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HYPERBOLIC EISENSTEIN SERIES FOR GEOMETRICALLY FINITE HYPERBOLIC SURFACES OF INFINITE VOLUME.

THÉRÈSE FALLIERO

ABSTRACT. Let M be a geometrically finite hyperbolic surface of infinite volume. After writing down the spectral decomposition for the Laplacian on 1-forms of M , we generalize the Kudla and Millson's construction of hyperbolic Eisenstein series (Invent Math 54:199-211, 1979) and other related results (see theorems 3.1, 4.2, 5.1).

INTRODUCTION.

The spectrum of the Laplace-Beltrami operator for a compact Riemann surface is discrete, it is no more the case when M is not compact. For example when you withdraw one point from M , it appears a continuous part in the spectrum whose spectral measure is described by an Eisenstein series. The study of the limiting behavior of the spectrum of the Laplace-Beltrami operator for a degenerating family of Riemann surfaces with finite area hyperbolic metrics have been used to explain this apparition (see for example [22], [15], [13]) and one of the motivation of this paper is the interest in a question of L.Ji in [15], p.308, concerning the approximation of Eisenstein series by suitable eigenfunctions of a degenerating family of hyperbolic Riemann surface. We hope to surround it via hyperbolic Eisenstein series (for results on degenerating Eisenstein series see, for example [18], [19], [8], [9]). What we really do here is to develop the suggestion in [17], to construct hyperbolic Eisenstein series and harmonic dual form in the infinite volume case: in this context we verify the convergence of hyperbolic Eisenstein series and the fact that it permits to realize a harmonic dual form to a simple closed geodesic on a geometrically finite hyperbolic surface of infinite volume (theorem 3.1) and in a similar way of an infinite geodesic joining a pair of punctures (theorem 4.2). We obtain a degeneration of hyperbolic Eisenstein series to horocyclic ones (theorem 5.1 and corollary 5.2):

Theorem 0.1. *Let $(S_l)_l$ a degenerating family of Riemann surfaces with infinite area hyperbolic metric and $\Omega_{c_l} = \Omega_l$ the hyperbolic Eisenstein series associated to the pinching geodesic c_l . For $\operatorname{Re} s > 0$, the family of 1-forms $\frac{1}{l^{s+1}}\Omega_l(s, \pi_l(.))$ converge uniformly on compact subsets of S_0 to $\frac{\Gamma(1+\frac{s}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{s}{2})} \operatorname{Im} \mathcal{E}_\infty(s+1, .)$.*

1. PRELIMINARY DEFINITIONS

Let us recall the standard analytic and geometric notations which will be used. In this paper a surface is a connected orientable two-dimensional manifold, without boundary unless

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otherwise specified. We denote by H the hyperbolic upper half-plane endowed with its standard metric of constant gaussian curvature -1 . A topologically finite surface is a surface homeomorphic to a compact surface with finitely many points exised and a geometrically finite hyperbolic surface M is a topologically finite, complete Riemann surface of constant curvature -1 . We will require that M is of infinite volume, then there exists a finitely generated, torsion free, discrete subgroup, Γ , of $PSL(2, \mathbb{R})$, unique up to conjugation, such that M is the quotient of H by Γ acting as Möbius transformations, Γ is a fuchsian group of the second kind and Γ has no elliptic elements different from the identity. The group Γ admits a finite sided polygonal fundamental domain in H . We recall now the description of the fundamental domain of $M = \Gamma \backslash H$ (see [1]). Let $L(\Gamma)$ be the limit set of Γ and $O(\Gamma) = \mathbb{R} \cup \{\infty\} - L(\Gamma)$. As $L(\Gamma)$ is closed in $\mathbb{R} \cup \{\infty\}$, $O(\Gamma)$ is open and so can be written as a countable union of $O(\Gamma) = \cup_{\alpha \in A} O_\alpha$ where the O_α are disjoint open intervals in $\mathbb{R} \cup \{\infty\}$. Then let $\Gamma_\alpha = \{\gamma \in \Gamma, \gamma(O_\alpha) = O_\alpha\}$.

This is an elementary hyperbolic subgroup of Γ . The fixed points of Γ_α are exactly the end-points of O_α . There is a finite subset $\{\alpha(1), \alpha(2), \dots, \alpha(n_f)\} \subset A$ so that, for $\alpha \in A$, O_α is conjugate to precisely one $O_{\alpha(j)}$ ($1 \leq j \leq n_f$). Let λ_α be the half-circle, lying in H , joining the end-points of O_α . Let Δ_α be the region in H bounded by O_α and λ_α . The Δ_α ($\alpha \in A$) are mutually disjoint.

Let P be the set of parabolic vertices of Γ , and for $p \in P$ let Γ_p be the parabolic subgroup of Γ fixing p . There is a finite subset $\{p(1), p(2), \dots, p(n_c)\} \subset P$ so that Γ_p is conjugate to precisely one $\Gamma_{p(j)}$ ($1 \leq j \leq n_c$). A circle lying in H and tangent to ∂H at p is called a horocycle at p . We can construct an open disc C_p determined by a horocycle at $p \in P$ so that:

- (i) if $p, q \in P, p \neq q$, then $C_p \cap C_q = \emptyset$,
- (ii) $g(C_p) = C_{g(p)}$ ($g \in \Gamma$),
- (iii) $C_p \cap \Delta_\alpha = \emptyset$ ($p \in P, \alpha \in A$).

If we consider the set $H - (\cup_{p \in P} C_p \cup \cup_{\alpha \in A} \Delta_\alpha)$, we see that it is invariant under Γ . We can find a finite-sided fundamental domain D for the action of Γ on this set; D is relatively compact in H .

Proposition 1.1. *There is a fundamental domain D for Γ of the form*

$$D = K^* \cup \cup_{j=1}^{n_f} D_{\alpha(j)} \cup \cup_{k=1}^{n_c} D_{p(k)}^*$$

where

- 1) K^* is relatively compact in H .
- 2) $D_{\alpha(j)}$ is a standard fundamental domain of $\Gamma_{\alpha(j)}$ on $\Delta_{\alpha(j)}$
- 3) $D_{p(k)}^*$ is a standard fundamental domain for $\Gamma_{p(k)}$ on $C_{p(k)}$.

We should note that $p \neq 0$ if and only if Γ is of the second kind.

The Nielsen region of the group Γ is the set $\tilde{N} = H - (\cup_{\alpha \in A} \Delta_\alpha)$, the truncated Nielsen region of Γ is $\tilde{K} = \tilde{N} - (\cup_{p \in P} C_p)$, $K = \Gamma \backslash \tilde{K}$ is called the compact core of M . So the surface $M = \Gamma \backslash H$ can be decomposed into a finite area surface with geodesic boundary N , called the Nielsen region, on which infinite area ends F_i are glued: the funnels. The Nielsen region N is itself decomposed into a compact surface K with geodesic and horocyclic boundary on

which non compact, finite area ends C_i are glued: the cusps. We have $M = K \cup C \cup F$, where $C = C_1 \cup \dots \cup C_{n_c}$ and $F = F_1 \cup \dots \cup F_{n_f}$.

A hyperbolic transformation $T \in PSL(2, \mathbb{R})$ generates a cyclic hyperbolic group $\langle T \rangle$. The quotient $C_l = \langle T \rangle \backslash H$ is a hyperbolic cylinder of diameter $l = l(T)$. By conjugation, we can identify the generator T with the map $z \mapsto e^l z$, and we define Γ_l to be the corresponding cyclic group. A natural fundamental domain for Γ_l would be the region $\mathcal{F}_l = \{1 \leq |z| \leq e^l\}$. The y -axis is the lift of the only simple closed geodesic on C_l , whose length is l . The standard funnel of $l > 0$, F_l , is the half hyperbolic cylinder $\Gamma_l \backslash H$, $F_l = (\mathbb{R}^+)_r \times (\mathbb{R} \backslash \mathbb{Z})_x$ with the metric $ds^2 = dr^2 + l^2 \cosh^2(r) dx^2$.

