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Higher order dilaton gravity: brane equations of motion in the covariant formulation

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Abstract

Dilaton gravity with general brane localized interactions is investigated. Models with corrections up to arbitrary order in field derivatives are considered. Effective gravitational equations of motion at the brane are derived in the covariant approach. Dependence of such brane equations on the bulk quantities is discussed. It is shown that the number of the bulk independent brane equations of motion depends strongly on the symmetries assumed for the model and for the background. Examples with two and four derivatives of the fields are presented in more detail.

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1 Introduction

Assuming that the space-time dimensionality is restricted to the number of currently known 4 dimensions, the Einstein tensor appearing in the equation of motion of the standard theory of gravity is the most general rank 2, divergence-free symmetric tensor depending on the metric and its first and second derivatives only, while being linear in the latter. As for the presently testable, gravity-related phenomena, all experiments seem to show no discrepancies between the General Relativity and observations¹. Nevertheless, a beyond the Standard Model theory is commonly pronounced necessary, with string theories amongst the most serious candidates. As compared to the standard Einstein theory of gravity, the effective Lagrangians obtained within the string theories framework contain corrections with higher powers of the Riemann tensor [2]. Moreover, the α' expansion in string theories predicts corrections of the higher order in derivatives not only for the metric tensor, but also for other fields, such as the dilaton. In this work we investigate models with higher order corrections for both the metric and the dilaton, as well as mixed gravity-dilaton interactions.

The order of the corrections is restricted by the dimensionality of the space-time. Specifically, the N -th power of the Riemann tensor can be included into a dilaton gravity Lagrangian only if the number of space-time dimensions, d , is sufficient, i.e. for $d \geq 2N$. For example, the 2nd order contribution quadratic in the Riemann tensor, known as the Gauss-Bonnet (GB) term [3], can be taken into account in a dilaton gravity theory already in 4 dimensions. On the other hand, in the classical, no-dilaton theories, the GB term is a full divergence in 4 dimensions. It becomes dynamically relevant in an at least 5-dimensional space-time only. For pure gravity the GB term was generalized to higher orders² by Lovelock [6]. In a previous paper by the present authors [7], the Einstein-Lovelock theory of gravity was generalized by coupling it to the dilaton. The appropriate action and equations of motion were constructed up to arbitrary order in derivatives of both the metric tensor and the dilaton.

The main motivation for the dilaton gravity theory constructed in [7] comes from string theories, which are considered to be the most promising attempt to quantize gravity and unify it with all other known interactions. The gravitational sectors of the effective field theories derived from string theories coincide with the Einstein theory of gravity only in the lowest order. Beyond such approximation they involve higher powers of the Riemann tensor as well as interactions with other fields with more than two derivatives. The dilaton gravity model considered in the present paper

¹However, there are claims that some alternative models could provide better explanation for the late-time cosmic acceleration and/or for the phenomena attributed usually to the existence of the Cold Dark Matter. (see e.g. [1] and references therein).

²Explicit formulae for the 3rd and 4th orders expressions are given in [4] and [5], respectively.

is closely connected to string theories - its interactions with up to four derivatives are exactly as those present in the effective theory derived from string theories and restricted to the gravity and the dilaton field (see e.g. [11]). Such correspondence has not been proven for interactions with six or more derivatives. However, there are considerable indications (see the discussion of the $O(d, d)$ symmetry in [7]) that our model may be a part of the effective string dilaton gravity action also at the level of more than four derivatives.

Dating back to the M-theory based work by Hořava and Witten [8], the idea of localizing the Standard Model on a brane embedded in a higher-dimensional space-time [9] gained quite a lot of attention. Corrections to the gravity interactions at the brane due to the bulk fields were investigated by many authors [10]. It seems interesting and important to consider what gravity would be induced at the brane, if the bulk action was given by a higher order dilaton gravity theory. Hence, the purpose of this work is to derive effective gravitational equations at the brane for the models of arbitrary higher order in derivatives, constructed in [7]. In order to keep full generality, the procedure will be carried out in the covariant approach. Starting from the d -dimensional ($d > 4$) higher order dilaton gravity theory, equations of motion in the effective $(d - 1)$ -dimensional theory, i.e. for a co-dimension 1 brane, will be derived.

For the standard lowest order gravity the effective equations of motion at the brane were derived in the covariant approach in [12]. That analysis was extended to the GB gravity in [13]. The effective equations at the brane for the lowest order dilaton gravity were derived in [14] (however, not in a fully covariant way). The covariant approach was employed in [15] for cosmological applications of dilaton gravity. Certain second order gravity models with branes, including either first or second order corrections for the scalar field, were analyzed for specific metrics in e.g. [16] and [17], respectively. Although brane models for arbitrary order Einstein-Lovelock gravity were investigated in [18], neither the dilaton field was included, nor the covariant approach was adopted. No comprehensive results for theories which simultaneously take into account interactions of the higher order in the Riemann tensor and involve the dilaton have been presented so far.

The main goal of the present work is to obtain the effective brane equations of motion for arbitrary order dilaton gravity models using the covariant approach. Due to the already pointed out relation of the higher order dilaton gravity model [7] to string theories, the covariant derivation of the effective brane equations in this setup will allow for studying in a more systematic way the potentially observable effects of string theories e.g. on specific cosmological models.

The rest of this paper is organized as follows: In section 2 we define the d -dimensional dilaton gravity models with corrections of the higher order in derivatives and with brane interactions whose exact nature we choose not to specify. The bulk equations of motion for these models are derived and rewritten in terms of quantities either projected on the brane, or perpendicular to it. Section

3 is devoted to the analysis of the junction conditions at the brane. The effective equations of motion at the brane for a general case with arbitrary order corrections are obtained in section 4. Construction of all possible brane equations is carefully discussed. The importance of the bulk \mathbb{Z}_2 symmetry for the form and existence of the effective equations is analyzed. In section 5 explicit results for the lowest order dilaton gravity and for the theory with up to four derivatives are presented. Some of the results for the latter, because of their complexity, are moved to the appendix. Section 6 contains our conclusions.

2 Higher dimensional equations of motion

We consider the d -dimensional dilaton gravity theory described by the following Lagrangian:

$$\mathcal{L} = e^{-\phi} \left[\sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \mathcal{T} \left(\left[\frac{1}{2} \mathcal{R}_{**}^{**} \oplus 2(\nabla\nabla)_*^* \phi \oplus (-1)(\partial\phi)^2 \right]^N \right) - V(\phi) + \mathcal{L}_B \delta_B \right]. \quad (1)$$

This is the Lagrangian (in the notation explained below) of the higher order dilaton gravity constructed in [7] and generalized by including a bulk scalar potential $V(\phi)$ and general brane interactions given by \mathcal{L}_B . The position of the brane³ is described by the Dirac delta type distribution δ_B . As was discussed in the Introduction, the main motivation for considering this dilaton gravity model is its close relation to string theories. Investigation of its properties in a brane scenario is a step towards studying e.g. the potentially observable cosmological effects of string theories.

The above Lagrangian is written in a very compact form using the following notation introduced in [7]: \mathcal{T} is a generalization of the ordinary trace. Acting on an arbitrary rank (m, m) tensor M it returns a number given by

$$\mathcal{T}(M) = \delta_{\rho_1 \rho_2 \dots \rho_m}^{\sigma_1 \sigma_2 \dots \sigma_m} M^{\rho_1 \rho_2 \dots \rho_m}_{\sigma_1 \sigma_2 \dots \sigma_m}, \quad (2)$$

where δ with m pairs of indices is the generalized Kronecker delta

$$\delta_{\rho_1 \rho_2 \dots \rho_m}^{\sigma_1 \sigma_2 \dots \sigma_m} = \det \begin{bmatrix} \delta_{\rho_1}^{\sigma_1} & \delta_{\rho_2}^{\sigma_1} & \dots & \delta_{\rho_m}^{\sigma_1} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \delta_{\rho_1}^{\sigma_m} & \delta_{\rho_2}^{\sigma_m} & \dots & \delta_{\rho_m}^{\sigma_m} \end{bmatrix}. \quad (3)$$

Asterisks are used as indices in the Lagrangian (1) to indicate ranks of tensors (e.g. to distinguish the Riemann tensor from the Ricci tensor or the curvature scalar). Under the generalized trace \mathcal{T} there are powers of “sums” (denoted by \oplus) - i.e. of linear combinations of different ranks tensors.

³The theory can include more branes, but we choose to focus our considerations on the brane for which we will be deriving the effective gravitational equations of motion.

This should be understood as a compact notation for the following operation: First, we perform the algebraic manipulations (sums and powers), treating all tensors as ordinary numbers. Then the generalized trace (2) of the obtained linear combination of tensors products should be understood as the corresponding linear combination of the generalized traces of tensor products. For example:

$$\begin{aligned} \mathcal{T} \left(\left[\frac{1}{2} \mathcal{R}_{**}^{**} \oplus 2(\nabla\nabla)_*^* \phi \right]^2 \right) &= \frac{1}{4} \mathcal{T}(\mathcal{R}_{**}^{**} \mathcal{R}_{**}^{**}) + 2 \mathcal{T}(\mathcal{R}_{**}^{**} (\nabla\nabla)_*^* \phi) + 4 \mathcal{T}((\nabla\nabla)_*^* \phi (\nabla\nabla)_*^* \phi) \\ &= \frac{1}{4} \delta_{\rho_1 \rho_2 \rho_3 \rho_4}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \mathcal{R}^{\rho_1 \rho_2}_{\sigma_1 \sigma_2} \mathcal{R}^{\rho_3 \rho_4}_{\sigma_3 \sigma_4} + 2 \delta_{\rho_1 \rho_2 \rho_3}^{\sigma_1 \sigma_2 \sigma_3} \mathcal{R}^{\rho_1 \rho_2}_{\sigma_1 \sigma_2} (\nabla^{\rho_3} \partial_{\sigma_3} \phi) + 4 \delta_{\rho_1 \rho_2}^{\sigma_1 \sigma_2} (\nabla^{\rho_1} \partial_{\sigma_1} \phi) (\nabla^{\rho_2} \partial_{\sigma_2} \phi), \end{aligned}$$

where, due to the properties of the Riemann tensor and the second covariant derivative of a scalar, we can unambiguously define: $\mathcal{R}_{\mu\nu}^{\rho\sigma} \equiv \mathcal{R}^{\rho\sigma}_{\mu\nu}$ and $(\nabla\nabla)_{\mu}^{\nu} \phi \equiv \nabla^{\nu} \partial_{\mu} \phi$.

With the above formulae it is easy to check that the number of higher order terms in the Lagrangian (1) is not arbitrary. The term in (1) with the generalized trace (2) of the highest rank tensor is proportional to

$$\mathcal{T} \left((\mathcal{R}_{**}^{**})^{N_{\max}} \right) = \delta_{\rho_1 \rho_2 \dots \rho_{(2N_{\max}-1)} \rho_{(2N_{\max})}}^{\sigma_1 \sigma_2 \dots \sigma_{(2N_{\max}-1)} \sigma_{(2N_{\max})}} \mathcal{R}^{\rho_1 \rho_2}_{\sigma_1 \sigma_2} \dots \mathcal{R}^{\rho_{(2N_{\max}-1)} \rho_{(2N_{\max})}}_{\sigma_{(2N_{\max}-1)} \sigma_{(2N_{\max})}}. \quad (4)$$

It is obvious from the definition (3) that, because of the antisymmetry in all indices of one type (covariant and contravariant), the $(2N_{\max}, 2N_{\max})$ rank generalized Kronecker delta (3) in eq. (4) is non-zero only if the number of space-time dimensions d is sufficiently large. Thus, there is an upper limit on the order of the interaction terms present in our Lagrangian: $N_{\max} \leq (d/2)$.

Using the results of [7], bulk equations of motion can be derived from the Lagrangian (1). They read

$$g_{\mu\nu} V(\phi) - \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} \bar{\mathcal{T}}_{\mu\nu} \left(\left[\frac{1}{2} \mathcal{R}_{**}^{**} \oplus 2(\nabla\nabla)_*^* \phi \oplus (-1)(\partial\phi)^2 \right]^N \right) - \tau_{\mu\nu} \delta_B = 0, \quad (5)$$

$$V(\phi) - V'(\phi) - \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} \mathcal{T} \left(\left[\frac{1}{2} \mathcal{R}_{**}^{**} \oplus 2(\nabla\nabla)_*^* \phi \oplus (-1)(\partial\phi)^2 \right]^N \right) - \tau_{\phi} \delta_B = 0, \quad (6)$$

where the brane localized terms, $\tau_{\mu\nu}$ and τ_{ϕ} , are calculated from the brane Lagrangian \mathcal{L}_B , namely

$$\tau_{\mu\nu} = h_{\mu\nu} \mathcal{L}_B - 2 \frac{\delta \mathcal{L}_B}{\delta h^{\mu\nu}}, \quad \tau_{\phi} = \mathcal{L}_B - \frac{\delta \mathcal{L}_B}{\delta \phi}. \quad (7)$$

In the tensor equation of motion (5) we introduced another generalization of the standard trace given by

$$\bar{\mathcal{T}}_{\mu}^{\nu}(M) = \delta_{\mu \rho_1 \rho_2 \dots \rho_m}^{\nu \sigma_1 \sigma_2 \dots \sigma_m} M^{\rho_1 \rho_2 \dots \rho_m}_{\sigma_1 \sigma_2 \dots \sigma_m}. \quad (8)$$

It maps arbitrary rank (m, m) tensors into rank $(1, 1)$ ones⁴.

