\epsilon_0)} y^\sigma \frac{dxdy}{y^2}\,.$$

Now (see, e.g. [27], p. 102):

**Lemma 5.7.** *The multiplicity of the projection map $H \to \Gamma\backslash H$ restricted to $B(z_0,\eta)$ with $2\eta < c_0$ is at most $M\rho^{-2}(z_0)$, where $M$ is a constant and $\rho(z_0)$ the injectivity radius at $z_0$.*

For the convenience of the reader, we give a detailed proof.

*Proof.* If $B(z_0,\eta)\cap B(\gamma z_0,\eta)\neq\emptyset$, $\gamma\in\Gamma$, then $d(z_0,\gamma z_0)<2\eta<c_0$ and $z_0$ is in a cusp region or the collar for a short geodesic. Let $c=c_0/2$. Then $\rho(z_0)<c$. Denote by $m(\eta)$ the multiplicity of the projection restricted to $B(z_0,\eta)$. As $2\eta+\rho(z_0)<3c$ and the $B(\gamma z_0,\rho(z_0))$ are disjoint, we have

$$m(\eta)\mu(B(z_0,\rho(z_0)) \le \mu(B(z_0,3c))\,.$$

Hence $m(\eta)\le \frac{\cosh 3c-1}{\cosh\rho(z_0)-1}\le 2(\cosh 3c-1)\rho(z_0)^{-2}$. □

Now we have

$$\begin{aligned}\sum_{\gamma\in w^{-1}(R-R_l)w} (\operatorname{Im}\gamma z_l)^\sigma &\le \frac{1}{\Lambda_{\epsilon_0}} m(\epsilon_0) \int_{\cup_{\gamma\in w^{-1}(R-R_l)w} B(\gamma z_l,\epsilon_0)} y^{\sigma-2}dxdy\\ &\le \frac{A(c_0)}{\Lambda_{\epsilon_0}}\rho^{-2}(z_l)\int_{\{z,0<\arg z<2\epsilon l,1\le|z|\le e^l\}} y^{\sigma-2}dxdy,\end{aligned}$$

with $A(c_0)=2(\operatorname{ch}\frac{3c_0}{2}-1)$.

Then

$$\sum_{w^{-1}(R-R_l)w} (\operatorname{Im}\gamma z_l)^\sigma \le \frac{A(c_0)}{\Lambda_{\epsilon_0}}\rho^{-2}(z_l)\frac{e^{l\sigma}-1}{\sigma}\frac{(2\epsilon l)^{\sigma-1}}{\sigma-1}\,. \tag{20}$$

Moreover, as $B(z_l,\epsilon_0)\subset\mathcal{F}_l$, $\rho(z_l)\ge\epsilon_0$ and the conclusion follows.

*Proof of Theorem 5.3.*

The main point of the proof is to use Vitali's theorem, as in the finite volume case, we follow [4], Proof of Theorem 4.2. More precisely, first we will recall a definition.

**Definition 5.1.** *A family $(f_n)$ of meromorphic functions on a domain $\mathcal{O}$ in $\mathbb{C}$ is called bounded in $\mathcal{O}$ if*

*(1) $\exists(z_m)_m$ a discrete subset in $\mathcal{O}$,*
*(2) $\forall K$ compact set of $\mathcal{O}\backslash\{z_m\}$, $\exists n(K)$, $\forall n\ge n(K)$, $f_n$ has no pole in $K$,*
*(3) $M_K=\sup_{n\ge n(K)}(\sup_{z\in K}|f_n(z)|)<+\infty$.*

As before, $\beta$ will denote a relatively compact set in the fundamental domain $\mathcal{F}$ of $S_0$. The family $(\frac{1}{l^s}\hat{f}_{l,q}(s,\pi(w)))_l$, where $\hat{f}_{l,q}(s,w)=\frac{1}{b_q(s)}\hat{a}_{l,q}(s,w)$, is a family of meromorphic functions on $\Sigma=\{\operatorname{Re}s>1/2\}\backslash\{s_1^q,...,s_n^q\}$. Now, by Theorem (5.2), for every compact $C\subset\{\operatorname{Re}s>1\}$ this family converges uniformly (a fortiori simply) on $C\times\beta\subset\{\operatorname{Re}s>1\}\times\mathcal{F}$ to

(1) $\frac{1}{b_q(s)}(E_{\infty,q}(s,.)+(-1)^q E_{0,q}(s,.))$ in case (1) of Theorem (5.2),
(2) $\frac{1}{b_q(s)}E_{\infty,q}(s,.)$ in case (2).

