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HYPERBOLIC EISENSTEIN SERIES FOR GEOMETRICALLY FINITE HYPERBOLIC SURFACES OF INFINITE VOLUME.

THÉRÈSE FALLIERO

ABSTRACT. Let M be a geometrically finite hyperbolic surface of infinite volume. After writing down the spectral decomposition for the Laplacian on 1-forms of M , we generalize the Kudla and Millson's construction of hyperbolic Eisenstein series and other related results (see theorems 3.1, 4.2, 5.1, 5.3).

INTRODUCTION.

The spectrum of the Laplace-Beltrami operator for a compact Riemann surface is discrete, it is no more the case when M is not compact. For example when we withdraw one point from M , it appears a continuous part in the spectrum whose spectral measure is described by an Eisenstein series. The study of the limiting behavior of the spectrum of the Laplace-Beltrami operator for a degenerating family of Riemann surfaces with finite area hyperbolic metric have been used to explain this apparition (see for example [22], [15], [13]). This paper finds one of its motivation in the general study of approximation to Eisenstein series (see for example the question of L.Ji in [15], p.308, line 15, concerning the approximation of Eisenstein series by suitable eigenfunctions of a degenerating family of hyperbolic Riemann surfaces). We hope to surround it via hyperbolic Eisenstein series (for results on degenerating Eisenstein series see, for example [18], [19], [8], [9]). What we really do here is to develop the suggestion in [17], to construct hyperbolic Eisenstein series and harmonic dual forms in the infinite volume case: in this context we verify the convergence of hyperbolic Eisenstein series and the fact that it permits to realize a harmonic dual form to a simple closed geodesic on a geometrically finite hyperbolic surface of infinite volume (theorem 3.1) and in a similar way of an infinite geodesic joining a pair of punctures (theorem 4.2). Moreover in section 5 we generalize the definition of hyperbolic Eisenstein series to the case of q -forms (see section 5 formula (10)). In particular we obtain a degeneration of hyperbolic Eisenstein series to horocyclic ones (theorem 5.3(2)). Since it is in the center of our motivation let's be a little more precise on this new result.

Let $(S_l)_l$ be a degenerating family of Riemann surfaces with infinite area hyperbolic metric, S_l having a funnel F_l whose boundary geodesic is denoted c_l . We denote $\Omega_{c_l} = \Omega_l$ the hyperbolic Eisenstein series of Kudla-Millson theory associated to the pinching geodesic c_l (see section 2), S_0 the component of the limiting surface

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containing the cusp ∞ (see section 5.1) and \mathcal{E}_∞ the horocyclic Eisenstein series associated to the limiting cusp (see formula (7)). For $s \in \{\operatorname{Re} s > -1/2\}$, except possibly a finite number of points in $(-1/2, 0)$, the family of meromorphic 1-forms $\frac{1}{t^{s+1}}\Omega_l(s, \cdot)$ converges uniformly on compact subsets of S_0 to $\frac{\Gamma(1+\frac{s}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{s}{2})} \operatorname{Im} \mathcal{E}_\infty(s+1, \cdot)$.

1. PRELIMINARY DEFINITIONS

Let us recall the standard analytic and geometric notations which will be used. In this paper a surface is a connected orientable two-dimensional manifold, without boundary unless otherwise specified. We denote by H the hyperbolic upper half-plane endowed with its standard metric of constant gaussian curvature -1 . A topologically finite (i.e. finite Euler characteristic) surface is a surface homeomorphic to a compact surface with finitely many points excised and a geometrically finite hyperbolic surface M is a topologically finite, complete Riemann surface of constant curvature -1 . We will require that M is of infinite volume, then there exists a finitely generated, torsion free, discrete subgroup, Γ , of $PSL(2, \mathbb{R})$, unique up to conjugation, such that M is the quotient of H by Γ acting as Möbius transformations, Γ is a Fuchsian group of the second kind and Γ has no elliptic elements. The group Γ admits a finite sided polygonal fundamental domain in H . We recall now the description of the fundamental domain of $M = \Gamma \backslash H$ (see [1]). Let $L(\Gamma)$ be the limit set of Γ , that is the set of limit points (in the Riemann-sphere topology) of all orbits Γz for $z \in H$ and $O(\Gamma) = \mathbb{R} \cup \{\infty\} - L(\Gamma)$. As $L(\Gamma)$ is closed in $\mathbb{R} \cup \{\infty\}$, $O(\Gamma)$ is open and so can be written as a countable union $O(\Gamma) = \cup_{\alpha \in A} O_\alpha$ where the O_α are disjoint open intervals in $\mathbb{R} \cup \{\infty\}$. Then let $\Gamma_\alpha = \{\gamma \in \Gamma, \gamma(O_\alpha) = O_\alpha\}$. This is an elementary hyperbolic subgroup of Γ . The fixed points of Γ_α are exactly the end-points of O_α . There is a finite subset $\{\alpha(1), \alpha(2), \dots, \alpha(n_f)\} \subset A$ so that, for $\alpha \in A$, O_α is conjugate to precisely one $O_{\alpha(j)}$ ($1 \leq j \leq n_f$). Let λ_α be the half-circle, lying in H , joining the end-points of O_α . Let Δ_α be the region in H bounded by O_α and λ_α . The Δ_α ($\alpha \in A$) are mutually disjoint.

Let P be the set of parabolic vertices of Γ , and for $p \in P$ let Γ_p be the parabolic subgroup of Γ fixing p . There is a finite subset $\{p(1), p(2), \dots, p(n_c)\} \subset P$ so that Γ_p is conjugate to precisely one $\Gamma_{p(j)}$ ($1 \leq j \leq n_c$). A circle lying in H and tangent to ∂H at p is called a horocycle at p . We can construct an open disc C_p determined by a horocycle at $p \in P$ so that:

- (i) if $p, q \in P, p \neq q$, then $C_p \cap C_q = \emptyset$,
- (ii) $\gamma(C_p) = C_{\gamma(p)}$ ($\gamma \in \Gamma$),
- (iii) $C_p \cap \Delta_\alpha = \emptyset$ ($p \in P, \alpha \in A$).

If we consider the set $H - (\cup_{p \in P} C_p \cup \cup_{\alpha \in A} \Delta_\alpha)$, we see that it is invariant under Γ . We can find a finite-sided fundamental domain D for the action of Γ on this set; D is relatively compact in H .

Proposition 1.1. *There is a fundamental domain D for Γ of the form*

$$D = K^* \cup \cup_{j=1}^{n_f} D_{\alpha(j)} \cup \cup_{k=1}^{n_c} D_{p(k)}^*$$

where

- 1) K^* is relatively compact in H ,

- 2) $D_{\alpha^{(j)}}$ is a standard fundamental domain of $\Gamma_{\alpha^{(j)}}$ on $\Delta_{\alpha^{(j)}}$,
3) $D_{p^{(k)}}^*$ is a standard fundamental domain for $\Gamma_{p^{(k)}}$ on $C_{p^{(k)}}$.

We should note that $p \neq 0$ if and only if Γ is of the second kind.

The Nielsen region of the group Γ is the set $\tilde{N} = H - (\cup_{\alpha \in A} \Delta_{\alpha})$, the truncated Nielsen region of Γ is $\tilde{K} = \tilde{N} - (\cup_{p \in P} C_p)$, $K = \Gamma \backslash \tilde{K}$ is called the compact core of M . So the surface $M = \Gamma \backslash H$ can be decomposed into a finite area surface with geodesic boundary N , called the Nielsen region, on which infinite area ends F_i are glued: the funnels. The Nielsen region N is itself decomposed into a compact surface K with geodesic and horocyclic boundary on which non compact, finite area ends C_i are glued: the cusps. We have $M = K \cup C \cup F$, where $C = C_1 \cup \dots \cup C_{n_c}$ and $F = F_1 \cup \dots \cup F_{n_f}$.

A hyperbolic transformation $T \in PSL(2, \mathbb{R})$ generates a cyclic hyperbolic group $\langle T \rangle$. The quotient $C_l = \langle T \rangle \backslash H$ is a hyperbolic cylinder of diameter $l = l(T)$. By conjugation, we can identify the generator T with the map $z \mapsto e^l z$, and we define Γ_l to be the corresponding cyclic group. A natural fundamental domain for Γ_l would be the region $\mathcal{F}_l = \{z \in H, 1 \leq |z| \leq e^l\}$. The y -axis is the lift of the only simple closed geodesic on C_l , whose length is l . The standard funnel of diameter $l > 0$, F_l , is the half hyperbolic cylinder $\Gamma_l \backslash H$, $F_l = (\mathbb{R}^+)_r \times (\mathbb{R} \backslash \mathbb{Z})_x$ with the metric $ds^2 = dr^2 + l^2 \cosh^2(r) dx^2$.

We can always conjugate a parabolic cyclic group $\langle T \rangle$ to the group Γ_{∞} generated by $z \mapsto z + 1$, so the parabolic cylinder is unique up to isometry. A natural fundamental domain for Γ_{∞} is $\mathcal{F}_{\infty} = \{0 \leq \text{Re } z \leq 1\} \subset H$. The standard cusp C_{∞} is the half parabolic cylinder $\Gamma_{\infty} \backslash H$, $C_{\infty} = ([0, \infty)_r \times (\mathbb{R} \backslash \mathbb{Z})_x$ with the metric $ds^2 = dr^2 + e^{-2r} dx^2$. The funnels F_i and the cusps C_i are isometric to the preceding standard models. We define the function r as the distance to the compact core K and the function ρ by

$$\rho(r) = \begin{cases} 2e^{-r} & \text{in } F \\ e^{-r} & \text{in } C \end{cases} .$$

We will adopt $(\rho, t) \in (0, 2] \times \mathbb{R}/l_j \mathbb{Z}$ as the standard coordinates for the funnel F_j , where t is arc length around the central geodesic at $\rho = 2$.

For the cusp our standard coordinates $(\rho, t) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$ are based on the model defined by the cyclic group Γ_{∞} . The cusp boundary is $y = 1$, so that $y = e^r$ and $\rho = 1/y$. We set $t \equiv x \pmod{\mathbb{Z}}$.

2. HYPERBOLIC EISENSTEIN SERIES ON A GEOMETRICALLY FINITE HYPERBOLIC SURFACE OF INFINITE VOLUME.

2.1. Return to Kudla and Millson hyperbolic Eisenstein series' definition.

In the following, M will denote an arbitrary Riemann surface and $L^2(M)$, the Hilbert space of square integrable 1-forms with inner product

$$(w_1, w_2) = \frac{1}{2} \int_M w_1 \wedge * \overline{w_2},$$

and corresponding norm $\|\cdot\|_{L^2}$. The pointwise norm of a 1-form w is defined by $w \wedge * \overline{w} = \|w\|^2 * 1$ where $*1$ is the volume form.

