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Towards Canonical Quantum Gravity for Geometries Admitting Maximally Symmetric Two-dimensional Surfaces

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Abstract

The 3+1 (canonical) decomposition of all geometries admitting two-dimensional space-like surfaces is exhibited. A proposal consisting of a specific re-normalization **Assumption** and an accompanying **Requirement** is put forward, which enables the canonical quantization of these geometries, through a generalization of Kuchar's quantization scheme in the case infinite degrees of freedom. The resulting Wheeler-deWitt equation is based on a re-normalized manifold parameterized by three smooth scalar functionals. The entire space of solutions to this equation is analytically given, exploiting the freedom left by the imposition of the **Requirement** and contained in the third functional.

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1 Introduction

Dirac's seminal work on his formalism for a self-contained treatment of systems with constraints [1], [2], [3], [4] has paved the way for a systematic treatment of constrained systems. Some of the landmarks in the study of constrained systems have been the connection between constraints and invariances [5], the extension of the formalism to describe fields with half-integer spin through the algebra of Grassmann variables [6] and the introduction of the BRST formalism [7]. All the classical results obtained so far have made up an armoury prerequisite for the quantization of gauge theories and there are several excellent reviews studying constraint systems with a finite number of degrees of freedom [8] or constraint field theories [9], as well as more general presentations [10], [11], [12], [13], [14]. In particular, the conventional canonical analysis approach of quantum gravity has been initiated by P.A.M. Dirac [15], P.G. Bergmann [16] and B.S. deWitt [17]. For a modern account see [18].

In the absence of a full theory of quantum gravity, it is reasonably important to address the quantization of (classes of) simplified geometries. The most elegant way to achieve a degree of simplification is to impose some symmetry. For example, the assumption of a G_3 symmetry group acting simply transitively on the surfaces of simultaneity, i.e. the existence of three independent space-like Killing Vector Fields (KVF), leads to classical and subsequently quantum homogeneous cosmology (see e.g. [19], [20]). The imposition of lesser symmetry, e.g. fewer KVF's, results in the various inhomogeneous cosmologies [21]. The canonical analysis under the assumption of spherical symmetry, which is a G_3 acting multiply transitively on two-dimensional space-like subsurfaces of the three-slices, has been first considered in [22], [23]. Quantum black holes have been treated, for instance, in [24] while in [25], [26] a lattice regularization has been employed to deal with the infinities arising due to the ill-defined nature of the quantum operator constraints. In this work we consider the quantization of all geometries admitting two-dimensional surfaces of maximal symmetry, i.e. spheres (constant positive curvature), planes (zero curvature) and Gauss-Bolyai-Lobachevsky (henceforth GBL) spaces (constant negative curvature). In the second section we give the reduced metrics, the space of classical

solutions and the Hamiltonian formulation of the reduced Einstein-Hilbert action principle, resulting in one (quadratic) Hamiltonian and one (linear) momentum first class constraint. In the third section we consider the quantization of this constraint system following Dirac's proposal of implementing the quantum operator constraints as conditions annihilating the wave-function [4]. Our guide-line is a conceptual generalization of the quantization scheme developed in [27], [28] for the case of constraint systems with finite degrees of freedom, to the present case, which is enabled through the use of a specific re-normalization **Assumption** and an associated **Requirement**. After the symmetry reduction, the system still represents an one-dimensional field theory since all remaining metric components depend on time and the radial coordinate. Nevertheless, we manage to extract a Wheeler-deWitt equation in terms of three smooth scalar functionals of the reduced geometries. The exploitation of a residual freedom left by the imposition of the **Requirement** enables us to acquire the entire solution space to this equation. Finally, some concluding remarks are included in the discussion, while the proof of the existence for the third functional is given in the Appendix.