We can always conjugate a parabolic cyclic group $\langle T \rangle$ to the group Γ_∞ generated by $z \mapsto z+1$, so the parabolic cylinder is unique up to isometry. A natural fundamental domain for Γ_∞ is $\mathcal{F}_\infty = \{0 \leq \operatorname{Re} z \leq 1\} \subset H$. The standard cusp C_∞ is the half parabolic cylinder $\Gamma_\infty \backslash H$, $C_\infty = ([0, \infty])_r \times (\mathbb{R} \backslash \mathbb{Z})_x$ with the metric $ds^2 = dr^2 + e^{-2r} dx^2$. The funnels F_i and the cusps C_i are isometric to the preceding standard models. We define the function r as the distance to the compact core K and the function ρ by

$$\rho(r) = \begin{cases} 2e^{-r} & \text{in } F \\ e^{-r} & \text{in } C \end{cases}.$$

We will adopt $(\rho, t) \in (0, 2] \times \mathbb{R}/l_j \mathbb{Z}$ as the standard coordinates for the funnel F_j , where t is arc length around the central geodesic at $\rho = 2$.

For the cusp our standard coordinates $(\rho, t) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$ are based on the model defined by the cyclic group Γ_∞ . The cusp boundary is $y = 1$, so that $y = e^r$ and $\rho = 1/y$. We set $t = x \pmod{\mathbb{Z}}$.

2. HYPERBOLIC EISENSTEIN SERIES ON A GEOMETRICALLY FINITE HYPERBOLIC SURFACE OF INFINITE VOLUME.

2.1. Return to Kudla and Millson hyperbolic Eisenstein series' definition. In the following, M will denote an arbitrary Riemann surface and $L^2(M)$, the Hilbert space of square integrable 1-forms with inner product

$$(w_1, w_2) = \frac{1}{2} \int_M w_1 \wedge \ast \overline{w_2},$$

and correspondig norm $\|\cdot\|$.

Let c be a simple closed curve on M . We may associate with c a real smooth closed differential n_c with compact support such that

$$\int_c \omega = \int_M \omega \wedge n_c,$$

for all closed differentials ω . Since every cycle c on M is a finite sum of cycles corresponding to simple closed curves, we conclude that to each such c , we can associated a real closed differential n_c with compact support such that (2.1) holds.

Let a and b be two cycles on the Riemann surface M . We define the intersection number of a and b by

$$a.b = \int_M n_a \wedge n_b.$$

In [17], Kudla and Millson construct the harmonic 1-form dual to a simple closed geodesic on a hyperbolic surface of finite volume M in terms of Eisenstein series. Let us recall the definition:

Definition 2.1. *Let η be a simple closed geodesic or an infinite geodesic joining p and q . An 1-form α is dual to η if for any closed 1-form with compact support, ω*

$$\int_M \omega \wedge \alpha = \int_\eta \omega .$$

Or equivalently: for any closed oriented cycle c' , we have

$$(1) \quad \int_{c'} \alpha = \eta \cdot c' .$$

Kudla and Millson construct a meromorphic family of forms on M , called hyperbolic Eisenstein series associated to an oriented simple closed geodesic c . Let \tilde{c} be a component of the inverse image of c in the covering $H \rightarrow M$ and Γ_1 the stabilizer of \tilde{c} in Γ . The hyperbolic Eisenstein series are expressed in Fermi coordinates in the following way for $\text{Re } s > 0$:

$$(2) \quad \Omega_c(s, z) = \Omega(s, z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \frac{dx_2}{(\cosh x_2)^{s+1}}, \quad .$$

with

$$k(s) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s}{2}\right)} .$$

By applying an element of $SL(2, \mathbb{R})$ we may assume \tilde{c} is the y -axis in H and that Γ_1 is generated by $\gamma_1 : z \mapsto e^l z$. The Fermi coordinates (x_1, x_2) associated to \tilde{c} are related to euclidean polar coordinates by

$$\begin{aligned} r &= e^{x_1} \\ \sin \theta &= \frac{1}{\cosh x_2} \end{aligned}$$

At the end of their paper they do the remark that “it is also interesting to consider the infinite volume case”.

2.2. The infinite volume case. We are going to verify that this definition retains a meaning in the case of a geometrically finite hyperbolic surface of infinite volume, $M = \Gamma \backslash H$.

Proposition 2.1. *The hyperbolic Eisenstein series $\Omega(s, z)$ converges for $\text{Re } s > 0$, uniformly on compact subsets of H , is bounded on M and represents a C^∞ closed form which is dual to c . Moreover it is an analytic function of s in $\text{Re } s > 0$.*

The proof in the infinite volume case is as straightforward as in these in the finite volume case ([17], [10]), but for the convenience of the reader we give some details.

Lemma 2.1. *Let K a compact of the fundamental domain D of Γ , there exists $\eta > 0$ such that for all $z_0 \in K$, $(B(\gamma z, \eta))_{\gamma \in \Gamma_1 \backslash \Gamma}$ are disjoint.*

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FOR GEOMETRICALLY

Let choose for fundamental domain of Γ_1 , $\mathcal{D}_1 = \{z \in H : 1 \leq |z| \leq \beta\}$. After passing to ordinary Euclidean polar coordinates (r, θ) we obtain with $\sigma = \operatorname{Re} s$:

$$\begin{aligned} \|\Omega(s, z)\| &\leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \setminus \Gamma} \frac{1}{(\operatorname{ch} x_2(\gamma z))^{\sigma+1}} \\ &\leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \setminus \Gamma} \left(\frac{y}{r}\right)^{\sigma+1} (\gamma z) \leq \frac{1}{|k(s)|} \sum_{[\gamma] \in \Gamma_1 \setminus \Gamma, \gamma z \in \mathcal{D}_1} y^{\sigma+1}(\gamma z) \end{aligned}$$

Moreover for all $z \in K$: $\int_{B(z, \eta)} y^{\sigma+1} \frac{dx dy}{y^2} = \Lambda_\eta y(z)^{\sigma+1}$. We note as in [17]: $R(T_1, T_2) = \{P \in \mathcal{D}_1 : T_1 < x_2(P) < T_2\}$. So for $T > 2\eta$:

$$\frac{1}{|k(s)|} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} \frac{1}{(\operatorname{ch}(x_2(\gamma z)))^{\sigma+1}} \leq \frac{1}{|k(s)|} \sum_{[\gamma] \in \Gamma_1 \setminus \Gamma, \gamma z \in \mathcal{D}_1 - R(-T, T)} y^{\sigma+1}(\gamma z)$$

We need the following:

Lemma 2.2. *Let $\gamma \in \Gamma_1 \setminus \Gamma$, $z, \zeta \in H$ such that for $\gamma z \notin R(-T, T)$, $\gamma \zeta \in B(\gamma z, \eta)$ then $\gamma \zeta \notin R(-T + 2\eta, T - 2\eta)$.*

Proof. As orthogonal projection P is 1- lipschitzien, we have $d(P\gamma z, P\gamma \zeta) \leq d(\gamma z, \gamma \zeta) \leq \eta$. If $x_2(\gamma z) \geq T$:

$$T \leq d(\gamma z, P\gamma z) \leq d(\gamma z, \gamma \zeta) + d(\gamma \zeta, P\gamma \zeta) + d(P\gamma z, P\gamma \zeta).$$

Then

$$T - 2\eta \leq d(\gamma \zeta, P\gamma \zeta).$$

□

Then

$$\begin{aligned} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} y^{\sigma+1}(\gamma z) &= \frac{1}{\Lambda_\eta} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} \int_{B(\gamma z, \eta)} y^{\sigma+1} \frac{dx dy}{y^2} \\ &\leq \frac{1}{\Lambda_\eta} \int_{R^c(-T+2\eta, T-2\eta)} y^{\sigma+1} \frac{dx dy}{y^2} \end{aligned}$$

where $R^c(-T + 2\eta, T - 2\eta)$ is the complementary in \mathcal{D}_1 of $R(-T + 2\eta, T - 2\eta)$. Note that if $\gamma z \notin R(-T, T)$ then $y(\gamma z) \leq \frac{\beta}{\operatorname{ch} T}$, so:

$$\begin{aligned} \sum_{\gamma \in \Gamma_1 \setminus \Gamma, \gamma z \notin R(-T, T)} y^{\sigma+1}(\gamma z) &\leq \frac{\beta}{\Lambda_\eta} \int_0^{\frac{\beta}{\operatorname{ch}(T-2\eta)}} y^{\sigma-1} dy \\ &\leq \frac{\beta}{\Lambda_\eta \sigma} \left(\frac{\beta}{\operatorname{ch}(T-2\eta)} \right)^\sigma. \end{aligned}$$

From this follow the uniform convergence of $\Omega(s, z)$ on compact subsets of H , uniformly on compact subsets of the half plane $\operatorname{Re} s > 0$. We next show that $\Omega(s, z)$ is bounded on D . For this we use a "very useful (and well-worn) fundamental lemma"(sic), see [14], p. 178, [12], p.27:

Proposition 2.2. *For any fuchsian group Γ , there exists $\mathcal{C}(q, \Gamma)$, such that for all $z \in H$*

$$\sum_{\gamma \in \Gamma} \frac{y(\gamma z)^q}{[1 + |\gamma z|]^{2q}} \leq \mathcal{C}(q, \Gamma)$$

The constant $\mathcal{C}(q, \Gamma)$ depending only of q and Γ .