Let c be a simple closed curve on M . We may associate with c a real smooth closed differential n_c with compact support such

$$(1) \quad \int_c \omega = \int_M \omega \wedge n_c,$$

for all closed differentials ω . Since every cycle c on M is a finite sum of cycles corresponding to simple closed curves, we conclude that to each such c , we can associate a real closed differential n_c with compact support such that (1) holds.

Let a and b be two cycles on the Riemann surface M . We define the intersection number of a and b by

$$a.b = \int_M n_a \wedge n_b.$$

In [17], Kudla and Millson construct the harmonic 1-form dual to a simple closed geodesic on a hyperbolic surface of finite volume M in terms of Eisenstein series. Let us recall the definition:

Definition 2.1. *Let η be a simple closed geodesic or an infinite geodesic joining p and q . A 1-form α is dual to η if for any closed 1-form with compact support, ω*

$$\int_M \omega \wedge \alpha = \int_\eta \omega.$$

Or equivalently: for any closed oriented cycle c' , we have

$$(2) \quad \int_{c'} \alpha = \eta \cdot c'.$$

Kudla and Millson construct a meromorphic family of forms on M , called hyperbolic Eisenstein series associated to an oriented simple closed geodesic c . Let \tilde{c} be a component of the inverse image of c in the covering $H \rightarrow M$ and Γ_1 the stabilizer of \tilde{c} in Γ . The hyperbolic Eisenstein series are expressed in Fermi coordinates in the following way for $\text{Re } s > 0$:

$$(3) \quad \Omega_c(s, z) = \Omega(s, z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \frac{dx_2}{(\cosh x_2)^{s+1}}, \quad k(s) := \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \frac{s}{2})}{\Gamma(1 + \frac{s}{2})}.$$

By applying an element of $SL(2, \mathbb{R})$ we may assume that \tilde{c} is the y -axis in H and that Γ_1 is generated by $\gamma_1 : z \mapsto e^l z$. The Fermi coordinates (x_1, x_2) associated to \tilde{c} are related to euclidean polar coordinates by

$$\begin{aligned} r &= e^{x_1}, \\ \sin \theta &= \frac{1}{\cosh x_2}. \end{aligned}$$

So we can write $\Omega(s, z) = \text{Im}(\Theta(s, z))$ with

$$(4) \quad \Theta(s, z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \left(\frac{y^s}{z|z|^s} dz \right).$$

At the end of their paper they do the remark that “it is also interesting to consider the infinite volume case”.

2.2. The infinite volume case. We are going to verify that this definition retains a meaning in the case of a geometrically finite hyperbolic surface of infinite volume, $M = \Gamma \backslash H$, Γ is a Fuchsian group of the second kind without elliptic elements. We identify M locally with its universal cover H . By $d(z, w)$ we denote the hyperbolic distance from $z \in H$ to $w \in H$. For $z_0 \in H$ and $\epsilon > 0$, by $B(z_0, \epsilon) \subset H$ we denote the radius ϵ hyperbolic metric ball centered at z_0 .

Proposition 2.1. *The hyperbolic Eisenstein series $\Omega(s, z)$ converges for $\text{Re } s > 0$, uniformly on compact subsets of H , is bounded on M and represents a C^∞ closed form which is dual to c . Moreover it is an analytic function of s in $\text{Re } s > 0$.*

The proof in the infinite volume case is as straightforward as in the finite volume case ([17], [10]), but for the convenience of the reader we give some details.

Proof. Recall first that if K is a compact of the fundamental domain D of Γ then there exists $\eta > 0$ such that for all $z_0 \in K$, the balls $(B(\gamma z_0, \eta))_{\gamma \in \Gamma_1 \setminus \Gamma}$ are disjoint.

Let us choose for fundamental domain of Γ_1 , $\mathcal{D}_1 = \{z \in H : 1 \leq |z| \leq \beta\}$. After passing to ordinary Euclidean polar coordinates (r, θ) , with $\sigma = \text{Re } s$, with $\|\Omega\|$ denotes the pointwise norm of Ω , we obtain:

$$\begin{aligned} \|\Omega(s, z)\| &\leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \setminus \Gamma} \frac{1}{(\text{ch } x_2(\gamma z))^{\sigma+1}} \\ &\leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \setminus \Gamma} \left(\frac{y}{r}\right)^{\sigma+1} (\gamma z) \leq \frac{1}{|k(s)|} \sum_{[\gamma] \in \Gamma_1 \setminus \Gamma, \gamma z \in \mathcal{D}_1} y^{\sigma+1}(\gamma z). \end{aligned}$$

Now y^σ is an eigenfunction of all the invariant integral operators on H . Let $k(z, z')$ be the point-pair invariant defined by $k(z, z') = 1$ or 0 according as the distance between z and z' is smaller than η . Then there exists Λ_η independent of z_0 so that

$$\int_{B(z_0, \eta)} y^\sigma \frac{dx dy}{y^2} = \int_H k(z_0, z) y^\sigma \frac{dx dy}{y^2}$$

and

$$\int_{B(z_0, \eta)} y^\sigma \frac{dx dy}{y^2} = \Lambda_\eta y(z_0)^\sigma.$$

This is a particular formulation of a more general result we will need further (see proposition 5.2).

We denote as in [17]: $R(T_1, T_2) = \{P \in \mathcal{D}_1 : T_1 < x_2(P) < T_2\}$. So for $T > 2\eta$:

$$\frac{1}{|k(s)|} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} \frac{1}{(\text{ch}(x_2(\gamma z)))^{\sigma+1}} \leq \frac{1}{|k(s)|} \sum_{[\gamma] \in \Gamma_1 \setminus \Gamma, \gamma z \in \mathcal{D}_1 \setminus R(-T, T)} y^{\sigma+1}(\gamma z).$$

We need the following:

Lemma 2.1. *Let $\gamma \in \Gamma_1 \setminus \Gamma$, $z, \zeta \in H$ such that for $\gamma z \notin R(-T, T)$, $\gamma \zeta \in B(\gamma z, \eta)$ then $\gamma \zeta \notin R(-T + 2\eta, T - 2\eta)$.*

Proof. Let $\pi : H \rightarrow \tilde{c}$ be the orthogonal projection on \tilde{c} . As π is 1-lipschitzien, we have for the hyperbolic distance $d(\pi \gamma z, \pi \gamma \zeta) \leq d(\gamma z, \gamma \zeta) \leq \eta$. If $x_2(\gamma z) \geq T$:

$$T \leq d(\gamma z, \pi \gamma z) \leq d(\gamma z, \gamma \zeta) + d(\gamma \zeta, \pi \gamma \zeta) + d(\pi \gamma z, \pi \gamma \zeta).$$

Then

$$T - 2\eta \leq d(\gamma \zeta, \pi \gamma \zeta).$$

□

Then

$$\begin{aligned} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} y^{\sigma+1}(\gamma z) &= \frac{1}{\Lambda_\eta} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} \int_{B(\gamma z, \eta)} y^{\sigma+1} \frac{dx dy}{y^2} \\ &\leq \frac{1}{\Lambda_\eta} \int_{R^c(-T+2\eta, T-2\eta)} y^{\sigma+1} \frac{dx dy}{y^2} \end{aligned}$$

where $R^c(-T+2\eta, T-2\eta)$ is the complementary in \mathcal{D}_1 of $R(-T+2\eta, T-2\eta)$. Note that if $\gamma z \notin R(-T, T)$ then $y(\gamma z) \leq \frac{\beta}{\text{ch } T}$, so:

$$\begin{aligned} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} y^{\sigma+1}(\gamma z) &\leq \frac{\beta}{\Lambda_\eta} \int_0^{\frac{\beta}{\text{ch}(T-2\eta)}} y^{\sigma-1} dy \\ &\leq \frac{\beta}{\Lambda_\eta \sigma} \left(\frac{\beta}{\text{ch}(T-2\eta)} \right)^\sigma. \end{aligned}$$

From this follows the uniform convergence of $\Omega(s, z)$ on compact subsets of H , uniformly on compact subsets of the half plane $\text{Re } s > 0$.

We next show that $\Omega(s, z)$ is bounded on D . For this we use a very useful fundamental lemma (see [14], p. 178, [12], p.27):

Proposition 2.2. *For any Fuchsian group Γ , there exists $\mathcal{C}(q, \Gamma)$, such that for all $z \in H$*

$$\sum_{\gamma \in \Gamma} \frac{y(\gamma z)^q}{[1 + |\gamma z|]^{2q}} \leq \mathcal{C}(q, \Gamma).$$

The constant $\mathcal{C}(q, \Gamma)$ depends only of q and Γ .

Let $z \in H$, there exists a system of representants S of $\Gamma_1 \setminus \Gamma$ such that for all $\gamma \in S$, $|\gamma z| \leq \beta$. Then:

$$\begin{aligned} \sum_{\Gamma_1 \setminus \Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1 + \beta)^{2(\sigma+1)}} &\leq \sum_{\Gamma_1 \setminus \Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1 + |\gamma z|)^{2(\sigma+1)}} \\ &\leq \sum_{\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1 + |\gamma z|)^{2(\sigma+1)}} \\ &\leq \mathcal{C}(\sigma + 1, \Gamma) \end{aligned}$$

and the result follows.

The fact that $\Omega(s, z)$ is dual to c follows straightly from the construction of Kudla and Millson. \square

3. SPECTRAL DECOMPOSITION AND ANALYTIC CONTINUATION.

The aim is to realize the injection $H_c^1 \rightarrow \mathcal{H}^1$, where H_c^1 is the first de Rham's cohomology group with compact support of M and \mathcal{H}^1 is the space of L^2 harmonic 1-forms of M . Recall that in our context $\dim \mathcal{H}^1 = \infty$ (see [2], p. 27).

We are going to prove, as in [17], the analytic continuation of the hyperbolic Eisenstein series. The essential difference with the finite volume case is the spectral decomposition of $L^2(M)$.