2 Possible Metrics, Classical Solutions and Hamiltonian Formulation

Our starting point is the two-dimensional spaces of positive, zero and negative constant curvature. Their line elements are respectively:

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad ds^2 = d\theta^2 + \theta^2 d\phi^2, \quad ds^2 = d\theta^2 + \sinh^2 \theta d\phi^2 \quad (2.1)$$

with an obvious range of the coordinates for each case. The corresponding (maximal) symmetry groups are generated by the following KVF's:

$$\xi_1 = \frac{\partial}{\partial \phi}, \quad \xi_2 = -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad \xi_3 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \quad (2.2)$$

$$\xi_1 = \frac{\partial}{\partial \phi}, \quad \xi_2 = -\cos \phi \frac{\partial}{\partial \theta} + \frac{\sin \phi}{\theta} \frac{\partial}{\partial \phi}, \quad \xi_3 = \sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \phi}{\theta} \frac{\partial}{\partial \phi} \quad (2.3)$$

$$\xi_1 = \frac{\partial}{\partial \phi}, \quad \xi_2 = -\cos \phi \frac{\partial}{\partial \theta} + \coth \theta \sin \phi \frac{\partial}{\partial \phi}, \quad \xi_3 = \sin \phi \frac{\partial}{\partial \theta} + \coth \theta \cos \phi \frac{\partial}{\partial \phi} \quad (2.4)$$

We next promote these KVF's to four-dimensional fields by adding to each and every of them the zero-sum $0 \frac{\partial}{\partial t} + 0 \frac{\partial}{\partial r}$. We then enforce these vector fields as symmetries of a generic space-time metric $g_{IJ}(t, r, \theta, \phi)$, i.e. we demand that its Lie derivative with respect to all three fields for each family vanishes. We thus arrive at three classes of

metrics, collectively described by the following line element:

$$ds^2 = \left(-\alpha(t, r)^2 + \frac{\beta(t, r)^2}{\gamma(t, r)^2} \right) dt^2 + 2\beta(t, r) dt dr + \gamma(t, r)^2 dr^2 + \psi(t, r)^2 d\theta^2 + \psi(t, r)^2 f(\theta)^2 d\phi^2 \quad (2.5)$$

where $f(\theta) = \sin \theta$ in the case of spherical symmetry, $f(\theta) = \theta$ in the case of plane symmetry, and $f(\theta) = \sinh \theta$ in the case of the *GBL* symmetry.

In order to attain the classical space of solution for these geometries one can, exploiting the freedom to change coordinates in the (t, r) subspace, bring the upper left block of the metric in conformally flat form and readily solve the vacuum Einstein's field equations. The result can be compactly summarized in the light-cone coordinates u, v by the following line elements:

$$ds^2 = 2\epsilon \frac{A + 2\psi(u, v)}{4\psi(u, v)} du dv + \psi(u, v)^2 d\theta^2 + \psi(u, v)^2 f(\theta)^2 d\phi^2, \quad (2.6)$$

where

$$\psi(u, v) = -\frac{A}{2} \left(1 + \text{ProductLog} \left(-\frac{\exp^{-\frac{\lambda(A+u+\epsilon v)}{A}}}{A} \right) \right),$$

$\epsilon = \pm 1$ and $\lambda = \pm 1$ for $f(\theta) = \sin \theta$, $\lambda = \pm i$ for $f(\theta) = \sinh \theta$; $\text{ProductLog}(z)$ is the principal solution for w to the equation $z = w \exp^w$. The corresponding to the plane symmetric case $f(\theta) = \theta$ line-element is given by:

$$ds^2 = 2\epsilon \frac{1}{\psi(u, v)} du dv + \psi(u, v)^2 d\theta^2 + \psi(u, v)^2 \theta^2 d\phi^2, \quad (2.7)$$

where

$$\psi(u, v) = \pm \sqrt{2u + 2\epsilon v}.$$

A somewhat more detailed derivation of the entire classical solution space, for $\Lambda \neq 0$ as well, is presented in Appendix A: As we see, for each type of symmetry, there exists one solution in which the metric components depend only on the radial coordinate (“point-like”) and another with only time dependence (“cosmological”). For example, in the case of spherical symmetry and $\Lambda = 0$, we have the well known Schwarzschild and Kantowski-Sachs metrics respectively.

Notice that in the cases of spherical and *GBL* symmetry the only degree of freedom left, apart from the cosmological constant Λ , is the *constant* A . This constant (which is to be identified with e.g. the Schwarzschild Mass in the corresponding case) is essential, as far as geometry is concerned: it can not be eliminated or even altered by a coordinate transformation, which is the “gauge” of any geometrical theory. In this sense this *constant* can, and must, be regarded as a *dynamical* degree of freedom, despite that there is no time evolution of it; the dynamical character of A is emanating from the fact that different values of it correspond to distinct geometries. The remaining case of plane

symmetry is not so rich, since it does not contain any such constant, and thus constitutes an isolated point in the classical vacuum solution space.