Let $z \in H$, there exists a system of representant S of $\Gamma_1 \backslash \Gamma$ such that for all $\gamma \in S$, $|\gamma z| \leq \beta$. Then:

$$\begin{aligned} \sum_{\Gamma_1 \backslash \Gamma} \frac{y^{\sigma+1}(\gamma z)}{(1 + \beta)^{2(\sigma+1)}} &\leq \sum_{\Gamma_1 \backslash \Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1 + |\gamma z|)^{2(\sigma+1)}} \\ &\leq \sum_{\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1 + |\gamma z|)^{2(\sigma+1)}} \\ &\leq \mathcal{C}(\sigma + 1, \Gamma) \end{aligned}$$

and the result.

The fact that $\Omega(s, z)$ is dual to c follows straightly from the construction of Kudla and Millson.

3. SPECTRAL DECOMPOSITION AND ANALYTIC CONTINUATION.

The aim is to realize the injection $H_c^1 \rightarrow \mathcal{H}^1$, where H_c^1 is the first de Rham's cohomology group with compact support of M and \mathcal{H}^1 is the space of L^2 harmonic 1-forms of M . Recall that in our context $\dim \mathcal{H}^1 = \infty$ (see [2], p. 27).

We are going to prove, as in [17], the analytic continuation of the hyperbolic Eisenstein series. The essential difference with the finite volume case is the spectral decomposition of $L^2(M)$.

3.1. Spectral theory. For any non-compact geometrically finite hyperbolic surface M , the essential spectrum of the (positive) Laplacian Δ_M defined by the hyperbolic metric on M (the Laplacian on functions) is $[1/4, \infty)$ and this is absolutely continuous. The discrete spectrum consists of finitely many eigenvalues in the range $(0, 1/4)$. In the finite-volume case one may also have embedded eigenvalues in the continuous spectrum, but these do not occur for infinite-volume surfaces. Then if M as infinite volume, the discrete spectrum of Δ_M is finite (possibly empty). The exponent of convergence δ of a fuchsian group Γ is defined to be the abscissa of convergence of the Dirichlet series:

$$\delta = \inf \left\{ s > 0, \sum_{T \in \Gamma} e^{-sd(z, Tw)} < \infty \right\}$$

for some $z, w \in \Gamma$.

Let Γ be a fuchsian group of the second kind and $L(\Gamma)$ be its limit set, then $0 < \delta < 1$ with $\delta > 1/2$ if Γ has parabolic elements. Patterson and Sullivan showed that δ is the Hausdorff dimension of the limit set when Γ is geometrically finite. Furthermore, if $\delta > 1/2$, then $\delta(1 - \delta)$ is the lowest eigenvalue of the Laplacian Δ_M . The connection to spectral theory was later extended to the case $\delta \leq 1/2$ by Patterson. In this case, the discrete spectrum of Δ_M is empty and δ is the location of the first resonance. For a detailed account of the spectral theory of infinite area surfaces, we refer the reader to [1].

3.2. Tensors and automorphic forms. This section introduces the notations used in the following subsection 3.3, section 5. Let M be a Riemann surface of finite Euler characteristic and carrying a metric ds^2 of constant curvature -1 . Let z be a local conformal variable and $ds^2 = \rho|dz|^2$. Let T^q denote the space of tensors $f(z)(dz)^q$ for q integer, on M . The covariant derivative ∇ sends T^q into $T^q \otimes (T^1 \oplus \bar{T}^1)$ and can be decomposed accordingly as $\nabla = \nabla_z^q \oplus \nabla_{\bar{z}}^q$, with $\nabla_z^q : T^q \rightarrow T^Q \otimes T^1 \cong T^{q+1}$, $\nabla_{\bar{z}}^q = \rho^q \partial \rho^{-q}$, $\nabla_q = \rho^{-1} \bar{\partial}$, $\nabla_q^z : T^q \rightarrow T^Q \otimes \bar{T}^1 \cong T^{q-1}$. The Laplacians Δ_q^+ and Δ_q^- on T^q are defined by $\Delta_q^+ = -\nabla_{q+1} \nabla^q$, $\Delta_q^- = -\nabla_{q-1} \nabla^q$. Thus $\Delta_0 = \Delta_0^\pm$ is the Laplacian on functions. The operators Δ_q^\pm are non-negative selfadjoint. We recall now the link between q -forms and automorphic forms of weight $2q$. We make use of the uniformization theorem. M may be realized as $H \backslash \Gamma$, where H is the upper half plane and γ a discrete subgroup of $PSL(2, \mathbb{R})$. Let $\tilde{\Gamma} \subset SL(2, \mathbb{R})$ be the groupe covering Γ under the projection $SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$. Using notably the notations in [11] and [7], let

Definition 3.1. Set

$$j_\gamma(z) = \frac{(cz + d)^2}{|cz + d|^2} = \frac{cz + d}{c\bar{z} + d} = \left(\frac{\gamma' z}{|\gamma' z|} \right)^{-1} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let \mathcal{F}_q be the space of all functions $f : H \rightarrow \mathbb{C}$ with

$$f(\gamma z) = j_\gamma(z)^q f(z), \quad \gamma \in \Gamma,$$

and if $\mathcal{D} = \Gamma \backslash H$ is the fundamental domain of Γ , define the Hilbert space $\mathcal{H}_q = \{f \in \mathcal{F}_q, \langle f, f \rangle_{\mathcal{D}} = \int_{\mathcal{D}} |f(z)|^2 d\mu(z) < \infty\}$ with $d\mu(z) = \frac{dx dy}{y^2}$ and the inner product $\langle f, g \rangle = \int_{\mathcal{D}} f(z) \overline{g(z)} d\mu(z)$. \mathcal{F}_q is isometric to T^q through the correspondance I :

$$T^q \ni f \rightarrow y^q f.$$

Under this correspondence, the operators $\nabla_q^z, \nabla_q^{\bar{z}}$ go over to the Maasz operators $L_q : \mathcal{F}_q \rightarrow \mathcal{F}_{q-1}$, $K_q : \mathcal{F}_q \rightarrow \mathcal{F}_{q+1}$ according to the diagram

$$\begin{array}{ccccc} T^{q-1} & \xleftarrow{\nabla_q^z} & T^q & \xrightarrow{\nabla_q^{\bar{z}}} & T^{q+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{q-1} & \xleftarrow{L_q} & \mathcal{F}_q & \xrightarrow{K_q} & \mathcal{F}_{q+1} \end{array}$$

where $L_q = (\bar{z} - z) \frac{\partial}{\partial \bar{z}} - q$ and $K_q = (z - \bar{z}) \frac{\partial}{\partial z} + q$. We have also:

$$\begin{aligned} L_q &= -2iy^{1+q} \frac{\partial}{\partial \bar{z}} y^{-q} = \overline{K_{-q}} \\ K_q &= 2iy^{1-q} \frac{\partial}{\partial z} y^q = \overline{L_{-q}}. \end{aligned}$$

We note

$$\begin{aligned} -L_{q+1} K_q &= -\Delta_{2q} + q(q+1) \\ -K_{q-1} L_q &= -\Delta_{2q} + q(q-1) \end{aligned}$$

with

$$\Delta_{2q} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2iqy \frac{\partial}{\partial x}.$$

¹Some authors called them of weight q or $-2q$

These second order differential operators are self-adjoint on \mathcal{F}_q . Furthermore the isometry I conjugates Δ_q^+ with $-\Delta_{2q} + q(q+1)$ and Δ_q^- with $-\Delta_{2q} + q(q-1)$.

We are first interested in the case $q = 2$. Let Δ_{Diff} the (positive) Laplacian on 1-forms on a geometrically finite hyperbolic surface, $\Delta_{\text{Diff}} = d\delta + \delta d$, $\delta = -*d*$ with $*$ the Hodge operator. In the following we note $\Delta_{\text{Diff}} = \Delta$. If ω is a 1-form in the holomorphic cotangent bundle, $\omega = f(z)dz$, then we define the image by the isometry I , $I(w) = I(fdz) = yf(z) = \tilde{f}(z)$. We have the relation: if $y\Delta(fdz) = -(\Delta_2 \tilde{f})dz$, in other words, with the preceding notations $\Delta = \Delta_1^-$.