We will show that for every compact set $K$ in $\Sigma$, there exists $l(K)$ such that for all $l$ less than $l(K)$, $s\mapsto \frac{1}{l^s}\hat{f}_{l,q}(s,\pi(.))$ has no pole on $K$ and $M_K=\sup_{l\le l(K)}(\sup_{s\in K}||\frac{1}{l^s}\hat{f}_{l,q}(s,\pi(.))||_{\infty,\beta})<+\infty$—and then we can conclude the result.

The outline of the proof being independent of $q$, we will omit writing $q$ and only indicate the changes needed in the case of general $q$ if necessary. In particular in this paragraph we can write $\Delta$ instead of $\Delta_{2q}$.

As in the finite volume case, we start from the equality

$$\frac{1}{l^s}\tilde{f}_l(s,z)=s(1-s)\int_{\mathcal{D}_l}G_s(z,z')\tilde{f}_l(s+2,z')/l^s\,d\mu(z')\,;$$

where, to simplify notation, we omit writing $q=0$; but we can write $G_s^l(z,z')$ to insist on the dependence in $l$. The definition and properties of $G_s^l(z,z')$ are given in Section 3.3. We start from Remark (5.3), the poles of $G_s^l(z,z')$ in $\operatorname{Re}s>1/2$ are simple, in finite number and in $(\frac{1}{2},1)$, they correspond to the small eigenvalues ($<1/4$) of the Laplacian on $S_l$. As $l$ tends to zero, these poles tend to the corresponding poles of the resolvent of the limit surface; in other words the small eigenvalues tend to the small eigenvalues (see [26],[7]). Now take $K$ a compact of $\Sigma$, there exists $l(K)$ such that for $l\le l(K)$, all the finite number of poles of $G_s^l(z,z')$ are in the open set $\Sigma\backslash K$, from the same Remark (5.3), $s\mapsto\frac{1}{l^s}\hat{f}_{l,q}(s,\pi(.))$ has no pole on $K$. We will suppose $C$ is a compact subset of $S_0$, $Y\subset S_0$ is such that $\pi_l(Y)$ is a standard collar, and $\pi_l(Y)\cap\pi_l(C)=\emptyset$, $X=S_0\backslash Y$, $K'$ a compact subset of $\Sigma'=\Sigma\backslash\{1\}$. Decompose the preceding integral and write

$$\frac{1}{l^s}\tilde{f}_l(s,\pi_l(w))=I_l+J_l\,,$$

where

$$I_l=\int_{\pi_l(Y)}G_s(\pi_l(w),z')\tilde{f}_l(s+2,z')/l^s\,d\mu(z')\,,\qquad J_l=\int_{\pi_l(X)}G_s(\pi_l(w),z')\tilde{f}_l(s+2,z')/l^s\,d\mu(z')\,.$$

Now the main step is to estimate $G_s(z,z',q)$. We apply [7] Corollary 1.3, p. 151:

**Proposition 5.2.** *If $\Delta_{2q}f=s(s-1)f$ in some non-Euclidean disc $B_r$ of radius $r$ about $z_0\in H$, then $f$ has the mean value property:*

$$f(z_0)=\frac{1}{m_q(r,s)}\int_{B_r}f(z)\left(\frac{z-\bar{z_0}}{z_0-\bar{z}}\right)^q d\mu(z)\ \text{ where }\ m_q(r,s)=2\pi\int_1^r \operatorname{sh}r P_{s,q}(r)\approx\pi r^2\ \text{ as }\ r\to 0\,;$$

*$P_{s,q}(r)=(1-\tanh^2\frac{r}{2})^s\,F(s-q,s+q,1;\tanh^2\frac{r}{2})$ and $F$ is the Gauss's hypergeometric function.*

to conclude that

$$\forall w\in C,\forall z'\in\pi_l(Y),\forall s\in K',|G_s(\pi_l(w),z',q)|\le O(1)||G_s(\pi_l(w),.,q)||_{L_2}\tag{21}$$

(see also [4] Lemme 4.4).
The next point is to show that

$$\forall(w,s)\in C\times K',||G_s(\pi_l(w),.,q)||_{L_2}=O(1),\tag{22}$$

the constants in question depend only on the compact sets. We can use [26], theorem 15(1), but we give here another presentation using the spectral decomposition.