One might wonder how this interpretation of “dynamical character” can be represented at the quantum level. The answer can be found in the local characterization of A in terms of invariant relations among geometrical quantities constructed out of the Riemann tensor and its covariant derivatives, which by the way ensure the essential character of A : Expressing e.g. (A.6) in phase-space and constructing its quantum operator analogue, we have a quantum dynamical evolution of A when we apply this operator on the states we find. This gives another type of evolution to *empty* spherical symmetric configurations, which does not make use of external matter source operators.

The Hamiltonian formulation of the system (2.5) proceeds, briefly, as follows (see, e.g., chapter 9 of [10]):

We first define the vectors

$$\eta^I = \frac{1}{\alpha(t, r)} \left(1, -\frac{\beta(t, r)}{\gamma(t, r)^2}, 0, 0 \right), \quad F^I = \eta_{;J}^J \eta^I - \eta_{;J}^I \eta^J,$$

where $I, J = 0, 1, 2, 3$ are space-time indices and “;” stands for covariant differentiation with respect to (2.5). Then, utilizing the Gauss-Codazzi equation, we eliminate all second time-derivatives from the Einstein-Hilbert action and arrive at an action quadratic in the velocities, $I = \int d^4x \sqrt{-g} (R - 2\Lambda - 2F_{;I}^I)$. The application of the Dirac algorithm results firstly in the primary constraints $P_\alpha \equiv \frac{\delta L}{\delta \dot{\alpha}} \approx 0$, $P^\beta \equiv \frac{\delta L}{\delta \dot{\beta}} \approx 0$ and the Hamiltonian

$$H = \int (N^o \mathcal{H}_o + N^i \mathcal{H}_i) dr, \quad (2.8)$$

where

$$N^o = \alpha(t, r), \quad N^1 = \frac{\beta(t, r)}{\gamma(t, r)^2}, \quad N^2 = 0, \quad N^3 = 0$$

and $\mathcal{H}_o, \mathcal{H}_i$ are given by

$$\mathcal{H}_o = \frac{1}{2} G^{\alpha\beta} \pi_\alpha \pi_\beta + V, \quad (2.9a)$$

$$\mathcal{H}_1 = -\gamma \pi'_\gamma + \psi' \pi_\psi, \quad \mathcal{H}_2 = 0, \quad \mathcal{H}_3 = 0, \quad (2.9b)$$

while the indices $\{\alpha, \beta\}$ take the values $\{\gamma, \psi\}$ and $' = \frac{d}{dr}$. The reduced Wheeler-deWitt super-metric $G^{\alpha\beta}$ reads

$$G^{\alpha\beta} = \begin{pmatrix} \frac{\gamma}{4\psi^2} & -\frac{1}{4\psi} \\ -\frac{1}{4\psi} & 0 \end{pmatrix}, \quad (2.10)$$

while the potential V is

$$V = -2\epsilon\gamma + 2\Lambda\gamma\psi^2 - 2\frac{\psi'^2}{\gamma} + 4\left(\frac{\psi\psi'}{\gamma}\right)' \quad (2.11)$$

with $\epsilon = \{1, 0, -1\}$ for the families (2.5) of two-dimensional subspaces with positive, zero or negative constant curvature, respectively. The requirement for preservation, in time, of the primary constraints leads to the secondary constraints

$$\mathcal{H}_o \approx 0, \quad \mathcal{H}_1 \approx 0 \quad (2.12)$$

At this stage, a tedious but straightforward calculation produces the following open Poisson bracket algebra of these constraints:

$$\begin{aligned} \{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} &= \left[\frac{1}{\gamma^2(r)} \mathcal{H}_1(r) + \frac{1}{\gamma^2(\tilde{r})} \mathcal{H}_1(\tilde{r}) \right] \delta'(r, \tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} &= \mathcal{H}_o(r) \delta'(r, \tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} &= \mathcal{H}_1(r) \delta'(r, \tilde{r}) - \mathcal{H}_1(\tilde{r}) \delta(r, \tilde{r})', \end{aligned} \quad (2.13)$$

indicating that they are first class and also signaling the termination of the algorithm. Thus, our system is described by (2.12) and the dynamical Hamilton-Jacobi equations $\frac{d\pi_\gamma}{dt} = \{\pi_\gamma, H\}$, $\frac{d\pi_\psi}{dt} = \{\pi_\psi, H\}$. One can readily check (as one must always do with reduced action principles) that these four equations, when expressed in the velocity phase-space with the help of the definitions $\frac{d\gamma}{dt} = \{\gamma, H\}$, $\frac{d\psi}{dt} = \{\psi, H\}$, are completely equivalent to the four independent Einstein's field equations satisfied by (2.5):

$$E_{IJ} \equiv R_{IJ} - \frac{1}{2} g_{IJ} R + \Lambda g_{IJ} = 0 \quad I, J = 0, 1, 2, 3. \quad (2.14)$$