3.3. Generalized eigenfunctions. We are going to give the spectral expansion in eigenforms of Δ ; we use [7], [20], [1]. For a finitely generated group of the second kind, for each cusp and for each funnel of the quotient there is a corresponding Eisenstein series, this is what we are going to develop now.

Proposition 3.1. *For $\text{Re } s > \delta$, the kernel of the resolvent $G_s(z, w, 1)$ for the self-adjoint operator Δ_2 acting on the Hilbert space $L_{2,2}$ of automorphic forms of weight 2, is given by the convergent series*

$$G_s(z, w, 1) = \sum_{\gamma \in \Gamma} j_\gamma(w) g_s(z, \gamma w, 1)$$

with $g_s(z, w, 1) = -\frac{w - \bar{z}}{z - \bar{w}} \frac{\Gamma(s+1)\Gamma(s-1)}{4\pi\Gamma(2s)} \sigma^{-s} F(s+1, s-1, s; \sigma^{-1})$ and F is the Gauss hypergeometric function.

For the funnel case, we identify z' with the standard coordinates (ρ', t') in the funnel F_j , and define

$$(3) \quad E_{j,1}^f(s, z, t') = \lim_{\rho' \rightarrow 0} \rho'^{-s} G_s(z, z', 1),$$

for $j = 1, \dots, n_f$. In the cusp C_j , with standard coordinates $z' = (\rho', t')$, we set

$$(4) \quad E_{j,1}^c(s, z) = \lim_{\rho' \rightarrow 0} \rho'^{1-s} G_s(z, z', 1),$$

for $j = 1, \dots, n_c$.

Let

$$P(z, \zeta) = \text{Im}(z)/|z - \zeta|^2$$

where $z \in H$ and $\zeta \in \mathbb{R}$ be the Poisson kernel. For $b \in O(\Gamma) = \mathbb{R} \cup \{\infty\} - L(\Gamma)$ define the Eisenstein series ([20], [3])

$$E_b(z, s, k) = \sum_{\gamma \in \Gamma} j(\gamma, z)^k P(\gamma(z), b)^s (\gamma(z), b)^k,$$

where $j(\gamma, z) = \gamma'(z)/|\gamma'(z)|$ and $(z, b) = (\bar{z} - b)/(z - b)$.

For the standard funnel F_l which corresponds to the region $\text{Re } z \geq 0$ in the model $C_l = \Gamma_l \backslash H$, we have (see [7] p.200):

$$\begin{aligned} \frac{1}{1-2s} E_{l,1}^f(s, z, x') &= \lim_{z' \rightarrow x'} (\text{Im } z')^{-s} G_s(z, z', 1) \\ &= -\frac{4^s}{4\pi} \frac{\Gamma(s+1)\Gamma(s-1)}{\Gamma(2s)} E_{x'}(z, s, 1). \end{aligned}$$

We note then

$$\mathcal{E}_{f_i}(s, z, x') = \frac{E_{i,1}^f(s, z, x')}{y} dz.$$

Recall 3.1. Recall the definition of the classical Eisenstein series. The stability group of a cusp \mathfrak{a} is an infinite cyclic group generated by a parabolic motion,

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma \mathfrak{a} = \mathfrak{a}\} = \langle \gamma_{\mathfrak{a}} \rangle ,$$

say. There exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We call $\sigma_{\mathfrak{a}}$ a scaling matrix of the cusp \mathfrak{a} , it is determined up to composition with a translation from the right. The Eisenstein series for the cusp \mathfrak{a} is then defined by:

$$E_{\mathfrak{a}}(z, s) = \sum_{\Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s ,$$

where s is a complex variable, $\operatorname{Re} s > 1$.

An Eisenstein series of weight 2 associated to a cusp \mathfrak{a} is the 1-form, defined for $\operatorname{Re} s > 1$, by:

$$\mathcal{E}_{\mathfrak{a}}(s, z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1} d(\sigma_{\mathfrak{a}}^{-1}\gamma z) = \frac{E_{\mathfrak{a},1}(s, z)}{y} dz ,$$

with

$$E_{\mathfrak{a},1}(s, z) = y \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1} ((\sigma_{\mathfrak{a}}^{-1}\gamma)'z) .$$

We now verify that it corresponds to the defining formula (4). For the standard cusp, we write

$$G_s(z, z', 1) = \sum_{\Gamma_{\infty} \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right) G_s^{\Gamma_{\infty}}(\gamma z, z', 1) ,$$

where $G_s^{\Gamma_{\infty}}(\gamma z, z', 1)$ is the resolvent kernel of the standard cusp for automorphic forms of weight 2. We use then [7] p. 155 (38), p.177 for $\operatorname{Im} z' > \operatorname{Im} \gamma z$, p. 172 (see also [1] p.72, p.102) to conclude that

$$\lim_{y' \rightarrow \infty} y'^{s-1} G_s(z, z', 1) = \sum_{\Gamma_{\infty} \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right) \frac{(\operatorname{Im} \gamma z)^s}{1 - 2s} = \frac{1}{1 - 2s} E_{\infty,1}(s, z) .$$

With the preceding notations we then have (see for example [7], [20]). For $w = f(z) dz$ square integrable, we have

$$\begin{aligned} w(z) &= \sum_{i=1}^m (w)_{\lambda_i}(z) + \frac{1}{4\pi i} \sum_{j=1}^{n_c} \int_{-\infty}^{+\infty} \langle w, \mathcal{E}_{c_j}(1/2 + it, \cdot) \rangle \mathcal{E}_{c_j}(1/2 + it, z) dt + \\ &\quad \frac{1}{4\pi i} \sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_j^2} \langle w, \mathcal{E}_{f_j}(1/2 + it, \cdot, b) \rangle \mathcal{E}_{f_j}(1/2 + it, z, b) db \right] dt. \end{aligned}$$

Remark 3.1. *One can easily deduce the formula for an arbitrary square integrable 1-form*

$$\begin{aligned}\Omega(z) &= f dz + g d\bar{z} = \sum_{i=1}^m (\Omega)_{\lambda_i}(z) + \\ &\frac{1}{4\pi i} \sum_{j=1}^{n_c} \int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot) \rangle \mathcal{E}_{c_j}(1/2 + it, z) + \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot)_{-1} \rangle \mathcal{E}_{c_j}(1/2 + it, z)_{-1} dt + \\ &\frac{1}{4\pi i} \sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \int_1^{\lambda_f^2} \langle \Omega, \mathcal{E}_{f_j}(1/2 + it, \cdot, b) \rangle \mathcal{E}_{f_j}(1/2 + it, z, b) + \langle \Omega, \mathcal{E}_{f_j}(1/2 + it, \cdot, b)_{-1} \rangle \mathcal{E}_{f_j}(1/2 + it, z, b)_{-1} db dt\end{aligned}$$

where, with obvious notations, $\mathcal{E}_{-1} = \overline{\mathcal{E}} = \frac{\overline{E}}{y} d\bar{z}$. To simplify we will write

$$\begin{aligned}\Omega(z) &= (\Omega)_{\lambda_i}(z) + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2 + it, \cdot)_{\pm} \rangle \mathcal{E}_{c_j}(1/2 + it, z)_{\pm} dt + \\ &\frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_f^2} \langle \Omega, \mathcal{E}_{f_k}(1/2 + it, \cdot, b)_{\pm} \rangle \mathcal{E}_{f_k}(1/2 + it, z, b)_{\pm} db \right] dt.\end{aligned}$$

3.4. Harmonic dual form. We are now going to see

Proposition 3.2. *The hyperbolic Eisenstein series Ω_c are square integrable.*

Proof. We consider a fundamental domain D contained in $\{z, 1 \leq |z| \leq e^l\}$ in which the segment (i, ie^l) represents the geodesic c . We note $C_\lambda = \{z \in D, d(z, c) = \lambda\}$ and $F_\lambda = \{z \in D, d(z, c) \geq \lambda\}$. Without loss of generality we can suppose that there is only one funnel on M and no cusps. Let V_λ the volume of $F_\lambda - F_{\lambda+1}$ there exists a constant c_1 such that $V_\lambda \geq c_1(sh(\lambda+1) - sh(\lambda))$. For $\text{Re } s = \sigma > 0$,

$$\|\Omega_c(s, z)\| = \|\Omega(s, z)\| \leq \frac{1}{|k(s)|} \sum_{\Gamma_1 \backslash \Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$$

Let $\eta(z) = \sum_{\Gamma_1 \backslash \Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$, we have

$$\begin{aligned}\int_D \|\Omega_c(s, z)\|^2 d\mu(z) &\leq \frac{1}{|k(s)|^2} \int_D \eta^2(z) d\mu(z) \\ &\leq \frac{1}{|k(s)|^2} \int_{1 \leq x_1 \leq e^l, -\infty < x_2 < +\infty} \eta(z) \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 dx_1 dx_2 \\ &\leq \frac{M}{|k(s)|^2} \int_{1 \leq x_1 \leq e^l, -\infty < x_2 < +\infty} \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2 dx_1 dx_2\end{aligned}$$

where $M > 0$ is such that $\forall z \in H, \eta(z) \leq M$.

