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# New Examples of Marginally Trapped Surfaces and Tubes in Warped Spacetimes

J.L. Flores<sup>\*</sup>, S. Haesen<sup>†</sup>, M. Ortega<sup>‡</sup>

<sup>\*</sup>*Departamento de Álgebra, Geometría y Topología,  
Facultad de Ciencias, Universidad de Málaga,  
Campus Teatinos, 29071 Málaga, Spain.*

<sup>†</sup>*Simon Stevin Institute for Geometry,  
Mina Krusemanweg 1, 5032 ME Tilburg, The Netherlands.*

<sup>‡</sup>*Departamento de Geometría y Topología,  
Facultad de Ciencias, Universidad de Granada,  
Avenida Fuentenueva s/n, 18071 Granada, Spain.*

## Abstract

In the present paper we provide new examples of marginally trapped surfaces and tubes in FLRW spacetimes by using a basic relation between these objects and CMC surfaces in 3-manifolds. We also provide a new method to construct marginally trapped surfaces in closed FLRW spacetimes, which is based on the classical Hopf map. The utility of this method is illustrated by providing marginally trapped surfaces crossing expanding and collapsing regions of a closed FLRW spacetime. The approach introduced in this paper is also extended to twisted spaces.

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*Key words.* Warped spacetimes, trapped surface, marginally trapped surface/tube, CMC surface, minimal surface, Hopf fibration, Clifford tori, twisted space.

## 1 Introduction

The notion of *trapped surface* was firstly introduced by Penrose [25] in order to study singularities in General Relativity. These surfaces, and their various relatives, have been extensively studied in recent years (just to mention a few, see e.g. [3, 5, 12, 19]), since they are central not only for singularity theorems, but also to understand the evolution of black holes, the cosmic censorship hypothesis, the Penrose inequality...

Trapped surfaces have the physical property that the two null congruences normal to the surface are both converging. From the mathematical point of view, the null converging condition means that the mean curvature vector, which measures the tension of the surface coming from the surrounding space, is a timelike vector everywhere on the surface. If, in addition, the mean curvature vector is future- or past-pointing all over the surface, the trapped surface is accordingly called *future-* or *past-trapped*.

In this paper, a *closed surface* is a compact surface without boundary, embedded in some other semi-Riemannian manifold. The existence of closed trapped surfaces has been investigated in several types of spacetimes. For example, the formation of these surfaces in several cosmological spacetimes have been studied in [13, 20].

A spacelike surface in a 4-dimensional Lorentzian manifold is called *marginally trapped*<sup>1</sup> if its mean curvature vector is null. When its mean curvature vector is zero all over the surface it is called *extremal*.

In order to gain some idea of the properties of marginally trapped surfaces in particular spacetimes, classification results were obtained for the case of having positive relative nullity in Lorentzian space forms [10] and in Robertson-Walker spaces [11]. In [14, 15, 16] marginally trapped surfaces invariant under symmetries of 4-dimensional Minkowski space were studied. A complete classification of spacelike surfaces in a 4-dimensional Lorentzian spacetime, containing the above cases, was recently given in [31].

Some results concerning the non-existence of closed marginally trapped surfaces can be also found in the literature. Among the classical ones, a result due to R. Penrose [25] implies the non-existence of closed marginally trapped surfaces in the Minkowski spacetime when it bounds a compact domain. In [21] the non-existence of closed marginally trapped surfaces is shown for strictly stationary spacetimes. Finally, in [8] the authors have shown the non-existence of marginally trapped surfaces bounding a domain and entering a region of a static spacetime where the Killing vector field is timelike, and with the additional assumptions of dominant energy condition and an outer untrapped barrier.

The main aim of this paper is to provide new examples of marginally trapped surfaces and tubes in warped spacetimes.

1. In Section 2, we establish some existence/non-existence results on trapped and marginally trapped surfaces in FLRW spacetimes (Corollaries 2.2, 2.4) by using a simple, but fundamental, relation between these surfaces and constant mean curvature surfaces in 3-manifolds (Theorem 2.1). In particular, we show the existence of closed marginally trapped surfaces with any genus in closed FLRW spacetimes.
2. In Section 3, we develop a method to construct marginally trapped surfaces in closed FLRW spacetimes, which is based on an extension of the classical Hopf map to a submersion between closed FLRW spacetimes of dimension 4 and 3. We illustrate it with a simple example in Subsection 3.4. In order to show the utility of this constructive method, in Subsection 3.5 we apply it to provide marginally trapped surfaces crossing expanding and collapsing regions of a closed FLRW spacetime.
3. Section 4 is devoted to studying *marginally trapped tubes*. They are defined as smooth hypersurfaces foliated by marginally trapped surfaces. Then, we give some existence/non-existence results for these objects in closed FLRW spacetimes (Corollaries 4.2, 4.3), and provide examples of them with any type of causal behavior, Subsection 4.1. Finally, in Section 5 we extend the approach introduced in this paper to twisted spaces (Theorem 5.1).

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<sup>1</sup>This definition may appear slightly modified in the bibliography.

## 2 Marginally trapped surfaces in $t$ -slices of warped spacetimes

In general, given an immersion  $\mathbf{x} : \bar{\Sigma}^n \rightarrow \Sigma^m$  of a  $n$ -dimensional semi-Riemannian manifold into another  $m$ -dimensional semi-Riemannian manifold, the second fundamental form and the mean curvature vector will be denoted by  $h_{\mathbf{x}}$  and  $\vec{H}_{\mathbf{x}} = \text{trace}(h_{\mathbf{x}})/n$ , respectively.

Let  $f : I \subset \mathbb{R} \rightarrow (0, \infty)$  be a smooth function ( $t \in I$ ),  $(M^3, g_3)$  a 3-dimensional Riemannian manifold and  $S$  a surface. Denote by  $\bar{M}_1^4 = I \times_f M^3$  the Lorentzian warped product manifold given by the product manifold  $I \times M^3$  endowed with metric  $\bar{g}_4 = -dt^2 + f^2 g_3$ . Let  $\varphi : S \rightarrow M^3$  be an immersion of  $S$  in  $M^3$ ,  $\psi : M^3 \rightarrow I \times_f M^3$  the embedding of  $M^3$  in  $I \times_f M^3$  and  $\phi : S \rightarrow I \times_f M^3$  the corresponding immersion of  $S$  in the warped product, both in the  $t$  slice ( $t$ -slice for short). According to a well-known result (see [9, p. 79]), the following relation holds between the corresponding second fundamental forms:

$$h_{\phi}(X, Y) = h_{\varphi}(X, Y) + h_{\psi}(X, Y), \quad \text{whereby } X, Y \in \mathfrak{X}(S). \quad (2.1)$$

The expression of  $h_{\psi}$  is also known (see e.g. [23, p. 344]):

$$h_{\psi}(X, Y) = -\bar{g}_4(X, Y) \frac{\text{grad}_{\bar{g}_4} f}{f} = \bar{g}_4(X, Y) \frac{f'}{f} \partial_t, \quad \text{where we have used } \text{grad}_{\bar{g}_4} f = -f' \partial_t.$$

Hence, we obtain  $h_{\phi}(X, Y) = h_{\varphi}(X, Y) + \bar{g}_4(X, Y) \frac{f'}{f} \partial_t$ . Taking one half of the trace of the above expression, using an orthonormal frame  $\{\partial_t, \{E_i\}_{i=1}^3\}$  w.r.t. the metric  $\bar{g}_4$ , i.e.  $E_i = \frac{e_i}{f}$  whereby  $\{e_i\}_{i=1}^3$  is the corresponding orthonormal frame w.r.t. the metric  $g_3$  on  $M^3$  (and  $S$ ), one obtains

$$\vec{H}_{\phi} = \frac{\vec{H}_{\varphi}}{f^2} + \frac{f'}{f} \partial_t, \quad (2.2)$$

where  $\vec{H}_{\phi}$  and  $\vec{H}_{\varphi}$  stand for the mean curvature vectors associated with  $h_{\phi}$  and  $h_{\varphi}$ , respectively.

Recall that a surface  $S$  is called of *constant mean curvature*, *CMC* for short, if the length of its mean curvature vector is a constant function.

**Theorem 2.1** *A surface  $S$  contained in a  $t_0$ -slice of  $\bar{M}_1^4 = I \times_f M^3$  is trapped (respectively, marginally trapped) iff it is a CMC surface in  $M^3$  with*

$$\|\vec{H}_{\varphi}\| < |f'(t_0)| \quad (\text{respectively, } \|\vec{H}_{\varphi}\| = |f'(t_0)|).$$

*Proof.* We compute the length of the mean curvature  $\vec{H}_{\phi}$ , by making use of (2.2):

$$f(t_0)^2 \bar{g}(\vec{H}_{\phi}, \vec{H}_{\phi}) = f(t_0)^2 \left\| \frac{\vec{H}_{\varphi}}{f(t_0)^2} + \frac{f'(t_0)}{f(t_0)} \partial_t \right\|^2 = \|\vec{H}_{\varphi}\|^2 - f'(t_0)^2.$$

This readily gives the results. ■

From this result one can deduce some simple consequences for *FLRW spacetimes*, i.e. warped spacetimes with fiber  $M^3 = \mathbb{R}^3, \mathbb{S}^3$  or  $\mathbb{H}^3$ . When the fiber is  $M^3 = \mathbb{S}^3$ , we will say that our FLRW is *closed*.

First, recall that the so-called Clifford tori  $C_u$  in  $\mathbb{S}^3$  are given by

$$C_u := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \cos(u), |z_2| = \sin(u)\}, \quad u \in (0, \pi/2).$$

These are closed surfaces in  $\mathbb{S}^3$  with constant mean curvature satisfying

$$\|\vec{H}_u\| = |2 \cot(2u)|. \quad (2.3)$$

Of course, other CMC tori can be obtained by applying to them isometries of  $\mathbb{S}^3$ . In addition, by making surgery on a finite number of Clifford tori, Butscher-Packard [6] obtained closed surfaces in  $\mathbb{S}^3$  that are also CMC with arbitrary genus. Moreover, as far as we know, these are the only known surfaces in  $\mathbb{S}^3$  which are closed, CMC, non-minimal and with arbitrary genus.