We end up this section by noting a few facts concerning the transformation properties of $\gamma(t, r)$, $\psi(t, r)$ and their derivatives under changes of the radial variable r of the form $r \rightarrow \tilde{r} = h(r)$. As it can easily be inferred from (2.5):

$$\tilde{\gamma}(\tilde{r}) = \gamma(r) \frac{dr}{d\tilde{r}}, \quad \tilde{\psi}(\tilde{r}) = \psi(r), \quad \frac{d\tilde{\psi}(\tilde{r})}{d\tilde{r}} = \frac{d\psi(r)}{dr} \frac{dr}{d\tilde{r}}, \quad (2.15)$$

where the t -dependence has been omitted for the sake of brevity. Thus, under the above coordinate transformations, ψ is a scalar, while γ , ψ' are covariant rank 1 tensors (one-forms), or, equivalently in one dimension, scalar densities of weight -1 . Therefore, the scalar derivative is not $\frac{d}{dr}$ but rather $\frac{d}{\gamma dr}$. Finally, if we consider an infinitesimal transformation $r \rightarrow \tilde{r} = r - \eta(r)$, it is easily seen that the corresponding changes induced on the basic fields are:

$$\delta \gamma(r) = (\gamma(r) \eta(r))', \quad \delta \psi(r) = \psi'(r) \eta(r) \quad (2.16)$$

i.e., nothing but the one-dimensional analogue of the appropriate Lie derivatives. With the use of (2.16), we can reveal the nature of the action of \mathcal{H}_1 on the basic configuration space variables as that of the generator of spatial diffeomorphisms:

$$\begin{aligned} \left\{ \gamma(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} &= (\gamma(r) \eta(r))', \\ \left\{ \psi(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} &= \psi'(r) \eta(r). \end{aligned} \tag{2.17}$$

Thus, we are justified to consider \mathcal{H}_1 as the representative, in phase-space, of an arbitrary infinitesimal re-parametrization of the radial coordinate. In the same manner we can also see that the action of the quadratic constraint \mathcal{H}_o on the basic configuration space variables is identical to an infinitesimal change of the time coordinate, see e.g. p. 21 in [29]. The qualitative difference in the corresponding proof is that the dynamical equations giving the time derivatives of the momenta are explicitly used; hence the terms generator of “time deformations” or “dynamical evolution” used for \mathcal{H}_o .

Thus, the linear and quadratic constraints are seen to be the generators of space-time diffeomorphisms, i.e. they represent in phase space the local “gauge” coordinate transformations; therefore the imposition of their proper quantum analogues will guarantee the invariance of the ensuing quantum theory under the aforementioned “gauge”. Our study is limited, at the present state, to this goal and is thus not concerned with global issues like boundary terms.

3 Quantization

We are now interested in attempting to quantize this Hamiltonian system following Dirac’s general spirit of realizing the classical first class constraints (2.12) as quantum operator constraint conditions annihilating the wave function. The main motivation behind such an approach is the justified desire to construct a quantum theory manifestly invariant under the “gauge” generated by the constraints. To begin with, let us first note that, despite the simplification brought by the imposition of the symmetry (2.2), (2.3), (2.4), the system is still a field theory in the sense that all configuration variables and canonical conjugate momenta depend not only on time (as is the case in homogeneous cosmology), but also on the radial coordinate r . Thus, to canonically quantize the system in the Schrödinger representation, we first realize the classical momenta as functional derivatives with respect to their corresponding conjugate fields

$$\hat{\pi}_\gamma(r) = -i \frac{\delta}{\delta \gamma(r)}, \quad \hat{\pi}_\psi(r) = -i \frac{\delta}{\delta \psi(r)}.$$