3.1. Spectral theory. For any non-compact geometrically finite hyperbolic surface M , the essential spectrum of the (positive) Laplacian Δ_M defined by the hyperbolic metric on M (the Laplacian on functions) is $[1/4, \infty)$ and this is absolutely continuous. The discrete spectrum consists of finitely many eigenvalues in the range $(0, 1/4)$. In the finite-volume case one may also have embedded eigenvalues in the continuous spectrum, but these do not occur for infinite-volume surfaces. Then if M as infinite volume, the discrete spectrum of Δ_M is finite (possibly empty). The exponent of convergence δ of a Fuchsian group Γ is defined to be the abscissa of convergence of the Dirichlet series:

$$\delta = \inf \left\{ s > 0, \sum_{T \in \Gamma} e^{-sd(z, Tw)} < \infty \right\}$$

for some $z, w \in H$ and $d(z, w)$ denotes again the hyperbolic distance from $z \in H$ to $w \in H$.

Let Γ be a Fuchsian group of the second kind and $L(\Gamma)$ be its limit set, then $0 < \delta < 1$ with $\delta > 1/2$ if Γ has parabolic elements. Patterson and Sullivan showed that δ is the Hausdorff dimension of the limit set when Γ is geometrically finite. Furthermore, if $\delta > 1/2$, then $\delta(1 - \delta)$ is the lowest eigenvalue of the Laplacian Δ_M . The connection to spectral theory was later extended to the case $\delta \leq 1/2$ by Patterson. In this case, the discrete spectrum of Δ_M is empty and δ is the location of the first resonance. For a detailed account of the spectral theory of infinite area surfaces, we refer the reader to [1].

3.2. Tensors and automorphic forms. This section introduces the notations used in the following subsection 3.3 and section 5. Let M be a geometrically finite hyperbolic surface. Let z be a local conformal coordinate and $ds^2 = \rho|dz|^2$ the Poincaré metric. Let $\omega_M = T^*M$ be the holomorphic cotangent bundle of M and for any integers n and m , let $\mathcal{E}^{r,s}(M, \omega_M^n \otimes \bar{\omega}_M^m)$ be the vector space of smooth differential forms of type (r, s) on M with values in the line bundle $\omega_M^n \otimes \bar{\omega}_M^m$. For an integer q , a q -form (or q -differential) is an element of $T^q = \mathcal{E}^{0,0}(M, \omega_M^q)$, the space of tensors of type q on M written locally as $f(z)(dz)^q$. M may be realized as $\Gamma \backslash H$, where H is the upper half plane and Γ a discrete subgroup of $PSL(2, \mathbb{R})$. The hyperbolic metric on M induces the natural scalar product

$$(\varphi, \psi) = \int_{\Gamma \backslash H} \varphi(z) \overline{\psi(z)} y^{2q-2} dx dy,$$

on T^q . Let \mathfrak{H}^q be the L^2 -closure of T^q with respect to this scalar product.

We recall now the link between q -forms and automorphic forms of weight¹ $2q$, called also q -automorphic forms for the following reason. As before we make use of the uniformization theorem.

Using notably the notations in [11] and [7], set

$$j_\gamma(z) = \frac{(cz + d)^2}{|cz + d|^2} = \frac{cz + d}{c\bar{z} + d} = \left(\frac{\gamma'z}{|\gamma'z|} \right)^{-1} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let \mathcal{F}_q be the space of all functions $f : H \rightarrow \mathbb{C}$ with

$$f(\gamma z) = j_\gamma(z)^q f(z), \quad \gamma \in \Gamma,$$

¹Some authors called them of weight q or $-2q$

and if $\mathcal{D} = \Gamma \backslash H$ is the fundamental domain of Γ , define the Hilbert space $\mathcal{H}_q = \{f \in \mathcal{F}_q, \langle f, f \rangle_{\mathcal{D}} = \int_{\mathcal{D}} |f(z)|^2 d\mu(z) < \infty\}$ with areal measure $d\mu(z) = \frac{dx dy}{y^2}$ and the inner product $\langle f, g \rangle = \int_{\mathcal{D}} f(z) \overline{g(z)} d\mu(z)$. An element in \mathcal{H}_q is called an automorphic form of weight $2q$. \mathcal{H}_q is isometric to \mathfrak{H}^q through the correspondence

$$I : \mathfrak{H}^q \ni f \mapsto y^q f \in \mathcal{H}_q.$$

Maass introduced the differential operators

$$\begin{aligned} L_q &= (\bar{z} - z) \frac{\partial}{\partial \bar{z}} - q & : \mathcal{F}_q \rightarrow \mathcal{F}_{q-1} \\ K_q &= (z - \bar{z}) \frac{\partial}{\partial z} + q & : \mathcal{F}_q \rightarrow \mathcal{F}_{q+1} \end{aligned}$$

We also have:

$$\begin{aligned} L_q &= -2iy^{1+q} \frac{\partial}{\partial \bar{z}} y^{-q} = \overline{K_{-q}}, \\ K_q &= 2iy^{1-q} \frac{\partial}{\partial z} y^q = \overline{L_{-q}}. \end{aligned}$$

We denote

$$\begin{aligned} -L_{q+1} K_q &= -\Delta_{2q} + q(q+1), \\ -K_{q-1} L_q &= -\Delta_{2q} + q(q-1) \end{aligned}$$

with

$$\Delta_{2q} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2iqy \frac{\partial}{\partial x}.$$

These second order differential operators are self-adjoint on \mathcal{H}_q .

Now, the metric and complex structure determine a covariant derivative

$$\nabla = \nabla^q \oplus \nabla_q : \mathcal{E}^{0,0}(M, \omega_M^q) \longrightarrow \mathcal{E}^{1,0}(M, \omega_M^q) \oplus \mathcal{E}^{0,1}(M, \omega_M^q)$$

on the line bundle ω_M^q . With the identifications $\mathcal{E}^{1,0}(M, \omega_M^q) \cong T^{q+1}$ and $\mathcal{E}^{0,1}(M, \omega_M^q) \cong T^{q-1}$, we have

$$\nabla^q : T^q \rightarrow T^{q+1}, \quad \nabla_q : T^q \rightarrow T^{q-1}.$$

Under the correspondence I , the operators ∇_q, ∇^q go over to the Maass operators according to the commutative diagram

$$\begin{array}{ccccc} \mathfrak{H}^{q-1} & \xleftarrow{\nabla_q} & \mathfrak{H}^q & \xrightarrow{\nabla_q} & \mathfrak{H}^{q+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}_{q-1} & \xleftarrow{L_q} & \mathcal{H}_q & \xrightarrow{K_q} & \mathcal{H}_{q+1} \end{array}$$

and so are given locally by $\nabla^q = 2i\rho^q \partial \rho^{-q}$, $\nabla_q = -2i\rho^{-1} \bar{\partial}$.

The Laplacians Δ_q^+ and Δ_q^- on T^q are defined by $\Delta_q^+ = -\nabla_{q+1} \nabla^q$, $\Delta_q^- = -\nabla^{q-1} \nabla_q$ and then the isometry I conjugates Δ_q^+ with $-\Delta_{2q} + q(q+1)$ and Δ_q^- with $-\Delta_{2q} + q(q-1)$. Thus $\Delta_0 = \Delta_0^\pm$ is the Laplacian on functions. The operators Δ_q^\pm are non-negative self adjoint.

We are first interested in the case $q = 2$. Let Δ_{Diff} the (positive) Laplacian on 1-forms on a geometrically finite hyperbolic surface, $\Delta_{\text{Diff}} = d\delta + \delta d$, $\delta = -*d*$ with $*$ the Hodge operator. In the following we denote $\Delta_{\text{Diff}} = \Delta$. If ω is a 1-form in the holomorphic cotangent bundle, $\omega = f(z) dz$, then we define the image by the isometry I , $I(\omega) = I(f dz) = yf(z) = \tilde{f}(z)$. We have the relation $y\Delta(f dz) = -(\Delta_2 \tilde{f}) dz$, in other words, with the preceding notations $\Delta = \Delta_1^-$.

3.3. Generalized eigenfunctions. We are going to give the spectral expansion in eigenforms of Δ ; we use [7], [20], [1]. With the notations of the section 1, for a finitely generated group of the second kind, for each cusp and for each funnel of the quotient there is a corresponding Eisenstein series, this is what we are going to develop now.

Proposition 3.1. *For $\text{Re } s > \delta$, the kernel of the resolvent $G_s(z, w, 1)$ for the self-adjoint operator Δ_2 acting on the Hilbert space \mathcal{H}_1 of automorphic forms of weight 2, is given by the convergent series*

$$G_s(z, w, 1) = \sum_{\gamma \in \Gamma} j_\gamma(w) g_s(z, \gamma w, 1)$$

with $g_s(z, w, 1) = -\frac{w - \bar{z} \Gamma(s+1)\Gamma(s-1)}{z - \bar{w} 4\pi\Gamma(2s)} \cosh^{-2s}(d(z, w)/2) F(s+1, s-1, s; \cosh^{-2}(d(z, w)/2))$ and F is the Gauss hypergeometric function.

For the funnel case, we identify z' with the standard coordinates (ρ', t') in the funnel F_j , and we denote

$$(5) \quad E_{j,1}^f(s, z, t') = \lim_{\rho' \rightarrow 0} \rho'^{-s} G_s(z, z', 1),$$

for $j = 1, \dots, n_f$. In the cusp C_j , with standard coordinates $z' = (\rho', t')$, we set

$$(6) \quad E_{j,1}^c(s, z) = \lim_{\rho' \rightarrow 0} \rho'^{1-s} G_s(z, z', 1),$$

for $j = 1, \dots, n_c$.

Let

$$P(z, \zeta) = \text{Im}(z)/|z - \zeta|^2$$

where $z \in H$ and $\zeta \in \mathbb{R}$, be the Poisson kernel. For $b \in O(\Gamma) = \mathbb{R} \cup \{\infty\} - L(\Gamma)$ define the Eisenstein series ([20], [3])

$$E_b(z, s, k) = \sum_{\gamma \in \Gamma} j(\gamma, z)^k P(\gamma(z), b)^s (\gamma(z), b)^k,$$

where $j(\gamma, z) = \gamma'(z)/|\gamma'(z)|$ and $(z, b) = (\bar{z} - b)/(z - b)$. This converges uniformly on compact subsets of H if $\text{Re}(s) > \delta$.

Proposition 3.2. *The series $E_b(z, s, k)$ can be continued to the whole complex plane as a meromorphic function in s .*

One verifies that $-\Delta_{2k} E_b(z, s, k) = s(1-s) E_b(z, s, k)$.

