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Lattice Refining LQC from an Isotropic Embedding of Anisotropic Cosmology

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Abstract. We demonstrate that it is possible to produce different isotropic embeddings of anisotropic Loop Quantum Cosmology, resulting in “lattice refinement” of the isotropic system. To introduce the general approach, we first use a simple model with only two anisotropic directions. We then employ the specific case of a Bianchi I model, to show how the method extends to three-dimensional systems. To concisely calculate the step-size of the resulting isotropic state, we define the “symmetric dual” of states and operators, for the two- and three-dimensional systems, respectively.

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1. Introduction

Loop Quantum Gravity (LQG) quantises canonical General Relativity, in a background independent and non-perturbative manner. It does so by considering the three-dimensional spatial slice to be a network of triads and then formulating the classical theory in terms of holonomies of the triad connections, around closed edges of the network, and fluxes of the triads, through the surfaces enclosed by these edges. Whilst the full, inhomogeneous theory has yet to be fully developed, significant progress has been made by applying the approach to symmetry reduced, mini-superspace models. In particular, by applying these principles of quantisation to Cosmology, one arrives at Loop Quantum Cosmology (LQC), which has proved itself especially successful [1].

Within LQC it has been possible, on the one hand, to explicitly show the avoidance of singularities that typically plague the classical versions of these cosmological theories [2, 3] and, on the other hand, to develop powerful effective theories, that capture first order corrections to classical cosmology [4]. In particular, it has been shown that, in order for the cosmological theory to be theoretically consistent [5, 6], the physical volume (rather than the scale factor, or the triad component) must be the fundamental variable used. This conclusion has been backed up by various phenomenological [7, 8] and theoretical [9] arguments. However, it is important to realise that since there is still lacking a complete derivation of LQC from the fundamental inhomogeneous and anisotropic theory, it is important to test LQC predictions for robustness (see e.g., Ref. [10]).

Various aspects of anisotropic cosmologies have been studied within LQC in the past [11, 12, 13], however the first full and consistent quantisation of a Bianchi I cosmology (the simplest of anisotropic cosmological models) was achieved in Ref. [14] ‡. Even more encouraging, the link back to the underlying full theory (LQG) has been strengthened, by considering the flux of the triads through surfaces consistent with the Bianchi I anisotropic case. With the quantisation of the Bianchi I model under control, it is possible to ask whether LQC features, obtained within the context of isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, are robust, at least with respect to this limited extension of the symmetries of the system.

In this paper, we will show that one can choose among different isotropic embeddings of the anisotropic Bianchi I model. We will then demonstrate that the choice of different embeddings has important consequences for the precise form of discretisation, produced by the loop quantisation procedure, in the isotropic sub-system. Note that, we use Bianchi I as an example of an anisotropic model. In other words, we do not imply that it is physically natural, nor that our results are constrained to hold only for this type of an anisotropic model.

In Section 2, the basics of LQC will be sketched, for both isotropic and anisotropic

‡ Performing a von Neumann stability analysis of the Hamiltonian constraint equation valid for anisotropic Bianchi I model, we have shown [15] that the the difference equation is unconditionally unstable.

models. In Section 3, it will be demonstrated that an isotropic embedding of an anisotropic system can be realised in various ways, leading to consequences for the form of the induced isotropic system. This will be first demonstrated for a prototype two-dimensional system. In Section 4, we will define a “symmetric dual” of operators and states, to apply our formalism. In Section 5, the approach will be extended to the three-dimensional Bianchi I model and it will be shown that, with certain additional caveats, the same conclusions can be drawn. Finally, we will demonstrate that “lattice refinement” [8, 16] can be motivated by alternative isotropic embeddings, while the difference from the standard quantisation of isotropic cosmology can be viewed simply as a different choice of basic observables.

2. Basics formalism of LQG/LQC

The full LQG theory is uniquely derived from the requirements that the theory is diffeomorphism invariant and satisfies the $SU(2)$ gauge freedom. In both, the full theory and in LQC, the fundamental variables are the holonomies of the $SU(2)$ connection, A_a^i , along a given edge, e , and the corresponding momentum, which turns out to be the flux of the triad through a two-surface S . The holonomies are given by

$$h_e(A) = \mathcal{P} \exp \int_e \dot{\gamma}^\mu(s) A_\mu^i(\gamma(s)) \tau_i ds , \quad (1)$$

where \mathcal{P} infers path ordering on the exponential, γ^μ is the tangent vector along the edge e , and τ_i are the basis of the $SU(2)$ Lie algebra. The corresponding momentum variable is

$$E(S, f) = \int_S \epsilon_{abc} E^{ci} f_i dx^a dx^b , \quad (2)$$

where f_i is an $SU(2)$ valued test function and E_i^a is the densitised triad; i, j, k, \dots are $SU(2)$ indices, whilst a, b, c, \dots are coordinate indices.

Restricting to isotropic and homogeneous cosmologies allows us to consider only straight edges along integral curves of the basis vectors, X_i^a , that produce the network. Then the connection is given by a (dynamic) multiple of the basis one forms, ω_a^i , namely $A_a^i = \tilde{c}(t)\omega_a^i$, whilst the triad is $E_i^a = \sqrt{{}^0g}\tilde{p}(t)X_i^a$, where 0g is the determinant of the fiducial metric[§] and \tilde{c} stands for the connection component. This is simply a consequence of the symmetries imposed, since isotropy ensures that there can be no angular dependence of the connection or triads, whilst homogeneity ensures they are the same at every spatial point (but not necessarily on different time slices).

With these symmetries, the holonomies become simply

$$\begin{aligned} h_i(A) &= \exp \left[\frac{-i\mu_0\sigma_i}{2} \tilde{c} \right] , \\ &= \cos \left(\frac{\mu_0\tilde{c}}{2} \right) + 2\tau_i \sin \left(\frac{\mu_0\tilde{c}}{2} \right) , \end{aligned} \quad (3)$$

[§] The fiducial metric is a complication that arises only for an open universe and is used to define the volume to which spatial integrals are restricted to ensure they remain finite. Physical results should not depend on this volume [17].

where τ_i , the basis of the SU(2) Lie algebra, are related to the Pauli matrices, σ_i , by $\tau_i = -i\sigma_i/2$ and μ_0 is the orientated length of the edge with respect to the fiducial metric. It is this μ_0 that is the ambiguity coming from the fact that we have imposed homogeneity on our system. In a full theory, the length of the edges would have some spectrum of values, in which case μ_0 would be the minimum such length, which has been shown to be non-zero [18]. Here we have treated μ_0 as a constant, however for consistent quantisation [6, 9, 7], it has been shown that it must vary as $\mu_0 \sim \mu^{-1/2}$. In order to include this varying parameter, one must change variables from the triad component to the volume [5], however here we will briefly sketch the derivation for the constant μ_0 case, with more details given in Ref. [5].

By analogue with the full theory, the kinematic Hilbert space is extended via the Bohr compactification of the real line [18]. An orthonormal basis for this Hilbert space is $\{|\mu\rangle\}$, where

$$\langle c|\mu\rangle = e^{i\frac{\mu c}{2}}. \quad (4)$$

Note that c is a re-definition of the connection component, so that the Poisson bracket of the connection component and the (re-defined) triad component is independent of the volume of the fiducial cell, namely $\{c, p\} = \kappa\gamma/3$, with $\kappa = 8\pi G$ and γ the Barbero-Immirzi parameter, fixed through a black hole entropy calculation and turns out to be $\gamma \approx 0.2375$.