We next have to decide on the initial space of state vectors. To elucidate our choice, let us consider the action of a momentum operator on some function of the configuration field variables, say

$$\hat{\pi}_\gamma(r) \gamma(\tilde{r})^2 = -2i \gamma(\tilde{r}) \delta(\tilde{r}, r).$$

The Dirac delta-function renders the outcome of this action a distribution rather than a function. Also, if the momentum operator were to act at the same point as the function, i.e. if $\tilde{r} = r$, then its action would produce a $\delta(0)$ and would therefore be ill-defined. Both of these unwanted features are rectified, as far as expressions linear in momentum operators are concerned, if we choose as our initial collection of states all *smooth functionals* (i.e., integrals over r) of the configuration variables $\gamma(r), \psi(r)$ and their derivatives of any order. Indeed, as we infer from the previous example,

$$\hat{\pi}_\gamma(r) \int d\tilde{r} \gamma(\tilde{r})^2 = -2i \int d\tilde{r} \gamma(\tilde{r}) \delta(\tilde{r}, r) = -2i\gamma(r);$$

thus the action of the momentum operators on all such states will be well-defined (no $\delta(0)$'s) and will also produce only local functions and not distributions. However, even so, $\delta(0)$'s will appear as soon as local expressions quadratic in momenta are considered, e.g.,

$$\hat{\pi}_\gamma(r) \hat{\pi}_\gamma(r) \int d\tilde{r} \gamma(\tilde{r})^2 = \hat{\pi}_\gamma(r) (-2i \int d\tilde{r} \gamma(\tilde{r}) \delta(\tilde{r}, r)) = \hat{\pi}_\gamma(r) (-2i\gamma(r)) = -2\delta(0).$$

Another problem of equal, if not greater, importance has to do with the number of derivatives (with respect to r) considered: A momentum operator acting on a smooth functional of degree n in derivatives of $\gamma(r), \psi(r)$ will, in general, produce a function of degree $2n$, e.g.,

$$\hat{\pi}_\gamma(r) \int d\tilde{r} \gamma''(\tilde{r})^2 = -2i \int d\tilde{r} \gamma''(\tilde{r}) \delta''(\tilde{r}, r) = -2i\gamma^{(4)}(r).$$

Thus, clearly, more and more derivatives must be included if we desire the action of momentum operators to keep us inside the space of integrands corresponding to the initial collection of smooth functionals; eventually, we have to consider $n \rightarrow \infty$. This, in a sense, can be considered as the reflection to the canonical approach, of the non-normalizability results existing in the so-called covariant approach. Loosely speaking, the way to deal with these problems is to regularize (i.e., render finite) the infinite distribution limits, and re-normalize the theory by, somehow, enforcing n to terminate at some finite value.

In the following, we are going to present a quantization scheme of our system which:

- (a) avoids the occurrence of $\delta(0)$'s
- (b) reveals the value $n = 1$ as the only *natural* (i.e. without *ad-hoc* cut-offs) possibility to obtain a closed space of state vectors
- (c) extracts a finite-dimensional Wheeler-deWitt equation governing the quantum dynamics.

The scheme closely parallels, conceptually, the quantization developed in [27], [28] for finite systems with one quadratic and a number of linear first class constraints. Therefore,

we deem it appropriate, and instructive, to present a brief account of the essentials of this construction.

To this end, let us consider a system described by a Hamiltonian of the form

$$\begin{aligned} H &\equiv \mu X + \mu^i \chi_i \\ &= \mu \left(\frac{1}{2} G^{AB}(Q^\Gamma) P_A P_B + U^A(Q^\Gamma) P_A + V(Q^\Gamma) \right) + \mu^i \phi_i^A(Q^\Gamma) P_A, \end{aligned} \quad (3.1)$$

where $A, B, \Gamma \dots = 1, 2, \dots, M$ count the configuration space variables and $i = 1, 2, \dots, N < (M - 1)$ numbers the super-momenta constraints $\chi_i \approx 0$, which along with the super-Hamiltonian constraint $X \approx 0$ are assumed to be first class:

$$\{X, X\} = 0, \quad \{X, \chi_i\} = X C_i + C_i^j \chi_j, \quad \{\chi_i, \chi_j\} = C_{ij}^k \chi_k, \quad (3.2)$$

where the first (trivial) Poisson bracket has been included only to emphasize the difference from (2.13).