⁴For $m = d$ the generalized Kronecker delta $\delta_{\mu \rho_1 \rho_2 \dots \rho_m}^{\nu \sigma_1 \sigma_2 \dots \sigma_m}$ should be replaced in (8) with the following combination: $\delta_{\mu}^{\nu} \delta_{\rho_1 \rho_2 \dots \rho_m}^{\sigma_1 \sigma_2 \dots \sigma_m} - \delta_{\mu}^{\sigma_1} \delta_{\rho_1 \rho_2 \dots \rho_m}^{\nu \sigma_2 \dots \sigma_m} - \dots - \delta_{\mu}^{\sigma_m} \delta_{\rho_1 \rho_2 \dots \rho_m}^{\sigma_1 \sigma_2 \dots \nu}$.

The main goal of the present work is to find the effective $(d-1)$ -dimensional equations of motion at the brane located at the support of δ_B . The first step in this direction is to identify parts “parallel” and “perpendicular” to the brane in all relevant tensors. In order to achieve this in a covariant way, we start with introducing a vector field n^μ normalized to 1 and perpendicular to the brane at its position. The choice of n^μ is not unique, due to the freedom in the bulk and the sign ambiguity at the brane. However, as we will see later, the effective brane equations of motion are unique. With any such a vector field n^μ we define the metric $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. This expression holds throughout the d -dimensional space-time but, restricted to the brane position, it just yields the metric induced on the brane. Subsequently, we divide the d -dimensional tensors into parts parallel and perpendicular to the vector field n^μ as

$$\mathcal{R}_{\mu\rho}^{\nu\sigma} = R_{\mu\rho}^{\nu\sigma} - 2K_{[\mu}^\nu K_{\rho]}^\sigma - 4n^{[\nu} D_{[\mu} K_{\rho]}^{\sigma]} - 4n_{[\mu} D^{[\nu} K_{\rho]}^{\sigma]} - 4n^{[\nu} n_{[\mu} \mathcal{L}_n K_{\rho]}^{\sigma]} + 4n^{[\nu} n_{[\mu} (KK)_{\rho]}^{\sigma]} , \quad (9)$$

$$\begin{aligned} \nabla^\nu \nabla_\mu \phi &= D^\nu D_\mu \phi + K_\mu^\nu \mathcal{L}_n \phi + n^\nu D_\mu \mathcal{L}_n \phi + n_\mu D^\nu \mathcal{L}_n \phi - n^\nu K_\mu^\sigma D_\sigma \phi - n_\mu K_\sigma^\nu D^\sigma \phi \\ &\quad + n^\nu n_\mu (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) , \end{aligned} \quad (10)$$

$$(\partial\phi)^2 = (D\phi)^2 + (\mathcal{L}_n \phi)^2 , \quad (11)$$

where $K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n h_{\mu\nu}$ is the extrinsic curvature of hypersurfaces orthogonal to n^μ , \mathcal{L}_n is the Lie derivative⁵ along n^μ , and $a^\lambda \equiv n^\rho \nabla_\rho n^\lambda$. It is important to distinguish between the d -dimensional tensors associated with the full metric $g_{\mu\nu}$, namely the Riemann tensor $\mathcal{R}_{\rho\sigma}^{\mu\nu}$ and the covariant derivative ∇_μ , and the $(d-1)$ -dimensional tensors associated with the metric $h_{\mu\nu}$ restricted to the brane position, i.e. $R_{\rho\sigma}^{\mu\nu}$ and D_μ .

The right hand sides of eqs. (9)-(11) are expressed almost entirely in terms of the vector field n^μ , Lie derivatives along n^μ and brane quantities orthogonal to n^μ , i.e. $R_{\rho\sigma}^{\mu\nu}$, $K_{\mu\nu}$ and D_μ . The term $a^\lambda \nabla_\lambda \phi = n^\rho (\nabla_\rho n^\lambda) (\nabla_\lambda \phi)$, containing the d -dimensional covariant derivatives, is the only exception. However, as we will see later, the derivation procedure for the effective gravitational equations at the brane will be constructed in such a way, that $a^\lambda \nabla_\lambda \phi$ will not appear in our final results.

The Riemann tensor $\mathcal{R}_{\mu\nu}^{\rho\sigma}$ and the tensor of the second covariant derivative of the dilaton $\nabla^\nu \nabla_\mu \phi$ appear in the bulk equations of motion (5) and (6) exclusively under the generalized traces (2) and (8), which involve full anti-symmetrization in all covariant and contravariant indices. Hence, the projection equations (9) and (10) can be simplified when used under those traces, namely

$$\mathcal{R}_{**}^{**} \rightarrow R_{**}^{**} - 2K_*^* K_*^* - 4(nn)_*^* (\mathcal{L}_n K_*^* - (KK)_*^*) - 8(nD)_*^* K_*^* , \quad (12)$$

$$(\nabla\nabla)_*^* \phi \rightarrow [(DD)_*^* \phi + K_*^* \mathcal{L}_n \phi] + (nn)_*^* (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) + 2[(nD)_*^* \mathcal{L}_n \phi - (nKD)_*^* \phi] , \quad (13)$$

⁵For a given tensor $M_{\sigma_1 \sigma_2 \dots \sigma_l}^{\rho_1 \rho_2 \dots \rho_m}$ and arbitrary direction v^μ , the Lie derivative along v^μ is defined as follows: $\mathcal{L}_v M_{\sigma_1 \sigma_2 \dots \sigma_l}^{\rho_1 \rho_2 \dots \rho_m} = v^\lambda \nabla_\lambda M_{\sigma_1 \sigma_2 \dots \sigma_l}^{\rho_1 \rho_2 \dots \rho_m} - \sum_{i=1}^m M_{\sigma_1 \sigma_2 \dots \sigma_l}^{\rho_1 \rho_2 \dots \lambda \dots \rho_m} \nabla_\lambda v^{\rho_i} + \sum_{j=1}^l M_{\sigma_1 \sigma_2 \dots \lambda \dots \sigma_l}^{\rho_1 \rho_2 \dots \rho_m} \nabla_{\sigma_j} v^\lambda$, thus e.g. $\mathcal{L}_n \phi = n^\lambda \nabla_\lambda \phi$.

where in order to make the formulae more compact we introduced the following notation: $(nn)_*^* \equiv n_* n^*$, $(\nabla\nabla)_*^* \equiv \nabla_* \nabla^*$, $(DD)_*^* \equiv D_* D^*$, $(KK)_*^* \equiv K_\lambda^* K_\lambda^*$, $(nD)_*^* \equiv \frac{1}{2} (n_* D^* + n^* D_*)$ and $(nKD)_*^* \equiv \frac{1}{2} (n_* K_\lambda^* D^\lambda + n^* K_\lambda^* D_\lambda)$.

Employing the dimensional reduction formulae (12), (13) and (11), the bulk equations of motion (5) and (6) can be rewritten as

$$g_{\mu\nu} V(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \bar{\mathcal{T}}_{\mu\nu} \left(\left[\left(\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right) \oplus (-1)(K_*^* \oplus (-1)\mathcal{L}_n \phi)^2 \right. \right. \\ \left. \left. \oplus 2(nn)_*^* ((KK)_*^* - \mathcal{L}_n K_*^*) \oplus 2(nn)_*^* (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) \right. \right. \\ \left. \left. \oplus (-4)(nD)_*^* K_*^* \oplus 4[(nD)_*^* \mathcal{L}_n \phi - (nKD)_*^* \phi] \right]^N \right) - \tau_{\mu\nu} \delta_B = 0, \quad (14)$$

$$V(\phi) - V'(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \mathcal{T} \left(\left[\left(\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right) \oplus (-1)(K_*^* \oplus (-1)\mathcal{L}_n \phi)^2 \right. \right. \\ \left. \left. \oplus 2(nn)_*^* ((KK)_*^* - \mathcal{L}_n K_*^*) \oplus 2(nn)_*^* (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) \right. \right. \\ \left. \left. \oplus (-4)(nD)_*^* K_*^* \oplus 4[(nD)_*^* \mathcal{L}_n \phi - (nKD)_*^* \phi] \right]^N \right) - \tau_\phi \delta_B = 0. \quad (15)$$

These equations remain defined and valid throughout the full d -dimensional space-time. They are smooth in the bulk, but contain discontinuous and singular (distribution-like) terms localized at the brane. The effective brane equations of motion can be obtained if the information from both smooth and singular parts of (14) and (15) is properly taken into account at the position of the brane.

3 Junction conditions

In order to find the junction conditions which have to be fulfilled at the brane, we shall define a 1-dimensional integration in the direction perpendicular to the brane. In order to achieve this, we will employ a special family of integral curves associated with the vector field n^μ . Specifically, let us define the curves $\gamma^\mu(\lambda)$ by requiring the vector field n^μ to be at each point tangent to γ^μ : $(d\gamma^\mu/d\lambda)(\lambda) = n^\mu(\gamma^\nu(\lambda))$. We choose the parameterization of these curves in such a way that each of them crosses the brane at the point $\gamma^\mu(0)$. For a scalar field $f(x^\mu)$, at each point x^μ , we consider the following integral:

$$\int_{\Gamma_x^\mu(\lambda_1, \lambda_2)} f(x^\mu) = \int_{\lambda_1}^{\lambda_2} f(\gamma_x^\mu(\lambda)) d\lambda, \quad (16)$$

where $\Gamma_x^\mu(\lambda_1, \lambda_2) = \{\gamma_x^\mu(\lambda) | \lambda \in [\lambda_1, \lambda_2]\}$ is a part of the curve $\gamma_x^\mu(\lambda)$ crossing the brane at the point x^μ . This integration is “inverse” to the Lie derivative⁶ in the following sense:

$$\int_{\Gamma_x^\mu(\lambda_1, \lambda_2)} \mathcal{L}_n f(x^\mu) = f(\gamma_x^\mu(\lambda_2)) - f(\gamma_x^\mu(\lambda_1)). \quad (17)$$

This integral can be generalized to the case of arbitrary tensor integrands by applying appropriate pullback and pushforward operations along the family of curves $\gamma^\mu(\lambda)$.

Junction conditions for a given point x_0^μ on the brane are obtained by integrating the d -dimensional equations of motion along (an infinitesimal part of) the curve $\gamma_{x_0}^\mu(\lambda)$ going through x_0^μ :

$$\int_{\perp B} F(x_0^\mu) \equiv \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{x_0}^\mu(-\epsilon, +\epsilon)} F(x_0^\mu), \quad (18)$$

where $F(x^\mu)$ is the scalar or the tensor given by the left hand sides of the bulk equations of motion (14) and (15), respectively⁷.

Only certain terms in the d -dimensional equations of motions will not yield zero during the above infinitesimal across-the-brane integration, namely the brane contributions involving $\tau_{\mu\nu}$ and τ_ϕ , explicitly proportional to δ_B , and the terms containing Lie derivatives (i.e. $\mathcal{L}_n^2 \phi$ and $\mathcal{L}_n K_{\mu\nu}$) of quantities which can be discontinuous at the brane ($\mathcal{L}_n \phi$ and $K_{\mu\nu}$, respectively). Hence, the relevant parts of the bulk equations of motions (14) and (15) yield

$$\sum_{N=1}^{N_{max}} \alpha_N N \int_{\perp B} \bar{\mathcal{T}}_{\mu\nu} \left((nn)_*^* \mathcal{L}_n (K_*^* \oplus (-1) \mathcal{L}_n \phi) \cdot \left[\left(\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right) \oplus (-1)(K_*^* \oplus (-1) \mathcal{L}_n \phi)^2 \right]^{N-1} \right) = \tau_{\mu\nu}, \quad (19)$$

$$\sum_{N=1}^{N_{max}} \alpha_N N \int_{\perp B} \mathcal{T} \left(\mathcal{L}_n (K_*^* \oplus (-1) \mathcal{L}_n \phi) \cdot \left[\left(\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right) \oplus (-1)(K_*^* \oplus (-1) \mathcal{L}_n \phi)^2 \right]^{N-1} \right) = \tau_\phi. \quad (20)$$

The integration of the terms containing the brane localized sources $\tau_{\mu\nu}$ and τ_ϕ was easy due to the obvious (defining) properties of the Dirac delta distribution δ_B . The remaining integrations require a more careful treatment. The integrands involve products of the distribution-like objects, $\mathcal{L}_n K_{\mu\nu}$ or $\mathcal{L}_n^2 \phi$, and the potentially discontinuous objects, $K_{\mu\nu}$ or $\mathcal{L}_n \phi$. Strictly speaking, such integrals

⁶From now on whenever we refer to a Lie derivative, it should be understood as the Lie derivative along the vector field n^μ .