The triad operator acts on these basis states as

$$\hat{p}|\mu\rangle = -i\frac{\kappa\gamma\hbar}{3}\frac{\partial}{\partial c}|\mu\rangle = \frac{\kappa\gamma\hbar}{6}|\mu||\mu\rangle. \quad (5)$$

Clearly, also the volume operator, $\hat{V} \equiv \hat{a}^3 = |\hat{p}|^{3/2}$, has eigenstates in this basis, namely

$$\hat{V}|\mu\rangle = V_\mu|\mu\rangle = \left(\frac{\kappa\gamma\hbar|\mu|}{6}\right)^{3/2}|\mu\rangle. \quad (6)$$

To calculate the eigenvalues of the inverse volume operator, the classical expression [18],

$$p^{-1} = (p^{L-1})^{1/(1-L)} = \left(\frac{3}{\kappa\gamma L}\{c, p^L\}\right)^{1/(1-L)}, \quad (7)$$

is used. This is a classical identity independent of L, but will be quantised to different operators for different L. Thus, L plays the rôle of a quantisation ambiguity.

It is now possible to quantise the classical Hamiltonian constraint, where the curvature is approximated via holonomies around a closed curve (which can not be shrunk to zero, due to the ‘‘area gap’’, given in terms of μ_0 ; it occurs by analogue with LQG), which gives

$$\hat{C}_{\text{grav}} = \frac{2i}{\kappa^2\hbar\gamma^3\mu_0^3}\text{tr}\sum_{i,j,k}\epsilon^{ijk}\left(\hat{h}_i^{(\mu_0)}\hat{h}_j^{(\mu_0)}\hat{h}_i^{(\mu_0)-1}\hat{h}_j^{(\mu_0)-1}\left[\hat{h}_k^{(\mu_0)-1}, \hat{V}\right]\text{sgn}(\hat{p})\right), \quad (8)$$

where, as in all quantum theories, there is an ambiguity in the factor ordering [9]; other quantum ambiguities have been fixed. Using Eq. (3), one can thus obtain a difference

equation [17],

$$\begin{aligned}
 & \left[\left| V_{\mu+5\mu_0} - V_{\mu+3\mu_0} \right| + \left| V_{\mu+\mu_0} - V_{\mu-\mu_0} \right| \right] \Psi_{\mu+4\mu_0}(\phi) - 4 \left| V_{\mu+\mu_0} V_{\mu-\mu_0} \right| \Psi_{\mu}(\phi) \\
 & + \left[\left| V_{\mu-3\mu_0} - V_{\mu-5\mu_0} \right| + \left| V_{\mu+\mu_0} - V_{\mu-\mu_0} \right| \right] \Psi_{\mu-4\mu_0}(\phi) \\
 & = -\frac{4\kappa^2 \gamma^3 \hbar \mu_0^3}{3} \mathcal{H}_{\phi}(\mu) \Psi_{\mu}(\phi) ,
 \end{aligned} \tag{9}$$

of step $4\mu_0$, for the wave-function coefficients $\Psi(\mu)$, defined by

$$|\Psi\rangle = \sum_{\mu} \Psi(\mu) |\mu\rangle . \tag{10}$$

Note that, the matter Hamiltonian $\hat{\mathcal{H}}_{\phi}$ is assumed to act diagonally on the basis states with eigenvalue $\mathcal{H}_{\phi}(\mu)$.

By adapting the network to the symmetries of a Bianchi I model, one can similarly derive the quantum Hamiltonian for this anisotropic system [14]. In this case, we have three triad components (p_1, p_2, p_3) , along the three directions of the anisotropic model. The gravitational sector of the kinematic Hilbert space consists of wave-functions $\Psi(p_1, p_2, p_3)$, satisfying

$$\Psi(p_1, p_2, p_3) = \Psi(|p_1|, |p_2|, |p_3|) . \tag{11}$$

Neglecting the details of the derivation of the Hamiltonian constraint, which are of no relevance for our present study, we draw the attention of the reader to the fact that in both, the isotropic and the Bianchi I case, the holonomies act on a state as shift vectors.

Let us define three dimensionless variables, λ_i , as

$$\lambda_i = \frac{\text{sgn}(p_i) \sqrt{|p_i|}}{(4\pi|\gamma| \sqrt{\Delta} l_{\text{Pl}}^3)^{1/3}} \quad \text{with } i = 1, 2, 3 ; \tag{12}$$

the quantum of area Δl_{Pl}^2 denotes the physical geometry, with $\Delta = 4\pi\gamma\sqrt{3}$. The Hamiltonian constraint for the wave-function $\Psi(\lambda_1, \lambda_2, \nu)$, where $\nu = 2\lambda_1\lambda_2\lambda_3$, is a difference equation. It reads [14]

$$\begin{aligned}
 \partial_{\phi}^2 \Psi(\lambda_1, \lambda_2, \nu; \phi) = & \frac{\pi G}{2} \sqrt{\nu} \left[(\nu + 2) \sqrt{\nu + 4} \Psi_4^+(\lambda_1, \lambda_2, \nu; \phi) \right. \\
 & - (\nu + 2) \sqrt{\nu} \Psi_0^+(\lambda_1, \lambda_2, \nu; \phi) \\
 & - (\nu - 2) \sqrt{\nu} \Psi_0^-(\lambda_1, \lambda_2, \nu; \phi) \\
 & \left. + (\nu - 2) \sqrt{|\nu - 4|} \Psi_4^-(\lambda_1, \lambda_2, \nu; \phi) \right] ,
 \end{aligned} \tag{13}$$

where $\Psi_{0,4}^{\pm}$ are defined as [14]

$$\begin{aligned}
 \Psi_0^{\pm}(\lambda_1, \lambda_2, \nu; \phi) = & \Psi \left(\frac{\nu \pm 2}{\nu} \cdot \lambda_1, \frac{\nu}{\nu \pm 2} \cdot \lambda_2, \nu; \phi \right) + \Psi \left(\frac{\nu \pm 2}{\nu} \cdot \lambda_1, \lambda_2, \nu; \phi \right) \\
 & + \Psi \left(\frac{\nu}{\nu \pm 2} \cdot \lambda_1, \frac{\nu \pm 2}{\nu} \cdot \lambda_2, \nu; \phi \right) + \Psi \left(\frac{\nu}{\nu \pm 2} \cdot \lambda_1, \lambda_2, \nu; \phi \right) \\
 & + \Psi \left(\lambda_1, \frac{\nu}{\nu \pm 2} \cdot \lambda_2, \nu; \phi \right) + \Psi \left(\lambda_1, \frac{\nu \pm 2}{\nu} \cdot \lambda_2, \nu; \phi \right) ,
 \end{aligned} \tag{14}$$