**Corollary 2.2** (*Existence result*). *There exist closed trapped and closed marginally trapped surfaces with arbitrary genus in closed FLRW spacetimes.*

*Proof.* From Theorem 2.1, any surface in  $M^3$  with constant mean curvature  $\|\vec{H}_\varphi\| = |f'(t_0)|$  can be seen as a marginally trapped surface in the  $t_0$ -slice of  $I \times_f M^3$ . Notice also that there exist closed CMC surfaces in  $M^3 = \mathbb{S}^3$  with arbitrary genus (standard spheres; Clifford tori; Butscher-Packard surfaces [6]). Therefore, there exist closed marginally trapped surfaces in  $I \times_f \mathbb{S}^3$  with arbitrary genus. From Theorem 2.1, any minimal surface  $S$  in  $M^3$  can be seen as a trapped surface in any  $t_0$ -slice of  $I \times_f M^3$  with  $f'(t_0) \neq 0$ . Notice also that there exist closed minimal surfaces in  $M^3 = \mathbb{S}^3$  with arbitrary genus [18]. Therefore, there exist closed trapped surfaces in  $I \times_f \mathbb{S}^3$  with arbitrary genus whenever  $f \neq cte$ . ■

**Remark 2.3** Standard spheres and embedded CMC tori can be chosen with any constant value of its mean curvature function, and so, there is no restriction for the warping function  $f$  in Corollary 2.2. However, the mean curvature function of a Butscher-Packard's surface has to be sufficiently small, due to the gluing process. Thus, in this case Corollary 2.2 only applies to warping functions of sufficiently small derivative. This observation must be also taken into account in Corollary 4.2.

**Corollary 2.4** (*Non-existence result*). *Let  $\overline{M}_1^4 = I \times_f M^3$  be a FLRW spacetime with fiber  $M^3 = \mathbb{H}^3$ . There are no closed marginally trapped surfaces contained in any  $t_0$ -slice such that  $|f'(t_0)| \leq 1$ .*

*Proof.* According to a result by do Carmo and Lawson [7], if  $S$  is a closed CMC surface in  $\mathbb{H}^3$ , it must be a geodesic sphere with mean curvature satisfying  $\|\vec{H}_\varphi\| > 1$ . Therefore, the proof directly follows from Theorem 2.1. ■

**Remark 2.5** Formula (2.2) implies that  $\vec{H}_\phi$  cannot be future-directed at  $t = t_0$  if  $f'(t_0) \leq 0$ , and so, the following result [29] is reobtained: *there are no future trapped (resp. marginally trapped) surfaces in any slice of collapsing (i.e.  $f'(t) \leq 0$  for all  $t$ ) warped spacetimes.* Analogously,  $\vec{H}_\phi$  cannot be past-directed at  $t = t_0$  if  $f'(t_0) \geq 0$ , hence: *there are no past trapped (resp. marginally trapped) surfaces in any slice of expanding (i.e.  $f'(t) \geq 0$  for all  $t$ ) warped spacetimes.*

**Remark 2.6** The stability result in [8] (which can be applied to more general surfaces than the ones contained in a  $t$ -slice, assumed some additional conditions) suggests that the marginally trapped surfaces found in this section should be unstable.

### 3 Marginally Trapped surfaces in closed FLRW spacetimes: A constructive method

In the present section we are going to construct marginally trapped surfaces, non-necessarily contained in a  $t$ -slice of warped spacetimes, by using the classical Hopf map. The price to pay is that we will need to restrict our ambient space to closed FLRW spacetimes.

Very roughly, the idea is as follows. We can see closed FLRW spacetimes  $I \times_f \mathbb{S}^3$  as a semi-Riemannian submersion over  $I \times_f \mathbb{S}^2(1/2)$  such that the lift of any curve in  $I \times_f \mathbb{S}^2(1/2)$  gives rise to a surface in  $I \times_f \mathbb{S}^3$  whose geometric properties depend on the base curve. Thus, by choosing appropriate curves in the base, we can obtain embedded surfaces in  $I \times_f \mathbb{S}^3$  with mean curvature vector as desired, i.e. spacelike, timelike or lightlike.

To develop our approach, first we need to recall some notions about semi-Riemannian submersions and the Hopf map.

#### 3.1 Semi-Riemannian submersions

Let  $\pi : (\mathbf{M}, g_M) \rightarrow (\mathbf{B}, g_B)$  be a surjective map between semi-Riemannian manifolds. Assume that  $\pi$  has maximal rank. The *fibers* are  $\pi^{-1}(b)$ , with  $b \in \mathbf{B}$ . A tangent vector to  $\mathbf{M}$  is called *vertical* (resp. *horizontal*) if it is tangent (resp. orthogonal) to the fibers. The *vertical part* of  $\pi$  at a point  $m \in \mathbf{M}$  is  $\ker(d\pi)_m \subset T_m\mathbf{M}$ . If for each point  $m \in \mathbf{M}$ ,  $\pi_*$  satisfies

$$g_M(u, v) = g_B(\pi_*u, \pi_*v), \quad (3.1)$$

for any horizontal tangent vectors  $u, v$  at  $m \in \mathbf{M}$ , then  $\pi$  is called a *semi-Riemannian submersion*. This lemma summarizes the basic properties of semi-Riemannian submersions, [22].

**Lemma 3.1** *Let  $\pi : (\mathbf{M}, g_M) \rightarrow (\mathbf{B}, g_B)$  be a semi-Riemannian submersion.*

- Given  $X \in \mathfrak{X}(\mathbf{B})$ , there exists a horizontal lift  $\tilde{X} \in \mathfrak{X}(\mathbf{M})$  of  $X$  such that  $\tilde{X}$  is horizontal and  $\pi_*\tilde{X} = X$
- Given a curve  $\gamma : I \rightarrow \mathbf{B}$ ,  $t_0 \in I$  and a point  $m \in \pi^{-1}(\gamma(t_0))$ , there exists a unique horizontal lift  $\tilde{\gamma} : I \rightarrow \mathbf{M}$  of  $\gamma$ , i.e. it satisfies  $\tilde{\gamma}(t_0) = m$ ,  $\pi \circ \tilde{\gamma} = \gamma$  and  $\tilde{\gamma}'$  is horizontal. In particular,  $\gamma$  is unitary if, and only if, so is  $\tilde{\gamma}$ .
- If  $\nabla^M$  and  $\nabla^B$  are the Levi-Civita connections of  $\mathbf{M}$  and  $\mathbf{B}$ , resp., then for any  $X, Y, Z \in \mathfrak{X}(\mathbf{B})$ ,  $g_M(\nabla_{\tilde{X}}^M \tilde{Y}, \tilde{Z}) = g_B(\nabla_X^B Y, Z)$ .

#### 3.2 The Hopf map and closed FLRW spacetimes

Let  $\mathbb{C}$  be the field of complex numbers, with  $i = \sqrt{-1}$  the complex unit,  $|z|$  the modulus of  $z \in \mathbb{C}$  and  $\bar{z}$  its complex conjugate. Firstly, the round 3-sphere in  $\mathbb{C}^2$  can be seen as  $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ , with standard metric  $g_3$ . Also, we can see the round 2-sphere of radius 1/2 as  $\mathbb{S}^2(1/2) = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1/4\}$ , with standard metric  $g_2$ . We recall the classical *Hopf map*

$$\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2), \quad \pi(z, w) = \left( z\bar{w}, \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 \right),$$

where  $\bar{\omega}$  is the complex conjugate of  $\omega$ . It is well-known that  $\pi$  is a Riemannian submersion with totally geodesic fibers. In fact, this Riemannian submersion  $\pi$  is the quotient map of the following isometry group action:

$$\mathbb{S}^1 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3, \quad (e^{i\theta}, (z, w)) \mapsto (e^{i\theta}z, e^{i\theta}w). \quad (3.2)$$

The fibers of  $\pi$  are the orbits of the action, i.e. given a point  $p = (z, w) \in \mathbb{S}^3$ , the orbit is  $\{e^{i\theta} \cdot p = (e^{i\theta}z, e^{i\theta}w) : e^{i\theta} \in \mathbb{S}^1\}$ , which is a big circle (geodesic) of  $\mathbb{S}^3$ . We also remark that the vertical part of  $\pi$  at  $p = (z, w) \in \mathbb{S}^3$  is spanned by  $ip = (iz, iw)$ . In other words,  $\ker(d\pi)_p = \text{Span}\{ip\}$ .

Given  $f : I \subset \mathbb{R} \rightarrow (0, \infty)$  a smooth function ( $t \in I$ ), consider the closed FLRW spacetime  $\bar{M}_1^4 = I \times_f \mathbb{S}^3$ . We also consider the *toy model*  $\bar{M}_1^3 = I \times_f \mathbb{S}^2(1/2)$ , i.e. the 3-dimensional Lorentzian manifold formed by the product manifold  $I \times \mathbb{S}^2(1/2)$  endowed with metric  $\bar{g}_3 = -dt^2 + f^2 g_2$ . Let  $\bar{\nabla}$ ,  $D$ ,  $\nabla$  and  $\nabla^2$  be the Levi-Civita connections of  $\bar{M}_1^4$ ,  $\mathbb{S}^3$ ,  $\bar{M}_1^3$  and  $\mathbb{S}^2(1/2)$ , resp. Note that the natural projection map of  $\bar{M}_1^4$  onto  $I$  is a semi-Riemannian submersion, whose horizontal part is spanned by  $\partial_t$ . Then, a vertical vector is orthogonal to  $\partial_t$ . Given a vector field  $X \in \mathfrak{X}(\mathbb{S}^3)$ , there exists a *vertical lift*  $\tilde{X}$  tangent to  $\bar{M}_1^4$  such that  $\tilde{X} \perp \partial_t$ . Given  $Z$  a tangent vector to  $\bar{M}_1^4$ ,  $\text{nor}(Z)$  is the orthogonal projection onto the horizontal part, whereas  $\text{tan}(Z)$  is the orthogonal projection onto the vertical part. Formally, there is a similar situation for  $\bar{M}_1^3$ , so that we can use the same notation. Thus, we obtain [23]:

**Lemma 3.2** *Let  $X, Y$  be tangent vector fields to  $\mathbb{S}^3$  (resp.  $\mathbb{S}^2(1/2)$ ) and  $\tilde{X}, \tilde{Y}$  be vertical lifts to  $\bar{M}_1^4$  (resp.  $\bar{M}_1^3$ ):*

1.  $\text{nor}(\bar{\nabla}_{\tilde{X}} \tilde{Y}) = -\frac{\bar{g}_4(\tilde{X}, \tilde{Y})}{f} \text{grad}_{\bar{g}_4}(f)$  (resp.  $\text{nor}(\nabla_{\tilde{X}} \tilde{Y}) = -\frac{\bar{g}_3(\tilde{X}, \tilde{Y})}{f} \text{grad}_{\bar{g}_3}(f)$ )
2.  $\text{tan}(\bar{\nabla}_{\tilde{X}} \tilde{Y})$  is the vertical lift of  $D_X Y$  (resp.  $\text{tan}(\nabla_{\tilde{X}} \tilde{Y})$  is the vertical lift of  $\nabla_X^2 Y$ ).