⁷Integral (18) is well defined for any point of the space-time. However, away from the brane the integrands are smooth and the integral vanishes in the $\epsilon \rightarrow 0$ limit. Non-trivial results are obtained only on the brane, where there are localized sources $\tau_{\mu\nu}$ and τ_ϕ .

have no mathematically unambiguous meaning, as distributions are defined by their integrals with smooth functions. Therefore, we need some kind of regularization to deal with such terms. Fortunately, there is an obvious way to regularize and calculate the integrals in eqs. (19) and (20). Let us first consider terms of the form $sf^k \mathcal{L}_n f$, where s is smooth, whereas f may be discontinuous at the brane. Employing the formula (17) and the Leibniz rule for the Lie derivative we get

$$\int_{\perp B} sf^k \mathcal{L}_n f = \frac{1}{k+1} \int_{\perp B} s \mathcal{L}_n (f^{k+1}) = \frac{1}{k+1} \int_{\perp B} \mathcal{L}_n (sf^{k+1}) = \frac{1}{k+1} [sf^{k+1}]_{\pm}. \quad (21)$$

The second equality follows from the fact that the integral of $(\mathcal{L}_n s)f^{k+1}$ vanishes in the $\epsilon \rightarrow 0$ limit. The square bracket with the \pm subscript was introduced to denote a jump in the value of some quantity when crossing the brane:

$$[f(x_0^\mu)]_{\pm} = [f(x_0^\mu)]_+ - [f(x_0^\mu)]_-, \quad (22)$$

while the square brackets with the subscripts $+$ and $-$ denote the limits of a given bulk quantity when approaching the brane from the “ $+$ ” and the “ $-$ ” sides, respectively:

$$[f(x_0^\mu)]_{+(-)} = \lim_{\epsilon \rightarrow 0^{+(-)}} f(\gamma_{x_0}^\mu(\epsilon)). \quad (23)$$

We intend to apply the regularization (21) to the integrals in eqs. (19) and (20). It is instructive to start from considering a simple example of such a calculation:

$$\begin{aligned} & \int_{\perp B} \mathcal{T} \left([K_*^* \oplus (-1)\mathcal{L}_n \phi] \mathcal{L}_n [K_*^* \oplus (-1)\mathcal{L}_n \phi] \right) \\ &= \int_{\perp B} \left[\delta_{\mu\rho}^{\nu\sigma} K_\nu^\mu \mathcal{L}_n K_\sigma^\rho - \delta_\mu^\nu (\mathcal{L}_n K_\nu^\mu \mathcal{L}_n \phi + K_\nu^\mu \mathcal{L}_n^2 \phi) + \mathcal{L}_n^2 \phi \mathcal{L}_n \phi \right] \\ &= \frac{1}{2} \delta_{\mu\rho}^{\nu\sigma} \int_{\perp B} \mathcal{L}_n (K_\nu^\mu K_\sigma^\rho) - \delta_\mu^\nu \int_{\perp B} \mathcal{L}_n (K_\nu^\mu \mathcal{L}_n \phi) + \frac{1}{2} \int_{\perp B} \mathcal{L}_n (\mathcal{L}_n \phi)^2 \\ &= \frac{1}{2} \delta_{\mu\rho}^{\nu\sigma} [K_\nu^\mu K_\sigma^\rho]_{\pm} - \delta_\mu^\nu [K_\nu^\mu \mathcal{L}_n \phi]_{\pm} + \frac{1}{2} [(\mathcal{L}_n \phi)^2]_{\pm} = \left[\mathcal{T} \left(\frac{1}{2} \{K_*^* \oplus (-1)\mathcal{L}_n \phi\}^2 \right) \right]_{\pm}. \end{aligned}$$

The same result is obtained if the regularization (21) is used when treating the formal sums of tensors of different ranks, under the generalized traces \mathcal{T} and $\overline{\mathcal{T}}_{\mu\nu}$, as ordinary functions. Performing similar calculations for all terms present in (19) and (20), we obtain the junction conditions as

$$\sum_{N=1}^{N_{max}} \alpha_N N \sum_{k=0}^{N-1} \frac{(-1)^k (N-1)!}{(2k+1)k!(N-1-k)!} \cdot \left[\overline{\mathcal{T}}_{\mu\nu} \left((nn)_*^* \left[\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right]^{N-1-k} \left(K_*^* \oplus (-1)\mathcal{L}_n \phi \right)^{2k+1} \right) \right]_{\pm} = \tau_{\mu\nu}, \quad (24)$$

$$\sum_{N=1}^{N_{max}} \alpha_N N \sum_{k=0}^{N-1} \frac{(-1)^k (N-1)!}{(2k+1)k!(N-1-k)!} \cdot \left[\mathcal{T} \left(\left[\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right]^{N-1-k} \left(K_*^* \oplus (-1)\mathcal{L}_n \phi \right)^{2k+1} \right) \right]_{\pm} = \tau_\phi. \quad (25)$$

These formulae can be rewritten in a slightly (but qualitatively) different form in the case of theories with the bulk \mathbb{Z}_2 symmetry and the brane located at the \mathbb{Z}_2 symmetry fixed point. Then, for any \mathbb{Z}_2 -odd quantity f , $[f]_- = -[f]_+$ and $[f]_{\pm}$ can be replaced with $2[f]_+$.

A comment on distinguishing both sides of the brane is in order here. We started our construction with the vector field n^μ , which at the brane is normal to it. The vector field $-n^\mu$ has the same property. Replacing n^μ with $-n^\mu$ (which corresponds to interchanging the “+” and the “-” sides of the brane) we have to change the sign of λ in the family of curves $\gamma_x^\mu(\lambda)$ used in (18) to define the integral $\int_{\perp B}$. From (22) it follows that this in turn changes the sign of $[\dots]_{\pm}$. The vector field n^μ enters linearly also the definitions of the Lie derivative \mathcal{L}_n and the extrinsic curvature $K_{\mu\nu}$. Hence, the combination $(K_*^* \oplus (-1)\mathcal{L}_n\phi)$ changes sign together with n^μ . Observe that only odd powers of this combination appear in the junction conditions (24) and (25). Thus, the change of sign in the definition (22) of $[\dots]_{\pm}$ is compensated by the change of sign of $\mathcal{L}_n\phi$ and $K_{\mu\nu}$. The left hand sides of (24) and (25) do not depend on the sign of n^μ - as it should be, as the brane localized interactions (7) appearing on the right hand sides of these equations do not depend on which side of the brane is called the “+” one.

The junction conditions (24) and (25) determine the jumps in the values of the extrinsic curvature, $K_{\mu\nu}$, and the Lie derivative of the scalar field, $\mathcal{L}_n\phi$, caused by the brane localized interactions described by $\tau_{\mu\nu}$ and τ_ϕ . Unfortunately, the solutions for $[K_{\mu\nu}]_{\pm}$ and $[\mathcal{L}_n\phi]_{\pm}$ (or $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$ in the case of \mathbb{Z}_2 -symmetric models) can be found explicitly only in some simple cases, for example when $N_{\max} = 1$ or when we are considering solutions which are highly symmetric.

4 Effective brane equations of motion

4.1 Definitions

In the previous section we found the junction conditions (24) and (25) which have to be fulfilled at the brane. Presently we would like to obtain $(d-1)$ -dimensional equations which we could call the “effective brane equations of motion”. First of all, a precise definition of such effective equations of motion at the brane has to be given. There are two obvious properties of these equations: First, they must follow from the full d -dimensional equations of motion (14) and (15). Second, they should describe the behavior of the quantities defined exactly on the brane, or infinitesimally close to it. We can obtain such equations in a very similar way to that employed in the junction conditions derivation. Specifically, for each point x_0^μ on the brane, we integrate the full d -dimensional equations of motion (14) and (15) over some infinitesimal interval “perpendicular” to the brane. However, the integration (16) has to be generalized in order to obtain something different than just the junction conditions (24) and (25). An obvious way to generalize the integral (16) is to use a weight function

$w(\lambda)$:

$$\int_{\lambda_1}^{\lambda_2} f(\gamma_x^\mu(\lambda)) w(\lambda) d\lambda, \quad (26)$$

where $\gamma_{x_0}^\mu(\lambda)$ is again the integral curve of the vector field n^μ intersecting the brane at the point x_0^μ . In fact, we need a family of weight functions $w_\epsilon(\lambda)$ such that the support of $w_\epsilon(\lambda)$ is included in the interval $(-\epsilon, +\epsilon)$. Any such a smooth function can be written as the following sum:

$$w_\epsilon(\lambda) = w_\epsilon^{(0)} + w_\epsilon^{(-)}(\lambda) + w_\epsilon^{(+)}(\lambda), \quad (27)$$

where $w_\epsilon^{(0)}$ is a constant (finite in the $\epsilon \rightarrow 0$ limit), $w_\epsilon^{(-)}(\lambda)$ has the support in $(-\epsilon, 0)$ and $w_\epsilon^{(+)}(\lambda)$ has the support in $(0, +\epsilon)$. The brane equations of motion are then obtained by integrating the d -dimensional equations of motion (14), (15) like in (26) and taking the limit of $\epsilon \rightarrow 0$, namely

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_{x_0}^\mu(-\epsilon, +\epsilon)} F(\gamma_{x_0}^\mu(\lambda)) w_\epsilon(\lambda) d\lambda = c_0 \int_{\perp B} F(x_0^\mu) + c_- \lim_{\epsilon \rightarrow 0^-} F(x_0^\mu + \epsilon n^\mu) + c_+ \lim_{\epsilon \rightarrow 0^+} F(x_0^\mu + \epsilon n^\mu), \quad (28)$$

where the coefficients c depend on the chosen w_ϵ :

$$c_0 = \lim_{\epsilon \rightarrow 0} w_\epsilon^{(0)}, \quad c_- = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^0 w_\epsilon^{(-)}(\lambda) d\lambda, \quad c_+ = \lim_{\epsilon \rightarrow 0} \int_0^{+\epsilon} w_\epsilon^{(+)}(\lambda) d\lambda. \quad (29)$$

By adopting different w_ϵ we obtain three independent $(d - 1)$ -dimensional equations. The first term on the r.h.s. of (28) is proportional to the corresponding junction condition discussed already in the previous section. The remaining two terms are proportional to two directional limits of the bulk equations (i.e. the limits of the bulk equations when approaching the brane from the “+” or the “-” side).

None of these equations can be called an effective brane equation of motion. The directional limits of the bulk equations have no explicit dependence on the brane localized quantities (7). The junction conditions do depend on the brane localized quantities, like the brane energy momentum tensor $\tau_{\mu\nu}$, but they determine the jumps in the bulk quantities values and not the dynamics of the brane quantities. However, one should not draw the conclusion that it is not possible to define any effective brane equations of motion, as there are more d -dimensional bulk equations than are needed for a $(d - 1)$ -dimensional brane gravity. In addition, one should remember that the metric tensor and the scalar field are continuous at the brane. The effective equations of motion at the brane are obtained by combining all available information: the junction conditions, the directional limits of all bulk equations and the continuity conditions. This will be explained in more detail in the next subsection - employing our model with the bulk equations of motion (14) and (15).

4.2 Construction – general case

We start with projecting the bulk tensor equation of motion (14) on the brane hypersurface and/or on the normal vector field n^μ . Multiplying (14) by $h_\rho^\mu h_\sigma^\nu$, using the fact that n^μ is orthogonal to all

brane directions, and taking appropriate limits of the bulk fields, we get the following two equations (one for each side of the brane):

$$\left[h_{\rho\sigma} V(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu} \left(\mathcal{M}^N \oplus 2N \left[((KK)_*^* - \mathcal{L}_n K_*^*) \oplus (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) \right] \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right) \right]_{-(+)} = 0, \quad (30)$$

where we introduced symbols \mathcal{M} and \mathcal{N} to denote the following formal combinations of the tensors of different ranks:

$$\mathcal{M} = \frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \oplus (-1)(K_*^* \oplus (-1)\mathcal{L}_n \phi)^2, \quad (31)$$

$$\mathcal{N} = [D_* K_*^* \oplus (K_*^\lambda D_\lambda \phi - D_* \mathcal{L}_n \phi)] [D^* K_*^* \oplus (K_\lambda^* D^\lambda \phi - D^* \mathcal{L}_n \phi)], \quad (32)$$

which will appear under the generalized traces \mathcal{T} and $\bar{\mathcal{T}}_{\mu\nu}$ in numerous formulae. The index structure of \mathcal{N} is different than that of all other objects appearing under \mathcal{T} and $\bar{\mathcal{T}}_{\mu\nu}$. So far all such objects had natural pairs (covariant-contravariant) of indices - corresponding to the pairs of indices present in the generalized Kronecker delta (3). In \mathcal{N} there is one unpaired covariant index in the first square bracket in (32) and one unpaired contravariant index in the second square bracket. It should be understood that these two indices form a pair under \mathcal{T} and $\bar{\mathcal{T}}_{\mu\nu}$. Thus, we have for example

$$\mathcal{T}(\mathcal{N}) = \delta_{\rho_1 \rho_2 \rho_3}^{\sigma_1 \sigma_2 \sigma_3} (D_{\sigma_1} K_{\sigma_2}^{\rho_2}) (D^{\rho_1} K_{\sigma_3}^{\rho_3}) + \delta_{\rho_1 \rho_2}^{\sigma_1 \sigma_2} (D_{\sigma_1} K_{\sigma_2}^{\rho_2}) (K_\lambda^{\rho_1} D^\lambda \phi - D^{\rho_1} \mathcal{L}_n \phi) + \dots$$