The last integral is $\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\sigma}{2})}{\Gamma(\frac{1}{2} + \frac{\sigma}{2})}(e^l - 1)$ and the result. \square

As before we have:

$$\Delta(\Omega(s, z)) + s(s+1)\Omega(s, z) = s(s+1)\Omega(s+2, z).$$

This formula has a consequence that, for fixed s with $\operatorname{Re} s > 0$, the function $\Delta^k(\Omega(s, z))$ is again square integrable for any $k > 0$.

Set $\operatorname{Re} s > 0$, with our convention of notations

$$(5) \quad \begin{aligned} \Omega(s, z) = & \Omega_0(z) + a_i(s)\varphi_i(z) + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} h_{\pm}^c(s, t) \mathcal{E}_c(1/2 + it, z)_{\pm} dt \\ & + \frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_f^2} H_{\pm}^f(s, t, b) \mathcal{E}_f(1/2 + it, z, b)_{\pm} db \right] dt. \end{aligned}$$

We obtain

$$H(s, t, b)[1/4 + t^2 + s(s+1)] = s(s+1)H(s+2, t, b),$$

where H corresponds to any H_f^{\pm} . From this we get a continuation of H to the region $\operatorname{Re} s > -1/2$ and we note that we have for all t and all b $H(0, t, b) = 0$.

Moreover for $\operatorname{Re} s > -1/2$, $\operatorname{Re}(s+2) > 0$ and we may substitute in (5) to obtain a continuation of $\Omega(s, z)$ to $\operatorname{Re} s > -1/2$. Thus we have proved the following theorem

Theorem 3.1. *$\Omega(s, z)$ has a meromorphic continuation to $\operatorname{Re} s > -1/2$ with $s = 0$ a regular point and $\Omega(0, z)$ is a harmonic form which is dual to c .*

Remark 3.2. 1) *Another way to see this:*

write $\Omega(s, z) = (\Delta + s(s+1))^{-1}(s(s+1)\Omega(s+2, z))$ and use the meromorphic continuation of the resolvent (see for example [1], [21]).

2) *With an analogue study of [17] (see also [16]) we can obtain a total description of the singularities of the hyperbolic Eisenstein series.*

4. THE CASE OF AN INFINITE GEODESIC JOINING TWO POINTS.

Without loss of generality we suppose the two cusps p and q to be 0 and ∞ respectively and, as the lift of the geodesic, we take the imaginary axis. Let η be the infinite geodesic $]p, q[$, can we do the same construction as Kudla and Millson? As in the finite volume case, the problem reduces to study the following series for $\operatorname{Re} s > 1$:

$$(6) \quad \hat{\eta}^s(z) = \frac{1}{k(s-1)} \sum_{\gamma \in \Gamma} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \right] = \operatorname{Im}(\theta^s(z)),$$

where

$$\theta^s(z) = \frac{1}{k(s-1)} \sum_{\gamma \in \Gamma} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right],$$

$$\text{and } k(s-1) = \frac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2 + s/2)}.$$

4.1. Some useful estimations. As usual we can suppose $\Gamma_{\infty} = \langle z \mapsto z+1 \rangle$ to be the stabilizer of ∞ in Γ and the stabilizer of 0, Γ_0 is then generated by $z \mapsto \frac{z}{-c_0^2 z + 1}$ (for some non zero constant c_0).

First of all we note that, contrary to the finite volume case, we have $\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \operatorname{Im}(\gamma z)$ convergent (see proposition 3.1 and formula 4). Another way to see this “by hand”:

Remark 4.1. We know that for $\operatorname{Re} s > \delta$, $\sum_{T \in \Gamma} e^{-sd(i,Tz)}$ converges, moreover there exists a constant $C > 0$ such that $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \operatorname{Im}^s(\gamma z) \leq C \sum_{T \in \Gamma} e^{-sd(i,Tz)}$, as in our case $\delta < 1$, we have the result.

As $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \left| \operatorname{Im}^s(\gamma z) \frac{c\bar{z}+d}{cz+d} \right| = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \operatorname{Im}^{\operatorname{Re} s}(\gamma z)$ we also deduce the convergence of $E_{\infty,1}(1, z)$.

With the notations of recall 3.1 and ρ the standard coordinate for the cusp \mathfrak{a} , we write the results of [1](p.110): $E_{\mathfrak{a}}(s, \cdot) = \rho^{-s}(1 - \chi_0(\rho)) + O(\rho_f^s \rho_c^{s-1})$, ρ is decomposed as $\rho_f \rho_c$ with $\rho_f = \rho$ in the funnels and $\rho_c = \rho$ in the cusps and we define $\chi_0 \in \mathcal{C}_0^\infty(X)$ such that

$$\chi_0 = \begin{cases} 1, & r \leq 0 \\ 0, & r \geq 1 \end{cases}.$$

Lemma 4.1. We have the following asymptotic behaviors for $\operatorname{Re} s > \delta$:

- 1) in a funnel for all cusp \mathfrak{a} , $E_{\mathfrak{a}}(s, z)$ is square integrable;
- 2) at $\mathfrak{a} = \infty$, $E_\infty(s, z) - y^s = O(y^{1-s})$ and $E_0(s, z) = O(y^{1-s})$;
- 3) near $\mathfrak{a} = 0$, $E_0(s, z) - y^s / (c_0^2 |z|^2)^s = O(y^{1-s} / (c_0^2 |z|^2)^{s-1})$ and $E_\infty(s, z) = O(y^{1-s} / (c_0^2 |z|^2)^{s-1})$.

4.2. Convergence of the Hyperbolic Eisenstein series and analytic continuation.

The calculus and results to prove the convergence of (6) adapt easily from the finite volume case. For the convenience of the lecture we recall the essential points.

We have $\| \sum_{\gamma \in \Gamma} \gamma * \left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \| \leq \sum_{\gamma \in \Gamma} \left(\frac{y}{|z|} \right)^\sigma (\gamma z)$, where $\sigma = \operatorname{Re} s > 1$ and if we denote by $\mathcal{S} = \sum_{\gamma \in \Gamma} \left(\frac{y}{|z|} \right)^\sigma (\gamma z)$, we have

$$\mathcal{S} = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) \sum_{n \in \mathbb{Z}} \frac{1}{|\gamma z + n|^\sigma}.$$

Let S_z a system of representatives of $\Gamma_\infty \setminus \Gamma$ such that $|\operatorname{Re} \gamma z| \leq 1/2$, then

$$\|\mathcal{S}\| \leq \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + 2 \sum_{\gamma \in S_z} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n - 1/2)^\sigma}.$$

We have

$$\begin{aligned} \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &= \sum_{\Gamma \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n \in \mathbb{Z}} \frac{1}{|ne\gamma z + 1|^\sigma} \\ &= \sum_{\gamma \in \Gamma \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + \sum_{\gamma \in \Gamma \setminus S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n \in \mathbb{Z}^*} \frac{1}{|ne|^\sigma [(x(\gamma z) + 1/ne)^2 + y^2(\gamma z)]^{\sigma/2}}. \end{aligned}$$

and for K a compact set in H there exists m in H such that

$$\forall z \in K, \forall \gamma \in \Gamma_0 \setminus S_z, |\gamma z| \geq |m| \text{ and } \operatorname{Im} \gamma z \geq \operatorname{Im} m.$$

So

$$\begin{aligned} \sum_{\gamma \in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &\leq \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} + \sum_{\gamma \in \Gamma_0 \setminus S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} \sum_{n \in \mathbb{Z}^*} \frac{1}{|ne|^\sigma (\operatorname{Im} m)^\sigma} \\ &\leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + 2 \sum_{n \in \mathbb{N}^*} \frac{1}{(ne)^\sigma} \frac{1}{|m|^\sigma (\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z); \end{aligned}$$

and finally

$$\begin{aligned} \|\mathcal{S}\| \leq & \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + 2 \sum_{n \in \mathbb{N}^*} \frac{1}{(ne)^\sigma} \frac{1}{|m|^\sigma (\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) + \\ & 2 \sum_{\Gamma_\infty \setminus \Gamma} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma}, \end{aligned}$$

the uniform convergence on all compact of H and all compact of $\operatorname{Re} s > 1$.