and

$$\begin{aligned}
 \Psi_4^\pm(\lambda_1, \lambda_2, \nu; \phi) = & \\
 & \Psi\left(\frac{\nu \pm 4}{\nu \pm 2} \cdot \lambda_1, \frac{\nu \pm 2}{\nu} \cdot \lambda_2, \nu \pm 4; \phi\right) + \Psi\left(\frac{\nu \pm 4}{\nu \pm 2} \cdot \lambda_1, \lambda_2, \nu \pm 4; \phi\right) \\
 & + \Psi\left(\frac{\nu \pm 2}{\nu} \cdot \lambda_1, \frac{\nu \pm 4}{\nu \pm 2} \cdot \lambda_2, \nu \pm 4; \phi\right) + \Psi\left(\frac{\nu \pm 2}{\nu} \cdot \lambda_1, \lambda_2, \nu \pm 4; \phi\right) \\
 & + \Psi\left(\lambda_1, \frac{\nu \pm 2}{\nu} \cdot \lambda_2, \nu \pm 4; \phi\right) + \Psi\left(\lambda_1, \frac{\nu \pm 4}{\nu \pm 2} \cdot \lambda_2, \nu \pm 4; \phi\right), \quad (15)
 \end{aligned}$$

respectively. Note that, ϕ plays the rôle of *internal* time, exactly as in the isotropic case. Thus, as one can easily check from Eq. (13), as far as the ν dependence is concerned, the steps are uniform, since the argument of the wave-function involves $\nu - 4, \nu, \nu + 4$, exactly as in the isotropic case. However, the dependence on λ_1, λ_2 is quite complicated, technically, to deal with.

Writing the step-size in terms of the triad components, implies

$$\begin{aligned}
 \Psi_{\text{iso}}(p) &\rightarrow \Psi_{\text{iso}}\left(p \pm \frac{K}{\sqrt{|p|}}\right) \\
 \text{and } \Psi(p_1, p_2, p_3) &\rightarrow \begin{cases} \Psi\left(p_1 \pm \tilde{K} \sqrt{\left|\frac{p_1}{p_2 p_3}\right|}, p_2, p_3\right) \\ \Psi\left(p_1, p_2 \pm \tilde{K} \sqrt{\left|\frac{p_2}{p_1 p_3}\right|}, p_3\right) \\ \Psi\left(p_1, p_2, p_3 \pm \tilde{K} \sqrt{\left|\frac{p_3}{p_1 p_2}\right|}\right), \end{cases} \quad (16)
 \end{aligned}$$

for some suitable constants K and \tilde{K} that are proportional to $\sqrt{\Delta} l_{\text{Pl}}^3$ [14]. Note that, typically the \tilde{K} constants, appearing in the argument of the anisotropic wave-function, will be different.

In both, the isotropic and anisotropic cases, “consistency” is derived entirely within the model, *i.e.*, the symmetries of the model are fixed from the outset and “consistency” is looked for, under these assumptions. This, of course, does not ensure that the model remains “consistent”, if it derived from a less symmetric approximation. In particular, the isotropic case is consistently quantised for the step-size of the holonomies given in Eq. (16), only if one does not allow additional degree of freedom (and similarly for the Bianchi I model), however this does not ensure that an isotropic embedding of the Bianchi I model will have the same step-size as the isotropic case, given in Eq. (16). However, with both models fully quantised, it is possible to consider such an isotropic embedding of the anisotropic Bianchi I model and explicitly evaluate the type of step-size that is induced on the effective isotropic system.

This will be done in some detail below, but one can immediately see that setting $p_1 = p_2 = p_3 = p$, will produce the expected form of step-size, however this embedding procedure does not lead to a viable isotropic theory, instead one has to use a projection of the anisotropic degrees of freedom [14]. The reason for this is the following: for $p_1 = p_2 = p_3 = p$, we have $\lambda_1 = \lambda_2 = \lambda_3 = (\nu/2)^{1/3}$. Thus, the state which satisfies

$p_1 = p_2 = p_3 = p$ is

$$\Psi \left(\left(\frac{\nu}{2} \right)^{1/3}, \left(\frac{\nu}{2} \right)^{1/3}, \nu \right) .$$

From Eq. (13), we see that applying the Hamiltonian to such a state involves states of the form $\tilde{\Psi}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\nu})$, which do not satisfy $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = (\tilde{\nu}/2)^{1/3}$. Thus, states for which $p_1 = p_2 = p_3 = p$, do not form a *super-selection* of states (i.e., the Hamiltonian does not preserve this property of states).

The purpose of this paper is to show that it is, in fact, possible to find a *super-selection* of states that are all (measured to be) isotropic. In particular, we will show that there are embeddings for which the expectation values of the variables p_1 , p_2 and p_3 are equal, and that are preserved under the action of the Hamiltonian. The Hamiltonian applied to such states does not produce the standard FLRW Hamiltonian of LQC, rather one finds a Hamiltonian constraint in which the discreteness length varies. Thus, one may view this as a form of lattice refinement on the isotropic sub-space. It is important to note however that the typical forms of lattice refining Hamiltonian used in the literature are not the same as those produced here; there is instead a simplified attempt to capture the missing underlying degrees of freedom (in this case the underlying anisotropic degrees of freedom).

3. The two-dimensional case

To demonstrate the approach, let us first consider an anisotropic wave-function that depends only on two directions, labelled 1, 2. The basic operators that are quantised are \hat{p}_i and \hat{h}_i , with $i = 1, 2$. Note that $p_i = a_i^2$, while h_i denote the holonomies.

In general, the wave-function is anisotropic, namely

$$|\Psi(p_1, p_2)\rangle \neq |\Psi(p_2, p_1)\rangle . \quad (17)$$

To be clear, the action of the operators \hat{p}_i is

$$\begin{aligned} \hat{p}_1 |\Psi(p_1, p_2)\rangle &= p_1 |\Psi(p_1, p_2)\rangle \\ \hat{p}_2 |\Psi(p_1, p_2)\rangle &= p_2 |\Psi(p_1, p_2)\rangle \end{aligned} \quad (18)$$

and

$$\begin{aligned} \hat{p}_1 |\Psi(p_2, p_1)\rangle &= p_2 |\Psi(p_2, p_1)\rangle \\ \hat{p}_2 |\Psi(p_2, p_1)\rangle &= p_1 |\Psi(p_2, p_1)\rangle , \end{aligned} \quad (19)$$

and the action of the holonomies \hat{h}_i on a states $|\Psi(p_1, p_2)\rangle$ and $|\Psi(p_2, p_1)\rangle$ are,

$$\begin{aligned} \hat{h}_1 |\Psi(p_1, p_2)\rangle &= |\Psi(p_1 + \delta_1(p_1, p_2), p_2)\rangle \\ \text{and } \hat{h}_2 |\Psi(p_1, p_2)\rangle &= |\Psi(p_1, p_2 + \delta_2(p_1, p_2))\rangle , \end{aligned} \quad (20)$$

and

$$\begin{aligned} \hat{h}_1 |\Psi(p_2, p_1)\rangle &= |\Psi(p_2 + \delta_1(p_1, p_2), p_1)\rangle \\ \text{and } \hat{h}_2 |\Psi(p_2, p_1)\rangle &= |\Psi(p_2, p_1 + \delta_2(p_1, p_2))\rangle, \end{aligned} \quad (21)$$

where δ_i is potentially a function of both p_1 and p_2 .