### 3.3 Constructing the surface

From now on, we will make use of Lemma 3.1, sometimes without indicating it explicitly. We define the projection  $\bar{\pi} : \bar{M}_1^4 \rightarrow \bar{M}_1^3$  as  $\bar{\pi}(t, p) := (t, \pi(p))$ .

**Lemma 3.3** *The map  $\bar{\pi}$  is a semi-Riemannian submersion with vertical part at  $(t, p)$  spanned by  $(0, ip)$ .*

*Proof.* We denote by  $\partial_t$  both, the vector field tangent to  $\bar{M}_1^4$  and  $\bar{M}_1^3$ . Then,

$$\bar{\pi}_* \partial_t |_{(t,p)} = \frac{d}{ds} \Big|_{s=0} \bar{\pi}(t + s, p) = \frac{d}{ds} \Big|_{s=0} (t + s, \pi(p)) = \partial_t |_{(t, \pi(p))}.$$

Therefore,

$$\bar{g}_3(\bar{\pi}_* \partial_t, \bar{\pi}_* \partial_t) = -1 = \bar{g}_4(\partial_t, \partial_t).$$

For any  $(t, p) \in \bar{M}_1^4$ , consider the curve  $\alpha(s) = (t, \cos(s)p + \sin(s)ip)$ . Taking into account that  $\ker(d\pi)_p = \text{Span}(ip)$ , we deduce:

$$\bar{\pi}_*(0, ip) = \frac{d}{ds} \Big|_{s=0} \bar{\pi}(\alpha(s)) = (0, \pi_*(ip)) = (0, 0).$$

Finally, take  $(0, X) \in T_{(t,p)}\overline{M}_1^4$  which is orthogonal to  $ip$ . Then,  $\overline{\pi}_*(0, X) = (0, \pi_*X)$ . Taking into account that  $\pi$  is a semi-Riemannian submersion, we deduce

$$\begin{aligned} \overline{g}_3(\overline{\pi}_*(0, X), \overline{\pi}_*(0, X)) &= \overline{g}_3((0, \pi_*X), (0, \pi_*X)) = f^2 g_2(\pi_*X, \pi_*X) \\ &= f^2 g_3(X, X) = \overline{g}_4((0, X), (0, X)). \quad \blacksquare \end{aligned}$$

We recall that the Hopf map  $\pi$  is the quotient map of the isometry group action (3.2). We call  $\Gamma_\theta : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ ,  $\Gamma_\theta(z, w) = (e^{i\theta}z, e^{i\theta}w)$ , which is an isometry of  $\mathbb{S}^3$ . We extend it to  $\overline{M}_1^4$  as follows. For each  $e^{i\theta} \in \mathbb{S}^1$ , we define the map

$$\overline{\Gamma}_\theta : \overline{M}_1^4 \rightarrow \overline{M}_1^4, \quad \overline{\Gamma}_\theta(t, p) = (t, \Gamma_\theta(p)).$$

Given  $(t, p) \in \overline{M}_1^4$ , consider  $T_t I \equiv \mathbb{R}$  and  $T_p \mathbb{S}^3 \subset \mathbb{C}^2$ . Thus, it is possible to let  $\overline{\Gamma}_\theta$  act on tangent vectors under these natural identifications.

**Lemma 3.4** *For each  $e^{i\theta} \in \mathbb{S}^1$ , the map  $\overline{\Gamma}_\theta$  is an isometry of  $\overline{M}_1^4$  with  $(\overline{\Gamma}_\theta)_* = \overline{\Gamma}_\theta$  under previous identifications.*

*Proof.* Firstly, we are going to show that  $(\overline{\Gamma}_\theta)_* = \overline{\Gamma}_\theta$ . Observe that:

$$(\overline{\Gamma}_\theta)_*(\partial_t |_{(t,p)}) = \frac{d}{ds} \Big|_{s=0} \overline{\Gamma}_\theta(t+s, p) = \frac{d}{ds} \Big|_{s=0} (t+s, \Gamma_\theta(p)) = \partial_t |_{(t, e^{i\theta}p)} = \overline{\Gamma}_\theta(\partial_t |_{(t,p)}).$$

On the other hand, given a curve  $\gamma$  in  $\mathbb{S}^3$  such that  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = X$ , we have

$$(\overline{\Gamma}_\theta)_*(0, X) = \frac{d}{ds} \Big|_{s=0} \overline{\Gamma}_\theta(0, \gamma(s)) = \frac{d}{ds} \Big|_{s=0} (0, e^{i\theta}\gamma(s)) = (0, e^{i\theta}X) = \overline{\Gamma}_\theta(0, X).$$

Finally, in order to prove that  $\overline{\Gamma}_\theta$  is isometry, notice that

$$\begin{aligned} \overline{g}_4((\overline{\Gamma}_\theta)_*(0, X), (\overline{\Gamma}_\theta)_*(0, X)) &= \overline{g}_4((0, e^{i\theta}X), (0, e^{i\theta}X)) = f^2 g_3(e^{i\theta}X, e^{i\theta}X) \\ &= f^2 g_3(X, X) = \overline{g}_4((0, X), (0, X)). \quad \blacksquare \end{aligned}$$

Since  $\mathbb{S}^2(1/2)$  is an orientable manifold,  $\overline{M}_1^3$  is also orientable. We choose the orientation on  $\overline{M}_1^3$  in such a way that for any local positive tangent frame  $\{X, Y\}$  on  $\mathbb{S}^2(1/2)$ , the set  $\{\partial_t, X, Y\}$  is a local positive frame on  $\overline{M}_1^3$ .

Let  $\alpha : J \subset \mathbb{R} \rightarrow \overline{M}_1^3$  be a unit spacelike Frenet curve with frenet apparatus  $\{T = \dot{\alpha}, N, B\}$  and  $\kappa > 0$ ,  $\tau$ . This means that the Frenet equations are

$$\nabla_T T = \epsilon_2 \kappa N, \quad \nabla_T N = \kappa T + \epsilon_3 \tau B, \quad \nabla_T B = -\epsilon_2 \tau N, \quad (3.3)$$

where  $\epsilon_2 = \overline{g}_3(N, N)$ ,  $\epsilon_3 = \overline{g}_3(B, B)$ ,  $\epsilon_2 = -\epsilon_3 = \pm 1$ , and  $\{T, N, B\}$  is a positive basis along  $\alpha$ . Consider  $\alpha(s) = (t(s), \alpha_2(s))$ , where  $t : J \rightarrow I$ ,  $\alpha_2 : J \rightarrow \mathbb{S}^2(1/2)$ . By Lemma 3.1, let  $\beta : J \subset \mathbb{R} \rightarrow \overline{M}_1^4$  be a horizontal lift of  $\alpha$ . Since  $\dot{\beta}$  is orthogonal to the vertical part of  $\overline{\pi}$ , we have

$$\overline{\pi} \circ \dot{\beta} = \alpha, \quad \dot{\beta} = (t, \beta_2), \quad \pi \circ \beta_2 = \alpha_2, \quad \dot{\beta}_2 \perp i\beta_2.$$

Now, we are able to construct a spacelike surface in  $\overline{M}_1^4$  with the help of  $\overline{\Gamma}_\theta$  and  $\beta$ . Define:

$$\phi : S = J \times \mathbb{S}^1 \rightarrow \overline{M}_1^4, \quad \phi(s, \theta) = \overline{\Gamma}_\theta(\beta(s)) = (t(s), e^{i\theta} \beta_2(s)). \quad (3.4)$$



It is clear that the derivatives of  $\phi$  are

$$\phi_s = (\dot{t}, e^{i\theta} \dot{\beta}_2), \quad \phi_\theta = (0, ie^{i\theta} \beta_2).$$

By using Lemma 3.3, the coefficients of the first fundamental form of  $\phi^* \bar{g}_4$  are the following:

$$\begin{aligned} E &= \bar{g}_4(\phi_s, \phi_s) = \bar{g}_4((\dot{t}, e^{i\theta} \dot{\beta}_2), (\dot{t}, e^{i\theta} \dot{\beta}_2)) = -\dot{t}^2 + f^2 g_3(e^{i\theta} \dot{\beta}_2, e^{i\theta} \dot{\beta}_2) \\ &= -\dot{t}^2 + f^2 g_3(\dot{\beta}_2, \dot{\beta}_2) = \bar{g}_4(\dot{\beta}, \dot{\beta}) = 1, \\ F &= \bar{g}_4(\phi_s, \phi_\theta) = \bar{g}_4((\dot{t}, e^{i\theta} \dot{\beta}_2), (0, ie^{i\theta} \beta_2)) = f^2 g_3(e^{i\theta} \dot{\beta}_2, ie^{i\theta} \beta_2) \\ &= f^2 g_3(\dot{\beta}_2, i\beta_2) = 0, \\ G &= \bar{g}_4(\phi_\theta, \phi_\theta) = \bar{g}_4((0, ie^{i\theta} \beta_2), (0, ie^{i\theta} \beta_2)) = f^2 g_3(ie^{i\theta} \beta_2, ie^{i\theta} \beta_2) \\ &= f^2 g_3(\beta_2, \beta_2) = f^2. \end{aligned}$$