From this ultimate inequality we conclude that θ^s is square integrable in the funnels as the Eisenstein series \mathcal{E}_∞ . To conclude, we have the following theorem:

Theorem 4.1. *For $\operatorname{Re} s > 1$, the Eisenstein series associated to the geodesic $\eta = (p, q)$ converge uniformly on all compact sets. It represents a \mathcal{C}^∞ closed form which is dual to η . For $\operatorname{Re} s > 1$ it satisfies the differential functional equation:*

$$\Delta \hat{\eta}^s = s(1-s)[\hat{\eta}^s - \hat{\eta}^{s+2}].$$

Now we want to prove the analytic continuation of $\hat{\eta}^s$ at $s = 1$. For this, first of all, we are going to show that $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ is square integrable. What we have to do is to investigate the Fourier expansion of θ^s , at each inequivalent cusp : 0 and ∞ , and to show that $y|\theta^s(z)|$ is bounded. We have the following proposition whose proof is identical in the finite volume (see [5]):

Proposition 4.1. *At ∞*

$$\theta^s(z) = \left(\frac{1}{i} + O(1/y)\right) dz ,$$

and at 0

$$\theta^s(z) = \left(-\frac{1}{ic_0^2 z^2} + O(1/y)\right) dz .$$

By proposition 4.1 and lemma 4.1, we conclude:

Proposition 4.2. *The 1-forms $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ and $\hat{\eta}^s(z) + \operatorname{Re}(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ are square integrable.*

Finally as in [5]:

Theorem 4.2. *The 1-form $\hat{\eta}^s$ has a meromorphic continuation to $\operatorname{Re} s > 1/2$, with $s = 1$ a regular point and $\hat{\eta}$ is the harmonic dual form to η .*

5. A CASE OF DEGENERATION.

A family of degenerating hyperbolic surfaces consists of a manifold M and a family $(g_l)_{l>0}$ of Riemannian metrics on M that meet the following assumptions: M is an oriented surface of negative Euler characteristic, and the metrics g_l are hyperbolic, chosen in such a way that there are finitely many closed curves c_i , geodesic with respect to all metrics, with the length l_i of each curve converging to 0 as l decreases. On the complement of the distinguished curves, the sequence of metrics is required to converge to a hyperbolic metric. More precisely there are finitely many disjoint open subsets $C_i \subset M$ that are diffeomorphic to cylinders $F_i \times J_i$

where $J_i \subset \mathbb{R}$ is a neighborhood of 0. The complement of $\bigcup_i C_i$ is relatively compact. The restriction of each metric g_l to $C_i = F_i \times J_i$ is a product metric

$$(x, a) \mapsto (l_i^2 + a^2)dx^2 + (l_i^2 + a^2)^{-1}da^2$$

and $l_i \rightarrow 0$ as $l \rightarrow 0$ (the curves $F_i \times \{0\} \subset C_i$ are closed geodesics of length l_i with respect to g_l). Let M_l denote the surface M equipped with the metric g_l if $l > 0$, and let $M_0 = M \setminus \bigcup_i c_i$ carry the limit metric $\lim_{l \rightarrow 0} g_l$. Note that M_0 is a complete hyperbolic surface by definition, which contains a pair of cusps for each i . Here consider a family of surfaces $S_l = \Gamma_l \backslash H$ degenerating to the surface S with only one geodesic c_l being pinched, Γ_l containing the transformation $\sigma_l(z) = e^l z$ corresponding to c_l . Let K_l be S_l minus C_l the standard collar for c_l . There exist homeomorphisms f_l from $S_l \setminus c_l$ to S_0 , with f_l tending to isometries C^2 -uniformly on the compact core $K_l \subset S_l$, define $\pi_l = f_l^{-1}$. Suppose that p is one of the two cusps of S arising from pinching c_l . Let $S_0 = \Gamma \backslash H$ be the component of S containing p and conjugate Γ to represent the cusp by the translation $w \mapsto w + 1$, in the following $p = \infty$.

Let for $\operatorname{Re} s > 1$ $\alpha_l(s, z) = \sum_{\gamma \in \langle \sigma_l \rangle \backslash \Gamma_l} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1} dz) \right]$ such that the hyperbolic Eisenstein series $\Omega_{c_l} = \Omega_l$ is related by $\Omega_l(s, z) = \frac{1}{k(s)} \alpha_l(s+1, z)$. Without loss of generality we suppose S_l having only one funnel F_1 . With the notations of the beginning, $S_l = K \cup (C_1 \cup \dots \cup C_{n_c}) \cup F_1$ and c_l is the one geodesic of the boundary of the compact core K , we consider the specific case of p the limit of the right side of the c_l -collar contained in $S_l \setminus F_1$.

Theorem 5.1. *Let $\operatorname{Re} s > 1$, the family of 1-forms $\frac{1}{l^s} \alpha_l(s, \pi_l(\cdot))$ converge uniformly on compact subsets of S_0 to $\operatorname{Im} \mathcal{E}_\infty(s, \cdot)$.*

It is a particular case of the theorem 5.2 bellow.

The sketch of the proof of this theorem is the same as in the finite volume case ([4], [22], see also [8]), but for the convenience of the reader we recall some material and results.

The following lemma can be found for example in [1]. The neighborhood of points within distance a of a geodesic γ ,

$$G_a = \{z \in K, d(z, \gamma) \leq a\},$$

is isometric for small a to a half-collar $[0, a] \times S^1$, $ds^2 = dr^2 + l^2 \cosh^2 r d\theta^2$.

Lemma 5.1. *Suppose that γ is a simple closed geodesic of length $l(\gamma)$ on a geometrically finite hyperbolic surface M . Then γ has a collar neighborhood of half-width d , such that*

$$\sinh(d) = \frac{1}{\sinh(l(\gamma)/2)}.$$

As a consequence, if η is any other closed geodesic intersecting γ transversally (still assuming γ is simple), then the lengths of the two geodesics satisfy the inequality

$$\sinh(l(\eta)/2) \geq \frac{1}{\sinh(l(\gamma)/2)}.$$

Lemma 5.2. *Let γ be a simple closed geodesic of length l on a complete hyperbolic surface M . If α is a simple closed geodesic that does not intersect γ , then*

$$\cosh d(\gamma, \alpha) \geq \coth(l/2).$$

A standard collar for a length l geodesic is a cylinder isometric to $\mathcal{C} \setminus \langle z \mapsto e^l z \rangle$ with $\mathcal{C} = \{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi - l\} \subset H$ with the restriction of the hyperbolic metric, and $\langle z \mapsto e^l z \rangle$ the cyclic group generated by the transformation $z \mapsto e^l z$. There is a constant c_0 (the short geodesic constant) such that each closed geodesic on S of length at most c_0 has a neighborhood isometric to the standard collar and each cusp for S has a neighborhood isometric to the standard cusp; furthermore, the collars for short geodesics and the cusp regions are all mutually disjoint.