The bulk dilaton equation of motion (15) is very similar to the projection (30) of the bulk tensor equation of motion on the brane (i.e. obtained by contracting with $h_{\rho}^{\mu} h_{\sigma}^{\nu}$) and reads

$$\left[V(\phi) - V'(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \mathcal{T} \left(\mathcal{M}^N \oplus 2N \left[((KK)_*^* - \mathcal{L}_n K_*^*) \oplus (\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi) \right] \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right) \right]_{-(+)} = 0. \quad (33)$$

The contractions of the bulk tensor equation of motion (14) with $h_{\rho}^{\mu} n^{\nu}$ and $n^{\mu} n^{\nu}$ are given by, respectively,

$$\left[\sum_{N=1}^{N_{max}} \alpha_N \bar{\mathcal{T}}_{\rho\sigma} \left(N \left[D^{\sigma} K_*^* \oplus (K_\lambda^{\sigma} D^\lambda \phi - D^{\sigma} \mathcal{L}_n \phi) \right] \mathcal{M}^{N-1} \right) \right]_{-(+)} = 0, \quad (34)$$

$$\left[V(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \mathcal{T}(\mathcal{M}^N) \right]_{-(+)} = 0. \quad (35)$$

Before discussing the properties of the above equations of motion, let us rewrite the tensor equation (30) in a somewhat different form. One of the reasons is that we would like to remove from our equations all terms containing the quantity $a^\lambda \nabla_\lambda \phi$, which is the only term being neither “parallel”, nor “perpendicular” to the brane. Another reason is that we would like to rewrite the tensor equation (30) in a form more suitable to compare our results with those presented so far in the literature. In order to achieve these goals, we conduct the following procedure: Using the decomposition (9) and the definition of the d -dimensional Weyl tensor $\mathcal{C}_{\mu\nu\rho\sigma}$:

$$\mathcal{C}_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} - \frac{2}{d-2} \left(g_{\mu[\rho} \mathcal{R}_{\sigma]\nu} - g_{\nu[\rho} \mathcal{R}_{\sigma]\mu} \right) + \frac{2}{(d-1)(d-2)} g_{\mu[\rho} g_{\sigma]\nu} \mathcal{R}, \quad (36)$$

we obtain the equation

$$\begin{aligned} \mathcal{L}_n K_{\mu\nu} - (KK)_{\mu\nu} &= \frac{1}{d-1} h_{\mu\nu} [(h\mathcal{L}_n K) - (KK)] - \frac{1}{d-3} [R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}] \\ &\quad + \frac{1}{(d-1)(d-3)} h_{\mu\nu} [R - K^2 + (KK)] - \frac{d-2}{d-3} E_{\mu\nu}, \end{aligned} \quad (37)$$

expressing the Lie derivative of the extrinsic curvature, $\mathcal{L}_n K_{\mu\nu}$, as a function of its trace, $(h\mathcal{L}_n K) \equiv h^{\mu\nu} \mathcal{L}_n K_{\mu\nu}$, and the following projection of the d -dimensional Weyl tensor:

$$E_{\mu\nu} = n^\alpha h_\mu^\beta n^\gamma h_\nu^\delta \mathcal{C}_{\alpha\beta\gamma\delta}. \quad (38)$$

After using eq. (37), the tensor equation (30) depends (explicitly) only on two second Lie derivatives of the bulk fields, $(h\mathcal{L}_n K)$ and $\mathcal{L}_n^2 \phi$, which appear always in the combinations $[(h\mathcal{L}_n K) - (KK)]$ and $[\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi]$. These two combinations can be calculated from a system of two linear algebraic equations consisting of the scalar equation (33) and the trace of the tensor equation (30) - both evaluated “next to the brane”. The solution of this system of equations reads

$$(h\mathcal{L}_n K) - (KK) = (d-1) \frac{b_0 B_1 - b_1 B_0}{b_0 b_2 - b_1^2}, \quad (39)$$

$$\mathcal{L}_n^2 \phi - a^\lambda \nabla_\lambda \phi = \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2}, \quad (40)$$

where

$$b_m = \sum_{N=1}^{N_{\max}} \alpha_N N \mathcal{T} \left((h_*^*)^m \mathcal{M}^{N-1} \right), \quad (41)$$

$$\begin{aligned} B_m &= \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} \mathcal{T} \left((h_*^*)^m \left[\mathcal{M}^N \oplus \frac{2N}{d-3} \mathcal{P} \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right] \right) \\ &\quad - (d-1)^m V(\phi) + (1-m) V'(\phi), \end{aligned} \quad (42)$$

$$\mathcal{P} = R_*^* - KK_*^* + (KK)_*^* - \frac{1}{d-1} h_*^* [R - K^2 + (KK)] + (d-2) E_*^*, \quad (43)$$

whereas \mathcal{M} and \mathcal{N} are defined in (31) and (32), respectively. Substituting eqs. (37), (39) and (40) into the projected tensor equation (30), we obtain a tensor equation of motion without explicit dependence on the second Lie derivatives of the bulk fields, namely

$$\left[h_{\rho\sigma} V - \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu} \left(\mathcal{M}^N \oplus \frac{2N}{d-3} \mathcal{P} \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right. \right. \\ \left. \left. \oplus 2N \left[h_{*} \frac{b_1 B_0 - b_0 B_1}{b_0 b_2 - b_1^2} \oplus \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2} \right] \mathcal{M}^{N-1} \right) \right]_{-(+)} = 0. \quad (44)$$

It is useful to reformulate this equation further in order to rewrite it as a sum of the ordinary lowest order Einstein equation and some corrections. The first step is very simple, we just have to multiply (44) by $(d-3)/(d-2)$ in order to get the usual normalization of the Ricci tensor. A second step is necessary, as the ratio of the coefficients in front of $h_{\rho\sigma} R$ and $R_{\rho\sigma}$ is different than the desired $-1/2$. To solve this problem we have to add a product of eq. (35) and the brane metric tensor $h_{\rho\sigma}$ with the appropriate coefficient. Finally, we obtain the following Einstein-like tensor equation of motion:

$$\left[\frac{(d-3)(2d-3)}{(d-1)(d-2)} h_{\rho\sigma} V - \frac{d-3}{d-2} \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu} \left(\mathcal{M}^N \oplus \frac{2N}{d-3} \mathcal{P} \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right. \right. \\ \left. \left. \oplus 2N \left[h_{*} \frac{b_1 B_0 - b_0 B_1}{b_0 b_2 - b_1^2} \oplus \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2} \right] \mathcal{M}^{N-1} \right) - \frac{d-3}{d-1} \sum_{N=1}^{N_{\max}} \frac{\alpha_N}{2} h_{\rho\sigma} \mathcal{T}(\mathcal{M}^N) \right]_{-(+)} = 0. \quad (45)$$

The above equation seems to be rather complicated, but, in fact, after calculating the generalized traces (2) and (8) it takes the most Einstein-like form we could obtain, as we shall show explicitly for $N_{\max} = 1, 2$. Moreover, it does not depend on the Lie derivative of the extrinsic curvature, $\mathcal{L}_n K_{\mu\nu}$, and on the second Lie derivative of the dilaton, $\mathcal{L}_n^2 \phi$. The whole dependence on these quantities, which cannot be restricted by the junction conditions, is encoded in the projection (38) of the bulk Weyl tensor on the brane, $E_{\mu\nu}$. This dependence is implicit via the quantities B_m and \mathcal{P} defined in (42) and (43), respectively. Due to the explicit bulk contribution, represented by $E_{\mu\nu}$, the Einstein-like equation of motion (45) does not form a closed system. Consequently, to describe fully the brane dynamics, the bulk solution usually would have to be known.

All the equations we wrote down so far in this subsection are directional limits of the bulk equations of motion (or their combinations). In order to get any effective brane equations of motion we need some dependence on the brane sources (7). Such a dependence can be introduced by taking into account the junction conditions, which determine the jumps in the values of $K_{\mu\nu}$ and $\mathcal{L}_n \phi$, i.e. $[K_{\mu\nu}]_{\pm}$ and $[\mathcal{L}_n \phi]_{\pm}$, in terms of the sources $\tau_{\mu\nu}$ and τ_{ϕ} . Let us now discuss what equations are obtained after substituting the solutions (not always known in a closed analytic form) of the

junction conditions (24) and (25) into the bulk equations (30), (33), (34) and (35)⁸.

The first obvious observation is that eqs. (30) and (33) give no useful effective brane equations of motion, as they involve second derivatives of the brane metric and the dilaton field along the vector field n^μ normal to the brane, i.e. $\mathcal{L}_n K_{\mu\nu}$ and $\mathcal{L}_n^2 \phi$, respectively. To show this more explicitly we consider the following construction: We choose some brane sources $\tau_{\mu\nu}$ and τ_ϕ , together with a metric $h_{\mu\nu}$ and a scalar field ϕ on the brane. More precisely, we choose $[h_{\mu\nu}]_+$ and $[\phi]_+$ (which is the same as $[h_{\mu\nu}]_-$ and $[\phi]_-$, as the fields are continuous) at one of the two hypersurfaces infinitesimally close to the brane. Due to the junction conditions, the brane sources $\tau_{\mu\nu}$ and τ_ϕ restrict (or in some cases determine) the first derivatives in the direction perpendicular to the brane, i.e. $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n \phi]_+$ (or $[K_{\mu\nu}]_-$ and $[\mathcal{L}_n \phi]_-$) at the chosen hypersurface “next to the brane”. The values of these fields (i.e. $[h_{\mu\nu}]_+$ and $[\phi]_+$) and their first derivatives constitute simply the boundary conditions for the quasi-linear second order differential eqs. (30) and (33). The Cauchy-Kowalewski theorem tells us that such boundary conditions problem can be solved at least in some neighborhood of the brane. The existence of the solutions does not require any relations between the brane fields and the brane sources. Without any additional assumptions (e.g. about the symmetries of the bulk/brane solutions) the eqs. (30) and (33) and the junction conditions (24) and (25) do not yield any effective brane equations of motion, as they are not sufficient to get any constraints on the brane fields $h_{\mu\nu}$ and ϕ in terms of the brane sources $\tau_{\mu\nu}$ and τ_ϕ .

From the above reasoning it is clear that in order to get any effective $(d - 1)$ -dimensional brane equations we need bulk equations of motion which do not involve the second derivatives perpendicular to the brane. Such equations, obtained from the d -dimensional tensor equation of motion by contracting at least one of its indices with the vector field n^μ normal to the brane, are given by (34) and (35). Let us start with the latter. The “+” side eq. (35) is just a consistency condition on the “+” side first Lie derivatives of the brane fields, $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n \phi]_+$. Using the relations

$$[K_{\mu\nu}]_- = [K_{\mu\nu}]_+ - [K_{\mu\nu}]_\pm, \quad [\mathcal{L}_n \phi]_- = [\mathcal{L}_n \phi]_+ - [\mathcal{L}_n \phi]_\pm,$$

the “-” side eq. (35) becomes another consistency condition on $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n \phi]_+$, this time depending on the brane sources $\tau_{\mu\nu}$ and τ_ϕ . It can be seen that in general, even in the simplest $N_{\max} = 1$ case, these two consistency conditions obtained from eq. (35) can be fulfilled, thus eq. (35) does not lead to any effective brane equations of motion. The situation may change if some assumptions about the bulk are made, e.g. if the bulk \mathbb{Z}_2 symmetry, relating $[K_{\mu\nu}]_+$ with $[K_{\mu\nu}]_-$, and thus with $[K_{\mu\nu}]_\pm$ and the brane sources $\tau_{\mu\nu}$ and τ_ϕ , is assumed. Such models will be discussed in the next subsection.

We have shown that in general eqs. (30), (33) and (35) cannot be used to obtain any effective

⁸For this discussion it is better to use eqs. (30), (33) and (35), than the derived from them eqs. (44) and (45).

brane equations. What remains to discuss now, are two eqs. (34) - one for each side of the brane. They were obtained from the full d -dimensional tensor equation of motion (14) by contracting one of the indices with n^μ and the other one with $h_{\mu\nu}$. The “+” side eq. (34) gives $(d - 1)$ additional consistency conditions on $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$. Together with those 2 resulting from eq. (35), we have $(d + 1)$ such conditions. Hence, the number of the conditions is smaller than the number of the (gauge independent) degrees of freedom in $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$, i.e. these conditions can be in general fulfilled.

The situation is more interesting for the “-” side eq. (34). One could try to use the same argument as for the “-” side eq. (35) and expect to get additional consistency conditions for $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$. However, this time that argument does not work. The reason is that the difference between the “+” and the “-” sides projections (34) is closely related to the junction conditions (24) and (25). After a somewhat tedious calculation, it can be shown that such a difference is equivalent to the following condition on the brane sources:

$$D^\mu\tau_{\mu\nu} + D^\mu\phi(h_{\mu\nu}\tau_\phi - \tau_{\mu\nu}) = 0. \quad (46)$$

This condition has the same simple form for all theories of the structure described by the Lagrangian (1) of arbitrary higher order in derivatives. It does relate the brane sources (7) to the brane metric (the covariant derivative on the l.h.s. is covariant with respect to the induced brane metric $h_{\mu\nu}$). However, its character depends crucially on the brane localized contribution \mathcal{L}_B to the Lagrangian (1). For a wide class of \mathcal{L}_B eq. (46) is not a dynamical brane equation of motion, because it does not involve the second derivatives (in the brane directions) of the brane fields. It is rather a consistency condition on the source terms⁹ $\tau_{\mu\nu}$ and τ_ϕ . Such consistency conditions are typical of gravity theories [19].