The physical state of the system is unaffected by the ‘‘gauge’’ transformations generated by (X, χ_i) , **but also** under the following three changes:

- (I) Mixing of the super-momenta with a non-singular matrix

$$\bar{\chi}_i = \lambda_i^j(Q^\Gamma) \chi_j$$

- (II) Gauging of the super-Hamiltonian with the super-momenta

$$\bar{X} = X + \kappa^{(Ai}(Q^\Gamma) \phi_i^{B)}(Q^\Gamma) P_A P_B + \sigma^i(Q^\Gamma) \phi_i^A(Q^\Gamma) P_A$$

- (III) Scaling of the super-Hamiltonian

$$\bar{X} = \tau^2(Q^\Gamma) X.$$

Therefore, the geometrical structures on the configuration space that can be inferred from the super-Hamiltonian are really equivalence classes under actions (I), (II) and (III); for example (II), (III) imply that the super-metric G^{AB} is known only up to conformal scalings and additions of the super-momenta coefficients $\bar{G}^{AB} = \tau^2(G^{AB} + \kappa^{(Ai} \phi_i^{B)})$. It is thus mandatory that, when we Dirac-quantize the system, we realize the quantum operator constraint conditions on the wave-function in such a way as to secure that the whole scheme is independent of actions (I), (II), (III). This is achieved by the following steps:

- (1) Realize the linear operator constraint conditions with the momentum operators to the right

$$\hat{\chi}_i \Psi = 0 \leftrightarrow \phi_i^A(Q^\Gamma) \frac{\partial \Psi(Q^\Gamma)}{\partial Q^A} = 0,$$

which maintains the geometrical meaning of the linear constraints and produces the $M - N$ independent solutions to the above equation $q^\alpha(Q^\Gamma)$, $\alpha = 1, 2, \dots, M - N$ called physical variables, since they are invariant under the transformations generated by the $\hat{\chi}_i$'s.

- (2) In order to make the final states physical with respect to the “gauge” generated by the quadratic constraint \hat{X} as well:

Define the induced structure $g^{\alpha\beta} \equiv G^{AB} \frac{\partial q^\alpha}{\partial Q^A} \frac{\partial q^\beta}{\partial Q^B}$ and realize the quadratic in momenta part of X as the conformal Laplace-Beltrami operator based on $g_{\alpha\beta}$. Note that in order for this construction to be self consistent, all components of $g_{\alpha\beta}$ must be functions of the physical coordinates q^γ . This can be proven to be so by virtue of the classical algebra the constraints satisfy (for specific quantum cosmology examples see [20]).

We are now ready to proceed with the quantization of our system, in close analogy to the scheme above outlined. In order to realize the equivalent to step 1, we first define the quantum analogue of $\mathcal{H}_1(r) \approx 0$ as

$$\hat{\mathcal{H}}_1(r)\Phi = 0 \leftrightarrow -\gamma(r) \left(\frac{\delta \Phi}{\delta \gamma(r)} \right)' + \psi'(r) \frac{\delta \Phi}{\delta \psi(r)} = 0. \quad (3.3)$$

As explained in the beginning of the section, the action of $\hat{\mathcal{H}}_1(r)$ on all smooth functionals is well defined, i.e., produces no $\delta(0)$'s. It can be proven that, in order for such a functional to be annihilated by this linear quantum operator, it must be scalar, i.e. have the form

$$\Phi = \int \gamma(\tilde{r}) L(\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(n)}) d\tilde{r} \quad (3.4a)$$

$$\Psi^{(0)} \equiv \psi(\tilde{r}), \quad \Psi^{(1)} \equiv \frac{\psi'(\tilde{r})}{\gamma(\tilde{r})}, \dots, \Psi^{(n)} \equiv \frac{1}{\gamma(\tilde{r})} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \psi(\tilde{r}) \right) \quad (3.4b)$$

where L is any function of its arguments. We note that, as it is discussed at the end of the previous section, $\frac{\psi'}{\gamma}$ is the only scalar first derivative of ψ , and likewise for the higher derivatives. The proof of this statement is analogous to the corresponding result concerning full gravity [30]: consider an infinitesimal re-parametrization of r $\tilde{r} = r - \eta(r)$. Under such a change, the left-hand side of (3.4a), being a number, must remain unaltered. If we calculate the change induced on the right-hand side we arrive at

$$0 = \int \left[L\delta\gamma + \gamma \frac{\delta L}{\delta \gamma} \delta\gamma + \gamma \frac{\delta L}{\delta \psi} \delta\psi \right] dr = \int [\gamma \hat{\mathcal{H}}_1(L)] \eta(r) dr, \quad (3.5)$$

where use of (2.16) and a partial integration has been made. Since this must hold for any $\eta(r)$, the result sought for is obtained.