Now to study the right side of the c_l -collar let $w = \frac{1}{l} \log z$, $z \in H$, and conjugate Γ_l by the map w to obtain $\tilde{\Gamma}_l$ acting on $\mathcal{S}_l = \{w, 0 < \operatorname{Im} w < \pi/l\}$. The hyperbolic metric on \mathcal{S}_l is $ds_l^2 = \left(\frac{l|dw|}{\sin(l \operatorname{Im} w)} \right)^2$, which tends uniformly on compact subsets to $\left(\frac{|dw|}{\operatorname{Im} w} \right)^2$. $\tilde{\Gamma}_l$ is a (non Möbius) group of deck transformations acting on \mathcal{S}_l ; the quotient $\tilde{\Gamma}_l \backslash \mathcal{S}_l$ is \mathcal{S}_l . Let \hat{f}_l be the restriction of f_l to the component $\mathcal{S}_l^{(r)}$ of $\mathcal{S}_l \setminus c_l$ containing the right half-collar for c_l . Let F_l be a lift of \hat{f}_l to the universal covers \mathcal{A}_l and H , where \mathcal{A}_l is the simply connected component of $H \setminus \pi^{-1}(c_l)$ which contained the standard right collar $\{z = re^{i\theta}, 1 \leq r \leq e^l, l < \theta < \pi/2\}$. More precisely [4](p.350), [22]:

Lemma 5.3. *The simply connected component \mathcal{A}_l contains $\{z = re^{i\theta}, 1 \leq r \leq e^l, lc(l) < \theta < \pi/2\}$ where $c(l) \rightarrow 0$, $l \rightarrow 0$.*

Start with the standard Γ fundamental domain $\mathcal{F} = \{w, 0 \leq \operatorname{Re} w < 1, \operatorname{Im} w \geq \operatorname{Im} A(w), \forall A \in \Gamma\}$. Set $D_l = F_l^{-1}(\mathcal{F})$, then D_l is a fundamental domain of \mathcal{S}_l . Divide the cosets of $\langle \sigma_l \rangle \backslash (\Gamma_l - \langle \sigma_l \rangle)$ into two classes $D = \{[A], A \in \Gamma_l, \inf \operatorname{Re} A(D_l) > 0\}$ and $G = \{[A], A \in \Gamma_l, \sup \operatorname{Re} A(D_l) < 0\}$.

Then \hat{f}_l has a lift \tilde{f}_l , a homeomorphism from a sub domain of \mathcal{S}_l to H : $\tilde{f}_l = F_l \circ w^{-1} : w(\mathcal{A}_l) \rightarrow H$. \tilde{f}_l induces a group homomorphism $\rho_l : \Gamma \rightarrow \tilde{\Gamma}_l$ by the rule $A \mapsto \tilde{f}_l^{-1} A \tilde{f}_l$, $A \in \Gamma$. We call $\rho_l(A) \in \tilde{\Gamma}_l$ the element corresponding to $A \in \Gamma$. Now by our normalizations for $\tilde{\Gamma}_l$ and Γ the translation $w \mapsto w + 1$ corresponds to itself. If we specify the further normalization $\tilde{f}_l(i) = i$ then the lifts \tilde{f}_l are uniquely determined and then we have [22]

Lemma 5.4. *The \tilde{f}_l tend uniformly on compact subsets to the identity, and thus for $A \in \Gamma$, the corresponding elements $\rho_l(A)$ tend uniformly on compact subsets to A .*

Divide the cosets $\langle z + 1 \rangle \backslash (\tilde{\Gamma}_l - \langle z + 1 \rangle)$ into two classes, the left and the right: for $\mathcal{F}_l = \tilde{f}_l^{-1}(\mathcal{F})$, $L = wGw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \inf \operatorname{Im} A(\mathcal{F}_l) > \pi/2l\}$ and $R = wDw^{-1} = \{[A], A \in \tilde{\Gamma}_l, \sup \operatorname{Im} A(\mathcal{F}_l) < \pi/2l\}$ (the line $\{\operatorname{Im} w = \pi/2l\}$ is a lift of c_l , and we write $[A]$ for the $\langle z + 1 \rangle$ coset of A). In particular the cosets $\langle z + 1 \rangle \backslash (\Gamma - \langle z + 1 \rangle)$ correspond to the right cosets of $\tilde{\Gamma}_l$: $\{[\rho_l(A)], A \in \Gamma, \langle w \mapsto w + 1 \rangle\} \subset R$. Then we can write, where χ_+ is the characteristic function of $\{\operatorname{Re} z > 0\}$ and χ_- the one of $\{\operatorname{Re} z \leq 0\}$,

$$\begin{aligned} A_{l,q}(s, z) &= \sum_{\langle \sigma_l \rangle \backslash \Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)} \right)^q \sin^{s-q} \theta(\gamma z) dz^q \\ &= y(z)^{s-q} (\chi_+ + \sum_D \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}) dz^q + y(z)^{s-q} (\chi_- + \sum_G \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}) dz^q \end{aligned}$$

and the q -form on $\mathcal{S}_l^{(r)}$: $a_{l,q}^R(s, z) = y(z)^{s-q} (\chi_+ + \sum_D \frac{\gamma'(z)^q}{\gamma(z)^q} \frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}}) dz^q = l^q (\sin l \operatorname{Im} w)^{s-q} (\chi_+ + \sum_R (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q}) dw^q$ where χ is the characteristic function of $w(\{\operatorname{Re} z > 0\})$.

Theorem 5.2. *Let $S_l = \Gamma_l \backslash H$ a family of geometrically finite hyperbolic surfaces degenerating to the surface S with only one geodesic c_l being pinched and S_l having only one funnel F_1 ; Γ_l contains the transformation $\sigma_l(z) = e^l z$ corresponding to c_l and the right half-collar for σ_l is in $S_l \setminus F_1$. Let S_0 be as above. For $q \in \mathbb{N}$, we associate to the pinching geodesic c_l , the q -form defined for $\operatorname{Re} s > 1$ by:*

$$A_{l,q}(s, z) = \sum_{\langle \sigma_l \rangle \backslash \Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)} \right)^q \sin^{s-q} \theta(\gamma z) dz^q.$$

Let $w = \frac{1}{l} \log z$ and write again with a little misuse of notation $A_{l,q}(s, w) = a_{l,q}(s, w) dw^q$ and the corresponding q -automorphic form (for $\tilde{\Gamma}_l$) $\tilde{a}_{l,q}(s, w) = \operatorname{Im}(w)^q a_{l,q}(s, w)$. Note $\tilde{a}_{l,q}^R$ the right half of $\tilde{a}_{l,q}$. Then we have $\frac{1}{l^s} \tilde{a}_{l,q}^R(s, \pi_l(\cdot))$ converge uniformly on compact subsets of S_0 and on compact subsets of $\operatorname{Re} s > 1$ to the Eisenstein series for weight q :

$$E_{\infty,q}(s, z) = \sum_{\Gamma_{\infty} \backslash \Gamma} (\operatorname{Im} \gamma z)^s \left(\frac{c\bar{z} + d}{cz + d} \right)^q.$$

Before proving this theorem we give some complements and its corollaries. Let for $\operatorname{Re} s > 1$, $b_q(s) = e^{i\pi q/2} \int_0^\pi \sin^{s-2} u e^{-iqu} du$. Note that $b_1(s) = k(s-1)$. The function b_q has the following properties (see e.g. [16]): $b_q(s+2) = \frac{s(s-1)}{s^2 - q^2} b_q(s)$, b_q admits a meromorphic continuation to all $s \in \mathbb{C}$, more precisely

$$b_q(s) = \pi 2^{-s+2} \frac{\Gamma(s+1)}{\Gamma(\frac{s+q}{2} + 1) \Gamma(\frac{s-q}{2} + 1)}.$$

In order to be consistent with the definition of Kudla and Millson's hyperbolic Eisenstein series, we may use the normalized q -automorphic forms

$$(7) \quad \Xi_{l,q}(s, z) = \frac{1}{b_q(s)} A_{l,q}(s, z).$$

We recall that the series (7) converges absolutely and locally uniformly for any $z \in H$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, and that it is invariant with respect to Γ . A straightforward computation shows that the series $A_{l,q}(s, z)$ satisfies the differential functional equation:

$$\Delta_q^\pm A_{l,q}(s, z) - s(1-s) A_{l,q}(s, z) = (s+q)(s-q) A_{l,q}(s+2, z),$$

and the series (7)

$$\Delta_q^\pm \Xi_{l,q}(s, z) - s(1-s) \Xi_{l,q}(s, z) = s(s-1) \Xi_{l,q}(s+2, z).$$

Proposition 5.1. *The series $A_{l,q}(s, z)$ (resp. $\Xi_{l,q}(s, z)$) admits a meromorphic continuation to all of \mathbb{C} .*

There are different ways to prove this; one is to use the differential functional equation (7) and to apply the method developped in [17] (see also [16]). More precisely let $\tilde{A}_{l,q}(s, z)$ (resp. $\tilde{\Xi}_{l,q}(s, z)$) the q -automorphic form associated to $A_{l,q}(s, z)$ (resp. $\Xi_{l,q}(s, z)$). We have

$$\Delta_{2q} \tilde{A}_{l,q}(s, z) + s(1-s) \tilde{A}_{l,q}(s, z) = (s+q)(q-s) \tilde{A}_{l,q}(s+2, z),$$

and

$$(8) \quad \Delta_{2q} \Xi_{l,q}(s, z) + s(1-s) \tilde{\Xi}_{l,q}(s, z) = s(1-s) \tilde{\Xi}_{l,q}(s+2, z);$$

in other words

$$\tilde{\Xi}_{l,q} = s(1-s) \int_{\mathcal{D}} G_s(z, z', q) \tilde{\Xi}_{l,q}(s+2, z') d\mu(z').$$

We precise another calculation we will develop further. We can rewrite $\tilde{A}_{l,q}(s, z)$ as

$$\tilde{A}_{l,q}(s, z) = \sum_{\Gamma_l \backslash \Gamma} \left(\frac{c\bar{z} + d}{cz + d} \right)^q \left(\frac{\overline{\gamma z}}{\gamma z} \right)^{q/2} \left(\frac{\operatorname{Im} \gamma z}{|\gamma z|} \right)^s$$

and using the Fourier development of $G_s(z, z', q)$ and the expansion of $\int_1^{e^l} G_s(z, iy', q) d \ln y'$ we obtain the result (see [7] corollary 4.2 p.188).