Therefore,  $\{U_1 = \phi_s, U_2 = (1/f)\phi_\theta\}$  is a globally defined orthonormal tangent frame to  $S$  in  $\bar{M}_1^4$ . We also need to construct an orthonormal normal frame. To do so, we use the isometries  $\bar{\Gamma}_\theta$  and the vectors  $N, B$  along  $\alpha$ . By Lemma 3.1, let  $\tilde{N}$  and  $\tilde{B}$  be horizontal lifts of  $N$  and  $B$ , resp., along  $\beta$ . Define

$$\eta_N, \eta_B : S \rightarrow T\bar{M}_1^4, \quad \eta_N = (\bar{\Gamma}_\theta)_* \tilde{N}, \quad \eta_B = (\bar{\Gamma}_\theta)_* \tilde{B}.$$

**Lemma 3.5** *The set  $\{\eta_N, \eta_B\}$  is a globally defined, orthonormal, normal frame to  $S$ .*

*Proof.* We note that

$$\phi_s = (\bar{\Gamma}_\theta)_* \dot{\beta}, \quad \phi_\theta = (\bar{\Gamma}_\theta)_*(0, i\beta_2).$$

Bearing in mind these two expressions, (3.1) and Lemma 3.4, we deduce

$$\begin{aligned} \bar{g}_4(\eta_N, \phi_s) &= \bar{g}_4((\bar{\Gamma}_\theta)_* \tilde{N}, (\bar{\Gamma}_\theta)_* \dot{\beta}) = \bar{g}_4(\tilde{N}, \dot{\beta}) = \bar{g}_3(N, T) = 0, \\ \bar{g}_4(\eta_N, \phi_\theta) &= \bar{g}_4((\bar{\Gamma}_\theta)_* \tilde{N}, (\bar{\Gamma}_\theta)_*(0, i\beta_2)) = \bar{g}_4(\tilde{N}, (0, i\beta_2)) = 0, \end{aligned}$$

where the last equality holds because  $\tilde{N}$  is horizontal and  $(0, i\beta_2)$  is vertical. On the other hand,

$$\begin{aligned} \bar{g}_4(\eta_B, \phi_s) &= \bar{g}_4((\bar{\Gamma}_\theta)_* \tilde{B}, (\bar{\Gamma}_\theta)_* \dot{\beta}) = \bar{g}_4(\tilde{B}, \dot{\beta}) = \bar{g}_3(B, T) = 0 \\ \bar{g}_4(\eta_B, \phi_\theta) &= \bar{g}_4((\bar{\Gamma}_\theta)_* \tilde{B}, (\bar{\Gamma}_\theta)_*(0, i\beta_2)) = \bar{g}_4(\tilde{B}, i\beta_2) = 0 \\ \bar{g}_4(\eta_N, \eta_N) &= \bar{g}_4((\bar{\Gamma}_\theta)_* \tilde{N}, (\bar{\Gamma}_\theta)_* \tilde{N}) = \bar{g}_4(\tilde{N}, \tilde{N}) = \bar{g}_3(N, N) = \epsilon_2. \end{aligned}$$

Similarly, we deduce  $\bar{g}_4(\eta_N, \eta_B) = 0$  and  $\bar{g}_4(\eta_B, \eta_B) = \epsilon_3$ . ■

Let  $h_\phi$  be the second fundamental form of the immersion  $\phi : S \rightarrow \bar{M}_1^4$ , and  $\vec{H}_\phi = \frac{1}{2} \text{trace}_{\bar{g}_4} h_\phi$  the corresponding mean curvature vector of  $S$  in  $\bar{M}_1^4$ . Then:

**Lemma 3.6** *The mean curvature of  $\phi$  is given by*

$$\begin{aligned} \vec{H}_\phi(s, e^{i\theta}) &= \frac{\epsilon_2}{2} \left( \kappa(s) + \frac{f'(t(s))}{f(t(s))} \bar{g}_3(\partial_t |_{\alpha(s)}, N(s)) \right) \cdot \eta_N(s) \\ &\quad + \frac{\epsilon_3}{2} \left( \frac{f'(t(s))}{f(t(s))} \bar{g}_3(\partial_t |_{\alpha(s)}, B(s)) \right) \cdot \eta_B(s). \end{aligned}$$

*Proof.* Since we already know  $U_1, U_2, \eta_N$  and  $\eta_B$ , we have

$$2\vec{H}_\phi = \sum_{i=1}^2 \{\epsilon_2 \bar{g}_4(h_\phi(U_i, U_i), \eta_N) \eta_N + \epsilon_3 \bar{g}_4(h_\phi(U_i, U_i), \eta_B) \eta_B\}.$$

We are going to compute all these four products. By Lemma 3.1 and (3.3)

$$\begin{aligned}\bar{g}_4(h_\phi(U_1, U_1), \eta_N) &= \bar{g}_4(\bar{\nabla}_{U_1} U_1, \eta_N) = \bar{g}_4(\bar{\nabla}_{\phi_s} \phi_s, \eta_N) \\ &= \bar{g}_4(\bar{\nabla}_{(\bar{\Gamma}_\theta)_*\dot{\beta}} (\bar{\Gamma}_\theta)_*\dot{\beta}, (\bar{\Gamma}_\theta)_*\tilde{N}) = \bar{g}_4(\bar{\nabla}_{\dot{\beta}} \dot{\beta}, \tilde{N}) \\ &= \bar{g}_3(\nabla_{\dot{\alpha}} \dot{\alpha}, N) = \bar{g}_3(\epsilon_2 \kappa N, N) = \kappa.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\bar{g}_4(h_\phi(U_1, U_1), \eta_B) &= \bar{g}_4(\bar{\nabla}_{U_1} U_1, \eta_B) = \bar{g}_4(\bar{\nabla}_{\dot{\beta}} \dot{\beta}, \tilde{B}) = \bar{g}_3(\nabla_{\dot{\alpha}} \dot{\alpha}, B) \\ &= \bar{g}_3(\epsilon_2 \kappa N, B) = 0.\end{aligned}$$

For  $U_2$ , we make use of Lemma 3.2. We consider the surface in  $\mathbb{S}^3$  given by

$$\xi : J \times \mathbb{S}^1 \rightarrow \mathbb{S}^3, \quad \xi(s, e^{i\theta}) = e^{i\theta} \beta_2(s).$$

Since  $\xi_\theta = ie^{i\theta} \beta_2(s)$ , it is clear that  $\phi_\theta = (0, \xi_\theta)$ . Also, note that  $U_2 = \frac{1}{f} \phi_\theta = \frac{1}{f}(0, \xi_\theta)$ . Since  $h_\phi(U_2, U_2) = \frac{1}{f^2} h_\phi(\phi_\theta, \phi_\theta)$ , we compute

$$\tan(\bar{\nabla}_{\phi_\theta} \phi_\theta) = \tan(\bar{\nabla}_{(0, \xi_\theta)} (0, \xi_\theta)) = (0, D_{\xi_\theta} \xi_\theta). \quad (3.5)$$

Thus, we have to compute  $D_{\xi_\theta} \xi_\theta$ . To do so, we recall that the position vector  $\chi : \mathbb{S}^3 \rightarrow \mathbb{C}^2$  is a unit normal vector field with second fundamental form  $h_\chi(X, Y) = -g_3(X, Y)\chi$  for any  $X, Y$  tangent to  $\mathbb{S}^3$ . Let  $\bar{D}$  be the Levi-Civita connection of  $\mathbb{C}^2$ . By the Gauss formula, and by the fact that  $\xi_\theta$  is unit,

$$D_{\xi_\theta} \xi_\theta = \bar{D}_{\xi_\theta} \xi_\theta - h_\chi(\xi_\theta, \xi_\theta) = \bar{D}_{\xi_\theta} \xi_\theta + \chi \circ \xi.$$

Now, we consider the curve in  $J \times \mathbb{S}^1$  given by  $\alpha(u) = (s, e^{i(\theta+u)})$ . Since  $\alpha(0) = (s, e^{i\theta})$  and  $\alpha'(0) = \partial_\theta|_{(s, e^{i\theta})}$ , we obtain

$$\bar{D}_{\xi_\theta} \xi_\theta = \frac{d}{du} \Big|_{u=0} \xi_\theta(\alpha(u)) = \frac{d}{du} \Big|_{u=0} \xi_\theta(s, e^{i(\theta+u)}) = \frac{d}{du} \Big|_{u=0} i e^{i(\theta+u)} \beta_2(s) = -e^{i\theta} \beta_2(s).$$

Finally, we see

$$D_{\xi_\theta} \xi_\theta = -e^{i\theta} \beta_2(s) + (\chi \circ \xi)(s, e^{i\theta}) = 0.$$

By (3.5), we see that  $\tan(\bar{\nabla}_{\phi_\theta} \phi_\theta) = 0$ . On the other hand, by Lemma 3.2

$$\text{nor}(\bar{\nabla}_{\phi_\theta} \phi_\theta) = \text{nor}(\bar{\nabla}_{(0, Z)} (0, Z)) = -\frac{\bar{g}_4((0, Z), (0, Z))}{f} \text{grad}_{\bar{g}_4}(f) = f g_3(Z, Z) f' \partial_t = f f' \partial_t.$$

Therefore, we obtain  $\bar{\nabla}_{\phi_\theta} \phi_\theta = f f' \partial_t$ . As a consequence, we get

$$\begin{aligned}h_\phi(U_2, U_2) &= \frac{\epsilon_2}{f^2} \bar{g}_4(\bar{\nabla}_{\phi_\theta} \phi_\theta, \eta_N) \eta_N + \frac{\epsilon_3}{f^2} \bar{g}_4(\bar{\nabla}_{\phi_\theta} \phi_\theta, \eta_B) \eta_B \\ &= \frac{\epsilon_2 f'}{f} \bar{g}_3(\partial_t, N) \eta_N + \frac{\epsilon_3 f'}{f} \bar{g}_3(\partial_t, B) \eta_B. \quad \blacksquare\end{aligned}$$

**Proposition 3.7** *Given  $(s, e^{i\theta}) \in S$ , the mean curvature vector of the immersion  $\phi$  is spacelike (resp. lightlike/zero, timelike) if, and only if,*

$$\epsilon_2 \left( \kappa + \frac{f'}{f} \bar{g}_3(\partial_t, N) \right)^2 + \epsilon_3 \left( \frac{f'}{f} \bar{g}_3(\partial_t, B) \right)^2 > 0 \quad (\text{resp. } = 0, < 0). \quad (3.6)$$

**Remark 3.8** Let us assume that the curve  $\alpha$  satisfies that for some  $s_0 \in J$ , the point  $t_0 = t(s_0)$  is such that  $f'(t_0) = 0$ . According to (3.6), the mean curvature vector of  $\phi$  is lightlike/zero (respectively, timelike or spacelike) iff  $\epsilon_2 \kappa^2(s_0) = 0$  (resp.  $< 0$  or  $> 0$ ). Moreover, in case the equality holds, the curvature at  $s_0$  must vanish, i.e.  $\kappa(s_0) = 0$ , and so our procedure is no longer valid at this point (recall that we assumed a Frenet basis with the assumption  $\kappa > 0$ ). As we will see later, sometimes we can overcome this difficulty by using a continuity argument.