3.1. Standard isotropic embedding

Typically, one can go to the isotropic case by defining the operator \hat{Q}_{sym} , as

$$\hat{Q}_{\text{sym}} |\Psi(p_1, p_2)\rangle \equiv |\Psi(p_1, p_1)\rangle = |\Psi(p_2, p_2)\rangle = |\Psi(p, p)\rangle. \quad (22)$$

In other words, the operator \hat{Q}_{sym} sets $p_1 = p_2$. It is easy to see that operating singly with either \hat{h}_1 , or \hat{h}_2 , on $|\Psi(p_1, p_2)\rangle$, before operating with \hat{Q}_{sym} , implies $\delta_i = 0$, e.g.,

$$\begin{aligned} \hat{Q}_{\text{sym}} \left[\hat{h}_1 |\Psi(p_1, p_2)\rangle \right] &= \hat{Q}_{\text{iso}} |\Psi(p_1 + \delta_1, p_2)\rangle \\ &= |\Psi(p_1 + \delta_1, p_1 + \delta_1)\rangle \\ &= |\Psi(p_2, p_2)\rangle \\ \Rightarrow p_1 + \delta_1 &= p_2, \end{aligned} \quad (23)$$

at least for singly defined wave-functions. Thus, the operation is not conserved under the action of a single holonomy, unless $\delta_1 = 0$. This can be understood easily, since a general anisotropic state can be always made isotropic by setting $p_1 = p_2$, but if we change either p_1 or p_2 , then the condition required to now make them equal will, of course, change.

If we consider only isotropic applications of the holonomies, e.g., $\hat{h}_1 \hat{h}_2$, then the operation can be conserved under the action of \hat{Q}_{sym} , provided $\delta_1 = \delta_2$. To be explicit,

$$\begin{aligned} \hat{Q}_{\text{sym}} \left[\hat{h}_1 \hat{h}_2 |\Psi(p_1, p_2)\rangle \right] &= \hat{Q}_{\text{sym}} |\Psi(p_1 + \delta_1, p_2 + \delta_2)\rangle \\ &= |\Psi(p_1 + \delta_1, p_1 + \delta_1)\rangle \\ &= |\Psi(p_2 + \delta_2, p_2 + \delta_2)\rangle, \\ \Rightarrow p_1 + \delta_1 &= p_2 + \delta_2, \end{aligned} \quad (24)$$

which preserves the condition $p_1 = p_2$ iff $\delta_1 = \delta_2$, for singly defined wave-functions.

So, we found that the value of p in either direction is the same and it changes in the same way, namely it is isotropic. Notice that, for consistency, this requires that we only consider isotropic applications of the holonomy operators (e.g., the Hamiltonian), which is physically acceptable since if our system is to end up isotropic, then it must evolve isotropically. However, in the Bianchi Hamiltonian (Eq. (13)) the holonomies act via multiplication, rather than addition, of the variables thus,

$$\begin{aligned} \hat{Q}_{\text{sym}} \left[\hat{h}_1 \hat{h}_2 |\Psi(p_1, p_2)\rangle \right] &= \hat{Q}_{\text{sym}} |\Psi(\delta_1 p_1, \delta_2 p_2)\rangle \\ &= |\Psi(\delta_1 p_1, \delta_1 p_1)\rangle \\ &= |\Psi(\delta_2 p_2, \delta_2 p_2)\rangle, \\ \Rightarrow \delta_1 p_1 &= \delta_2 p_2, \end{aligned} \quad (25)$$

which again preserves the condition $p_1 = p_2$ iff $\delta_1 = \delta_2$, (again for singly defined wavefunctions). The full Bianchi Hamiltonian (Eq. (13)) contains linear combinations of operations of the form of Eq. (25) that do not satisfy $\delta_1 = \delta_2$. Thus, the condition $p_1 = p_2$ is not preserved. In order for the standard ‘‘improved dynamics’’ of the isotropic model to be recovered, one must project out the anisotropic degrees of freedom [14].

3.2. Alternative isotropic embedding

There is also another approach one could follow to forming an isotropic system from the fully (two-dimensional) anisotropic state $|\Psi(p_1, p_2)\rangle$. Let us define the symmetrisation operator by

$$\hat{Q}_{\text{sym}}|\Psi(p_1, p_2)\rangle = \frac{1}{2} \left[|\Psi(p_1, p_2)\rangle + |\Psi(p_2, p_1)\rangle \right]. \quad (26)$$

Thus, whilst (for example) $\hat{p}_1 \left[\hat{Q}_{\text{sym}}|\Psi(p_1, p_2)\rangle \right]$ does not appear to be isotropic, the physically expectation value of \hat{p}_1 between two such states is. To be clear,

$$\hat{p}_1 \left[\hat{Q}_{\text{sym}}|\Psi(p_1, p_2)\rangle \right] = \frac{1}{2} \left[p_1 |\Psi(p_1, p_2)\rangle + p_2 |\Psi(p_2, p_1)\rangle \right], \quad (27)$$

is not symmetric, however the inner product,

$$\begin{aligned} & \langle \Psi(p_1, p_2) | \hat{Q}_{\text{sym}} | \hat{p}_1 | \hat{Q}_{\text{sym}} \Psi(p_1, p_2) \rangle \\ &= \frac{1}{4} \left[p_1 + p_1 \langle \Psi(p_1, p_2) | \Psi(p_2, p_1) \rangle + p_2 \langle \Psi(p_2, p_1) | \Psi(p_1, p_2) \rangle + p_2 \right] \\ &= \frac{(p_1 + p_2)}{4} \left[1 + \langle \Psi(p_1, p_2) | \Psi(p_2, p_1) \rangle \right] \\ &= \langle \Psi(p_1, p_2) | \hat{Q}_{\text{sym}} | \hat{p}_2 | \hat{Q}_{\text{sym}} \Psi(p_1, p_2) \rangle, \end{aligned} \quad (28)$$

is clearly symmetric. Note that, we used $\langle X|Y\rangle = \langle Y|X\rangle$ and the notation

$$\langle \Psi(p_1, p_2) | \hat{Q}_{\text{sym}} | \equiv \left(\hat{Q}_{\text{sym}} | \Psi(p_1, p_2) \rangle \right)^\dagger. \quad (29)$$

Thus, the measured scale factor along either direction is the same.

We can now consider what the action of an holonomy is and, in particular, whether it is conserved under the symmetrisation operator. Consider, for example, the operation of \hat{h}_1 :

$$\hat{Q}_{\text{sym}} \left[\hat{h}_1 | \Psi(p_1, p_2) \rangle \right] = \frac{1}{2} \left[|\Psi(p_1 + \delta_1, p_2)\rangle + |\Psi(p_2, p_1 + \delta_1)\rangle \right]. \quad (30)$$

We can again check that this state is symmetric, in that the expectation values of the two p_i 's are the same, namely

$$\begin{aligned} & \langle \Psi(p_1, p_2) | \hat{h}_1 | \hat{Q}_{\text{sym}} | \hat{p}_1 | \hat{Q}_{\text{sym}} \hat{h}_1 \Psi(p_1, p_2) \rangle \\ &= \frac{p_1 + p_2 + \delta_1}{4} \left[1 + \langle \Psi(p_1 + \delta_1, p_2) | \Psi(p_2, p_1 + \delta_1) \rangle \right], \\ &= \langle \Psi(p_1, p_2) | \hat{h}_1 | \hat{Q}_{\text{sym}} | \hat{p}_2 | \hat{Q}_{\text{sym}} \hat{h}_1 \Psi(p_1, p_2) \rangle. \end{aligned} \quad (31)$$

The above conclusion is however only valid, if the symmetrisation operator is always the last one to be taken on the state. This has some undesirable features, in particular,

$$\hat{O} \left(\hat{Q}_{\text{sym}} |\Psi\rangle \right) \neq \hat{Q}_{\text{sym}} \left(\hat{O} |\Psi\rangle \right) , \quad (32)$$

namely that the symmetrisation operation does not commute with other operators, which is problematic, since the operation of a generic operator (\hat{O}) on a symmetric state, does not, on its own, lead to a symmetric state. In the next section, we will show how that this problem is avoided provided we are interested only in “symmetric operators”. In order to define such operators, it is useful to define a process we refer to as the “symmetric dual”.