Remark 3.1. *Thus if $\delta < 1/2$, $E_b(z, s, k)$ is analytic in a neighbourhood of $\text{Re}(s) = 1/2$.*

For the standard funnel F_l which corresponds to the region $\text{Re } z \geq 0$ in the model $C_l = \Gamma_l \backslash H$, we have (see [7] p.200):

$$\begin{aligned} E_{l,1}^f(s, z, x') &= \lim_{z' \rightarrow x'} (\text{Im } z')^{-s} G_s(z, z', 1) \\ &= -\frac{4^s \Gamma(s+1)\Gamma(s-1)}{4\pi \Gamma(2s)} E_{x'}(z, s, 1). \end{aligned}$$

We write then

$$\mathcal{E}_{f_l}(s, z, x') = (1-2s) \frac{E_{l,1}^f(s, z, x')}{y} dz.$$

Remark 3.2. Recall the definition of the classical Eisenstein series. The stabilizer of a cusp \mathfrak{a} is an infinite cyclic group generated by a parabolic motion,

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma \mathfrak{a} = \mathfrak{a}\} = \langle \gamma_{\mathfrak{a}} \rangle ,$$

say. There exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$, called a scaling matrix of the cusp \mathfrak{a} , such that $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}}^{-1} \gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\sigma_{\mathfrak{a}}$ is determined up to composition with a translation from the right. The Eisenstein series for the cusp \mathfrak{a} is then defined by:

$$E_{\mathfrak{a}}(z, s) = \sum_{\Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s ,$$

where s is a complex variable with $\operatorname{Re} s > \delta$.

Definition 3.1. In a similar way we define an Eisenstein series of weight $2q$ associated to a cusp \mathfrak{a} as the automorphic form of weight $2q$, for $\operatorname{Re} s > \delta$:

$$E_{\mathfrak{a},q}(s, z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s j_{\sigma_{\mathfrak{a}}^{-1} \gamma}(z)^{-q} = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s \left(\frac{(\sigma_{\mathfrak{a}}^{-1} \gamma)' z}{|(\sigma_{\mathfrak{a}}^{-1} \gamma)' z|} \right)^q .$$

It can be continued to the whole complex plane as a meromorphic function in s .

We call a horocyclic Eisenstein series the 1-form corresponding to the Eisenstein series of weight 2 associated to a cusp \mathfrak{a} , $E_{\mathfrak{a},1}$ and defined for $\operatorname{Re} s > 1$, by:

$$(7) \quad \mathcal{E}_{\mathfrak{a}}(s, z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1} \gamma z)^{s-1} d(\sigma_{\mathfrak{a}}^{-1} \gamma z) = \frac{E_{\mathfrak{a},1}(s, z)}{y} dz .$$

We now verify that it corresponds to the defining formula (6). For the standard cusp, we write

$$G_s(z, z', 1) = \sum_{\Gamma_{\infty} \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right) G_s^{\Gamma_{\infty}}(\gamma z, z', 1) ,$$

where $G_s^{\Gamma_{\infty}}(z, z', 1)$ is the resolvent kernel of the standard cusp for automorphic forms of weight 2. We use then [7] p. 155 (38), p.177 for $\operatorname{Im} z' > \operatorname{Im} \gamma z$, p. 172 (see also [1] p.72, p.102) to conclude that

$$\lim_{y' \rightarrow \infty} y'^{s-1} G_s(z, z', 1) = \sum_{\Gamma_{\infty} \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right) \frac{(\operatorname{Im} \gamma z)^s}{1 - 2s} = \frac{1}{1 - 2s} E_{\infty,1}(s, z) .$$

Recall the decomposition (see section 1) $M = K \cup_{j=1}^{n_c} C_j \cup_{j=1}^{n_f} F_j$ and denote \mathcal{E}_{C_j} the horocyclic Eisenstein series associated to the cusp C_j .

With the preceding notations we then have (see for example [7], [20])

Proposition 3.3. For $w = f(z) dz$ square integrable,

$$w(z) = \sum_{i=1}^m (w)_{\lambda_i}(z) + \frac{1}{4\pi i} \sum_{j=1}^{n_c} \int_{-\infty}^{+\infty} \langle w, \mathcal{E}_{C_j}(1/2 + it, \cdot) \rangle \mathcal{E}_{C_j}(1/2 + it, z) dt + \frac{1}{4\pi i} \sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_j^2} \langle w, \mathcal{E}_{F_j}(1/2 + it, \cdot, b) \rangle \mathcal{E}_{F_j}(1/2 + it, z, b) db \right] dt ;$$

where the first sum in the right member is the projection of w on the discrete spectrum.

Remark 3.3. One can easily deduce the formula for an arbitrary square integrable 1-form

$$\begin{aligned}\Omega(z) &= f dz + g d\bar{z} = \sum_{i=1}^m (\Omega)_{\lambda_i}(z) + \\ &\frac{1}{4\pi i} \sum_{j=1}^{n_c} \int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot) \rangle \mathcal{E}_{c_j}(1/2 + it, z) + \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot)_{-1} \rangle \mathcal{E}_{c_j}(1/2 + it, z)_{-1} dt \\ &+ \frac{1}{4\pi i} \sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \int_1^{\lambda_j^2} \langle \Omega, \mathcal{E}_{f_j}(1/2 + it, \cdot, b) \rangle \mathcal{E}_{f_j}(1/2 + it, z, b) db dt \\ &+ \frac{1}{4\pi i} \sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \int_1^{\lambda_j^2} \langle \Omega, \mathcal{E}_{f_j}(1/2 + it, \cdot, b)_{-1} \rangle \mathcal{E}_{f_j}(1/2 + it, z, b)_{-1} db dt\end{aligned}$$

where, with obvious notations, $\mathcal{E}_{-1} = \bar{\mathcal{E}} = \frac{\bar{E}}{y} d\bar{z}$. To simplify we will write

$$\begin{aligned}\Omega(z) &= (\Omega)_{\lambda_i}(z) + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot)_{\pm} \rangle \mathcal{E}_{c_j}(1/2 + it, z)_{\pm} dt + \\ &\frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_j^2} \langle \Omega, \mathcal{E}_{f_k}(1/2 + it, \cdot, b)_{\pm} \rangle \mathcal{E}_{f_k}(1/2 + it, z, b)_{\pm} db \right] dt.\end{aligned}$$

3.4. Harmonic dual form. We are now going to see

Proposition 3.4. The hyperbolic Eisenstein series Ω_c are square integrable.

Proof. We consider a fundamental domain D contained in $\{z, 1 \leq |z| \leq e^l\}$ in which the segment (i, ie^l) represents the geodesic c . We denote $C_\lambda = \{z \in D, d(z, c) = \lambda\}$ and $F_\lambda = \{z \in D, d(z, c) \geq \lambda\}$. Without loss of generality we can suppose that there is only one funnel on M and no cusps. Let V_λ the volume of $F_\lambda - F_{\lambda+1}$ there exists a constant c_1 such that $V_\lambda \geq c_1(\sinh(\lambda+1) - \sinh(\lambda))$. For $\text{Re } s = \sigma > 0$, $\|\Omega_c(s, z)\| = \|\Omega(s, z)\| \leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \backslash \Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$. Let $\eta(z) = \sum_{\Gamma_1 \backslash \Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$. We know from the section 2 that there exists a constant $K > 0$ such that $\forall z \in H$, $\eta(z) \leq K$.

We have

$$\begin{aligned}\int_D \|\Omega_c(s, z)\|^2 d\mu(z) &\leq \frac{1}{|k(s)|^2} \int_D \eta^2(z) d\mu(z) \\ &\leq \frac{1}{|k(s)|^2} \int_{1 \leq x_1 \leq e^l, -\infty < x_2 < +\infty} \eta(z) \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 dx_1 dx_2 \\ &\leq \frac{K}{|k(s)|^2} \int_{1 \leq x_1 \leq e^l, -\infty < x_2 < +\infty} \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 dx_1 dx_2.\end{aligned}$$

The last integral is $\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\sigma}{2})}{\Gamma(\frac{1}{2} + \frac{\sigma}{2})}(e^l - 1)$ and the result follows. \square

As in [17] we verify:

$$\Delta(\Omega(s, z)) + s(s+1)\Omega(s, z) = s(s+1)\Omega(s+2, z).$$

This formula has a consequence that, for fixed s with $\operatorname{Re} s > 0$, the function $\Delta^k(\Omega(s, z))$ is again square integrable for any $k > 0$.
Set $\operatorname{Re} s > 0$, with our convention of notations

$$(8) \quad \begin{aligned} \Omega(s, z) = & \Omega_0(z) + a_i(s)\varphi_i(z) + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} h_{\pm}^c(s, t)\mathcal{E}_c(1/2 + it, z)_{\pm} dt \\ & + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_f^2} H_{\pm}^f(s, t, b)\mathcal{E}_f(1/2 + it, z, b)_{\pm} db \right] dt. \end{aligned}$$

We obtain

$$H(s, t, b)[1/4 + t^2 + s(s + 1)] = s(s + 1)H(s + 2, t, b),$$

where H corresponds to any H_{\pm}^f . From this we get a continuation of H to the region $\operatorname{Re} s > -1/2$ and we note that for all t and all b we have $H(0, t, b) = 0$. Moreover for $\operatorname{Re} s > -1/2$, $\operatorname{Re}(s + 2) > 0$ and we may substitute in (8) to obtain a continuation of $\Omega(s, z)$ to $\operatorname{Re} s > -1/2$. Thus we have proved the following theorem

Theorem 3.1. $\Omega(s, z)$ has a meromorphic continuation to $\operatorname{Re} s > -1/2$ with $s = 0$ a regular point and $\Omega(0, z)$ is a harmonic form which is dual to c .

Remark 3.4. 1) Another way to see this:

write $\Omega(s, z) = (\Delta + s(s + 1))^{-1}(s(s + 1)\Omega(s + 2, z))$ and use the meromorphic continuation of the resolvent (see for example [1], [21]).

2) With an analogue study of [17] (see also [16]) we can obtain a total description of the singularities of the hyperbolic Eisenstein series.

4. THE CASE OF AN INFINITE GEODESIC JOINING TWO POINTS.

Without loss of generality we suppose the two cusps p and q to be 0 and ∞ respectively and, as the lift of the geodesic, we take the imaginary axis. Let η be the infinite geodesic $]p, q[$, can we do the same construction as Kudla and Millson? As in the finite volume case, the problem reduces to study the following series for $\operatorname{Re} s > 1$:

$$(9) \quad \hat{\eta}^s(z) = \frac{1}{k(s-1)} \sum_{\gamma \in \Gamma} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \right] = \operatorname{Im}(\theta^s(z)),$$

where

$$\theta^s(z) = \frac{1}{k(s-1)} \sum_{\gamma \in \Gamma} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right],$$

$$\text{and } k(s-1) = \frac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2 + s/2)}.$$

4.1. Some useful estimates. As usual we can suppose $\Gamma_{\infty} = \langle z \mapsto z + 1 \rangle$ to be the stabilizer of ∞ in Γ and the stabilizer of 0, Γ_0 is then generated by $z \mapsto \frac{z}{-c_0^2 z + 1}$ (for some non zero constant c_0).