Finally, we point out that given a curve  $\alpha_2 : J \subset \mathbb{R} \rightarrow \mathbb{S}^2(1/2)$ , the lift of  $\alpha_2$  to  $\mathbb{S}^3$  via the Hopf projection  $\pi$  is classically called a *Hopf tube*. If, in addition,  $J = \mathbb{R}$ ,  $\alpha_2$  is periodic and  $\alpha_2$  has no self-intersection points (in other words, the image of  $\alpha_2$  is homeomorphic to a circle), the Hopf tube is an (embedded) Clifford torus in  $\mathbb{S}^3$ . This clearly extends to our curves  $\alpha$  and immersions  $\phi$ ; that is, if the image of  $\alpha : \mathbb{R} \rightarrow \overline{M}_1^3$  is homeomorphic to a circle, the associated lift  $\phi$  is a torus without boundary embedded in  $\overline{M}_1^4$ .

### 3.4 A simple example

Let  $\gamma : \tilde{J} \subset \mathbb{R} \rightarrow \mathbb{S}^2(1/2)$  be a Frenet unit curve. Let  $\nabla^2$  be the Levi-Civita connection of  $\mathbb{S}^2(1/2)$ . The Frenet apparatus of  $\gamma$  is  $\{d\gamma/d\tilde{s}, \mathbf{n}\}$  with geodesic curvature  $c$ . In other words,

$$\nabla_{d\gamma/d\tilde{s}}^2 d\gamma/d\tilde{s} = c\mathbf{n}, \quad \nabla_{d\gamma/d\tilde{s}}^2 \mathbf{n} = -c \cdot d\gamma/d\tilde{s},$$

and  $\{d\gamma/d\tilde{s}, \mathbf{n}\}$  is a positive basis for the usual orientation on  $\mathbb{S}^2(1/2)$ . Given  $t_0 \in I$ , let us define  $\alpha_2 : J \subset \mathbb{R} \rightarrow \mathbb{S}^2(1/2)$ ,  $\alpha_2(s) = \gamma(s/f(t_0))$ , and the curve  $\alpha : J \rightarrow \overline{M}_1^3$ ,  $\alpha(s) = (t_0, \alpha_2(s))$ . Simple computations give  $\dot{\alpha} = (0, \dot{\alpha}_2) = \left(0, \frac{1}{f} \frac{d\gamma}{d\tilde{s}}\right)$ . We call  $\mathbf{v} = \frac{1}{f} \frac{d\gamma}{d\tilde{s}}$ . Note that  $(0, \mathbf{v})$  is unit for  $\bar{g}_3$ , and therefore,  $\alpha$  is a spacelike unit curve in  $\overline{M}_1^3$ . Then, we easily obtain

$$\nabla_{\mathbf{v}}^2 \mathbf{v} = \frac{c}{f^2} \mathbf{n}, \quad \nabla_{\mathbf{v}}^2 \mathbf{n} = -c\mathbf{v}.$$

We put  $T = \dot{\alpha} = (0, \mathbf{v})$ . From Lemma 3.2, we have  $\nabla_T T = \nabla_{(0, \mathbf{v})} (0, \mathbf{v}) = \tan\left(\nabla_{(0, \mathbf{v})} (0, \mathbf{v})\right) + \text{nor}\left(\nabla_{(0, \mathbf{v})} (0, \mathbf{v})\right) = \left(0, \nabla_{\mathbf{v}}^2 \mathbf{v}\right) - \frac{\bar{g}_3((0, \mathbf{v}), (0, \mathbf{v}))}{f} \text{grad}_{\bar{g}_3} f = \left(\frac{f'}{f}, \frac{c}{f^2} \mathbf{n}\right)$ . The square  $\bar{g}_3$ -norm of  $\nabla_T T$  is  $\bar{g}_3(\nabla_T T, \nabla_T T) = -\frac{(f')^2}{f^2} + f^2 g_2\left(\frac{c}{f^2} \mathbf{n}, \frac{c}{f^2} \mathbf{n}\right) = \frac{c^2 - (f')^2}{f^2}$ . In order to obtain a Frenet curve, we must assume  $c^2 - (f')^2 \neq 0$  everywhere. We define

$$\Delta := c^2 - (f')^2, \quad \delta := \text{sign}(\Delta) = \pm 1, \quad \kappa := \frac{\sqrt{\delta \Delta}}{f}. \quad (3.7)$$

According to previous computations and notation, we have

$$\nabla_T T = \delta \frac{\sqrt{\delta \Delta}}{f} \left( \frac{f' \delta}{\sqrt{\delta \Delta}}, \frac{\delta c \mathbf{n}}{f \sqrt{\delta \Delta}} \right).$$

Therefore, we can choose

$$N = \left( \frac{f' \delta}{\sqrt{\delta \Delta}}, \frac{\delta c \mathbf{n}}{f \sqrt{\delta \Delta}} \right), \quad B = \left( \frac{c}{\sqrt{\delta \Delta}}, \frac{f' \mathbf{n}}{f \sqrt{\delta \Delta}} \right).$$

In fact, it is easy to check that  $\{T, N, B\}$  is an orthonormal basis along  $\alpha$ , with  $\epsilon_2 = \delta = -\epsilon_3$ . Let us check that it is also positive. We recall that we should compare it with  $\{\partial_t, \gamma', \mathbf{n}\}$ , bearing in mind (3.7); that is to say, we compute

$$\det(T, N, B) = \begin{vmatrix} 0 & \frac{f'\delta}{\sqrt{\delta\Delta}} & \frac{c}{\sqrt{\delta\Delta}} \\ 1/f & 0 & 0 \\ 0 & \frac{\delta c}{f\sqrt{\delta\Delta}} & \frac{f'}{f\sqrt{\delta\Delta}} \end{vmatrix} = \frac{-1}{f^2\Delta} \begin{vmatrix} f' & c \\ c & f' \end{vmatrix} = \frac{1}{f^2} > 0.$$

With this curve  $\alpha$ , we construct an immersion  $\phi$  as in (3.4). By (3.6), we can study the causal character of the mean curvature vector  $\vec{H}_\phi$  of  $\phi$ . Thus, we have

$$\begin{aligned} \epsilon_2 \left( \kappa + \frac{f'}{f} \bar{g}_3(\partial_t, N) \right)^2 + \epsilon_3 \left( \frac{f'}{f} \bar{g}_3(\partial_t, B) \right)^2 &= \delta \left( \frac{\sqrt{\delta\Delta}}{f} - \frac{\delta(f')^2}{f\sqrt{\delta\Delta}} \right)^2 - \delta \frac{(f')^2}{f^2} \left( \frac{c}{\sqrt{\delta\Delta}} \right)^2 = \\ &= \delta \frac{(c^2 - 2(f')^2)^2 - (f')^2 c^2}{\delta f^2 \Delta}. \end{aligned}$$

This means that  $\vec{H}_\phi$  is spacelike (resp. lightlike/zero, timelike) if, and only if,

$$\mathcal{S} := \delta \left( (c^2 - 2f'(t_0)^2)^2 - f'(t_0)^2 c^2 \right) > 0 \quad (\text{resp. } = 0, < 0).$$

For instance, if the chosen level  $t_0$  gives rise to a critical slide  $f'(t_0) = 0$ , the mean curvature  $\vec{H}_\phi$  is always spacelike.  $\blacksquare$

**Proposition 3.9** *There exist infinitely many embedded tori  $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \overline{M}_1^4$  which are trapped, marginally trapped or untrapped (whenever  $f$  is not constant everywhere).*

*Proof.* We resort to previous example. Pick a point  $t_0$  such that, say,  $f'(t_0) > 0$ , and consider a curve  $\gamma$  with constant geodesic curvature  $c$  (i.e., the curve  $\gamma$  is a small circle of  $\mathbb{S}^2(1/2)$ ). If  $c = 2f'(t_0)$  holds, simple computations show  $\delta = 1$  and  $\mathcal{S} = 0$ . This means that the embedding  $\phi$  is a marginally trapped surface. On the other hand, if  $c > 2f'(t_0)$ , we have  $\mathcal{S} > 0$ , which is the spacelike case. Finally, if we take  $f'(t_0) < c < 2f'(t_0)$ , then the mean curvature vector of  $\phi$  is timelike.  $\blacksquare$

**Remark 3.10** Previous proposition can be directly obtained from Theorem 2.1, just by considering suitable CMC tori in  $M^3 = \mathbb{S}^3$ .