4. The “symmetric dual”

Here, we will define the “symmetric dual” of operators and states. Note that, the symmetrised version of an operator must be used even if $p_1 = p_2$ (we refer the reader to the previous section), since one has to ensure that only symmetric products of the holonomies are taken.

Define the symmetrised version of a state, $|\Psi\rangle_{\text{sym}}$, as

$$|\Psi\rangle_{\text{sym}} \equiv \frac{1}{A} [|\Psi\rangle + |\Psi\rangle^{\text{S}}] , \quad (33)$$

$$= \frac{1}{A} [|\Psi(p_1, p_2)\rangle + |\Psi(p_2, p_1)\rangle] , \quad (34)$$

where by $|\Psi\rangle^{\text{S}}$ we denote the symmetric “dual” of a state $|\Psi(p_1, p_2)\rangle$. Similarly, define the symmetrised version of an operator, \hat{O}_{sym} , as

$$\hat{O}_{\text{sym}} \equiv \frac{1}{B} [\hat{O} + \hat{O}^{\text{S}}] , \quad (35)$$

where the operation of the “dual” operator is defined by requiring,

$$\left(\hat{O} |\Psi\rangle \right)^{\text{S}} = \hat{O}^{\text{S}} |\Psi\rangle^{\text{S}} , \quad (36)$$

and noting that $(|\Psi\rangle^{\text{S}})^{\text{S}} = |\Psi\rangle$, by definition. We thus get

$$\hat{O}_{\text{sym}} |\Psi\rangle_{\text{sym}} = \frac{1}{AB} \left[\hat{O} |\Psi\rangle + \hat{O} |\Psi\rangle^{\text{S}} + \hat{O}^{\text{S}} |\Psi\rangle + \hat{O}^{\text{S}} |\Psi\rangle^{\text{S}} \right] , \quad (37)$$

$$= \frac{1}{A} \left[|\tilde{\Psi}\rangle + |\tilde{\Psi}\rangle^{\text{S}} \right] , \quad (38)$$

where

$$|\tilde{\Psi}\rangle = \frac{1}{B} \left[\hat{O} |\Psi\rangle + \hat{O}^{\text{S}} |\Psi\rangle \right] . \quad (39)$$

Thus, we find that the operation of a symmetric operator on a symmetric state is (as expected) a symmetric state. This can be visualised on the two-dimensional (p_1, p_2) plane. The case of $p_1 = p_2$ is the diagonal line, and if we restrict ourselves to states that lie on this line, then we must also restrict to operators that act *diagonally*, as one can see in Fig. 1. However, if we consider a symmetric superposition of states, then in order

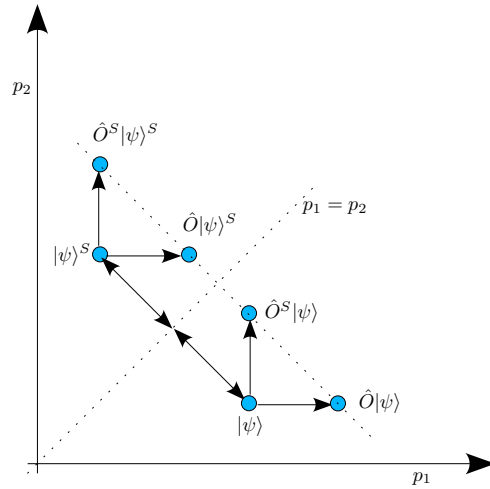


Figure 1. Visualisation of the symmetrisation of states.

for the operation of a general operator \hat{O} to give another symmetric state, one should require the operation to be symmetric. One easy way of doing this, is to define the dual operator, \hat{O}^S , and act with the symmetric combination $\hat{O}_{\text{sym}} \sim (\hat{O} + \hat{O}^S)$. Note that, this procedure is by no means unique, however such symmetric operators maintain the symmetric subspace made up of states $|\Psi\rangle_{\text{sym}}$, *i.e.*, if we restrict ourselves to considering only such operators, then we have a *super-selection* of states $|\Psi\rangle_{\text{sym}}$.

In Fig. 1, we have taken the operator \hat{O} to act on a state by shifting (p_1, p_2) . This is, of course, motivated by the fact that we want to consider holonomies, which we will assume act as shift operators, as in Eq. (20). Consider the restrictions placed on $\delta_1(p_1, p_2)$ and $\delta_2(p_1, p_2)$, by requiring that

$$\hat{h}_1 \hat{h}_2 = \hat{O} = \hat{O}^S, \quad (40)$$

i.e., that the action of a holonomy along each direction, is a symmetric operation in the sense defined above. As we will see, does not mean that the state is “shifted” diagonally. We have

$$\hat{h}_1 \hat{h}_2 |\Psi\rangle = \hat{h}_1 \hat{h}_2 |\Psi(p_1, p_2)\rangle = |\Psi(p_1 + \delta_1(p_1, p_2), p_2 + \delta_2(p_1, p_2))\rangle$$

and

$$\hat{h}_1 \hat{h}_2 |\Psi\rangle^S = \hat{h}_1 \hat{h}_2 |\Psi(p_2, p_1)\rangle = |\Psi(p_2 + \delta_1(p_2, p_1), p_1 + \delta_2(p_2, p_1))\rangle.$$

Thus, to satisfy

$$\left[\hat{O} |\Psi \right]^S = \hat{O}^S |\Psi\rangle^S = \hat{O} |\Psi\rangle^S, \quad (41)$$

one should require that

$$\begin{aligned} |\Psi(p_2 + \delta_2(p_1, p_2), p_1 + \delta_1(p_1, p_2))\rangle = \\ |\Psi(p_2 + \delta_1(p_2, p_1), p_1 + \delta_2(p_2, p_1))\rangle, \end{aligned} \quad (42)$$

which implies

$$\delta_2(p_1, p_2) = \delta_1(p_2, p_1) . \quad (43)$$

For illustration, let us consider some simple examples. For $\delta_1 = \delta_2 = \text{const.}$, the pair of holonomies acts diagonally, with a constant step. A more interesting example is

$$\delta_1 = \frac{1}{\sqrt{p_1}} = \frac{1}{\sqrt{p_2}} = \delta_2 ,$$

which is valid only when $p_1 = p_2$, namely on the diagonal line. This is the analogue of the “new” quantisation (or “improved” quantisation) approach. Motivated by the three-dimensional anisotropic results of Ref. [14], one can take

$$\delta_1(p_1, p_2) = \sqrt{\frac{p_1}{p_2}} , \quad \delta_2(p_1, p_2) = \sqrt{\frac{p_2}{p_1}} .$$

Clearly there are many other cases one may consider.