Theorem 5.3. *Let λ_k , $1 \leq k \leq n$ the eigenvalues of the Laplace-Beltrami operator on S_0 such that $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < 1/4 \leq \lambda_{n+1}$ and the corresponding $s_k = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_k}$ with $\operatorname{Re} s \geq 1/2$. The family $(\frac{1}{l^s} \tilde{a}_{l,q}^R(s, \pi_l(\cdot)))_l$ converge uniformly on compact subsets of S_0 and on compact subsets of $\Omega = \{\operatorname{Re} s > 1/2\} \setminus \{s_1, \dots, s_n\}$ to $E_{\infty,q}(\cdot, s)$.*

Corollary 5.1. *In the case the geodesic c_l is not separating $(\frac{1}{l^s} \tilde{a}_{l,q}(s, \pi_l(\cdot)))_l$ converge uniformly on compact subsets of S_0 and Ω to $E_{\infty,q} - E_{0,q}$.*

Corollary 5.2. *In the case c_l is the geodesic boundary of a funnel $(\frac{1}{l^s} \tilde{a}_{l,q}(s, \pi_l(\cdot)))_l$ converge uniformly on compact subsets of S_0 and Ω to $E_{\infty,q}$.*

Remark 5.1. *We refer also to the result of J.Fay [6] final remark p.201-202.*

Proof. (theorem 5.2)

We have $\frac{1}{l^s} \tilde{a}_{l,q}^R(s, w) = \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} (\chi + \sum_R (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q}) = \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} (\chi + \sum_{G_l} (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q}) + \frac{1}{l^{s-q}} (\operatorname{Im} w)^q (\sin l \operatorname{Im} w)^{s-q} (\sum_{R-G_l} (\tilde{\gamma}' w)^q |\tilde{\gamma}' w|^{s-q})$.

Because of lemma 5.4, the principal problem lies in estimating the second sum: it remains to show that

$$\lim_{l \rightarrow 0} \sum_{R-G_l} |\tilde{\gamma}' w|^\sigma = 0,$$

where $\sigma = \operatorname{Re} s$. It is enough to demonstrate the convergence for β a relatively compact set in the fundamental domain \mathcal{F} . Given $\epsilon > 0$ denote G the set of cosets and representatives for $\langle z+1 \rangle \backslash \Gamma$ such that $\sup \operatorname{Im} A(\mathcal{F}) < \epsilon$ for $[A] \notin G$ and let G_l be the corresponding cosets of $\langle z+1 \rangle \backslash \Gamma_l$ with corresponding representatives. The set G is finite.

The cosets G of Γ satisfy (modulo the $\langle z+1 \rangle$ action) $\{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w > \epsilon\} \subset \cup_{A \in G} A(\mathcal{F})$; thus for l sufficiently small the cosets G_l satisfy (modulo the $\langle z+1 \rangle$ action) $\{0 \leq \operatorname{Re} w < 1, 2\epsilon < \operatorname{Im} w < \pi/2l\} \subset \cup_{A \in G_l} A(\mathcal{F}_l)$ (a consequence of the convergence on compact subsets of the \tilde{f}_l and that \mathcal{F}_l contains the right-half-collar for c_l , $\{0 \leq \operatorname{Re} w < 1, c(l) \leq \operatorname{Im} w < \pi/2l\}$). Now for a right coset $[A] \in R - G_l$ then $A(\mathcal{F}_l)$ lies below the c_l geodesic $\{\operatorname{Im} w = \pi/2l\}$ and is disjoint modulo the $\langle z+1 \rangle$ action from $\{0 \leq \operatorname{Re} w < 1, 2\epsilon < \operatorname{Im} w < \pi/2l\}$, since the latter is covered by the G_l cosets. Thus for $[A] \in R - G_l$, modulo the $\langle z+1 \rangle$ action, then $A(\mathcal{F}_l) \subset \{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\}$. For $w \in \beta$ we write

$$\sum_{R-G_l} |\tilde{\gamma}' w|^\sigma = \sum_{w^{-1}(R-G_l)w} |z_l \frac{\gamma' z_l}{\gamma z_l}|^\sigma \leq |z_l|^\sigma \sum_{w^{-1}(R-G_l)w} |\gamma' z_l|^\sigma,$$

where $w = \frac{1}{l} \log z_l$ and $\gamma \mathcal{F}_l \subset w^{-1}(\{0 \leq \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\})$. We deduce

$$\sum_{R-G_l} |\gamma' w|^\sigma \leq \frac{1}{\sin^\sigma(l \operatorname{Im} w)} \sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l).$$

Let $\epsilon_0 \in]0, \sinh^{-1} 1[$ such that for $l \leq l_0(\epsilon)$ and $w \in \beta$, $B(z_l, \epsilon_0) \subset \mathcal{F}_l$. Now y^s is an eigenfunction of all the invariant integral operators on H . Let $k(z, z')$ be the point-pair invariant defined by $k(z, z') = 1$ or 0 according as the distance between z and z' is smaller than ϵ_0 . Then there exists Λ_{ϵ_0} independent of z_0 so that

$$\int_{B(z_0, \epsilon)} y^\sigma \frac{dx dy}{y^2} = \int_H k(z_0, z) y^\sigma \frac{dx dy}{y^2}$$

and

$$\int_{B(z_0, \epsilon)} y^\sigma \frac{dx dy}{y^2} = \Lambda_{\epsilon_0} y(z_0)^\sigma.$$

So

$$\sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma = \frac{1}{\Lambda_{\epsilon_0}} \sum_{w^{-1}(R-G_l)w} \int_{B(\gamma z_l, \epsilon_0)} y^\sigma \frac{dx dy}{y^2}.$$

Now [22]:

Lemma 5.5. *The multiplicity of the projection map $H \rightarrow H \setminus \Gamma$ restricted to $B(z_0, \eta)$ with $2\eta < c_0$ is at most $M\rho^{-2}(z_0)$ where M a constant and $\rho(z_0)$ the injectivity radius at z_0 .*

Proof. If $B(z_0, \eta) \cap B(\gamma z_0, \eta) \neq \emptyset$, $\gamma \in \Gamma$, then $d(z_0, \gamma z_0) < 2\eta < c_0$ and z_0 is in a cusp region or the collar for a short geodesic. Let $c = c_0/2$, then we have $\rho(z_0) < c$. Set $m(\eta)$ the multiplicity of the projection restricted to $B(z_0, \eta)$. As $2\eta + \rho(z_0) < 3c$ and the $B(\gamma z_0, \rho(z_0))$ are disjoint we have

$$m(\eta) \mu(B(z_0, \rho(z_0))) \leq \mu(B(z_0, 3c)).$$

$$\text{So } m(\eta) \leq \frac{\cosh 3c - 1}{\cosh \rho(z_0) - 1} \leq 2(\cosh 3c - 1) \rho(z_0)^{-2}. \quad \square$$

Then we have

$$\sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma \leq \frac{A(c_0)}{\Lambda_{\epsilon_0}} \rho^{-2}(z_l) \frac{e^{l\sigma} - 1}{\sigma} \frac{(2\epsilon l)^{\sigma-1}}{\sigma - 1}.$$

Moreover as $B(z_l, \epsilon_0) \subset \mathcal{F}_l$, $\rho(z_l) \geq \epsilon_0$ and the conclusion. \square

Proof. (corollary 5.2) We remark that σ_l being the geodesic in the funnel, then $D = \langle \sigma_l \rangle \setminus (\Gamma_l - \langle \sigma_l \rangle)$ and $G = \emptyset$. \square

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