The situation changes for more complicated brane Lagrangians \mathcal{L}_B . Specifically, if \mathcal{L}_B contains e.g. localized kinetic terms for the gravity [20] or for the scalar [21], the condition (46) becomes a dynamical brane equation of motion involving the second derivatives (in the brane directions) of the brane fields $h_{\mu\nu}$ and ϕ . Certainly even in such a case we do not get a full system of the brane gravitational equations of motion. The reason is obvious: the condition (46) is a $(d - 1)$ -dimensional vector of equations, while the brane gravitational equations should have a $(d - 1)$ -dimensional tensor character.

The condition (46) seems to be of the lowest order, as all the higher order terms present in the bulk Lagrangian (1) “canceled-out” in its derivation when we combined the higher order junction conditions (24) and (25) with the higher order bulk equation (34). However, the condition (46) may

⁹In the case without the dilaton, i.e. in the case of the Einstein-Lovelock theory of gravity, eq. (46) reduces to the requirement that the brane tensor source $\tau_{\mu\nu}$ must be covariantly conserved.

involve terms with more than two derivatives of the fields if appropriate interactions are present in the brane Lagrangian \mathcal{L}_B .

Let us summarize the general case without \mathbb{Z}_2 or any other symmetry imposed on the bulk Lagrangian (1) or the bulk background solution. We have the following system of equations:

- (i) Junction conditions (24) and (25), which determine (not always explicitly) the jumps in the values of the extrinsic curvature and the Lie derivative of the dilaton: $[K_{\mu\nu}]_{\pm}$ and $[\mathcal{L}_n\phi]_{\pm}$.
- (ii) System of equations for $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$ given by the “+” side eq. (34) together with both eqs. (35). The number of these equations is smaller than the number of (gauge-independent) degrees of freedom in $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$, so that system of equations in general has solutions.
- (iii) Consistency condition (46) on the brane sources $\tau_{\mu\nu}$ and τ_ϕ .

The first two systems of equations, (i) and (ii), are not restrictive enough to obtain any effective brane equations of motion relating the dynamics of the brane fields to the brane sources (7). One can choose any brane fields, i.e. induced metric $h_{\mu\nu}$ and dilaton ϕ , and any brane sources, $\tau_{\mu\nu}$ and τ_ϕ , satisfying the consistency condition (iii). In general, there exist values of $[K_{\mu\nu}]_{-(+)}$ and $[\mathcal{L}_n\phi]_{-(+)}$ which fulfill eqs. (i) and (ii) for any such choice. On the other hand, the consistency condition (iii) may have a character of a dynamical brane equation of motion. This depends crucially on the form of the brane localized interactions described by \mathcal{L}_B .

4.3 Construction with \mathbb{Z}_2 symmetry

The bulk \mathbb{Z}_2 symmetry is employed in many papers on brane models. We will show now that such a symmetry not only simplifies the calculations, but can also change qualitatively the problem of the existence of the effective brane equations of motion.

The bulk \mathbb{Z}_2 symmetry relates the fields at the “+” and the “−” sides of the brane. We have e.g.

$$[K_{\mu\nu}]_+ = -[K_{\mu\nu}]_- = \frac{1}{2} [K_{\mu\nu}]_{\pm}, \quad [\mathcal{L}_n\phi]_+ = -[\mathcal{L}_n\phi]_- = \frac{1}{2} [\mathcal{L}_n\phi]_{\pm}. \quad (47)$$

In such a case, the “−” side eqs. (30), (33)-(35) coincide with the “+” side ones. Moreover, the junction conditions (24) and (25) determine the extrinsic curvature $K_{\mu\nu}$ and the Lie derivative of the dilaton $\mathcal{L}_n\phi$ on both sides of the brane. The junction conditions are given by equations of order $(2N_{\max} - 1)$, so in general they cannot be solved explicitly for models of the higher order in derivatives. They can be solved analytically in some cases of highly symmetric configurations. Otherwise we have to solve them numerically. Substituting the (explicit or not) solutions of the junction conditions, $[K_{\mu\nu}]_+(\tau_{\mu\nu}, \tau_\phi)$ and $[\mathcal{L}_n\phi]_+(\tau_{\mu\nu}, \tau_\phi)$, into eqs. (35) we obtain the following

effective brane equation of motion:

$$V(\phi) - \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} \mathcal{T} \left(\left[\frac{1}{2} R_{**}^{**} \oplus 2(DD)_*^* \phi \oplus (-1)(D\phi)^2 \right. \right. \\ \left. \left. \oplus (-1) \left([K_*^*]_+(\tau_{\mu\nu}, \tau_\phi) \oplus (-1)[\mathcal{L}_n \phi]_+(\tau_{\mu\nu}, \tau_\phi) \right)^2 \right]^N \right) = 0. \quad (48)$$

This is the first equation we found which has the character of an effective brane equation of motion even for simple brane Lagrangians \mathcal{L}_B . It involves the brane sources (7), as well as the second derivatives (along the brane directions) of the brane metric, $R_{\mu\nu}^{\rho\sigma}$, and the dilaton, i.e. $D^\nu D_\mu \phi$ and $(D\phi)^2$. The covariant derivatives of the dilaton can be eliminated using the condition (46). This way we obtain one equation relating the dynamics of the brane metric $h_{\mu\nu}$ to the brane sources $\tau_{\mu\nu}$ and τ_ϕ . It is the only such equation which appears in our theory - if we assume nothing else but the \mathbb{Z}_2 symmetry in the bulk.

One effective equation of motion in a $(d-1)$ -dimensional brane gravity is not much. The induced brane metric $h_{\mu\nu}$ has $d(d-1)/2$ gauge independent degrees of freedom. Nevertheless, even one equation of motion can be very important if we restrict our attention to highly symmetric brane solutions. For example, any maximally symmetric space-time (like de Sitter space-time used to describe inflation) is fully determined by just one parameter - the curvature scalar. Similarly, the cosmologically important Friedmann-Robertson-Walker space-time depends on one function of time only - the cosmic scale factor, for each sign of the brane spatial curvature.

Despite the fact that there is only one “true” effective brane equation (48), it is sometimes convenient to write down the full tensor Einstein-like equation of motion for the brane fields $h_{\mu\nu}$ and ϕ . It follows straightforwardly from eq. (45) and is of the same structure, namely

$$\frac{(d-3)(2d-3)}{(d-1)(d-2)} h_{\rho\sigma} V - \frac{d-3}{d-2} \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} h_\rho^\mu h_\sigma^\nu \bar{T}_{\mu\nu} \left(\mathcal{M}^N \oplus \frac{2N}{d-3} \mathcal{P} \mathcal{M}^{N-1} \oplus 8N(1-N) \mathcal{N} \mathcal{M}^{N-2} \right. \\ \left. \oplus 2N \left[h_*^* \frac{b_1 B_0 - b_0 B_1}{b_0 b_2 - b_1^2} \oplus \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2} \right] \mathcal{M}^{N-1} \right) - \frac{d-3}{d-1} \sum_{N=1}^{N_{max}} \frac{\alpha_N}{2} h_{\rho\sigma} \mathcal{T}(\mathcal{M}^N) = 0, \quad (49)$$

with the metric $h_{\mu\nu}$ and the dilaton ϕ taken on the brane and all (implicit) terms involving $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n \phi]_+$ (with the bulk \mathbb{Z}_2 symmetry it does not matter which side of the brane we choose) replaced with the appropriate solutions of the junction conditions (24) and (25). Obviously, the above effective brane equation does not constitute a closed system. The solution of the equations of motion for the bulk gravity has to be known to fully describe the gravity induced on the brane. However, the total bulk dependence of the above equation is described by the projected Weyl tensor $E_{\mu\nu}$, given by (38), which enters (implicitly - via \mathcal{P} and B_m defined in (43) and (42), respectively) in a quite complicated and in general non-linear way.

Our effective brane equation (49) has one feature which is unusual for the standard equations of motion, but quite typical of the brane ones. Specifically, there are no terms linear in the brane energy-momentum tensor $\tau_{\mu\nu}$. The reason is as follows: The brane sources (in our model: $\tau_{\mu\nu}$ and τ_ϕ) appear explicitly in the junction conditions only, which determine the jumps in the values of the Lie derivatives (in our model: the extrinsic curvature, $K_{\mu\nu}$, and the Lie derivative of the dilaton, $\mathcal{L}_n\phi$) of the brane fields (the induced metric tensor $h_{\mu\nu}$ and the dilaton field ϕ , respectively). The left hand sides of the junction conditions (24) and (25) are given by polynomials of the order $(2N_{\max} - 1)$ with only odd powers of the Lie derivatives. Although in general there can be several different solutions of these equations, we are interested only in those which vanish in the limit of vanishing sources (in the absence of sources the jumps in the values of the Lie derivatives have to vanish). Such solutions can be written as series in $\tau_{\mu\nu}$ and τ_ϕ with vanishing constant terms. On the other hand, only even powers of the Lie derivatives ($K_{\mu\nu}$ and $\mathcal{L}_n\phi$) are present in the (bulk) equations of motion. Consequently, in our effective brane equation (49) the terms involving the sources are at least quadratic (or bilinear) in $\tau_{\mu\nu}$ and τ_ϕ . Moreover, among the bilinear terms there is no term proportional to $\tau_\phi\tau_{\mu\nu}$. As we will show in the next section, such term is absent in the case of $N_{\max} = 1$. It cannot appear for higher N_{\max} as well, as the higher order corrections can change only those terms in the junction conditions solutions which are of higher order in the sources (we recall that one should consider only such solutions which have no constant terms when expanded in the sources).

In the standard Einstein equation the energy-momentum tensor appears linearly only. In order to have such a term in our effective brane equations of motion (48) and (49) we have to rewrite $\tau_{\mu\nu}$ as a sum of the energy-momentum tensor $\tilde{\tau}_{\mu\nu}$ associated with the fields we are interested in (e.g. the Standard Model fields which are usually assumed to be localized on a brane) and some “cosmological constant” $\tilde{\lambda}$ term: $\tau_{\mu\nu} = \tilde{\tau}_{\mu\nu} + h_{\mu\nu}\tilde{\lambda}$. The result of such a redefinition will be discussed in more detail in the next section, which is devoted to the dilaton gravity with $N_{\max} = 1$ and $N_{\max} = 2$.

5 Examples

The results presented in the previous sections are valid for models with corrections of arbitrarily high orders. In this section we write down those results explicitly¹⁰ for two simplest cases of $N_{\max} = 1$ and $N_{\max} = 2$. Although models with higher order corrections are the main topic of this work, we nevertheless want to present the results also for the lowest order theory. There are two reasons. First: the complexity of the calculations grows rapidly with the order of corrections.

¹⁰Due to their complexity, explicit formulae leading to the $N_{\max} = 2$ Einstein-like brane equation are moved to the appendix.

Thus, it is more instructive to discuss the main features of our procedure and results in the simplest situation with $N_{\max} = 1$. Second: we have obtained new results even for the lowest order theory. In our approach we treat the dilaton field on the same footing as the metric tensor, which has not been done before. The results on the effective brane equations for the dilaton gravity obtained so far in the literature were not fully satisfactory¹¹.

In the present work we give only the general formulae for $N_{\max} = 1$ and $N_{\max} = 2$. Presentation and discussion of some specific examples are postponed to a future publication.

5.1 $N_{\max} = 1$

Let us illustrate the main features of the construction of the effective brane equations of motion by considering a simple example with $N_{\max} = 1$. In this case the “+” and the “−” sides limits of the bulk tensor equation of motion projected on the brane (30) have the form of

$$\left[\left\{ R_{\mu\nu} + (DD)_{\mu\nu}\phi - \frac{1}{2}h_{\mu\nu} [R + 2(DD)\phi - (D\phi)^2] + \alpha_1^{-1}V(\phi)h_{\mu\nu} \right\} + \left\{ 2(KK)_{\mu\nu} - K_{\mu\nu}(K - \mathcal{L}_n\phi) - \frac{1}{2}h_{\mu\nu} \left(3(KK) - (K - \mathcal{L}_n\phi)^2 - 2a^\lambda\nabla_\lambda\phi \right) \right\} - \left\{ \mathcal{L}_n K_{\mu\nu} - h_{\mu\nu}((h\mathcal{L}_n K) - \mathcal{L}_n^2\phi) \right\} \right]_{-(+)} = 0, \quad (50)$$

whereas the “+” and the “−” sides limits of the dilaton equation of motion (33) read

$$\left[\left\{ R + 2(DD)\phi - (D\phi)^2 - 2\alpha_1^{-1}(V(\phi) - V'(\phi)) \right\} + \left\{ 3(KK) - (K - \mathcal{L}_n\phi)^2 - 2a^\lambda\nabla_\lambda\phi \right\} - 2 \left\{ (h\mathcal{L}_n K) - \mathcal{L}_n^2\phi \right\} \right]_{-(+)} = 0. \quad (51)$$

Both of these equations are genuine bulk equations of motion. They just determine the second Lie derivatives of the fields ($\mathcal{L}_n K_{\mu\nu}$ and $\mathcal{L}_n^2\phi$ in the third curly bracket in each equation) in terms of the fields (the metric tensor $h_{\mu\nu}$ and the dilaton field ϕ , together with their derivatives along the brane directions, in the first curly bracket) and their first Lie derivatives¹² ($K_{\mu\nu}$ and $\mathcal{L}_n\phi$ in the second curly bracket). The values of all the quantities in the first two curly brackets in each equation are

¹¹For example, in ref. [14] not all the calculations were carried out in a fully covariant way. Terms containing the combination $a^\lambda\nabla_\lambda\phi$ were removed by a gauge choice. Moreover, the influence of the bulk scalar field on the brane dynamics was taken into account only in some approximation and its bulk behavior was not eliminated from the brane gravitational equations.