The question then is, what would be the measured step-size between states separated by this symmetric operator. We have

$$\begin{aligned} \text{sym} \langle \Psi(p_1, p_2) | \hat{p}_1 | \Psi(p_1, p_2) \rangle_{\text{sym}} &= \text{sym} \langle \Psi(p_1, p_2) | \hat{p}_2 | \Psi(p_1, p_2) \rangle_{\text{sym}} \\ &= \frac{p_1 + p_2}{A^2} \left[1 + \langle \Psi(p_1, p_2) | \Psi(p_2, p_1) \rangle \right] \\ &= \frac{p_1 + p_2}{A^2} \left[1 + \langle \Psi | \Psi \rangle^S \right] \\ &= p_1 + p_2 , \end{aligned} \quad (44)$$

where in the last line, we used the fact that the symmetrised state should be normalised, and we have chosen

$$\langle \Psi | \Psi \rangle^S = \langle \Psi(p_1, p_2) | \Psi(p_2, p_1) \rangle = A^2 - 1 . \quad (45)$$

Defining,

$$|\psi(p_1, p_2)\rangle_{\text{sym}} \equiv \hat{h}_1 \hat{h}_2 | \Psi(p_1, p_2) \rangle_{\text{sym}} , \quad (46)$$

and taking the inner product, we find

$$\begin{aligned} \text{sym} \langle \psi(p_1, p_2) | \hat{p}_1 | \psi(p_1, p_2) \rangle_{\text{sym}} &= \text{sym} \langle \psi(p_1, p_2) | \hat{p}_2 | \psi(p_1, p_2) \rangle_{\text{sym}} \\ &= \frac{p_1 + p_2 + \delta_1(p_1, p_2) + \delta_1(p_2, p_1)}{A^2} \left[1 + \left\langle \Psi(\tilde{p}_1, \tilde{p}_2) \middle| \Psi(\tilde{p}_2, \tilde{p}_1) \right\rangle \right] , \end{aligned} \quad (47)$$

where

$$(\tilde{p}_1, \tilde{p}_2) = (p_1 + \delta_1(p_1, p_2), p_2 + \delta_2(p_1, p_2))$$

and

$$(\tilde{p}_2, \tilde{p}_1) = (p_2 + \delta_1(p_2, p_1), p_1 + \delta_2(p_2, p_1)) .$$

We can then evaluate what the measured step-size, Δ_{sym} , would be:

$$\begin{aligned} \Delta_{\text{sym}} &\equiv \text{sym} \langle \psi(p_1, p_2) | \hat{p}_i | \psi(p_1, p_2) \rangle_{\text{sym}} - \text{sym} \langle \Psi(p_1, p_2) | \hat{p}_i | \Psi(p_1, p_2) \rangle_{\text{sym}} \\ &= (p_1 + p_2)(X - 1) + (\delta_1(p_1, p_2) + \delta_2(p_1, p_2))X , \end{aligned} \quad (48)$$

where X is defined as

$$X = \frac{1 + \langle \tilde{\Psi}(\tilde{p}_1, \tilde{p}_2) | \tilde{\Psi}(\tilde{p}_2, \tilde{p}_1) \rangle}{1 + \langle \tilde{\Psi}(p_1, p_2) | \tilde{\Psi}(p_2, p_1) \rangle}. \quad (49)$$

Thus, we have

$$X = \frac{1 + \langle \tilde{\chi} | \tilde{\chi} \rangle^S}{1 + \langle \tilde{\Psi} | \tilde{\Psi} \rangle^S}, \quad (50)$$

and we used $|\tilde{\chi}\rangle \equiv |\tilde{\Psi}(\tilde{p}_1, \tilde{p}_2)\rangle$ to emphasise the possibility of a difference from $|\tilde{\Psi}(p_1, p_2)\rangle$. The normalisation of the symmetrised state gives the denominator (see, Eq. (45)), but we get $X = 1$ iff the norm of every state (or every state in the superselection, if one is possible) is the same. In particular, if we were to restrict ourselves to $p_1 = p_2 \equiv p$, this would imply that $\tilde{p}_1 = \tilde{p}_2$ and hence that $|\tilde{\chi}\rangle = |\tilde{\chi}\rangle^S$ and $|\tilde{\Psi}\rangle = |\tilde{\Psi}\rangle^S$. This gives $X = 1$, which means $\Delta = \delta_1(p_1, p_2) + \delta_1(p_2, p_1) = 2\delta(p)$. However, for general (“symmetric dual”) states, X can be greater, or less, than unity. Thus, the step-size would include a different dependence on p_1 and p_2 ; it can be seen as a “lattice refinement” analogue.

In Section 5, we will extend to three directions and explicitly use the Bianchi I model of Ref. [14] to evaluate the step-sizes for the two different symmetrisation schemes. Indeed, the above can be used directly for the case that two of the directions are set equal to each other. For example, if the restriction $p_1 = p_2 \neq p_3$ is made, then we are restricted to a sub-class of Bianchi I models, which are essentially two-dimensional. Such cylindrically symmetric models would be useful for investigating black holes, where the expectation value of \hat{p} is different along the radial and angular directions.

Intuitively, what we have is that the use of superpositions to symmetrise a state, means that (for symmetric operators, such as pairs of holonomies with the correct type of step) we can find symmetric states which however have different step-sizes than in the standard case. Essentially, the standard case is just a special case of the above procedure, when the two states used in the superposition are the same.

To be clear, we have chosen to normalise the states, such that

$$\langle \Psi(p_1, p_2) | \Psi(p_2, p_1) \rangle = A^2 - 1. \quad (51)$$

However, the restriction $\delta_1(p_1, p_2) = \delta_2(p_2, p_1)$, is not sufficient to simultaneously specify the normalisation of

$$\langle \Psi[p_1 + \delta_1(p_1, p_2), p_2 + \delta_2(p_1, p_2)] | \Psi[p_2 + \delta_1(p_2, p_1), p_1 + \delta_2(p_2, p_1)] \rangle.$$

For $p_1 = p_2$,

$$\begin{aligned} \Delta_{\text{sym}} &= \delta_1(p_1, p_2) + \delta_2(p_1, p_2) \\ &= 2\delta(p), \end{aligned} \quad (52)$$

where the last equality follows from the fact that $\delta_1(p_1, p_2) = \delta_2(p_2, p_1) \equiv \delta(p)$. For any other choice of (p_1, p_2) , such that $p_1 \neq p_2$, this is not the case. Whilst we cannot rule out the possibility that the (symmetric) state evolves towards a state made up of a superposition of two even more anisotropic states, this seems unlikely.

If we take that, at least for large $p_1 + p_2$, we have $|\Psi(p_1, p_2)\rangle \rightarrow |\Psi(p, p)\rangle$, then we expect $X \searrow 1$ (*i.e.*, the inner product of the “symmetric dual” states tends to 1 from above; it can be understood by thinking of the states as vectors in a vector space). Thus,

$$\Delta_{\text{sym}} \geq \delta_1(p_1, p_2) + \delta_2(p_1, p_2) , \quad (53)$$

whilst tending to equality as p_1 and p_2 become large and tend to each other.