### 3.5 Marginally trapped surfaces crossing expanding and collapsing regions

Consider the toy model  $\overline{M}_1^3 = I \times_f \mathbb{S}^2(1/2)$  associated to a closed FLRW spacetime  $\overline{M}_1^4 = I \times_f \mathbb{S}^3$ . Let  $\alpha : J \subset \mathbb{R} \rightarrow \overline{M}_1^3$ ,  $s \mapsto (t(s), \alpha_2(s))$  be a curve with  $\dot{t} = h(t)$ ; in particular,

$$\ddot{t}(s) = h'(t)\dot{t}(s) = h'(t)h(t).$$

If we impose the vector field  $\dot{\alpha} = T = h(t)\partial_t + \dot{\alpha}_2$  to be unitary, i.e.

$$\bar{g}_3(\dot{\alpha}, \dot{\alpha}) = -h(t)^2 + f(t)^2 g_2(\dot{\alpha}_2, \dot{\alpha}_2) = 1,$$

we deduce,

$$g_2(\dot{\alpha}_2, \dot{\alpha}_2) = \frac{1 + h(t)^2}{f(t)^2}.$$

Consider the unitary reparametrization  $\tilde{\alpha}_2$  of  $\alpha_2$ , i.e.

$$\dot{\alpha}_2 = \rho \dot{\tilde{\alpha}}_2, \quad \rho = \frac{\sqrt{1 + h(t)^2}}{f(t)}, \quad g_2(\dot{\tilde{\alpha}}_2, \dot{\tilde{\alpha}}_2) = 1 \quad (\Rightarrow g(\dot{\tilde{\alpha}}_2, \dot{\tilde{\alpha}}_2) = f(t)^2)$$

Then,  $T$  can be rewritten as

$$T = h(t)\partial_t + \rho \dot{\tilde{\alpha}}_2.$$

Moreover

$$\begin{aligned} \nabla_{\dot{\tilde{\alpha}}_2} \dot{\tilde{\alpha}}_2 &= \nabla_{\dot{\tilde{\alpha}}_2}^2 \dot{\tilde{\alpha}}_2 - \frac{\bar{g}_3(\dot{\tilde{\alpha}}_2, \dot{\tilde{\alpha}}_2)}{f(t)} \text{grad}_{\bar{g}_3}(f) = \nabla_{\dot{\tilde{\alpha}}_2}^2 \dot{\tilde{\alpha}}_2 + f(t)f'(t)\partial_t \\ \nabla_{\dot{\tilde{\alpha}}_2} \partial_t &= \nabla_{\partial_t} \dot{\tilde{\alpha}}_2 = \frac{f'(t)}{f(t)} \dot{\tilde{\alpha}}_2, \quad \nabla_{\partial_t} \partial_t = 0. \end{aligned}$$

Therefore, we deduce:

$$\begin{aligned} \frac{DT(s)}{ds} &= \frac{D}{ds} (\dot{t}(s)\partial_t + \rho \dot{\tilde{\alpha}}_2) = \ddot{t}(s)\partial_t + \dot{t}(s)\frac{D\partial_t}{ds} + \dot{\rho}\dot{\tilde{\alpha}}_2 + \rho\frac{D\dot{\tilde{\alpha}}_2}{ds} \\ &= \ddot{t}(s)\partial_t + \dot{t}(s)^2\nabla_{\partial_t}\partial_t + \dot{t}(s)\nabla_{\dot{\tilde{\alpha}}_2}\partial_t + \dot{\rho}\dot{\tilde{\alpha}}_2 + \rho \left( \dot{t}(s)\nabla_{\partial_t}\dot{\tilde{\alpha}}_2 + \nabla_{\dot{\tilde{\alpha}}_2}\dot{\tilde{\alpha}}_2 \right) \\ &= h'(t)h(t)\partial_t + h(t)\rho\nabla_{\dot{\tilde{\alpha}}_2}\partial_t + \dot{\rho}\dot{\tilde{\alpha}}_2 + h(t)\rho\nabla_{\partial_t}\dot{\tilde{\alpha}}_2 + \rho^2\nabla_{\dot{\tilde{\alpha}}_2}\dot{\tilde{\alpha}}_2 \\ &= h'(t)h(t)\partial_t + 2h(t)\rho\frac{f'(t)}{f(t)}\dot{\tilde{\alpha}}_2 + \dot{\rho}\dot{\tilde{\alpha}}_2 + \rho^2\nabla_{\dot{\tilde{\alpha}}_2}\dot{\tilde{\alpha}}_2 \\ &= h'(t)h(t)\partial_t + \left( 2h(t)\rho\frac{f'(t)}{f(t)} + \dot{\rho} \right) \dot{\tilde{\alpha}}_2 + \rho^2 \left( \nabla_{\dot{\tilde{\alpha}}_2}^2 \dot{\tilde{\alpha}}_2 + f(t)f'(t)\partial_t \right) \\ &= (h'(t)h(t) + \rho^2 f(t)f'(t)) \partial_t + \left( 2h(t)\rho\frac{f'(t)}{f(t)} + \dot{\rho} \right) \dot{\tilde{\alpha}}_2 + \rho^2 \nabla_{\dot{\tilde{\alpha}}_2}^2 \dot{\tilde{\alpha}}_2, \end{aligned}$$

where the partner function of  $\dot{\tilde{\alpha}}_2$  becomes

$$\begin{aligned} 2h(t)\rho\frac{f'(t)}{f(t)} + \dot{\rho} &= 2h(t)\rho\frac{f'(t)}{f(t)} - h(t)\frac{(1 + h(t)^2)f'(t) - f(t)h(t)h'(t)}{f(t)^2\sqrt{1 + h(t)^2}} \\ &= \left( h(t)h'(t) + (1 + h(t)^2)\frac{f'(t)}{f(t)} \right) \frac{h(t)}{f(t)\sqrt{1 + h(t)^2}}. \end{aligned}$$

Next, let  $\{\dot{\tilde{\alpha}}_2, \mathcal{N}\}$  be some orthonormal basis of  $TS^2(1/2)$  along  $\tilde{\alpha}_2$ . Assume that  $\tilde{\alpha}_2$  is a circle (non-necessarily maximum) in  $S^2(1/2)$ . Then,

$$\nabla_{\dot{\tilde{\alpha}}_2}^2 \dot{\tilde{\alpha}}_2 = k\mathcal{N}.$$

In conclusion, we deduce

$$\nabla_T T = \left( hh' + (1 + h^2)\frac{f'}{f} \right) \left( \partial_t + \frac{h}{\sqrt{1 + h^2}f} \dot{\tilde{\alpha}}_2 \right) + \frac{(1 + h^2)}{f^2} k\mathcal{N}.$$

In particular, if we recall that  $\nabla_T T = \epsilon_2 \kappa N$ , we have

$$\begin{aligned}\bar{g}_3(\nabla_T T, \nabla_T T) &= \left( hh' + (1+h^2) \frac{f'}{f} \right)^2 \left( \frac{-1}{1+h^2} \right) + (1+h^2)^2 \frac{k^2}{f^2} \\ &= \epsilon_2 \kappa^2.\end{aligned}$$

Therefore, from the last equality in this expression:

$$\begin{aligned}\kappa^2 &= \epsilon_2 \left( \frac{(1+h^2)^2 k^2}{f^2} - \frac{1}{1+h^2} \left( hh' + (1+h^2) \frac{f'}{f} \right)^2 \right) \\ &= \epsilon_2 (1+h^2) \left( \frac{(1+h^2)k^2}{f^2} - \left( (\ln \sqrt{1+h^2})' + (\ln f)' \right)^2 \right).\end{aligned}\tag{3.8}$$

Next, we are going to compute the vector field  $B$ . To this aim, we write it as:

$$B = \bar{A} \partial_t + \bar{B} \dot{\alpha}_2 + \bar{C} N$$

By imposing  $\bar{g}_3(T, B) = -h\bar{A} + \bar{B}\sqrt{1+h^2}f = 0$  we deduce

$$\bar{B} = \frac{h}{\sqrt{1+h^2}f} \bar{A}.\tag{3.9}$$

On the other hand, by imposing  $\bar{g}_3(N, B) = 0$  we have  $\bar{g}_3(\nabla_T T, B) = 0$ , and so,

$$\left( hh' + (1+h^2) \frac{f'}{f} \right) \left( -\bar{A} + \frac{hf}{\sqrt{1+h^2}} \bar{B} \right) + (1+h^2)k\bar{C} = 0.\tag{3.10}$$

Taking into account (3.9) in (3.10) we deduce:

$$0 = - \left( \frac{hh'}{1+h^2} + \frac{f'}{f} \right) \bar{A} + (1+h^2)k\bar{C}.$$

Therefore, if  $k \neq 0$  we have

$$\bar{C} = \frac{1}{(1+h^2)2k} \left( hh' + (1+h^2) \frac{f'}{f} \right) \bar{A}.\tag{3.11}$$

Next, we impose  $g_3(B, B) = -\bar{A}^2 + f^2(\bar{B}^2 + \bar{C}^2) = \epsilon_3$ . Taking into account (3.8), (3.9) and (3.11) in this expression, one deduces:

$$\begin{aligned}\bar{A}^2 &= \frac{\epsilon_3}{-\frac{1}{1+h^2} + \frac{f^2}{(1+h^2)^4 k^2} \left( hh' + (1+h^2) \frac{f'}{f} \right)^2} = - \frac{\epsilon_3 k^2 (1+h^2)^2}{f^2 \left( \frac{(1+h^2)k^2}{f^2} - ((\ln \sqrt{1+h^2})' + (\ln f)')^2 \right)} \\ &= \frac{k^2 (1+h^2)^3}{f^2 \kappa^2}.\end{aligned}$$

Next, we impose that the surface  $S$  generated by the curve  $\alpha$  is marginally trapped (3.6):

$$\begin{aligned}\left( \kappa + \frac{f'}{f} \bar{g}_3(\partial_t, N) \right)^2 &= \left( \frac{f'}{f} \bar{g}_3(\partial_t, B) \right)^2 \\ \left( \kappa + \frac{f'}{f \epsilon_2 \kappa} \bar{g}_3(\partial_t, \nabla_T T) \right)^2 &= (f'/f)^2 \bar{A}^2 \\ (\epsilon_2 \kappa^2 - (\ln f)' (hh' + (1+h^2)(\ln f)'))^2 &= ((\ln f)')^2 \kappa^2 \bar{A}^2.\end{aligned}$$

If we develop both members of the equation, we deduce:

$$2((\ln f)')^2 + \left( 3(\ln \sqrt{1+h^2})' + \epsilon \frac{\sqrt{1+h^2}k}{f} \right) (\ln f)' + ((\ln \sqrt{1+h^2})')^2 - \frac{(1+h^2)k^2}{f^2} = 0,$$

which is a second order equation for  $(\ln f)'$ . The discriminant of this equation is:

$$D = \left( (\ln \sqrt{1+h^2})' + \frac{3\epsilon\sqrt{1+h^2}k}{f} \right)^2.$$

Therefore, the solutions are

$$(\ln f)'_{\pm} = \frac{-3(\ln \sqrt{1+h^2})' - \frac{\epsilon\sqrt{1+h^2}k}{f} \pm \left( (\ln \sqrt{1+h^2})' + \frac{3\epsilon\sqrt{1+h^2}k}{f} \right)}{4},$$

that is,

$$\begin{cases} f'_+ = -\frac{1}{2}(\ln \sqrt{1+h^2})' f_+ + \frac{\epsilon}{2}\sqrt{1+h^2}k \\ f'_- = -(\ln \sqrt{1+h^2})' f_- - \epsilon\sqrt{1+h^2}k. \end{cases}$$

The solution  $(\ln f)'_-$  implies  $\kappa = 0$  (recall (3.8)). So, take the solution  $(\ln f)'_+$  with  $\epsilon = +1$ . Take also  $t(s) = \sin(s)$ , and thus,  $h(t) = \pm\sqrt{1-t^2}$ . In this case, the warping function  $f$  must satisfy the differential equation

$$f'(t) = \frac{t}{2(2-t^2)}f(t) + \frac{\sqrt{2-t^2}}{2}k, \quad f(0) = f_0 > 0. \quad (3.12)$$

So, according to previous construction, we will obtain a marginally trapped surface crossing expanding and collapsing regions along a closed FLRW spacetime with, say,  $I = (-1.3, 1.3)$ , if, in addition, the following two properties hold for  $f_0$  big enough:

- (i)  $f(t) > 0$  for all  $t \in [-1.3, 1.3]$ , and
- (ii)  $f'(t)$  changes its sign along  $(-1, 1)$ .