¹²The last term in the second curly bracket is a mixture of the first Lie derivative of the dilaton with its derivatives along the brane directions. Hence, a part of that term should be moved to the first curly bracket. However, this subtlety does not change any further reasoning. Moreover, as was already mentioned, this slightly problematic term, $a^\lambda\nabla_\lambda\phi$, does not appear in our final results due to the appropriately designed derivation of the effective brane equations.

just the Cauchy boundary conditions for the corresponding d -dimensional second order differential equations. Without other equations or any additional assumptions about the bulk solution, these boundary conditions can be arbitrary. Thus eqs. (50) and (51) do not give any constraints on the brane fields $h_{\mu\nu}$ and ϕ in terms of the brane sources $\tau_{\mu\nu}$ and τ_ϕ . To get any effective brane equations of motion we have to analyze the junction conditions and the bulk tensor equation of motion with at least one index contracted with that of the vector n^μ normal to the brane.

The junction conditions (24) and (25) are very simple in the case of $N_{\max} = 1$. Solving them we can express the jumps in the values of the extrinsic curvature, $K_{\mu\nu}$, and the Lie derivative of the scalar field, $\mathcal{L}_n\phi$, in terms of the brane sources, $\tau_{\mu\nu}$ and τ_ϕ , as

$$[K_{\mu\nu}]_{\pm} = \alpha_1^{-1}(h_{\mu\nu}\tau_\phi - \tau_{\mu\nu}), \quad (52)$$

$$[\mathcal{L}_n\phi]_{\pm} = \alpha_1^{-1}((d-2)\tau_\phi - \tau). \quad (53)$$

Subsequently, we consider two vector eqs. (34) - one for each side of the brane. The difference of those two equations together with the junction conditions (52) and (53) gives the consistency condition (46) for the sources, namely

$$D^\mu\tau_{\mu\nu} + D^\mu\phi(h_{\mu\nu}\tau_\phi - \tau_{\mu\nu}) = 0. \quad (54)$$

As the second combination of the “-” and the “+” side eqs. (34) we take the “+” one. Hence, we obtain the following condition for the “+” side quantities:

$$D^\mu[K_{\mu\nu}]_+ - D^\mu\phi[K_{\mu\nu}]_+ - D_\nu[K]_+ + D_\nu[\mathcal{L}_n\phi]_+ = 0. \quad (55)$$

Finally, we have to take into account two scalar eqs. (35) with $N_{\max} = 1$. The brane curvature R and the dilaton ϕ , together with the first and the second (covariant with respect to the brane metric $h_{\mu\nu}$) derivatives of the latter all are continuous at the brane. Thus their contributions cancel in the difference of the “-” and the “+” side eqs. (35). Such a difference reduces to the equality $0 = [(K - \mathcal{L}_n\phi)^2]_{\pm} - [(KK)]_{\pm}$, which, after employing the junction conditions (52) and (53), can be rewritten as

$$\tau^{\mu\nu}[K_{\mu\nu}]_+ - \tau_\phi[\mathcal{L}_n\phi]_+ + \frac{1}{2}\alpha_1^{-1}((\tau\tau) - 2\tau\tau_\phi + (d-2)\tau_\phi^2) = 0, \quad (56)$$

where $(\tau\tau) \equiv \tau^{\mu\nu}\tau_{\mu\nu}$. The “+” side eq. (35) reads

$$R + 2(DD)\phi - (D\phi)^2 - 2\alpha_1^{-1}V(\phi) = ([K]_+ - [\mathcal{L}_n\phi]_+)^2 - [(KK)]_+. \quad (57)$$

The last six eqs. (52)-(57) are the only equations which can yield the effective brane equations of motion in a general case. The junction conditions (52) and (53) determine the jumps in the values of

the Lie derivatives of the metric tensor and the dilaton field: $[K_{\mu\nu}]_{\pm}$ and $[\mathcal{L}_n\phi]_{\pm}$, respectively. These quantities do not appear in any of the remaining four equations. Equation (54) is a consistency condition on the brane sources $\tau_{\mu\nu}$ and τ_ϕ - corresponding to the covariant conservation of the energy momentum tensor in the standard theory of gravity. The remaining equations establish conditions for the directional limits of the values of the first derivatives (normal to the brane) of the brane fields - evaluated at one of the “sides of the brane”, which we chose to be the “+” side. Specifically, eq. (56) relates $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$ to the brane sources $\tau_{\mu\nu}$ and τ_ϕ , while eqs. (55) and (57) relate them to the brane fields $h_{\mu\nu}$ and ϕ . Equations (55)-(57) provide us with $(d+1)$ relations, i.e. less than the number of the (gauge-independent) degrees of freedom in $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$. Thus, in general, eqs. (55)-(57) can be solved for arbitrary brane fields and sources. The junction conditions (52) and (53) do not change this situation, as without any assumptions on the bulk solution (as e.g. the already mentioned and usually employed bulk \mathbb{Z}_2 symmetry) the quantities $[K_{\mu\nu}]_{\pm}$, $[\mathcal{L}_n\phi]_{\pm}$, $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$ are all independent.

The only equation which involves the brane quantities exclusively is given by the formula (54). However, it is usually considered as a consistency condition on the brane sources and not as a dynamical equation of motion. As we pointed out in the previous section, it can yield a dynamical equation if the brane Lagrangian \mathcal{L}_B is complicated enough, e.g. when it involves brane localized kinetic terms for the bulk fields.

Let us now check the implications of some usually assumed features of the bulk solution. The by far most popular assumption of this kind is the already discussed bulk \mathbb{Z}_2 symmetry with the fixed point at the brane position. With this symmetry it is enough to consider eqs. (34) and (35) on one “side of the brane” only. The corresponding equations on the other side of the brane are fulfilled automatically. Hence, in the case of $N_{\max} = 1$ we are left with eqs. (52)-(54) and (57). Employing $[K_{\mu\nu}]_{\pm} = 2[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_{\pm} = 2[\mathcal{L}_n\phi]_+$ relations, which are due to the bulk \mathbb{Z}_2 symmetry, together with the junction conditions (52) and (53), we can rewrite eq. (57) in the following form:

$$R + 2(DD)\phi - (D\phi)^2 - 2\alpha_1^{-1}V(\phi) = \frac{1}{4}\alpha_1^{-2} [-(\tau\tau) + 2\tau\tau_\phi - (d-2)\tau_\phi^2]. \quad (58)$$

It determines the brane curvature scalar R in terms of the brane sources $\tau_{\mu\nu}$ and τ_ϕ (the dilaton field derivative can be obtained from the consistency condition (54)).

We shall now construct explicitly the brane tensor equation of motion for $N_{\max} = 1$ - following the procedure for arbitrary N_{\max} , described at the end of subsection 4.2. We start with eliminating the second derivatives normal to the brane, $\mathcal{L}_n K_{\mu\nu}$ and $\mathcal{L}_n^2\phi$, from the bulk tensor equation (50) and the bulk dilaton equation of motion (51). The parameters b_m and B_m , defined in (41) and

(42), can be easily calculated:

$$b_m = \alpha_1 \frac{(d-1)!}{(d-1-m)!}, \quad (59)$$

$$B_m = \alpha_1 \left\{ \frac{1}{2}(d-3)^m [R - K^2 + (KK)] + (d-2)^m [(DD)\phi + K\mathcal{L}_n\phi] - \frac{1}{2}(d-1)^m [(D\phi)^2 + (\mathcal{L}_n\phi)^2] \right\} - (d-1)^m V + (1-m)V'. \quad (60)$$

Employing the above explicit formulae on the parameters b_m and B_m , and the definitions (31) and (43), with $K_{\mu\nu}$ and $\mathcal{L}_n\phi$ given by their limits $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$ obtained from the junction conditions (52) and (53) supplemented by the relations (47) due to the bulk \mathbb{Z}_2 symmetry, the brane tensor equation (49) reduces to

$$R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R = -\frac{d-3}{d-2}[(DD)_{\mu\nu}\phi - h_{\mu\nu}(DD)\phi] - \frac{d-3}{d-1}h_{\mu\nu} \left[\frac{1}{2}(D\phi)^2 + \alpha_1^{-1}V(\phi) \right] - E_{\mu\nu} + \frac{1}{4}\alpha_1^{-2} \left[\frac{1}{d-2}\tau\tau_{\mu\nu} - (\tau\tau)_{\mu\nu} + h_{\mu\nu} \left(\frac{1}{2}(\tau\tau) - \frac{1}{(d-1)(d-2)}\tau^2 - \frac{d-3}{d-1}\tau\tau_\phi + \frac{(d-2)(d-3)}{2(d-1)}\tau_\phi^2 \right) \right]. \quad (61)$$

This equations is truly of the form of the $(d-1)$ -dimensional Einstein equation with some corrections (which was not apparent for the general N_{\max} formula (49)). Three specific types of contributions can be discerned on its r.h.s. There are terms with the explicit ϕ -dependence, typical of the gravity theories with scalar fields. Moreover, the tensor $E_{\mu\nu}$ represents the bulk influence on the dynamics at the brane. Finally, the last square bracket contains the contributions from the brane sources $\tau_{\mu\nu}$ and τ_ϕ . These contributions are quadratic in the brane energy-momentum tensor $\tau_{\mu\nu}$ (which is typical of the brane models) and its dilaton counterpart τ_ϕ . In order to have terms linear in some energy-momentum tensor (as is the case in the standard Einstein gravity) we have to rewrite $\tau_{\mu\nu}$ as the already mentioned sum of the energy-momentum tensor $\tilde{\tau}_{\mu\nu}$ associated with the fields we are interested in and some ‘‘cosmological constant’’ $\tilde{\lambda}$ term: $\tau_{\mu\nu} = \tilde{\tau}_{\mu\nu} + h_{\mu\nu}\tilde{\lambda}$. With such a decomposition we get the Einstein-like brane equation of motion as

$$R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R = 8\pi\tilde{G}\tilde{\tau}_{\mu\nu} - \frac{d-3}{d-2}[(DD)_{\mu\nu}\phi - h_{\mu\nu}(DD)\phi] - \frac{d-3}{d-1}h_{\mu\nu} \left[\frac{1}{2}(D\phi)^2 + \alpha_1^{-1}V(\phi) \right] - E_{\mu\nu} + \frac{1}{4}\alpha_1^{-2} \left[\frac{1}{d-2}\tilde{\tau}\tilde{\tau}_{\mu\nu} - (\tilde{\tau}\tilde{\tau})_{\mu\nu} + \frac{1}{2}h_{\mu\nu}(\tilde{\tau}\tilde{\tau}) - \frac{1}{(d-1)(d-2)}h_{\mu\nu}\tilde{\tau}^2 + \left(\frac{d-3}{d-2}\tilde{\lambda} - \frac{d-3}{d-1}\tau_\phi \right) h_{\mu\nu}\tilde{\tau} + \left(\frac{d-3}{2}\tilde{\lambda}^2 - (d-3)\tilde{\lambda}\tau_\phi + \frac{(d-2)(d-3)}{2(d-1)}\tau_\phi^2 \right) h_{\mu\nu} \right], \quad (62)$$

where we introduced

$$\tilde{G} \equiv \frac{-(d-3)\tilde{\lambda}}{32(d-2)\pi\alpha_1^2}, \quad (63)$$

which can be interpreted as the effective brane Newton’s constant. The contributions to eq. (62) which are proportional to the brane metric, $h_{\mu\nu}$, and depend neither on the scalar field ϕ , nor on

the tensor $\tilde{\tau}_{\mu\nu}$, should be interpreted as the effective brane cosmological constant:

$$\tilde{\Lambda} = \frac{d-3}{d-1} \alpha_1^{-1} V|_{\phi=0} - \frac{1}{4} \alpha_1^{-2} \left(\frac{d-3}{2} \tilde{\lambda}^2 - (d-3) \tilde{\lambda} \tau_\phi|_{\phi=0} + \frac{(d-2)(d-3)}{2(d-1)} \tau_\phi^2|_{\phi=0} \right). \quad (64)$$

It depends on the brane Lagrangian \mathcal{L}_B (via τ_ϕ) and on the way in which we divide the brane energy-momentum tensor $\tau_{\mu\nu}$ into its “standard” part $\tilde{\tau}_{\mu\nu}$ and the “cosmological constant” term $h_{\mu\nu} \tilde{\lambda}$. In addition, a part of $E_{\mu\nu}$ proportional to $h_{\mu\nu}$ can be also treated as a contribution to the effective brane cosmological constant.

The effective brane equation of motion (62) has the same tensor character as the standard Einstein equation. This certainly does not mean that we found another bulk-independent effective brane equation of motion - in addition to (58). It is just a convenient way to parameterize the bulk influence by a single geometric quantity: the projection of the bulk Weyl tensor on the brane, $E_{\mu\nu}$. Observe that there is no additional bulk influence due to the presence of the dilaton field¹³.