For the case of an isotropic system, made up of ever more anisotropic states, the converse is true, namely,

$$\Delta_{\text{sym}} \leq \delta_1(p_1, p_2) + \delta_2(p_1, p_2) , \quad (54)$$

whilst not necessarily tending to equality as p_1 and p_2 become large. Note that here we also require Eq. (43) to hold, which is what ensures that the measured step-size, Δ_{sym} , is indeed symmetric under interchange of p_1, p_2 .

In conclusion, in general the step-size can be either larger or smaller (and has a different dependence on (p_1, p_2)) than in the $p_1 = p_2$ case. For the case of a state made up of a superposition of ever more isotropic states, the measured step-size tends to the *standard* step-size as p_1, p_2 become large. This is precisely what is modelled by “lattice refinement”^{||}, albeit in an heuristic way, except that we might expect the “lattice refinement” to approach the special case of the “new” quantisation refinement rate, for large scales.

5. The three-dimensional case

In this section, we demonstrate a concrete example of the above procedure for the case of an anisotropic Bianchi I model, recently consistently derived in Ref. [14]. In this case the states are defined on the orthonormal basis $|\lambda_1, \lambda_2, \lambda_3\rangle$, where λ_i are essentially the scale factors along the three directions of the Bianchi model.

In the standard approach, isotropic states are taken to be those in which the dependence on λ_1, λ_2 and λ_3 is “integrated out”, or more precisely, defining the volume of the state as $\nu = 2\lambda_1\lambda_2\lambda_3$ to eliminate one of the directions (λ_3), the map

$$|\Psi(\lambda_1, \lambda_2, \nu)\rangle \rightarrow \left| \sum_{\lambda_1, \lambda_2} \Psi(\lambda_1, \lambda_2, \nu) \right\rangle \equiv |\Psi(\nu)\rangle , \quad (55)$$

produces isotropic states. This projection produces exactly isotropic system with the equations given in terms of the volume (called the “ ν quantisation”). Here we will show that it is also possible to have a super-selection of states that are isotropic and which evolve according to an alternative quantisation procedure which may be modelled by lattice refinement. We are going to work with the three scale factors λ_i , to demonstrate that there is an ambiguity in exactly what the measured volume of such isotropic states

^{||} Here and throughout we use “lattice refinement” to refer to any quantisation in which the lattice *defined with respect to μ* is dynamically varying. In particular, then, “new” quantisation is simply a special case of such “lattice refinement”.

would be. Firstly, we want to find a symmetric superposition of states that is isotropic, in the sense defined in Section 3.2.

Consider the state

$$|\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle \equiv \frac{1}{A} \left[|\Psi(\lambda_1, \lambda_2, \lambda_3)\rangle + |\Psi(\lambda_3, \lambda_1, \lambda_2)\rangle + |\Psi(\lambda_2, \lambda_3, \lambda_1)\rangle \right], \quad (56)$$

with the additional restriction

$$\begin{aligned} & \langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_3, \lambda_1, \lambda_2) \rangle \\ &= \langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle \\ &= \langle \Psi(\lambda_3, \lambda_1, \lambda_2) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle \\ &\equiv f, \end{aligned} \quad (57)$$

on the anisotropic states being used. This additional requirement is not present in the two-dimensional case, since there is only one cross-correlator between two anisotropic states, which is trivially equal to itself. In the three-dimensional system however, discussed here, we have three cross-correlators, which need to be equal if the state is to be isotropic. This is a restriction that would be present in any dimension higher than 2.

The expectation values of the scale factors along each direction of such a state are

$$\langle \hat{\lambda}_i \rangle = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}. \quad (58)$$

Note that we have chosen A to normalise the state. It is worth mentioning that, by choosing a different state, the conditions required for the expectation values to match change. In particular, for the state

$$\begin{aligned} |\chi(\lambda_1, \lambda_2, \lambda_3)\rangle \equiv \frac{1}{A} \left[& \langle \Psi(\lambda_3, \lambda_1, \lambda_2) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle |\Psi(\lambda_1, \lambda_2, \lambda_3)\rangle \right. \\ & + \langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_3, \lambda_1, \lambda_2) \rangle |\Psi(\lambda_2, \lambda_3, \lambda_1)\rangle \\ & \left. + \langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle |\Psi(\lambda_3, \lambda_1, \lambda_2)\rangle \right], \end{aligned} \quad (59)$$

to be isotropic, we require

$$\begin{aligned} & |\langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_3, \lambda_1, \lambda_2) \rangle|^2 \\ &= |\langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle|^2 \\ &= |\langle \Psi(\lambda_3, \lambda_1, \lambda_2) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle|^2. \end{aligned} \quad (60)$$

With such a state, one also finds that $\langle \hat{\lambda}_i \rangle = (\lambda_1 + \lambda_2 + \lambda_3)/3$, which explicitly demonstrates the non-uniqueness of the choice of isotropic embedding.

For both these states, the measured scale factor is equal in each direction and is given by the average of the scale factors of the underlying, anisotropic states. However, the measured volume of such a state, found using the volume operator,

$$\hat{\nu} |\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle = 2\lambda_1 \lambda_2 \lambda_3 |\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle, \quad (61)$$

is just $\nu = 2\lambda_1 \lambda_2 \lambda_3$, which is *not* the cube of the measured scale factor. Thus, whilst it is the eigenvalue of the anisotropically defined volume operator, it is not necessarily what

we would measure as the volume. Essentially, this is because whilst the scale factors λ_i are measured to be equal in each direction, they are not eigenvalues of the state, *i.e.*, $\hat{\lambda}_i|\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle \neq \lambda_i|\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle$. However, both the average and the product (*i.e.*, the volume) of the scale factors are eigenvalues. It is this ambiguity, that leads to the possibility of deviations from the standard isotropic case. In particular, for our choice of isotropic states, we have essentially defined the projection,

$$|\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle \rightarrow \tilde{\Psi}(\langle\lambda\rangle, \nu(\langle\lambda\rangle)), \quad (62)$$

where the state depends only on the average, $\langle\lambda\rangle$, of the three scale factors, and their product (*i.e.*, the volume) can be viewed as a complicated function of this average.

The Hamiltonian found in Ref. [14] is linear and can be written as a symmetric combination of three operators that act on each λ_1 , λ_2 and λ_3 equally. Thus the Hamiltonian operator satisfies our definition of a “symmetric operator”, Eq. (35) and hence we can directly apply the above states to it. Alternatively one can directly evaluate the Hamiltonian acting on a symmetric state and show that it contains only states that are themselves symmetric.

As it is explained in Ref. [14], the Hamiltonian acts on uniform steps along the ν direction, just as in the “new quantisation” of isotropic cosmologies. Just as in the standard case, one finds that the Hamiltonian, acting on symmetric states, produces a *super-selection* of points along the ν direction, that are equally spaced, however from Eq. (62) it is clear that this no longer implies a uniform spacing in either $\langle\lambda\rangle$ (the measured scale factor) or $\langle\lambda\rangle^3$ (the measured volume). To deduce the step-size in the measured scale factor (which would give the step-size in the holonomies that should be used in the case of isotropic LQC), one would have to invert the function $\nu(\langle\lambda\rangle)$. The use of “lattice refinement” in isotropic LQC is a heuristic first approximation to this, and whilst the power law like refinements typically used in the literature [8] are rather crude, they have the advantage of being analytically tractable.