In effect, we have, using the notation in Section 3.3, in particular Proposition (3.3), where $\{\varphi_n\}$ is a complete orthonormal basis of eigenfunctions of $\Delta$ with corresponding eigenvalues $\{\lambda_n\}$ and where we don't note the dependence on $l$ of the eigenfunctions, cuspidal and funnel Eisenstein series:

**Lemma 5.8.** *For $a > 1$*

$$G_s^l(\pi_l(w),z) = G_a^l(\pi_l(w),z) + \sum_n \left[\frac{1}{s(1-s)-\lambda_n} - \frac{1}{a(1-a)-\lambda_n}\right] \varphi_n(\pi_l(w))\overline{\varphi_n(z)} +$$

$$\frac{1}{4\pi}\sum_j \int_{-\infty}^{+\infty} \left[\frac{1}{s(1-s)-(1/4+t^2)} - \frac{1}{a(1-a)-(1/4+t^2)}\right] E_j^c(\frac{1}{2}+it,\pi_l(w))\overline{E_j^c(\frac{1}{2}+it,z)}\,dt +$$

$$\frac{1}{4\pi}\sum_k \int_{-\infty}^{+\infty} \left[\frac{1}{s(1-s)-(1/4+t^2)} - \frac{1}{a(1-a)-(1/4+t^2)}\right] \int_1^{e^{l_k}} E_k^f(\frac{1}{2}+it,\pi_l(w),b)\overline{E_k^f(\frac{1}{2}+it,z,b)}\,db\,dt\,,$$

*all the sums are finite.*

*Proof.* We can use and follow the method in for example [13] (and the references therein), [1], p. 103, to prove it, but not to be too long we can also remark as in [16], p. 87-88, [1], Theorem 6.2, p. 82, that, with $E_\alpha$ corresponding to any $E_j^c$, $E_k^f$

$$\Delta E_\alpha(\frac{1}{2}+it,) + (1/4+t^2)E_\alpha(\frac{1}{2}+it,) = 0,$$

and $E_\alpha(\frac{1}{2}+it,.) \in \rho^{-1/2}L^2(S_l)$.

So for $u$, $\operatorname{Re} u > 1$ ($\operatorname{Re} u - 1/2 > 1/2$)

$$R_u(\Delta + u(1-u))E_\alpha = E_\alpha.$$

We deduce that,

$$\left(\overline{G_u^l(\pi_l(w),)}, E_\alpha(\frac{1}{2}+it,)\right) = \frac{1}{\bar{u}(1-\bar{u})-(1/4+t^2)}\overline{E_\alpha(\frac{1}{2}+it,\pi_l(w))}$$

and in a similar way as $\varphi_n \in L^2(S_l)$,

$$\frac{1}{u(1-u)-s_n(1-s_n)}\varphi_n(\pi_l(w),b) = \int_{S_l} G_u^l(\pi_l(w),z)\varphi_n(z)\,d\mu(z)\,.$$

Applying Proposition 3.3 to $\overline{G_s^l(\pi_l(w),)} - \overline{G_a^l(\pi_l(w),)}$ by complex conjugation we obtain the lemma for $\operatorname{Re} s > 1$ and by analytic continuation, to $\operatorname{Re} s > 1/2$. □

From this we now deduce, writing $\lambda_u = u(1-u)$,

**Lemma 5.9.**

$$||G_s^l(\pi_l(w),.) - G_a^l(\pi_l(w),.)||_{L^2}^2 = |\lambda_a-\lambda_s|^2 \sum_n \frac{|\varphi_n(\pi_l(w)|^2}{|\lambda_s-\lambda_n|^2|\lambda_a-\lambda_n|^2} +$$

$$|\lambda_a-\lambda_s|^2 \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{|E_j^c(1/2+it,\pi_l(w))|^2}{|\lambda_s-(1/4+t^2)|^2|\lambda_a-(1/4+t^2)|^2}\,dt +$$

$$|\lambda_a-\lambda_s|^2 \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{|\lambda_s-(1/4+t^2)|^2|\lambda_a-(1/4+t^2)|^2} \int_1^{e^{l_k}} |E_k^f(1/2+it,\pi_l(w),b)|^2\,db\,dt\,.$$

So there exists a constant $M$, depending only on the compact set $K'$ of $\Sigma'$, such that

$$||G_s(\pi_l(w),.,l) - G_a(\pi_l(w),.,l)||^2_{L^2} \leq \frac{|\lambda_a - \lambda_s|^2}{M} ||G_a(\pi_l(w),.,l)||^2_{L^2} .$$

The fact that we have for $a > 1$, $||G_a(\pi_l(w),.,q)||_{L_2} = O(1)$, the constants in question depend only on the compact sets, is an adaptation of [13] or [14].

Now we are going to prove that $I_l = O(1)$ and $J_l = O(l^2)$.