First of all we note that, contrary to the finite volume case, we have

Lemma 4.1. The series $\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \operatorname{Im}(\gamma z)$ is convergent (see proposition 3.1 and formula 6).

Another way to see this is “by hand”: We know that for $\text{Re } s > \delta$, $\sum_{T \in \Gamma} e^{-sd(i,Tz)}$ converges, moreover there exists a constant $C > 0$ such that $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}^s(\gamma z) \leq C \sum_{T \in \Gamma} e^{-sd(i,Tz)}$, as in our case $\delta < 1$, we have the result.

As $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \left| \text{Im}^s(\gamma z) \frac{c\bar{z}+d}{cz+d} \right| = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}^{\text{Re } s}(\gamma z)$ we also deduce the convergence of the series representing $E_{\infty,1}(1, z)$ (3.2). With the notations of the recall 3.1 and ρ the standard coordinate for the cusp \mathfrak{a} , we re-write the results of [1](p.110) in the following way: $E_{\mathfrak{a}}(s, \cdot) = \rho^{-s}(1 - \chi_0(\rho)) + O(\rho_f^s \rho_c^{s-1})$, ρ is decomposed as $\rho_f \rho_c$ with $\rho_f = \rho$ in the funnels and $\rho_c = \rho$ in the cusps and we define $\chi_0 \in \mathcal{C}_0^\infty(X)$ such that

$$\chi_0 = \begin{cases} 1, & r \leq 0 \\ 0, & r \geq 1 \end{cases} .$$

Lemma 4.2. *We have the following asymptotic behaviors for $\text{Re } s > \delta$:*

- 1) *in a funnel, for all cusp \mathfrak{a} , $E_{\mathfrak{a}}(s, z)$ is square integrable;*
- 2) *at $\mathfrak{a} = \infty$, $E_\infty(s, z) - y^s = O(y^{1-s})$ and $E_0(s, z) = O(y^{1-s})$;*
- 3) *near $\mathfrak{a} = 0$, $E_0(s, z) - y^s/(c_0^2|z|^2)^s = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$ and $E_\infty(s, z) = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$.*

4.2. Convergence of the Hyperbolic Eisenstein series and analytic continuation. The computations to prove the convergence of (9) are easily adapted from the finite volume case. For the convenience of the reader we recall the essential points.

We have $\|\sum_{\gamma \in \Gamma} \gamma * \left(\frac{y}{|z|}\right)^{s-1} \text{Im}(z^{-1}dz)\| \leq \sum_{\gamma \in \Gamma} \left(\frac{y}{|z|}\right)^\sigma (\gamma z)$, where $\sigma = \text{Re } s > 1$

and if we denote by $\mathcal{S} = \sum_{\gamma \in \Gamma} \left(\frac{y}{|z|}\right)^\sigma (\gamma z)$, we have

$$\mathcal{S} = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) \sum_{n \in \mathbb{Z}} \frac{1}{|\gamma z + n|^\sigma} .$$

Let S_z be a system of representatives of $\Gamma_\infty \setminus \Gamma$ such that $|\text{Re } \gamma z| \leq 1/2$, then

$$\|\mathcal{S}\| \leq \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + 2 \sum_{\gamma \in S_z} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma} .$$

We have

$$\begin{aligned} \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &= \sum_{\Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n \in \mathbb{Z}} \frac{1}{|-nc_0^2 \gamma z + 1|^\sigma} \\ &= \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n \in \mathbb{Z}^*} \frac{1}{|nc_0^2|^\sigma [(x(\gamma z) - 1/nc_0^2)^2 + y^2(\gamma z)]^{\sigma/2}} . \end{aligned}$$

For K a compact set in H there exists m in H such that

$$\forall z \in K, \forall \gamma \in \Gamma_0 \setminus S_z, |\gamma z| \geq |m| \text{ and } \text{Im } \gamma z \geq \text{Im } m .$$

So

$$\begin{aligned} \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &\leq \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} + \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} \sum_{n \in \mathbb{Z}^*} \frac{1}{|nc_0^2|^\sigma (\text{Im } m)^\sigma} \\ &\leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + 2 \sum_{n \in \mathbb{N}^*} \frac{1}{(nc_0^2)^\sigma} \frac{1}{|m|^\sigma (\text{Im } m)^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) ; \end{aligned}$$

and finally, for all z in K

$$\begin{aligned} \|\mathcal{S}\| \leq & \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + 2 \sum_{n \in \mathbb{N}^*} \frac{1}{(nc_0^2)^\sigma} \frac{1}{|m|^\sigma (\text{Im } m)^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + \\ & 2 \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma}, \end{aligned}$$

the uniform convergence on all compact of H and all compact of $\text{Re } s > 1$.

From this ultimate inequality we conclude that θ^s is square integrable in the funnels as the Eisenstein series \mathcal{E}_∞ . To conclude, we have the following theorem:

Theorem 4.1. *For $\text{Re } s > 1$, the Eisenstein series associated to the geodesic $\eta = (p, q)$ converges uniformly on all compact sets. It represents a \mathcal{C}^∞ closed form which is dual to η . For $\text{Re } s > 1$ it satisfies the differential functional equation:*

$$\Delta \hat{\eta}^s = s(1-s)[\hat{\eta}^s - \hat{\eta}^{s+2}].$$

Now we want to prove the analytic continuation of $\hat{\eta}^s$ at $s = 1$. For this, first of all, we are going to show that $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ is square integrable. As we have shown that θ^s is square integrable in the funnels, what we have to do is to investigate the Fourier expansion of θ^s , at each inequivalent cusps: 0 and ∞ , and to show that $\|\theta^s\|$ is bounded at the cusps. As in the case of finite volume case we have ([5]):

Proposition 4.1. *At ∞*

$$\theta^s(z) = \left(\frac{1}{i} + O(1/y)\right) dz ,$$

and at 0

$$\theta^s(z) = \left(-\frac{1}{ic_0^2 z^2} + O(1/y)\right) dz .$$

By proposition 4.2 and lemma 4.1, we conclude:

Proposition 4.2. *The 1-forms $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ and $\hat{\eta}^s(z) + \text{Re}(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ are square integrable.*

Finally as in [5]:

Theorem 4.2. *The 1-form $\hat{\eta}^s$ has a meromorphic continuation to $\text{Re } s > 1/2$, with $s = 1$ a regular point and $\hat{\eta}$ is a harmonic form which is dual to η .*

5. DEGENERATIONS.

5.1. Background material and the main results. A family of degenerating hyperbolic surfaces consists of a manifold M and a family $(g_l)_{l>0}$ of Riemannian metrics on M that satisfy the following assumptions: M is an oriented surface of negative Euler characteristic and the metrics g_l are hyperbolic, chosen in such a way that there are finitely many closed curves c_i , geodesic with respect to all metrics, with the length l_i of each curve converging to 0 as l decreases. On the complement of the distinguished curves, the sequence of metrics is required to converge to a hyperbolic metric. More precisely there are finitely many disjoint open subsets $C_i \subset M$ that are diffeomorphic to cylinders $F_i \times J_i$ where $J_i \subset \mathbb{R}$ is a neighborhood

of 0. The complement of $\bigcup_i C_i$ is relatively compact. The restriction of each metric g_l to $C_i = F_i \times J_i$ is a product metric

$$(x, a) \mapsto (l_i^2 + a^2)dx^2 + (l_i^2 + a^2)^{-1}da^2$$

and $l_i \rightarrow 0$ as $l \rightarrow 0$ (the curves $F_i \times \{0\} \subset C_i$ are closed geodesics of length l_i with respect to g_l). Let M_l denote the surface M equipped with the metric g_l if $l > 0$ and let $M_0 = M \setminus \bigcup_i c_i$ carry the limit metric $\lim_{l \rightarrow 0} g_l$. Note that M_0 is a complete hyperbolic surface by definition, which contains a pair of cusps for each i . Here we consider a family of surfaces $S_l = \Gamma_l \setminus H$ degenerating to the surface S with only one geodesic c_l being pinched, Γ_l containing the transformation $\sigma_l(z) = e^l z$ corresponding to c_l . Let K_l be S_l minus C_l the standard collar for c_l . There exist homeomorphisms f_l from $S_l \setminus c_l$ to S , with f_l tending to isometries C^2 -uniformly on the compact core $K_l \subset S_l$; define $\pi_l = f_l^{-1}$. Suppose that p is one of the two cusps of S arising from pinching c_l . Let $S_0 = \Gamma \setminus H$ be the component of S containing p and conjugate Γ to represent the cusp by the translation $w \mapsto w + 1$, in the following $p = \infty$.

Let for $\operatorname{Re} s > 1$, $\alpha_l(s, z) = \sum_{\gamma \in \langle \sigma_l \rangle \setminus \Gamma_l} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \right]$ such that the hyperbolic Eisenstein series $\Omega_{c_l} = \Omega_l$ is related by $\Omega_l(s, z) = \frac{1}{k(s)} \alpha_l(s + 1, z)$. Without loss of generality we suppose S_l having only one funnel F_1 . With the notations of the beginning, $S_l = K \cup (C_1 \cup \dots \cup C_{n_c}) \cup F_1$ and c_l is the one geodesic of the boundary of the compact core K . We consider the specific case of p , the limit of the right side of the c_l -collar, contained in $S_l \setminus F_1$.

Theorem 5.1. *Let $\operatorname{Re} s > 1$, the family of 1-forms $\frac{1}{l^s} \alpha_l(s, \pi_l(\cdot))$ converges uniformly on compact subsets of S_0 to $\operatorname{Im} \mathcal{E}_\infty(s, \cdot)$.*

It is a particular case of the theorem 5.2 below.

The sketch of the proof of this theorem follows these of the finite volume case ([4], [22], see also [8]). First of all we recall some material and results.

The following lemma can be found for example in [1]. The neighborhood of points within distance a of a geodesic γ , where $d(z, \gamma)$ is the hyperbolic distance from z to γ ,

$$G_a = \{z \in K, d(z, \gamma) \leq a\},$$

is isometric for small a to a half-collar $[0, a] \times S^1$, $ds^2 = dr^2 + l^2 \cosh^2 r d\theta^2$.