There are many ways that such isotropic states can be produced from superpositions of anisotropic states. Certainly, those given in Eqs. (56) and (59) are not the most general ones, however they serve to demonstrate the possibility that whilst the underlying, anisotropic system requires specific, well defined holonomy step-sizes, the isotropic version contains an ambiguity in the precise value of the step-size that one can take. Here, we have focused only on the Bianchi I model, for which a full understanding of the quantisation exists, however one expects similar effects to occur when starting from other anisotropic and inhomogeneous models. In particular, the motivation for lattice refinement in LQC comes from the fact that one expects that the additional degrees of freedom present in the underlying, full theory, will not exactly produce the symmetry reduced models of cosmology. Here we have shown that the set of states that are measured to be isotropic form a super-selection of states within the Bianchi 1 model, which have a non-constant measured volume step. In this sense we have shown that the anisotropic degrees of freedom of the Bianchi 1 model, that are missing in the symmetric FLRW case, can produce a non-standard evolution of the FLRW volume,

which is exactly what is modelled by lattice refinement. It is important to note however that the Hamiltonian governing the evolution of these isotropic states is not the standard FLRW Hamiltonian that is usually considered in lattice refinement models. Thus whilst it may be hoped that lattice refinement models capture some of the consequences of a varying discreteness size, the use of the standard Hamiltonian makes this a rather crude approach.

Finally, it is important to note that in Eq. (62), we chose to view the measured scale factor of the isotropic state as the primary variable, however one could have chosen to view the eigenvalue of the volume operator as fundamental, in which case the isotropic volume of the state would be ν and the isotropic Hamiltonian would indeed produce a *super-selection* with uniform steps in volume. In this case however the scale factor of the state ($\langle\lambda\rangle$) would be related to the volume through a complicated function (essentially the inverse of $\nu(\langle\lambda\rangle)$). This would seem to be largely a matter of taste, however it is possible that future work may choose one view point over the other.

6. Conclusions

Historically, the quantisation of FLRW cosmologies within LQC, considered the triad element to be the fundamental variable that is to be quantised [18]. However, it was realised that this leads to an instability within the theory at large scales [11], which is particularly problematic for inflation [7]. It was later shown that these problems are cured [7] for the “new” quantisation approach in which the physical volume is considered as the fundamental object to be quantised ¶ [5].

In terms of the basic triad component p , the discretisation of the Hamiltonian occurs with step-size given by $p^{\mathcal{A}}$, where $\mathcal{A} = 0$ for the original quantisation approach and $\mathcal{A} = -1/2$ for the “new” quantisation. These two possibilities are the two limits in which, either only the labels of the edge spins, or the number of vertices dynamically change in the underlying lattice (respectively for the “old” and “new” quantisation approaches). Whilst the behaviour of the full Hamiltonian constraint on a general lattice has yet to be derived from LQG, it is expected that the constraint on \mathcal{A} should be $0 < \mathcal{A} < -1/2$, since the Hamiltonian acts by a combination of both processes. A full derivation of this from an underlying LQG point of view, is not yet possible, and here we do not restrict the value of \mathcal{A} and refer to the general case as “lattice refinement” ⁺. There has been much work (and significant debate) on whether “lattice refinement” for a general \mathcal{A} is theoretically consistent [5, 6, 9] and strictly working within the symmetry reduced FLRW models, it appears that only for $\mathcal{A} = -1/2$ (“new” quantisation) is this the case [6].

¶ Although the onset of inflation within LQC may remain problematic, even with the use of the “new” quantisation [19]; a question which has been recently addressed in Ref. [20].

⁺ The term “lattice refinement” is also sometimes used to include only the cases with $-1/2 < \mathcal{A} < 0$ case, which explicitly excluded the “new” quantisation approach. Here, as previously, we use the term “lattice refinement” in its most general sense, to refer all quantisations in which the step-size is given by $p^{\mathcal{A}}$, which includes both the “new” and “old” quantisations as special cases.

However, here we have shown that, given a (fully consistent) quantum anisotropic model, it is possible to find isotropic states for which the discrete step-size of the isotropically embedded Hamiltonian constraint is not (necessarily) that of “new” quantisation. This demonstrates the possibility of “lattice refinement” being due to the degrees of freedom that are absent in the isotropic model. It is of course possible to embed* the “new” quantisation approach within the anisotropic models and it may be that this is the more natural procedure, however without an *a priori* knowledge of which embedding process to follow, the conservative approach would be to consider both.

We have shown that the difference between the two procedures is essentially due to what one considers to be more fundamental, the volume of the underlying states (ν) or the measured volume of the symmetric state ($\langle\lambda\rangle^3$), which are not necessarily equal. Choosing ν leads to the “new” quantised Hamiltonian of isotropic cosmology, whilst choosing $\langle\lambda\rangle^3$ results in some kind of different “lattice refinement”. This “lattice refinement” is significantly more complicated than the single power law behaviour, p^A , usually considered, however one may hope to capture some of its effects using the simple model.

It is important to note that here we have started from a Bianchi I model and looked for isotropic embeddings, however similar ideas should apply to more general models. We are not suggesting that the Bianchi I model is the underlying geometry of our universe, however it is sufficient to demonstrate the ambiguities associated with finding an effective isotropic cosmology within a model that has more degrees of freedom. This can be thought of as the source of the “lattice refinement” ambiguity of LQC, however more work is needed to find out if the simple power law model is a sufficiently accurate approximation of this ambiguity. More work is also necessary to see how this new formulation of “lattice refinement”, in particular the new relationship between the effective scale factor, $\langle\lambda\rangle$, and the volume, ν , affects the consistency of the isotropic theory.

Finally, one may be concerned with the fact that simply deciding which variables are the more fundamental one, can have important implications for the large scale physics of the theory, however this is not an unusual occurrence for quantum systems. In all quantum systems, we start from a classical action and follow specific procedures to produce the quantum version of the theory. Precisely which classically equivalent action to choose is a source of quantum ambiguities. From a theoretical point of view, one may hope to have the choice picked out from a more complete understanding of the underlying theory, whilst on a practical level, it is often more useful to parametrise our ignorance and fix these parameters from experiments and observations. In the latter case one simply assumes that the approach is an effective theory, approximating some, as yet unknown, full underlying theory, which is sufficiently accurate for the current data. In the case of LQC (and indeed classical cosmology) we are constantly working within the framework of an effective theory of universal dynamics, in which (it

* Here the term “embedding” is used in a loose sense. The FRLW model is in fact a “projection” of a system with more degrees of freedom.

appears) the symmetry reduction is typically a valid approximation. In this context, “lattice refinement” is simply an effective parametrisation of the fact that cosmological symmetries are not exact symmetries of the physical universe. Whilst measuring any consequences of this effective parametrisation seems unlikely, it is possible that the largest scales of the Cosmic Microwave Background (which correspond to the earliest scales to leave the horizon) might retain a signature of this LQC regime, from which ambiguities such as “lattice refinement” may be fixed.

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