5.2 $N_{\max} = 2$

The formulae for arbitrary order of corrections given in sections 3 and 4 can be employed to obtain more explicit equations for any given N_{\max} . However, the complexity of the resulting expressions grows rapidly with N_{\max} . The effective brane equations of motion become quite intricate already for the $N_{\max} = 2$ case. Nevertheless, they can be still obtained even in the most general case, i.e. without any additional assumptions on the bulk background. We impose only the usual \mathbb{Z}_2 symmetry. As was already underlined, the $N_{\max} = 2$ case is equivalent to the appropriate subset of higher order interaction terms of the effective action derived from string theories.

In the $N_{\max} = 2$ case the junction conditions (24) and (25) take the following form:

$$\begin{aligned} \tau_{\mu\nu} = & 2 \left[\alpha_1 [h_{\mu\nu} (K - \mathcal{L}_n \phi) - K_{\mu\nu}] \right. \\ & + 2 \alpha_2 \left\{ [h_{\mu\nu} (K - \mathcal{L}_n \phi) - K_{\mu\nu}] [R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2] \right. \\ & \quad - 2 h_{\mu\nu} K_{\rho\sigma} [R^{\rho\sigma} + (KK)^{\rho\sigma} + (DD)^{\rho\sigma}\phi] - 2(K - \mathcal{L}_n\phi) [R_{\mu\nu} + (DD)_{\mu\nu}\phi + (KK)_{\mu\nu}] \\ & \quad + 2 K_{\mu\rho} [R_\nu^\rho + (KK)_\nu^\rho + (DD)_\nu^\rho\phi] + 2 K_{\nu\rho} [R_\mu^\rho + (KK)_\mu^\rho + (DD)_\mu^\rho\phi] \\ & \quad \left. \left. + 2 K^{\rho\sigma} [R_{\mu\rho\nu\sigma} - K_{\mu\rho} K_{\nu\sigma}] + \frac{2}{3} [h_{\mu\nu} (K - \mathcal{L}_n\phi)^3 + 2(KKK)] \right\} \right]_+, \quad (65) \end{aligned}$$

$$\begin{aligned} \tau_\phi = & 2 \left[\alpha_1 (K - \mathcal{L}_n\phi) + 2 \alpha_2 \left\{ (K - \mathcal{L}_n\phi) [R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2] \right. \right. \\ & \quad \left. \left. + \frac{2}{3} (K - \mathcal{L}_n\phi)^3 - 2K_{\mu\nu} [R^{\mu\nu} + \frac{1}{3}(KK)^{\mu\nu} + (DD)^{\mu\nu}\phi] \right\} \right]_+, \quad (66) \end{aligned}$$

¹³This differs from the results previously presented in the literature. For example, it is claimed in [15] that in the effective brane equations of motion it is easier to remove the dependence on the projected bulk Weyl tensor than the dependence on the bulk dilaton field. Our analysis clearly indicates the opposite.

where (KKK) denotes the trace of the third power of the extrinsic curvature: $K_\rho^\sigma K_\sigma^\lambda K_\lambda^\rho$. There are two new features as compared to the lowest order case. First: the junction conditions (65) and (66) are no longer linear in $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$. They are now third order equations for these quantities and their tensor structure is much more complicated. Thus, obtaining an explicit result is considerably more difficult. Second: solving these junction conditions (explicitly or not) yields the jumps in the values of $K_{\mu\nu}$ and $\mathcal{L}_n\phi$ as functions not only of the brane sources $\tau_{\mu\nu}$ and τ_ϕ , but also of the brane curvature R and the dilaton ϕ .

The simplest form of the effective brane equation of motion is given by the scalar eq. (35). In the model with the bulk \mathbb{Z}_2 symmetry and $N_{\max} = 2$ it reads:

$$\begin{aligned} & V - \frac{1}{2}\alpha_1 [R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2] \\ & - \frac{1}{2}\alpha_2 \left\{ [R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2]^2 \right. \\ & \quad - 4[R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu} + (DD)_{\mu\nu}\phi + K_{\mu\nu}\mathcal{L}_n\phi] \cdot \\ & \quad \cdot [R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (DD)^{\mu\nu}\phi + K^{\mu\nu}\mathcal{L}_n\phi] \\ & \quad \left. + [R_{\mu\nu\rho\sigma} - K_{\mu\rho}K_{\nu\sigma} + K_{\mu\sigma}K_{\nu\rho}] [R^{\mu\nu\rho\sigma} - K^{\mu\rho}K^{\nu\sigma} + K^{\mu\sigma}K^{\nu\rho}] \right\} = 0, \end{aligned} \quad (67)$$

with $K_{\mu\nu}$ (and all its contractions) and $\mathcal{L}_n\phi$ replaced by their “next to the brane” values $[K_{\mu\nu}]_+$ and $[\mathcal{L}_n\phi]_+$, given by the solutions of the junction conditions (65) and (66). This is the only bulk-independent brane equation of motion in addition to the consistency condition (46) on the brane sources $\tau_{\mu\nu}$ and τ_ϕ . It is also possible to derive the Einstein-like effective equation of motion, as discussed at the end of subsection 4.2. However, the result is truly complicated and we postpone presenting the appropriate formulae to the appendix. Those formulae indicate that employing the brane Einstein-like effective equation for a general case is rather problematic. However, it can be significantly simplified if we restrict our considerations to the situations with sufficient symmetries. This is quite typical of all theories of gravity. Usually only solutions with some specific symmetry properties are looked for. An alternative way to investigate some highly symmetric solutions is to use the simplest form of the effective brane equation (67) instead of the Einstein-like form discussed in the appendix.

6 Conclusions

The starting point of the present analysis has been given by the d -dimensional higher order dilaton gravity constructed previously [7] as a generalization of the Einstein-Lovelock theory and remaining in the close relation to the effective action in string theories if restricted to the gravity and the dilaton field. It was supplemented by a co-dimension 1 brane with general brane localized inter-

actions \mathcal{L}_B included into the Lagrangian (1). The effective brane equations of motion for such a theory were constructed and discussed. All calculations were performed in the covariant approach.

In order to obtain the effective brane equations of motion one has to start from the full bulk equations of motion (14), (15) and attempt to eliminate all quantities evaluated away from the brane position. In general this is not possible and the effective brane equations do not form a closed system. The dynamics of the brane fields $h_{\mu\nu}$ (the induced brane metric tensor) and ϕ (the dilaton field) depends usually not only on the brane sources $\tau_{\mu\nu}$ and τ_ϕ defined in (7), but also on the bulk gravity solution.

On the basis of the full bulk equations of motion derived from the Lagrangian density defining our model, we can obtain three types of equations involving fields determined either on or infinitesimally close to the brane. These equations represent the junction conditions and two directional limits of the bulk equations established when the brane is approached from the “+” and the “-” sides, respectively. The junction conditions (24) and (25) relate the brane fields, $h_{\mu\nu}$ and ϕ , and the sources, $\tau_{\mu\nu}$ and τ_ϕ , to the across-the-brane jumps in the values of the extrinsic curvature and the Lie derivative (along the vector field n^μ orthonormal to the brane) of the scalar field: $[K_{\mu\nu}]_\pm$ and $[\mathcal{L}_n\phi]_\pm$, respectively. The directional limits of the bulk equations (given by eqs. (30), (33)-(35)) involve the brane fields ($h_{\mu\nu}$ and ϕ), together with their first ($K_{\mu\nu}$ and $\mathcal{L}_n\phi$) and second ($\mathcal{L}_n K_{\mu\nu}$ and $\mathcal{L}_n^2\phi$) Lie derivatives, but do not include the brane sources $\tau_{\mu\nu}$ and τ_ϕ . An important point should be underlined: the number of the d -dimensional bulk gravitational equations of motion is bigger than the number of equations necessary for a $(d-1)$ -dimensional brane gravity. These extra equations (corresponding to the consistency conditions for a Cauchy problem with the boundary conditions defined infinitesimally close to the brane) play a crucial role, as they do not depend on the second Lie derivatives of the fields, i.e. $\mathcal{L}_n K_{\mu\nu}$ and $\mathcal{L}_n^2\phi$.

Genuine effective brane equations of motion should relate the values of the fields, $h_{\mu\nu}$ and ϕ , evaluated on the brane, to the brane sources, $\tau_{\mu\nu}$ and τ_ϕ . In addition, they should not depend on the bulk configuration. The crucial question is: How many such brane equations can be obtained by combining information from all bulk equations mentioned in the previous paragraph? The answer depends strongly on the symmetries we assume for the bulk theory and for the bulk and/or brane solutions. The minimal number of the effective brane equations is obtained when no such symmetries are assumed. The only effective brane equation in such a case is given by formula (46). However, this equation is usually treated not as a dynamical brane equation of motion, but rather as a consistency condition on the brane sources (7). In the pure gravity case (i.e. without the dilaton) it reduces simply to the covariant conservation of the brane localized energy-momentum tensor. However, there are models in which this “consistency” condition yields a dynamical equation of motion. This is the case when e.g. brane localized kinetic terms are present in the brane localized

Lagrangian \mathcal{L}_B .

The number of the effective brane equations of motion increases when we restrict the model by imposing some symmetries. The bulk \mathbb{Z}_2 symmetry with the fixed point coinciding with the brane position is particularly frequently employed. In such a case, an additional effective brane equation of motion (48) appears. It is obtained from the bulk tensor equation of motion if both indices are contracted with the vector field n^μ normal to the brane - after applying the appropriate junction conditions. This is the only effective brane equation of motion in models for which the “consistency” condition (46) is not dynamical. The importance of this equation depends on the class of solutions we are interested in. For example, it is all we need when considering maximally symmetric brane solutions.

In many models the number of the effective brane equations of motion is (much) smaller than the number of independent components of the Einstein equation in a $(d - 1)$ -dimensional space-time. Nevertheless, it is useful to derive an Einstein-like (tensor) brane equation. Such an equation for the higher (arbitrary) order dilaton gravity with the bulk \mathbb{Z}_2 symmetry is given by eq. (49). The entire dependence of the brane dynamics on the bulk gravity solution is encoded in the Weyl tensor projected on the brane, $E_{\mu\nu}$, which appears in eq. (49) implicitly via the parameters defined by formulae (42) and (43). It should be stressed that, contrary to some previous claims, those effective brane gravitational equations do not depend on the bulk scalar solution.

Our general results obtained for corrections up to order $2N_{\max}$ (arbitrary, as long as it is not higher than the space-time dimensionality) in derivatives are presented in a very compact notation based on the generalizations (2) and (8) of the trace operation. Those results are rewritten explicitly in the conventional notation for the two simplest cases of $N_{\max} = 1$ and $N_{\max} = 2$. Although the analysis of the higher order theories is the main topic of our work, two reasons motivated us to address also the lowest order theory (i.e. with $N_{\max} = 1$), which has been already considered by other authors. First, it is useful as the simplest illustration of our general, non-trivial procedure. Second, we improved the analysis presented so far in the literature even for this lowest order theory. The resulting explicit formulae for the effective brane Newton’s and cosmological constants are given by eqs. (63) and (64), respectively.

The case of $N_{\max} = 2$ is the Einstein-Gauss-Bonnet gravity interacting with a scalar field (which is also self-interacting) via terms with up to four derivatives. Similarly to the $N_{\max} = 1$ case, the effective brane equations of motion do not involve quantities dependent on the bulk scalar solution. The total bulk influence enters again through the projected Weyl tensor $E_{\mu\nu}$. The effective brane equations of motion for such a theory have not been presented before. They are quite lengthy when the generalized traces are explicitly calculated (all necessary formulae are collected in the appendix). Although the effective brane equations are rather complicated for a general case,

they simplify substantially if highly symmetric branes are considered. Applications of the derived equations for such symmetric models are postponed to a future publication.