Lemma 5.1. *Suppose that γ is a simple closed geodesic of length $l(\gamma)$ on a geometrically finite hyperbolic surface M . Then γ has a collar neighborhood of half-width d , such that*

$$\sinh(d) = \frac{1}{\sinh(l(\gamma)/2)}.$$

As a consequence, if η is any other closed geodesic intersecting γ transversally (still assuming γ is simple), then the lengths of the two geodesics satisfy the inequality

$$\sinh(l(\eta)/2) \geq \frac{1}{\sinh(l(\gamma)/2)}.$$

Lemma 5.2. *Let γ be a simple closed geodesic of length l on a complete hyperbolic surface M . If α is a simple closed geodesic that does not intersect γ then, where*

$d(\gamma, \alpha)$ is the hyperbolic distance of γ to α

$$\cosh d(\gamma, \alpha) \geq \coth(l/2).$$

A standard collar for a length l geodesic is a cylinder isometric to $\mathcal{C} \setminus \langle z \mapsto e^l z \rangle$ with $\mathcal{C} = \{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi - l\} \subset H$ with the restriction of the hyperbolic metric, and $\langle z \mapsto e^l z \rangle$ the cyclic group generated by the transformation $z \mapsto e^l z$. There is a constant k_0 (the short geodesic constant) such that each closed geodesic on S_l of length at most k_0 has a neighborhood isometric to the standard collar and each cusp for S_l has a neighborhood isometric to the standard cusp; furthermore, the collars for short geodesics and the cusp regions are all mutually disjoint.

Now to study the right side of the c_l -collar let $w = \frac{1}{l} \log z$, with the principal branch $z \in H$, and conjugate Γ_l by the map w to obtain $\tilde{\Gamma}_l$ acting on $\mathcal{S}_l = \{w, 0 < \operatorname{Im} w < \pi/l\}$. The hyperbolic metric on \mathcal{S}_l is $ds_l^2 = \left(\frac{l|dw|}{\sin(l \operatorname{Im} w)}\right)^2$, which tends uniformly on compact subsets to $\left(\frac{|dw|}{\operatorname{Im} w}\right)^2$. $\tilde{\Gamma}_l$ is a (non Möbius) group of deck transformations acting on \mathcal{S}_l ; the quotient $\tilde{\Gamma}_l \backslash \mathcal{S}_l$ is S_l . Let \hat{f}_l be the restriction of f_l to the component $\mathcal{S}_l^{(r)}$ of $S_l \setminus c_l$ containing the right half-collar for c_l . Let F_l be a lift of \hat{f}_l to the universal covers \mathcal{A}_l and H , where \mathcal{A}_l is the simply connected component of $H \setminus \pi^{-1}(c_l)$ which contains the standard right collar $\{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi/2\}$. More precisely [4](p.350), [22]:

Lemma 5.3. *The simply connected component \mathcal{A}_l contains $\{z = re^{i\theta}, 1 \leq r \leq e^l, lc(l) < \theta < \pi/2\}$ where $c(l) \rightarrow 0, l \rightarrow 0$.*

Start with the standard Γ fundamental domain $\mathcal{F} = \{w \in H, 0 \leq \operatorname{Re} w < 1, \operatorname{Im} w \geq \operatorname{Im} A(w), \forall A \in \Gamma\}$. Set $D_l = F_l^{-1}(\mathcal{F})$, then D_l is a fundamental domain of S_l . Divide the cosets of $\langle \sigma_l \rangle \backslash (\Gamma_l - \langle \sigma_l \rangle)$ into two classes $D = \{[A], A \in \Gamma_l, \inf \operatorname{Re} A(D_l) > 0\}$ and $G = \{[A], A \in \Gamma_l, \sup \operatorname{Re} A(D_l) < 0\}$.

Then \hat{f}_l has a lift \tilde{f}_l , a homeomorphism from a subdomain of \mathcal{S}_l to H : $\tilde{f}_l = F_l \circ w^{-1} : w(\mathcal{A}_l) \rightarrow H$. The homeomorphism \tilde{f}_l induces a group homomorphism $\rho_l : \Gamma \rightarrow \tilde{\Gamma}_l, A \mapsto \tilde{f}_l^{-1} A \tilde{f}_l, A \in \Gamma$. We call $\rho_l(A) \in \tilde{\Gamma}_l$ the element corresponding to $A \in \Gamma$. Now by our normalizations for $\tilde{\Gamma}_l$ and Γ the translation $w \mapsto w + 1$ corresponds to itself. If we specify the further normalization $\tilde{f}_l(i) = i$ then the lifts \tilde{f}_l are uniquely determined and then we have (see e.g. [22] p.107):

Lemma 5.4. *The \tilde{f}_l tend uniformly on compact subsets to the identity, and thus for $A \in \Gamma$, the corresponding elements $\rho_l(A)$ tend uniformly on compact subsets to A .*

For $q \in \mathbb{N}$, we associate to the pinching geodesic c_l , the q -form defined for $\operatorname{Re} s > 1$ by:

$$A_{l,q}(s, z) = \sum_{\langle \sigma_l \rangle \backslash \Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)} \right)^q \sin^{s-q} \theta(\gamma z) dz^q.$$

Divide the cosets $\langle z \mapsto z + 1 \rangle \backslash (\tilde{\Gamma}_l - \langle z \mapsto z + 1 \rangle)$ into two classes, the left and the right: for $\mathcal{F}_l = \tilde{f}_l^{-1}(\mathcal{F})$, $L = wGw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \inf \operatorname{Im} A(\mathcal{F}_l) > \pi/2l\}$ and $R = wDw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \sup \operatorname{Im} A(\mathcal{F}_l) < \pi/2l\}$ (the line $\{\operatorname{Im} w = \pi/2l\}$ is a lift of c_l , and we write $[A]$ for the $\langle z \mapsto z + 1 \rangle$ coset of A). In particular the cosets

$\langle z \mapsto z + 1 \rangle \backslash (\Gamma - \langle z \mapsto z + 1 \rangle)$ correspond to the right cosets of $\tilde{\Gamma}_l$: $\{[\rho_l(A)], A \in \Gamma, \langle w \mapsto w + 1 \rangle\} \subset R$. Then we can write, where χ_+ is the characteristic function of $\{\operatorname{Re} z > 0\}$ and χ_- the one of $\{\operatorname{Re} z \leq 0\}$,

$$\begin{aligned} A_{l,q}(s, z) &= \sum_{\langle \sigma_l \rangle \backslash \Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)} \right)^q \sin^{s-q} \theta(\gamma z) dz^q \\ &= y(z)^{s-q} (\chi_+ + \sum_D \frac{\gamma'(z)^q |\gamma'(z)|^{s-q}}{\gamma(z)^q |\gamma(z)|^{s-q}}) dz^q + y(z)^{s-q} (\chi_- + \sum_G \frac{\gamma'(z)^q |\gamma'(z)|^{s-q}}{\gamma(z)^q |\gamma(z)|^{s-q}}) dz^q \end{aligned}$$

and the q -form on $\mathcal{S}_l^{(\tau)}$:

$$\begin{aligned} A_{l,q}^R(s, z) &= y(z)^{s-q} (\chi_+ + \sum_D \frac{\gamma'(z)^q |\gamma'(z)|^{s-q}}{\gamma(z)^q |\gamma(z)|^{s-q}}) dz^q \\ &= l^q (\sin l \operatorname{Im} w)^{s-q} (\chi + \sum_R (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q}) dw^q, \end{aligned}$$

where χ is the characteristic function of $w(\operatorname{Re} z > 0\}$).

Theorem 5.2. *Let $S_l = \Gamma_l \backslash H$ be a family of geometrically finite hyperbolic surfaces degenerating to the surface S with only one geodesic c_l being pinched and with S_l having only one funnel F_1 ; Γ_l contains the transformation $\sigma_l(z) = e^l z$ corresponding to c_l and the right half-collar for c_l is in $S_l \setminus F_1$. Let $S_0 = \Gamma \backslash H$ be the component of S containing p , the cusp arising from the right half-collar for c_l , $p = \infty$. Let $w = \frac{1}{l} \log z$ and write again with a little misuse of notation $A_{l,q}(s, w) = a_{l,q}(s, w) dw^q$ and the corresponding q -automorphic form (for $\tilde{\Gamma}_l$) $\hat{a}_{l,q}(s, w) = \operatorname{Im}(w)^q a_{l,q}(s, w)$. Denote $\hat{a}_{l,q}^R$ the right half of $\hat{a}_{l,q}$. Then the family $(\frac{1}{l^s} \hat{a}_{l,q}^R(s, \pi_l(\cdot)))_l$ converges uniformly on compact subsets of S_0 and on compact subsets of $\operatorname{Re} s > 1$ to the Eisenstein series for weight $2q$:*

$$E_{\infty,q}(s, w) = \sum_{\Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma w)^s \left(\frac{c\bar{w} + d}{cw + d} \right)^q.$$

Remark 5.1. *We have the analogue result: if we denote $\hat{a}_{l,q}^L$ the left half of $\hat{a}_{l,q}$, then $\frac{1}{l^s} \hat{a}_{l,q}^L(s, \pi_l(\cdot))$ converges to $(-1)^q E_{\mathfrak{a},q}(s, \cdot)$, where $E_{\mathfrak{a},q}(s, \cdot)$ is the Eisenstein series of weight $2q$ (see definition 3.1) associated to \mathfrak{a} , the other cusp arising from the left half-collar for c_l .*

Before proving this theorem we give some complements and its corollaries.

For $\operatorname{Re} s > 1$ let $b_q(s) = e^{i\pi q/2} \int_0^\pi (\sin u)^{s-2} e^{-iqu} du$. Note that $b_1(s) = k(s-1)$. The function b_q has the following properties (see e.g. [16]): $b_q(s+2) = \frac{s(s-1)}{s^2-q^2} b_q(s)$, b_q admits a meromorphic continuation to all $s \in \mathbb{C}$, more precisely

$$b_q(s) = \pi 2^{-s+2} \frac{\Gamma(s-1)}{\Gamma(\frac{s+q}{2}) \Gamma(\frac{s-q}{2})}.$$

In order to be consistent with the definition of Kudla and Millson's hyperbolic Eisenstein series, we may use the normalized q -automorphic forms

$$(10) \quad \Xi_{l,q}(s, z) = \frac{1}{b_q(s)} A_{l,q}(s, z).$$

We have indeed $\Xi_{l,1}(s, z) = \Theta(s - 1, z)$ defined in (4) section 2.1.