For (i), first we are going to prove that  $f(t) > 0$  on  $[0, 1.3]$ . By contradiction, we assume there exists  $t_1 \in [0, 1.3]$  such that  $f(t_1) \leq 0$ . Since  $f'(0) = k > 0$ ,  $f$  has to start to decrease at a certain point, and therefore there exists  $t_0 \in (0, t_1)$  such that  $f'(t_0) = 0$ ,  $f(t_0) > 0$ . This is in contradiction to the fact that

$$f'(t_0) = \frac{t_0}{2(2-t_0^2)}f(t_0) + \frac{\sqrt{2-t_0^2}}{2}k > 0.$$

Next, we focus on  $[-1.3, 0]$ . Let  $g(t)$  be the solution of the problem:

$$g'(t) = \frac{\sqrt{2-t^2}}{2}k, \quad g(0) = f_0.$$

It is straightforward to check that

$$g(t) = \frac{1}{4}kt\sqrt{2-t^2} + \frac{k}{2}\arcsin(t/\sqrt{2}) + f_0 \quad \text{on } [-1.3, 0].$$

In particular, we notice that  $g(t) > 0$  on  $[-1.3, 0]$  whenever  $f_0$  is big enough. We are going to show that, under these conditions,  $g(t) \leq f(t)$  on  $[-1.3, 0]$ . To this aim, define

$$\Gamma = \{t \in [-1.3, 0] : g(t) \leq f(t)\}.$$

Since  $g(0) = f_0 = f(0)$ , it is  $0 \in \Gamma \neq \emptyset$ . Moreover,  $\Gamma = (g-f)^{-1}((-\infty, 0])$  is closed in  $[-1.3, 0]$ . In order to show that  $\Gamma = [-1.3, 0]$ , it suffices to prove that  $\Gamma$  is open, or, equivalently, if  $[t, 0] \subset \Gamma$  then  $[t-\epsilon, 0] \subset \Gamma$  for some  $\epsilon > 0$ . So, assume that  $[t, 0] \subset \Gamma$ . Then,  $f(t) \geq g(t)$ , and thus,

$$f'(\bar{t}) - g'(\bar{t}) = \frac{\bar{t}}{2(2-\bar{t}^2)} f(\bar{t}) < 0, \quad \text{for all } \bar{t} \in [t, 0].$$

Therefore,

$$\int_t^0 f'(\bar{t}) d\bar{t} < \int_t^0 g'(\bar{t}) d\bar{t}, \quad \text{and thus, } f(t) > g(t).$$

By continuity, there exists  $\epsilon > 0$  such that  $f(\bar{t}) \geq g(\bar{t})$  for all  $\bar{t} \in [t-\epsilon, 0]$ , and so,  $[t-\epsilon, 0] \subset \Gamma$ . Summarizing, we have proved that property (i) above holds whenever  $f_0$  is big enough.

For property (ii), observe that  $f'(0) > 0$ . Moreover,  $f(-1) \geq g(-1) \nearrow \infty$  if  $f_0 \nearrow \infty$ . Therefore,

$$f'(-1) = -\frac{f(-1)}{2} + \frac{k}{2} < 0 \quad \text{if } f_0 \text{ is big enough.}$$

Hence, property (ii) also holds whenever  $f_0$  is big enough.

In conclusion, we have proved the existence of a closed FLRW spacetime admitting a marginally trapped surface crossing expanding and collapsing regions.

**Remark 3.11** Two important subtleties have been omitted in previous development:

1. Notice that our approach breaks down at the points where function  $h$  becomes zero, i.e. for  $t(s) = \pm 1$ ; and so, we cannot ensure, a priori, that our surface is marginally trapped at the corresponding points. This difficulty is overcome just by noting that the continuity of the length of the mean curvature vector, joined to the fact that this length is zero at the rest of the points, ensures that it is also zero here.
2. A similar argument shows that our surface is marginally trapped at the points where  $f'$  vanishes (which must exist by property (ii)). From (3.12) it is straightforward to check that these points are isolated in  $I$ , and so, again a continuity argument on the length of the mean curvature vector ensures that the surface is also marginally trapped there.

## 4 Marginally trapped tubes in FLRW spacetimes

In the present paper we propose the following definition of marginally trapped tube (compare with the definition of MOTT in [1]):

**Definition 4.1** *A smooth manifold  $\mathcal{G}$  which admits a foliation  $\{\mathcal{S}_\lambda : \lambda \in \Lambda\}$ , is a marginally trapped tube in a spacetime  $\mathcal{M}$  if there is a smooth immersion of codimension 1,  $\Phi : \mathcal{G} \rightarrow \mathcal{M}$ , such that:*

**(A)** *Each  $\Phi(\mathcal{S}_\lambda)$  ( $\lambda \in \Lambda$ ) is a marginally trapped surface in  $\mathcal{M}$ , and*



**(B)**  $\Phi(\mathcal{S}_\lambda) \cap \Phi(\mathcal{S}_\mu) = \emptyset$  for any  $\mu, \lambda \in \Lambda$ ,  $\mu \neq \lambda$ .

The second condition is required in order to avoid self-intersections in the direction of propagation of the tube.

With this definition in mind, we are going to apply our approach to obtain some information about marginally trapped tubes in FLRW spacetimes. We begin with the following direct consequences of the corresponding existence/non-existence results for marginally trapped surfaces in Section 2 (Cor. 2.2, 2.4):

**Corollary 4.2** (*Existence result*). *There exist marginally trapped tubes whose  $t$ -sections are formed by closed marginally trapped surfaces of any genus in closed ( $M^3 = \mathbb{S}^3$ ) FLRW spacetimes.*

**Corollary 4.3** (*Non-existence result*). *Let  $\overline{M}_1^4 = I \times_f M^3$  be a FLRW spacetime with fiber  $M^3 = \mathbb{H}^3$ . There are no marginally trapped tubes, with  $t$ -sections formed by closed marginally trapped surfaces, crossing  $t_0$ -slices with  $|f'(t_0)| \leq 1$ .*

Next, we are going to give examples of marginally trapped tubes with any type of causality in closed FLRW spacetimes.

#### 4.1 Examples of marginally trapped tubes with different causality

Given a closed FLRW spacetime  $\overline{M}_1^4 = I \times_f \mathbb{S}^3$ , first we are going to construct a marginally trapped tube foliated by Clifford tori (see Section 2) and defined for any time.

We define the smooth function

$$h : I \rightarrow (0, \pi/2), \quad h(t) = \frac{1}{2} \operatorname{arccot} \left( \frac{f'(t)}{2} \right) = \frac{\pi}{4} - \frac{1}{2} \arctan \left( \frac{f'(t)}{2} \right),$$

whose derivative is

$$h'(t) = \frac{-f''(t)}{4 + f'(t)^2}.$$

Next, we define the embedding

$$\begin{aligned} \phi : I \times \mathbb{S}^1 \times \mathbb{S}^1 &\rightarrow \overline{M}_1^4, \\ \phi(t, e^{i\theta}, e^{i\nu}) &= \left( t, e^{i\theta} \cos(h(t)), e^{i\nu} \sin(h(t)) \right). \end{aligned}$$

We notice that for each  $t \in I$ , the surface  $\phi(t, -, -) : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \overline{M}_1^4$ , is a Clifford torus embedded in the  $t$ -slice. By comparing with the expression of the Clifford torus, we see that the length of the mean curvature of the torus (see (2.3)) at  $t$  is  $\|\vec{H}_u\|$ , with  $u = h(t)$ . A straightforward computation shows

$$\|\vec{H}_u\|_{u=h(t)} = |2 \cot(2u)|_{u=h(t)} = |f'(t)|.$$

By Theorem 2.1, for each  $t$ , the torus is a marginally trapped surface.

Next, we pay attention to the causal character of the embedding  $\phi$ . In this sense, we are going to compute the first fundamental form induced by  $\bar{g}_4$  on our surface. To this aim, first we compute the derivatives

$$\begin{aligned}\phi_t &= \left(1, -e^{i\theta} \sin(h(t))h'(t), e^{i\nu} \cos(h(t))h'(t)\right), \\ \phi_\theta &= \left(0, ie^{i\theta} \cos(h(t)), 0\right), \\ \phi_\nu &= \left(0, 0, ie^{i\nu} \sin(h(t))\right).\end{aligned}$$

A straightforward computation shows

$$\phi^* \bar{g}_4 \equiv \begin{pmatrix} (f(t)h'(t))^2 - 1 & 0 & 0 \\ 0 & (f(t)\cos(h(t)))^2 & 0 \\ 0 & 0 & (f(t)\sin(h(t)))^2 \end{pmatrix}.$$

Since  $0 < h(t) < \pi/2$  for any  $t \in I$ , the derivatives  $\phi_\theta$  and  $\phi_\nu$  are always spacelike. Thus, everything depends on the derivative  $\phi_t$ . By recalling the expressions of  $\phi_t$  and  $h'(t)$ , we obtain

$$z(t) := \bar{g}_4(\phi_t, \phi_t) = -1 + \left(\frac{f(t)f''(t)}{4 + f'(t)^2}\right)^2. \quad (4.1)$$

This expression can take any value, positive, negative or zero, depending only on the warping function  $f$ . We show a list of particular cases:

1. We choose non-negative real constants  $a$  and  $b$  such that  $a^2 = 4 + b^2$ . In particular,  $a > b$ , which makes the function  $f : I = \mathbb{R} \rightarrow (0, \infty)$ ,  $f(t) = a \cosh(t) + b \sinh(t)$  well-defined. A simple computation shows  $z(t) = 0$ . Therefore,  $\phi_t$  is everywhere lightlike, and so is the corresponding marginally trapped tube.
2. We choose real constants  $c_1, c_2 > 0$ . Then, the function  $f : \mathbb{R} \rightarrow (0, \infty)$ ,  $f(t) = \frac{4+c_1^2}{4c_2}t^2 + c_1t + c_2$  is well-defined. A simple computation shows  $z(t) = -3/4$ . This implies that  $\phi_t$  is everywhere timelike, and so is the corresponding marginally trapped tube.
3. We define the function  $f : (-1, 1) \rightarrow (0, \infty)$ ,  $f(t) = \frac{2}{1-t^2}$ . Now, we see

$$z(t) + 1 = \left(\frac{f(t)f''(t)}{4 + f'(t)^2}\right)^2 = \frac{6t^2 + 2}{t^8 - 4t^6 + 6t^4 + 1}.$$

If we show that  $z(t) \geq 1$ , then we obtain that  $\phi_t$  is always spacelike, and thus, the marginally trapped tube is also spacelike. By simple computations, we have that for any  $t \in (-1, 1)$ ,  $z(t) + 1 \geq 2 \iff 0 \geq t^2(t^6 - 4t^4 + 6t^2 - 3)$ . Standard computations give that the only real roots of the equality are  $t = 0, \pm 1$ . This readily proves  $z(t) \geq 1$  for any  $t \in (-1, 1)$ .

4. We define the function  $f : \mathbb{R} \rightarrow (0, \infty)$ ,  $f(t) = 3 + \cos(2t)$ . A straightforward computation gives

$$z(t) = -1 + \left(\frac{f(t)f''(t)}{4 + f'(t)^2}\right)^2 = -1 + \frac{4 \cos^2(2t)(3 + \cos(2t))^2}{(3 - \cos(4t))^2}.$$

It is easy to check  $z(0) = 15$  and  $z(\pi/4) = -1$ . Therefore, the marginally trapped tube changes its causal character with time.

## 5 Marginally trapped surfaces in $t$ -slices of twisted spaces

Assume that the warping function  $f$  also depends on the fiber  $M^3$ , i.e.  $f : I \times M^3 \rightarrow (0, \infty)$ . Denote again by  $\overline{M}_1^4 = I \times_f M^3$  the Lorentzian twisted manifold given by the product manifold  $I \times M^3$  endowed with metric  $\overline{g}_4 = -dt^2 + f^2 g_3$ . As in Section 2, let  $\varphi : S \rightarrow M^3$  be an immersion of  $S$  in  $M^3$ ,  $\psi : M^3 \rightarrow I \times_f M^3$  the embedding of  $M^3$  in  $I \times_f M^3$  and  $\phi : S \rightarrow I \times_f M^3$  the corresponding immersion of  $S$  in the twisted product, both in a  $t$ -slice. Again from [9, p. 79], we have the following relation between the corresponding second fundamental forms:

$$h_\phi(X, Y) = h_\varphi(X, Y) + h_\psi(X, Y), \quad \text{whereby } X, Y \in \mathfrak{X}(S).$$

From [26, Prop. 2], the second fundamental form  $h_\psi$  is

$$h_\psi(X, Y) = -\overline{g}_4(X, Y) \frac{\text{grad}_{\overline{g}_4} f}{f} = \overline{g}_4(X, Y) \left( \frac{\partial_t f}{f} \partial_t - \frac{1}{f^3} \text{grad}_{g_3} f \right),$$

where we have used that

$$\text{grad}_{\overline{g}_4} f \left( = \sum_{ij} \overline{g}_4^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \right) = -\partial_t f \partial_t + \frac{1}{f^2} \text{grad}_{g_3} f.$$

Hence,

$$h_\phi(X, Y) = h_\varphi(X, Y) + \overline{g}_4(X, Y) \left( \frac{\partial_t f}{f} \partial_t - \frac{1}{f^3} \text{grad}_{g_3} f \right).$$

Taking one half of the trace of the above expression, using an orthonormal frame  $\{\partial_t, \{E_i\}_{i=1}^3\}$  w.r.t. the metric  $\overline{g}_4$ , i.e.  $E_i = \frac{e_i}{f}$  whereby  $\{e_i\}_{i=1}^3$  is the corresponding orthonormal frame w.r.t. the metric  $g_3$  on  $M^3$  (and  $S$ ), one obtains

$$\vec{H}_\phi = \frac{1}{f^2} \left( \vec{H}_\varphi - \frac{1}{f} \text{grad}_{g_3} f \right) + \frac{\partial_t f}{f} \partial_t, \quad (5.1)$$

where  $\vec{H}_\phi$  and  $\vec{H}_\varphi$  stand for the mean curvature associated to  $h_\phi$  and  $h_\varphi$ , respectively.

**Theorem 5.1** *A surface  $S$  contained in a  $t_0$ -slice of twisted  $\overline{M}_1^4 = I \times_f M^3$  is marginally trapped iff its mean curvature vector satisfies:*

$$\left\| \vec{H}_\varphi(\cdot) - \frac{1}{f(t_0, \cdot)} \text{grad}_{g_3} f(t_0, \cdot) \right\| = |\partial_t f(t_0, \cdot)|.$$

*On the other hand,  $S$  is trapped iff*

$$\left\| \vec{H}_\varphi(\cdot) - \frac{1}{f(t_0, \cdot)} \text{grad}_{g_3} f(t_0, \cdot) \right\| < |\partial_t f(t_0, \cdot)|.$$

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## References

- [1] L. Andersson, M. Mars, W. Simon, *Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes*, : Adv. Theor. Math. Phys. Volume 12, Number 4 (2008), 853-888.
- [2] J. K. Beem, P. E. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, 2nd Edition, Pure App. Math. **202**, Marcel Dekker, New York, 1996.
- [3] R. Beig, N. Murchadha, *Vacuum Spacetimes with Future Trapped Surfaces*, Class. Quant. Grav. **13** (1996), 739-752.
- [4] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys **46:4** (1991), 1-45.
- [5] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Diff. Geom. **59** (2001), 177-267; G. Huisken, T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Diff. Geom. **59** (2001), 353-437.
- [6] A. Butscher, F. Pacard, *Doubling Constant Mean Curvature Tori in  $\mathbb{S}^3$* , Ann. Scuola Norm. Sup. Pisa Cl. Sci., (5) Vol. V (2006), 611-638.
- [7] M. P. do Carmo, H. B. Lawson Jr, *On Alexandrov–Bernstein theorems in hyperbolic space*, Duke Math. J. **50** (1983), 995-1003.
- [8] A. Carrasco, M. Mars, *On marginally outer trapped surfaces in stationary and static spacetimes*, Class. Quant. Grav. **25** (2008), 055011 (19pp).
- [9] B.-Y. Chen, *Geometry of Submanifolds*, Pure App. Math. **22**. Marcel Dekker, Inc., New York, 1973.
- [10] B.-Y. Chen, J. Van der Veken, *Marginally trapped surfaces in Lorentzian space with positive relative nullity*, Class. Quant. Grav. **24** (2007), 551-563.
- [11] B.-Y. Chen, J. Van der Veken, *Spatial and Lorentzian surfaces in Robertson-Walker spacetimes*, J. Math. Phys. **48** (2007), 073509 (12pp).
- [12] M. Dafermos, *The interior of charged black holes and the problem of uniqueness in general relativity*, Comm. Pure Appl. Math. **58** (2005), 445-504.
- [13] G. Ellis, *Closed trapped surfaces in cosmology*, Gen. Relat. Grav. **35** (2003), 1309-1319.
- [14] S. Haesen, M. Ortega, *Boost invariant marginally trapped surfaces in Minkowski 4-space*, Class. Quant. Grav. **24** (2007), 5441-5452.
- [15] S. Haesen, M. Ortega, *Marginally trapped surfaces in Minkowski 4-space invariant under a rotation subgroup of the Lorentz group*, Gen. Rel. Grav. **41** (2009), 1819-1834.
- [16] S. Haesen, M. Ortega, *Screw Invariant Marginally Trapped Surfaces in Minkowski 4-space*, J. Math. Anal. Appl. **355** (2009), 639-648.
- [17] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space–Time*, Cambridge University Press, London, 1973.

- [18] H.B. Lawson, *Complete Minimal Surfaces in  $S^3$* , Ann. Math. **92** (1970), 335-374.
- [19] L. Lehner, *Numerical Relativity: a review*, Class. Quant. Grav. **18** (2001), R25-86.
- [20] E. Malec, N. Murchadha, *Trapped surfaces in cosmological spacetimes*, Phys. Rev. D, **54** (1996), 4799-4804.
- [21] M. Mars, J. M. M. Senovilla, *Trapped surfaces and symmetries*, Class. Quant. Grav. **20** (2003), L293-L300.
- [22] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459-469.
- [23] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Inc., New York, 1983.
- [24] R. Penrose, *Techniques of Differential Topology in Relativity*, SIAM, Philadelphia, Pa., 1972.
- [25] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57-59.
- [26] R. Ponge, H. Reckziegel, *Twisted Products in Pseudo-Riemannian*, Geom. Dedicata **48** (1993), 15-25.
- [27] A. Romero, M. Sánchez, *On the completeness of certain families of semi-Riemannian manifolds*, Geom. Dedicata **53** (1994), 103-117.
- [28] R. K. Sachs, H. Wu, *General Relativity for Mathematicians*, Graduate Texts in Mathematics, Springer, New York, 1977.
- [29] J.M.M. Senovilla, *Novel results on trapped surfaces*, Proc. 2nd Conference on Mathematics of Gravitation, Warsaw, Poland, 8pp.
- [30] J. M. M. Senovilla, *Trapped surfaces, horizons and exact solutions in higher dimensions*, Class. Quant. Grav. **19** (2002), L113-L119.
- [31] J. M. M. Senovilla, *Classification of spacelike surfaces in spacetime*, Class. Quant. Grav. **24** (2007), 3091-3124.
- [32] K. Stromberg, *An Introduction to classical real analysis*, Chapman and Hall, London, 1996.