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Appendix

In the appendix we collect the formulae appearing in the Einstein-like effective brane equation (49) for the case of $N_{\max} = 2$. This equation, after writing down explicitly the sum over N , takes the form of

$$\begin{aligned} & \frac{(d-3)(2d-3)}{(d-1)(d-2)} h_{\rho\sigma} V - \frac{1}{2} \alpha_1 \left\{ \frac{d-3}{d-2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{M}) + \frac{2}{d-3} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{P}) + \frac{d-3}{d-1} h_{\rho\sigma} \mathcal{T}(\mathcal{M}) \right. \\ & \quad \left. + 2(d-3) \frac{b_1 B_0 - b_0 B_1}{b_0 b_2 - b_1^2} h_{\rho\sigma} + \frac{2}{d-2} \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2} h_{\rho\sigma} \right\} \\ & - \frac{1}{2} \alpha_2 \left\{ \frac{d-3}{d-2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{M}^2) + \frac{4}{d-3} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{P}\mathcal{M}) + \frac{d-3}{d-1} h_{\rho\sigma} \mathcal{T}(\mathcal{M}^2) - \frac{16(d-3)}{d-2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{N}) \right. \\ & \quad \left. + \frac{4(d-3)}{d-2} \frac{b_1 B_0 - b_0 B_1}{b_0 b_2 - b_1^2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(h_*^* \mathcal{M}) + \frac{4(d-3)}{d-2} \frac{b_1 B_1 - b_2 B_0}{b_0 b_2 - b_1^2} h_{\rho}^{\mu} h_{\sigma}^{\nu} \bar{\mathcal{T}}_{\mu\nu}(\mathcal{M}) \right\} = 0. \quad (\text{A.1}) \end{aligned}$$

Performing the summations over N in definitions (41) and (42), we obtain the following expressions for the parameters b_m and B_m :

$$b_0 = \alpha_1 + 2 \alpha_2 \mathcal{T}(\mathcal{M}), \quad (\text{A.2})$$

$$b_1 = \alpha_1(d-1) + 2 \alpha_2 \mathcal{T}(h_*^* \mathcal{M}), \quad (\text{A.3})$$

$$b_2 = \alpha_1(d-2)(d-1) + 2 \alpha_2 \mathcal{T}((h_*^*)^2 \mathcal{M}), \quad (\text{A.4})$$

$$B_0 = -V + V' + \alpha_1 \mathcal{T}(\mathcal{M}) + \alpha_2 \left[\mathcal{T}(\mathcal{M}^2) + \frac{2}{d-3} \mathcal{T}(\mathcal{P}\mathcal{M}) - 8 \mathcal{T}(\mathcal{N}) \right], \quad (\text{A.5})$$

$$B_1 = -(d-1)V + \alpha_1 \mathcal{T}(h_*^* \mathcal{M}) + \alpha_2 \left[\mathcal{T}(h_*^* \mathcal{M}^2) + \frac{2}{d-3} \mathcal{T}(h_*^* \mathcal{P}\mathcal{M}) - 8 \mathcal{T}(h_*^* \mathcal{N}) \right]. \quad (\text{A.6})$$

Various generalized traces present in the above formulae should be replaced with the following explicit expressions:

$$\mathcal{T}(\mathcal{M}) = R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2, \quad (\text{A.7})$$

$$\begin{aligned} \mathcal{T}(\mathcal{M}^2) &= [R - K^2 + (KK) + 2(DD)\phi + 2K\mathcal{L}_n\phi - (D\phi)^2 - (\mathcal{L}_n\phi)^2]^2 \\ &\quad - 4 [R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu} + (DD)_{\mu\nu}\phi + K_{\mu\nu}\mathcal{L}_n\phi] \\ &\quad \cdot [R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (DD)^{\mu\nu}\phi + K^{\mu\nu}\mathcal{L}_n\phi] \\ &\quad + [R_{\mu\nu\rho\sigma} - K_{\mu\rho}K_{\nu\sigma} + K_{\mu\sigma}K_{\nu\rho}] [R^{\mu\nu\rho\sigma} - K^{\mu\rho}K^{\nu\sigma} + K^{\mu\sigma}K^{\nu\rho}], \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \mathcal{T}(\mathcal{PM}) &= \frac{2}{d-1} [R - K^2 + (KK)] [R - K^2 + (KK) + (DD)\phi + K\mathcal{L}_n\phi] \\ &\quad - 2 [R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu} + (DD)_{\mu\nu}\phi + K_{\mu\nu}\mathcal{L}_n\phi] \\ &\quad \cdot [R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (d-2)E^{\mu\nu}], \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} \mathcal{T}(\mathcal{N}) &= [D_\mu K - D_\rho K_\mu^\rho + K_\mu^\rho D_\rho\phi - D_\mu\mathcal{L}_n\phi] [D^\mu K - D^\sigma K_\sigma^\mu + K_\sigma^\mu D^\sigma\phi - D^\mu\mathcal{L}_n\phi] \\ &\quad - (D_\mu K_\rho^\sigma)(D^\mu K_\sigma^\rho) + (D_\mu K_\rho^\sigma)(D^\rho K_\sigma^\mu), \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \mathcal{T}(h_*^*\mathcal{M}) &= (d-3)[R - K^2 + (KK)] + 2(d-2)[(DD)\phi + K\mathcal{L}_n\phi] \\ &\quad - (d-1)[(D\phi)^2 + (\mathcal{L}_n\phi)^2], \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \mathcal{T}(h_*^*\mathcal{M}^2) &= (d-5)[R - K^2 + (KK)]^2 + 2(d-4)[R - K^2 + (KK)] [(DD)\phi + K\mathcal{L}_n\phi] \\ &\quad + (d-3)\left(4[(DD)\phi + K\mathcal{L}_n\phi]^2 - [R - K^2 + (KK)] [(D\phi)^2 + (\mathcal{L}_n\phi)^2]\right) \\ &\quad - 4(d-2)[(DD)\phi + K\mathcal{L}_n\phi] [(D\phi)^2 + (\mathcal{L}_n\phi)^2] + (d-1)[(D\phi)^2 + (\mathcal{L}_n\phi)^2]^2 \\ &\quad - 4(d-5)[R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}] [R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu}] \\ &\quad - 8(d-4)[R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}] [(DD)^{\mu\nu}\phi + K^{\mu\nu}\mathcal{L}_n\phi] \\ &\quad - 4(d-3)[(DD)_{\mu\nu}\phi + K_{\mu\nu}\mathcal{L}_n\phi] [(DD)^{\mu\nu}\phi + K^{\mu\nu}\mathcal{L}_n\phi] \\ &\quad + (d-5)[R_{\mu\nu\rho\sigma} - K_{\mu\rho}K_{\nu\sigma} + K_{\mu\sigma}K_{\nu\rho}] [R^{\mu\nu\rho\sigma} - K^{\mu\rho}K^{\nu\sigma} + K^{\mu\sigma}K^{\nu\rho}], \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} \mathcal{T}(h_*^*\mathcal{PM}) &= \frac{2(d-4)}{d-1} [R - K^2 + (KK)]^2 + \frac{2(d-3)}{d-1} [R - K^2 + (KK)] [(DD)\phi + K\mathcal{L}_n\phi] \\ &\quad - 2\left((d-4)[R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}] + (d-3)[(DD)_{\mu\nu}\phi + K_{\mu\nu}\mathcal{L}_n\phi]\right) \\ &\quad \cdot [R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (d-2)E^{\mu\nu}], \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \mathcal{T}(h_*^*\mathcal{N}) &= (d-4)\left([D_\mu K - D_\rho K_\mu^\rho] [D^\mu K - D^\sigma K_\sigma^\mu] - (D_\mu K_\rho^\sigma)(D^\mu K_\sigma^\rho) + (D_\mu K_\rho^\sigma)(D^\rho K_\sigma^\mu)\right) \\ &\quad + 2(d-3)K_\mu^\nu [(D_\nu K)(D^\mu\phi) - (D_\rho K_\nu^\rho)(D^\mu\phi)] \\ &\quad + (d-2)[K_\mu^\rho D_\rho\phi - D_\mu\mathcal{L}_n\phi] [K_\sigma^\mu D^\sigma\phi - D^\mu\mathcal{L}_n\phi], \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \mathcal{T}((h_*^*)^2\mathcal{M}) &= (d-4)(d-3)[R - K^2 + (KK)] + 2(d-3)(d-2)[(DD)\phi + K\mathcal{L}_n\phi] \\ &\quad - (d-2)(d-1)[(D\phi)^2 + (\mathcal{L}_n\phi)^2], \end{aligned} \quad (\text{A.15})$$

$$h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{M}) = h_{\rho\sigma} \mathcal{T}(\mathcal{M}) - 2[R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma} + (DD)_{\rho\sigma}\phi + K_{\rho\sigma}\mathcal{L}_n\phi], \quad (\text{A.16})$$

$$h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{P}) = \frac{1}{d-1}h_{\rho\sigma}[R - K^2 + (KK)] - [R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma}], \quad (\text{A.17})$$

$$\begin{aligned} h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{M}^2) &= h_{\rho\sigma} \mathcal{T}(\mathcal{M}^2) - 4[R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma} + (DD)_{\rho\sigma}\phi + K_{\rho\sigma}\mathcal{L}_n\phi] \mathcal{T}(\mathcal{M}) \\ &\quad + 8[R_{\rho\mu} - KK_{\rho\mu} + (KK)_{\rho\mu} + (DD)_{\rho\mu}\phi + K_{\rho\mu}\mathcal{L}_n\phi] \\ &\quad \cdot [R^\mu{}_\sigma - KK^\mu{}_\sigma + (KK)^\mu{}_\sigma + (DD)^\mu{}_\sigma\phi + K^\mu{}_\sigma\mathcal{L}_n\phi] \\ &\quad + 8[R_{\rho\sigma\mu\nu} - K_{\rho\mu}K_{\sigma\nu} + K_{\rho\nu}K_{\sigma\mu}][R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (DD)^{\mu\nu}\phi + K^{\mu\nu}\mathcal{L}_n\phi] \\ &\quad - 4[R_{\rho\mu\nu\lambda} - K_{\rho\nu}K_{\mu\lambda} + K_{\rho\lambda}K_{\mu\nu}][R_\sigma{}^{\mu\nu\lambda} - K_\sigma{}^\nu K^{\mu\lambda} + K_\sigma{}^\lambda K^{\mu\nu}], \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{N}) &= h_{\rho\sigma} \mathcal{T}(\mathcal{N}) + 2(D_\mu K_\rho^\nu)(D^\mu K_{\nu\sigma}) \\ &\quad - 2[D_\mu K - D_\nu K_\mu^\nu][D^\mu K_{\rho\sigma} - D_\rho K_\sigma^\mu] + [D_\rho K_\mu^\nu - D_\mu K_\rho^\nu][D_\sigma K_\nu^\mu - D_\nu K_\sigma^\mu] \\ &\quad - [D_\rho K - D_\mu K_\rho^\mu + K_\rho^\mu D_\mu\phi - D_\rho\mathcal{L}_n\phi][D_\sigma K - D_\nu K_\sigma^\nu + K_\sigma^\nu D_\nu\phi - D_\sigma\mathcal{L}_n\phi], \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(h_*^*\mathcal{M}) &= h_{\rho\sigma} \left((d-4)[R - K^2 + (KK)] + 2(d-3)[(DD)\phi + K\mathcal{L}_n\phi] \right. \\ &\quad \left. - (d-2)[(D\phi)^2 + (\mathcal{L}_n\phi)^2] \right) \\ &\quad - 2(d-3)[R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma}] - 2(d-2)[(DD)_{\rho\sigma}\phi + K_{\rho\sigma}\mathcal{L}_n\phi], \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned} h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{PM}) &= h_{\rho\sigma} \mathcal{T}(\mathcal{PM}) - \frac{1}{d-1}h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(h_*^*\mathcal{M}) - [R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma} + (d-2)E_{\rho\sigma}] \\ &\quad - 2[R - K^2 + (KK)][R_{\rho\sigma} - KK_{\rho\sigma} + (KK)_{\rho\sigma} + (DD)_{\rho\sigma}\phi + K_{\rho\sigma}\mathcal{L}_n\phi] \\ &\quad + 2[R_{\rho\mu} - KK_{\rho\mu} + (KK)_{\rho\mu} + (d-2)E_{\rho\mu}][R_\sigma^\mu - KK_\sigma^\mu + (KK)_\sigma^\mu + (DD)_\sigma^\mu\phi + K_\sigma^\mu\mathcal{L}_n\phi] \\ &\quad + 2[R_\sigma^\mu - KK_\sigma^\mu + (KK)_\sigma^\mu + (d-2)E_\sigma^\mu][R_{\rho\mu} - KK_{\rho\mu} + (KK)_{\rho\mu} + (DD)_{\rho\mu} + K_{\rho\mu}\mathcal{L}_n\phi] \\ &\quad + 2[R_{\rho\mu\sigma\nu} - K_{\rho\mu}K_{\sigma\nu} + K_{\rho\nu}K_{\sigma\mu}][R^{\mu\nu} - KK^{\mu\nu} + (KK)^{\mu\nu} + (d-2)E^{\mu\nu}]. \end{aligned} \quad (\text{A.21})$$

By substituting eqs. (A.2)-(A.21) into (A.1) and employing the solutions of the junction conditions (65) and (66), the effective brane equation of motion in the Einstein-like form can be obtained. However, the result is quite intricate and will not be given here. Moreover, the formulae collected in this appendix should be compared with the relatively simple eq. (67). This shows how big price, in terms of complication, has to be paid in order to derive the effective brane equation of motion in the Einstein-like form, instead of confining to eq. (67).

Similarly to the $N_{\max} = 1$ case, the whole bulk dependence of the brane dynamics is encoded in the projected Weyl tensor $E_{\mu\nu}$. However, this dependence is quite complicated for $N_{\max} = 2$. Specifically, $E_{\mu\nu}$ enters the brane Einstein-like equation of motion (A.1) through the generalized

trace $h_\rho^\mu h_\sigma^\nu \overline{\mathcal{T}}_{\mu\nu}(\mathcal{PM})$, appearing in eq. (A.1) and given by (A.21), as well as through the generalized traces $\mathcal{T}(\mathcal{PM})$ and $\mathcal{T}(h_*^* \mathcal{PM})$, present in the definitions of B_0 and B_1 and given by (A.9) and (A.13), respectively. Although the projected Weyl tensor $E_{\mu\nu}$ enters these formulae only linearly, it is involved in intricate contractions with other tensors. Furthermore, these tensors contain the extrinsic curvature $K_{\mu\nu}$, i.e. a complicated solution of the junction conditions (65) and (66) - which in general cannot be solved explicitly.

As for $N_{\max} > 2$, it is in principle possible to write down the Einstein-like effective brane equation of motion explicitly, but they become practically intractable.

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