We recall that the series (10) converges absolutely and locally uniformly for any $z \in H$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, and that it is invariant with respect to Γ . A straightforward computation shows that the series $A_{l,q}(s, z)$ satisfies the functional differential equation:

$$\Delta_q^\pm A_{l,q}(s, z) - s(1-s)A_{l,q}(s, z) = (s+q)(s-q)A_{l,q}(s+2, z),$$

and the series (10)

$$\Delta_q^\pm \Xi_{l,q}(s, z) - s(1-s)\Xi_{l,q}(s, z) = s(s-1)\Xi_{l,q}(s+2, z).$$

Proposition 5.1. *The series $A_{l,q}(s, z)$ (resp. $\Xi_{l,q}(s, z)$) admits a meromorphic continuation to all of \mathbb{C} .*

Proof. There are different ways to prove this; one is to use the functional differential equation (10) and to apply the method developed in [17] (see also [16]). More precisely let $\tilde{a}_{l,q}(s, z)$ (resp. $\tilde{f}_{l,q}(s, z)$) the q -automorphic form associated to $A_{l,q}(s, z)$ (resp. $\Xi_{l,q}(s, z)$). We have

$$\Delta_{2q}\tilde{a}_{l,q}(s, z) + s(1-s)\tilde{a}_{l,q}(s, z) = (s+q)(q-s)\tilde{a}_{l,q}(s+2, z),$$

and

$$(11) \quad \Delta_{2q}\tilde{f}_{l,q}(s, z) + s(1-s)\tilde{f}_{l,q}(s, z) = s(1-s)\tilde{f}_{l,q}(s+2, z);$$

in other words

$$\tilde{f}_{l,q}(s, z) = s(1-s) \int_{\mathcal{D}} G_s(z, z', q) \tilde{f}_{l,q}(s+2, z') d\mu(z').$$

Another way is to use [7]. We precise another calculation we will develop further. We can rewrite $\tilde{a}_{l,q}(s, z)$ as

$$\tilde{a}_{l,q}(s, z) = \sum_{\Gamma_l \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right)^q \left(\frac{\bar{\gamma}z}{\gamma z} \right)^{q/2} \left(\frac{\operatorname{Im} \gamma z}{|\gamma z|} \right)^s$$

and using the Fourier development of $G_s(z, z', q)$ and the expansion of $\int_1^{e^t} G_s(z, iy', q) d \ln y'$ we obtain the result (see [7] corollary 4.2 p.188). \square

We can then refine the theorem 5.2 as

Theorem 5.3. *Let λ_k^q , $1 \leq k \leq n$, be the eigenvalues of the laplacian Δ_{2q} on S_0 such that $0 < \lambda_1^q < \lambda_2^q < \dots < \lambda_n^q < 1/4 \leq \lambda_{n+1}^q$ and the corresponding $s_k^q = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_k^q}$ with $\operatorname{Re} s_k^q \geq 1/2$ and let $\Omega = \{\operatorname{Re} s > 1/2\} \setminus \{s_1^q, \dots, s_n^q\}$.*

- (1) *In the case the geodesic c_l is not separating, without loss of generality we can suppose that the two limiting cusps are represented by ∞ and 0 , the family $(\frac{1}{t^s} \hat{f}_{l,q}(s, \pi_l(\cdot)))_l$ converges uniformly on compact subsets of S_0 and on compact subsets of Ω to $\frac{1}{b_q(s)} E_{\infty, q}(s, \cdot) + \frac{(-1)^q}{b_q(s)} E_{0, q}(s, \cdot)$.*
- (2) *In the case c_l is the geodesic boundary of a funnel, the family $(\frac{1}{t^s} \hat{f}_{l,q}(s, \pi_l(\cdot)))_l$ converges uniformly on compact subsets of S_0 and Ω to $\frac{1}{b_q(s)} E_{\infty, q}(s, \cdot)$.*

Remark 5.2. *We remark that as σ_l is the geodesic in the funnel, then $D = \langle \sigma_l \rangle \setminus (\Gamma_l - \langle \sigma_l \rangle)$ and $G = \emptyset$.*

We have seen that in the infinite volume case $s = 1$ is a regular value for $E_{\infty,q}(s, z)$ for every $q \in \mathbb{N}$. In the finite volume case we know that $s = 1$ is a simple pole for $E_{\infty,0}(s, z)$, which is no more the case for $q \geq 1$:

Remark 5.3. For $q \in \mathbb{N}^*$, $s = 1$ is a regular value of $E_{\infty,q}(s, z)$.

Proof. It comes from the Fourier development of $E_{\infty,q}(s, z)$, we have ([7], p.175)

$$E_{\infty,q}(s, z) = y^s + \varphi_q(s)y^{1-s} + \sum_{m \neq 0} a_m(y, s)_q e^{2i\pi mx},$$

with $\varphi_q(s) = \frac{\Gamma^2(s)}{\Gamma(s-q)\Gamma(s+q)} e^{-\pi i q} \varphi_0(s)$. The function φ_0 has a simple pole at $s = 1$ (with residue $\text{Vol}(M_0)/\pi$) and $1/\Gamma(s-q)$ has a zero at $s = 1$. \square

We know give the poles of $b_q(s)$:

Lemma 5.5. For an even q , the poles of $b_q(s)$ are simples and at the numbers $s = 1 - 2k$, $k \in \mathbb{N}$.

For an odd q , the poles of $b_q(s)$ are simples and at the numbers $s = -2k$, $k \in \mathbb{N}$.

From the preceding results and theorem (5.3) we deduce

Corollary 5.1. With the preceding notations,

- (1) Suppose q is odd. If the geodesic c_l is not separating (resp. the geodesic boundary of a funnel) the family $(\frac{1}{l^s} \hat{a}_{l,q}(s, \pi_l(\cdot)))_l$ converges uniformly on compact subsets of S_0 and on compact subsets of Ω to $E_{\infty,q}(s, \cdot) + (-1)^q E_{0,q}(s, \cdot)$ (resp. $E_{\infty,q}(s, \cdot)$).
- (2) Suppose q is even, we have the same preceding results replacing Ω by $\Omega \setminus \{1\}$.

Remark 5.4. For $q = 1$ we obtain the result announced in the introduction.

5.2. Proofs of the main theorems and final remarks.

Proof of theorem 5.2.

It is enough to demonstrate the convergence for β a relatively compact set in the fundamental domain \mathcal{F} . Given $\epsilon > 0$ denote G_ϵ the set of cosets and representatives for $\langle z \mapsto z + 1 \rangle \backslash \Gamma$ such that $\sup \text{Im } A(\mathcal{F}) < \epsilon$ for $[A] \notin G_\epsilon$ and let R_l be the corresponding cosets of $\langle z \mapsto z + 1 \rangle \backslash \tilde{\Gamma}_l$ with corresponding representatives. The set G_ϵ is finite.

The cosets G_ϵ of Γ satisfy (modulo the $\langle z \mapsto z + 1 \rangle$ action) $\{0 \leq \text{Re } w < 1, \text{Im } w > \epsilon\} \subset \cup_{A \in G_\epsilon} A(\mathcal{F})$; thus for l sufficiently small the cosets R_l satisfy (modulo the $\langle z \mapsto z + 1 \rangle$ action) $\{0 \leq \text{Re } w < 1, 2\epsilon < \text{Im } w < \pi/2l\} \subset \cup_{A \in R_l} A(\mathcal{F}_l)$ (a consequence of the convergence on compact subsets of the \tilde{f}_l and that \mathcal{F}_l contains the right half-collar for c_l , $\{0 \leq \text{Re } w < 1, c(l) \leq \text{Im } w < \pi/2l\}$). Now for a right coset $[A] \in R - R_l$ then $A(\mathcal{F}_l)$ lies below the c_l geodesic $\{\text{Im } w = \pi/2l\}$ and is disjoint modulo the $\langle z \mapsto z + 1 \rangle$ action from $\{0 \leq \text{Re } w < 1, 2\epsilon < \text{Im } w < \pi/2l\}$, since the latter is covered by the R_l cosets. Thus for $[A] \in R - R_l$, modulo the $\langle z \mapsto z + 1 \rangle$

action, then $A(\mathcal{F}_l) \subset \{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\}$. For $w \in \beta$ we write

$$\begin{aligned} \frac{1}{l^s} \hat{a}_{l,q}^R(s, w) &= \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} \left(\chi + \sum_R (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q} \right) \\ &= \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} \left(\chi + \sum_{R_l} (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q} \right) \\ &+ \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} \left(\sum_{R-R_l} (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q} \right). \end{aligned}$$

Because of lemma 5.4, the principal problem lies in estimating the second sum: it remains to show that

$$\lim_{l \rightarrow 0} \sum_{R-R_l} |\tilde{\gamma}' w|^\sigma = 0,$$

where $\sigma = \operatorname{Re} s$.

$$\sum_{R-R_l} |\tilde{\gamma}' w|^\sigma = \sum_{w^{-1}(R-R_l)w} |z_l \frac{\gamma' z_l}{\gamma z_l}|^\sigma \leq |z_l|^\sigma \sum_{w^{-1}(R-R_l)w} |\gamma' z_l|^\sigma,$$

where $w = \frac{1}{l} \log z_l$ and $\gamma \mathcal{F}_l \subset w^{-1}(\{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\})$. We deduce

$$\sum_{R-R_l} |\tilde{\gamma}' w|^\sigma \leq \frac{1}{\sin^\sigma(l \operatorname{Im} w)} \sum_{w^{-1}(R-R_l)w} \operatorname{Im}^\sigma(\gamma z_l).$$

Let $\epsilon_0 \in]0, \sinh^{-1} 1[$ such that for $l \leq l_0(\epsilon)$ and $w \in \beta$, $B(z_l, \epsilon_0) \subset \mathcal{F}_l$. Then there exists Λ_{ϵ_0} independent of z_0 so that

$$\int_{B(z_0, \epsilon_0)} y^\sigma \frac{dx dy}{y^2} = \Lambda_{\epsilon_0} y(z_0)^\sigma.$$

So

$$\sum_{w^{-1}(R-R_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma = \frac{1}{\Lambda_{\epsilon_0}} \sum_{w^{-1}(R-R_l)w} \int_{B(\gamma z_l, \epsilon_0)} y^\sigma \frac{dx dy}{y^2}.$$

Now (see e.g. [22] p.102):

Lemma 5.6. *The multiplicity of the projection map $H \rightarrow \Gamma \backslash H$ restricted to $B(z_0, \eta)$ with $2\eta < c_0$ is at most $M\rho^{-2}(z_0)$, where M is a constant and $\rho(z_0)$ the injectivity radius at z_0 .*

We give a detailed proof for the convenience of the reader.