With the same proof as [4] Lemma 4.3, we have $\int_{\pi_l(Y)} |\tilde{f}_l(s+2,z)|\, d\mu(z) = O(1) l^{\mathrm{Re}\, s}$ true for $\mathrm{Re}\, s > 1/2$.

Recall that $S_l = R \cup C_1 \cup ... \cup C_{n_c} \cup F_1$ where here, we denote by $R$ the compact core. $J_l = \int_{\pi_l(X)\cap R} G_s(z,z')\tilde{f}_l(s+2,z')/l^s\, d\mu(z') + \int_{\pi_l(X)\cap(C_1\cup...\cup C_{n_c}\cup F_1)} G_s(z,z')\tilde{f}_l(s+2,z')/l^s\, d\mu(z')$ .

By Theorem 5.2 and (22), $\int_{\pi_l(X)\cap K} G_s(z,z')\tilde{f}_l(s+2,z')/l^s\, d\mu(z') = O(l^2)$.

For $z' \in C_1 \cup ... \cup C_{n_c} \cup F_1$ we have the same estimates as in (21). Now let's verify that we have also $\int_{R\cap(C_1\cup...\cup C_{n_c}\cup F_1)} \tilde{f}_l(s+2,z')/l^s\, d\mu(z') = O(1)$.

We have

$$\begin{aligned} \int_{R\cap(C_1\cup...\cup C_{n_c}\cup F_1)} \tilde{f}_l(s+2,z')\, d\mu(z') &\leq \frac{2}{|k(s+1)|} \int_{1\leq r\leq e^r} (\sin\theta)^{\mathrm{Re}\, s} \frac{dr d\theta}{r} \\ &\leq \frac{2}{|k(s+1)|} l \times l^{\mathrm{Re}\, s+1} \end{aligned}$$

This allows concluding that $I_l = O(1)$ and $J_l = O(l^2)$. This ends like in [5].

**Remark 5.8.** *1) We refer also to the result of Fay* [6]*, final remark, p. 201–202. 2) It is possible to establish a link between hyperbolic Eisenstein series and generalized eigenfunctions (I thank F. Naud for this suggestion) as presented in Borthwick (*[1]*, p. 68)*

*Let $E_{f_l}(s,z,t)$ correspond to the funnel with pinching boundary geodesic. We use the first point of this remark and its notations to write $E_{f_l}(s,z,t) = b^s E^f(s,z,b) = \sum_{n\in\mathbb{Z}} F_{n,0}(z,s) b^{\bar{n}}$, where $b = e^t$ and $\bar{n} = 2i\pi n/l$; and conclude that as $l \to 0$,*

$$\frac{1}{l}\int_0^l E_{f_l}(s,z,t)\, dt \to E_\infty(s,z) .$$

**Application.** One of the applications we can think about is in studying the degeneration of the residues of the hyperbolic Eisenstein series. Let us give an example: we look at the case of a degenerating family of compact Riemann surfaces, a non-separating geodesic being pinched, with the family of scalar-valued hyperbolic Eisenstein series degenerating. Remember that we obtained that the series $\frac{1}{l^s} A_l(s,z)$ converges uniformly on compact subsets of $S_0$ to $E_\infty(s,z) + E_0(s,z)$. The last sum of Eisenstein series has no poles on $[1/2,1]$ except at $s = 1$ and a finite number of $s_k = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_k}$ where the $\lambda_k$ correspond to the residual spectrum. In other words, if $(\lambda_k(l))$ converges to $\lambda_k$, where $\lambda_k$ is a small cuspidal eigenvalue, then $Res(\frac{1}{l^s} A_l(s,z)) \to 0$, otherwise $Res(\frac{1}{l^s} A_l(s,z)) \to Res(E_\infty(s,z) + E_0(s,z))$. There are many obstacles to carrying this calculation further. First of all, we are only dealing with small eigenvalues and so here there is no hope to characterize the embedded eigenvalues through degeneration. Moreover, we need to take into account the multiplicity of an eigenvalue $\lambda_k(l)$. So the easiest result we can obtain is a characterization of the residual spectrum (recall that an eigenvalue in the

residual spectrum is simple) with $\alpha_k(l) = \int_{c_l} \psi_{k,l}(z)\, d\mu(z)$ where the eigenfunction $\psi_{k,l}$ is associated to the eigenvalue $\lambda_k(l)$; if $\lambda_k$ is a pole of $E_\infty(s,z) + E_0(s,z)$, then $\alpha_k(l) = O(l^{1/2+\sqrt{1/4-\lambda_k(l)}})$, and otherwise $\alpha_k(l) = o(l^{1/2+\sqrt{1/4-\lambda_k(l)}})$.

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