Proof. If $B(z_0, \eta) \cap B(\gamma z_0, \eta) \neq \emptyset$, $\gamma \in \Gamma$, then $d(z_0, \gamma z_0) < 2\eta < c_0$ and z_0 is in a cusp region or the collar for a short geodesic. Let $c = c_0/2$, then we have $\rho(z_0) < c$. Set $m(\eta)$ the multiplicity of the projection restricted to $B(z_0, \eta)$. As $2\eta + \rho(z_0) < 3c$ and the $B(\gamma z_0, \rho(z_0))$ are disjoint we have

$$m(\eta) \mu(B(z_0, \rho(z_0))) \leq \mu(B(z_0, 3c)).$$

So $m(\eta) \leq \frac{\cosh 3c - 1}{\cosh \rho(z_0) - 1} \leq 2(\cosh 3c - 1)\rho(z_0)^{-2}$. \square

Then we have

$$\sum_{w^{-1}(R-R_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma \leq \frac{A(c_0)}{\Lambda_{\epsilon_0}} \rho^{-2}(z_l) \frac{e^{l\sigma} - 1}{\sigma} \frac{(2\epsilon l)^{\sigma-1}}{\sigma - 1}.$$

Moreover as $B(z_l, \epsilon_0) \subset \mathcal{F}_l$, $\rho(z_l) \geq \epsilon_0$ and the conclusion follows.

Proof of theorem 5.3.

The main point of the proof is to use Vitali's theorem as in the finite volume case. More precisely, first recall the

Definition 5.1. *A family (f_n) of meromorphic functions on a domain \mathcal{O} in \mathbb{C} is said bounded in \mathcal{O} if*

- (1) $\exists(z_m)_m$ a discrete subset in \mathcal{O} ,
- (2) $\forall K$ compact set of $\mathcal{O} \setminus \{z_m\}$, $\exists n(K)$, $\forall n \geq n(K)$, f_n has no pole in K ,
- (3) $M_K = \sup_{n \geq n(K)} (\sup_{z \in K} |f_n(z)|) < +\infty$.

As before β will denote a relatively compact set in the fundamental domain \mathcal{F} of S_0 . The family $(\frac{1}{l^s} \hat{f}_{l,q}(s, \pi(w)))_l$, where $\hat{f}_{l,q}(s, w) = \frac{1}{b_q(s)} \hat{a}_{l,q}(s, w)$ is a family of meromorphic functions on $\Omega = \{\text{Re } s > 1/2\} \setminus \{s_1^q, \dots, s_n^q\}$; because of theorem (5.2), for every compact $C \subset \{\text{Re } s > 1\}$ it converges uniformly (a fortiori simply) on $C \times \beta \subset \{\text{Re } s > 1\} \times \mathcal{F}$ to

- (1) $\frac{1}{b_q(s)} (E_{\infty,q}(s, \cdot) + (-1)^q E_{0,q}(s, \cdot))$ in case (1) of theorem (5.2),
- (2) $\frac{1}{b_q(s)} E_{\infty,q}(s, \cdot)$ in case (2).

We'll show that for every K compact set in Ω , there exists $l(K)$ such that for all l less than $l(K)$, $s \mapsto \frac{1}{l^s} \hat{f}_{l,q}(s, \pi(\cdot))$ has no pole on K and $M_K = \sup_{l \leq l(K)} (\sup_{z \in K} \|\frac{1}{l^s} \hat{f}_{l,q}(s, \pi(\cdot))\|_{\infty, \beta}) < +\infty$ then we can conclude.

The frame of the proof being independent of q , we will suppose $q = 0$ and only indicate the notable changes in the general q case, when there are. As in the finite volume case we start from the equality

$$\tilde{f}_l(s, z)/l^s = s(1-s) \int_{\mathcal{D}} G_s(z, z') \tilde{f}_l(s+2, z')/l^s d\mu(z');$$

where, to simplify the notations, we omit to write $q = 0$.

We will denote C a compact subset of S_0 , $Y \subset S_0$ such that $\pi_l(Y)$ is a standard collar and $\pi_l(Y) \cap \pi_l(C) = \emptyset$, $X = S_0 \setminus Y$, K' a compact subset of $\Omega' = \Omega \setminus \{1\}$. Cut the preceding integral as $I_l + J_l$ where

$$I_l = \int_{\pi_l(Y)} G_s(z, z') \tilde{f}_l(s+2, z')/l^s d\mu(z'), \quad J_l = \int_{\pi_l(X)} G_s(z, z') \tilde{f}_l(s+2, z')/l^s d\mu(z').$$

The main step is now to estimate $G_s(z, z', q)$. We apply [7] Corollary 1.3 p.151:

Proposition 5.2. *If $\Delta_{2q} f = s(s-1)f$ in some non-Eucliden disc B_r of radius r about $z_0 \in H$, then f has the mean value property:*

$$f(z_0) = \frac{1}{m_q(r, s)} \int_{B_r} f(z) \left(\frac{z - \bar{z}_0}{z_0 - \bar{z}} \right)^q d\mu(z) \text{ where } m_q(r, s) = 2\pi \int_1^r \text{sh } r P_{s,q}(r) \approx \pi r^2 \text{ as } r \rightarrow 0;$$

to conclude that $\forall w \in C, \forall z' \in \pi_l(Y), \forall s \in K', |G_s(\pi_l(w), z', q)| \leq O(1) \|G_s(\pi_l(w), \cdot, q)\|_{L_2}$ (see also [4] Lemme 4.4).

The next point is to show that $\forall (w, s) \in C \times K', \|G_s(\pi_l(w), \cdot, q)\|_{L_2} = O(1)$ (see also [4] Lemme 4.5), the constants in question depending only of the compact sets. For this we use the following spectral decomposition:

Lemma 5.7. *For $a > 1$*

$$G_s(\pi_l(w), z, l) = G_a(\pi_l(w), z, l) + \sum_n \left[\frac{1}{s(1-s) - \lambda_n} - \frac{1}{a(1-a) - \lambda_n} \right] \varphi_n(\pi_l(w) \overline{\varphi_n(z)}) +$$

$$\frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\frac{1}{s(1-s) - (1/4 + t^2)} - \frac{1}{a(1-a) - (1/4 + t^2)} \right] E_c(\pi_l(w), \frac{1}{2} + it) \overline{E_c(z, \frac{1}{2} + it)} dt +$$

$$\frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\frac{1}{s(1-s) - (1/4 + t^2)} - \frac{1}{a(1-a) - (1/4 + t^2)} \right] \int_1^{\lambda_f^2} E_f(\pi_l(w), \frac{1}{2} + it, b) \overline{E_f(z, \frac{1}{2} + it, b)} db dt.$$

From which we deduce, with the notation $\lambda_u = u(1-u)$

Lemma 5.8.

$$\|G_s(\pi_l(w), \cdot, l) - G_a(\pi_l(w), \cdot, l)\|_{L^2}^2 = |\lambda_a - \lambda_s|^2 \sum_n \frac{|\varphi_n(\pi_l(w))|^2}{|\lambda_s - \lambda_n(l)|^2 |\lambda_a - \lambda_n(l)|^2} +$$

$$|\lambda_a - \lambda_s|^2 \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{|E_c(\pi_l(w), 1/2 + it)|^2}{|\lambda_s - (1/2 + t^2)|^2 |\lambda_a - (1/2 + t^2)|^2} dt +$$

$$|\lambda_a - \lambda_s|^2 \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{|\lambda_s - (1/2 + t^2)|^2 |\lambda_a - (1/2 + t^2)|^2} \int_1^{\lambda_f^2} |E_f(\pi_l(w), 1/2 + it, b)|^2 db dt.$$

So there exists a constant M depending only on the compact K' of Ω' such that

$$\|G_s(\pi_l(w), \cdot, l) - G_a(\pi_l(w), \cdot, l)\|_{L^2}^2 \leq \frac{|\lambda_a - \lambda_s|^2}{M} \|G_a(\pi_l(w), \cdot, l)\|_{L^2}^2.$$

This allows for concluding that $I_l = O(1)$ and $J_l = O(l^2)$.

Remark 5.5. 1) We refer also to the result of J.Fay [6] final remark p.201-202.

2) It is possible to establish a link between hyperbolic Eisenstein series and generalized eigenfunctions- I thank F.Naud for this suggestion- and to use the first remark to conclude that as $l \rightarrow 0$

$$\int_0^l E_l^f(s, z, t) e^{ts} dt \rightarrow E_\infty(s, z).$$

Application. One of the application we can think about is in studying the degeneration of the residues of the hyperbolic Eisenstein series. Let us give an example: we look at the case of a degenerating family of compact Riemann surfaces, a non-separating geodesic being pinched, with the family of scalar-valued hyperbolic Eisenstein series degenerating. Remember that we obtained that the series $\frac{1}{l^s} A_l(s, z)$ converges uniformly on compact of S_0 to $E_\infty(s, z) + E_0(s, z)$. The last sum of Eisenstein series has no poles on $[1/2, 1]$ except at $s = 1$ and for a finite number of $s_k = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_k}$ where the λ_k correspond to the residual spectrum. In other words if $(\lambda_k(l))$ converges to λ_k where λ_k is a small cuspidal eigenvalue, then $Res(\frac{1}{l^s} A_l(s, z)) \rightarrow 0$, otherwise $Res(\frac{1}{l^s} A_l(s, z)) \rightarrow Res(E_\infty(s, z) + E_0(s, z))$. There are many restrictions to go ahead with calculation. First of all we are only dealing with small eigenvalues and so here there is no hope to characterize embedded eigenvalues through degeneration. Moreover we need to take into account the multiplicity of an eigenvalue $\lambda_k(l)$. So the easiest result we can obtained is a characterization of the residual spectrum -remember that an eigenvalue in the residual spectrum is simple- with $\alpha_k(l) = \int_{c_l} \psi_{k,l}(z) d\mu(z)$ where the eigenfunction $\psi_{k,l}$ is associated to the eigenvalue $\lambda_k(l)$, if λ_k is a pole of $E_\infty(s, z) + E_0(s, z)$ then $\alpha_k(l) = O(l^{1/2 + \sqrt{1/4 - \lambda_k(l)}})$ and otherwise $\alpha_k(l) = o(l^{1/2 + \sqrt{1/4 - \lambda_k(l)}})$.

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