We now try to realize step 2 of the program previously outlined. We have to define the equivalent of Kuchař's induced metric on the so far space of “physical” states Φ described by (3.4a) which are the analogues, in our case, of Kuchař's physical variables q^α . Let us start our investigation by considering **one** initial candidate of the above form.

Then, generalizing the partial to functional derivatives, the induced metric will be given by

$$g^{\Phi\Phi} = G^{\alpha\beta} \frac{\delta\Phi}{\delta x^\alpha} \frac{\delta\Phi}{\delta x^\beta}, \quad \text{where } x^\alpha = \{\gamma, \psi\} \quad (3.6)$$

and $G^{\alpha\beta}$ is given by (2.10). Note that this metric is well defined since it contains only first functional derivatives of the state vectors, as opposed to any second order functional derivative operator that might be considered as a quantum analogue of the kinetic part of \mathcal{H}_o . Nevertheless, $g^{\Phi\Phi}$ is a local function and not a smooth functional. It is thus clear that, if we want the induced metric $g^{\Phi\Phi}$ to be composed out of the ‘‘physical’’ states annihilated by $\hat{\mathcal{H}}_1$, we must establish a correspondence between local functions and smooth functionals. A way to achieve this is to adopt the following ansatz:

Assumption: *We assume that, as part of the re-normalization procedure, we are permitted to map local functions to their corresponding smeared expressions e.g., $\psi(r) \leftrightarrow \int d\tilde{r}\psi(\tilde{r})$.*

Let us be more specific, concerning the meaning of the above Assumption. Let \mathcal{F} be the space which contains all local functions, and define the equivalence relations

$$\sim: \{f_1(r) \sim f_2(\tilde{r}), \tilde{r} = g(r)\}, \quad \approx: \{h_1(r) \approx h_2(\tilde{r}) \frac{d\tilde{r}}{dr}, \tilde{r} = g(r)\} \quad (3.7)$$

for scalars and densities respectively.

Now let $\mathcal{F}_o = \{f \in \mathcal{F}, \text{ mod } (\sim, \approx)\}$ and \mathcal{F}_I the space of the smeared functionals. We define the one to one maps $\mathfrak{G}, \mathfrak{G}^{-1}$

$$\mathfrak{G} : \mathcal{F}_o \mapsto \mathcal{F}_I : \quad \psi(r) \mapsto \int \psi(\tilde{r}) d\tilde{r}, \quad \mathfrak{G}^{-1} : \mathcal{F}_I \mapsto \mathcal{F}_o : \quad \int \psi(\tilde{r}) d\tilde{r} \mapsto \psi(r) \quad (3.8)$$

The necessity to define the maps $\mathfrak{G}, \mathfrak{G}^{-1}$ on the equivalence classes and not on the individual functions, stems out of the fact that we are trying to develop a quantum theory of the geometries (2.5) and not of their coordinate representations. If we had tried to define the map \mathfrak{G} from the original space \mathcal{F} to \mathcal{F}_I we would end up with states which would not be invariant under spatial coordinate transformations (\mathbf{r} -reparameterizations). Indeed, one can make a correspondence between local functions and smeared expressions, but smeared expressions *must* contain another arbitrary smearing function, say $s(r)$. Then the map between functions and smeared expressions is one to one (as is also the above map) and is given by multiplying by $s(r)$ and integrating over r ; while the inverse map is given by varying w.r.t. $s(r)$. However, this would be in the opposite direction from that which led us to the states (3.4a) by imposition of the linear operator constraint. As an example consider the action of this operator on one particular case of the states (3.4a), containing the structure $s(r)$:

$$\hat{\mathcal{H}}_1(r) \int s(\tilde{r}) \gamma(\tilde{r}) \psi(\tilde{r}) d\tilde{r} = -s'(r) \gamma(r) \psi(r) \neq 0 \quad \text{for arbitrary } s(r) \quad (3.9)$$

Thus, every foreign to the geometry structure $s(r)$ is not allowed to enter the physical states.

Now, after the correspondence has been established, we can come to the basic property the induced metric must have. In the case of finite degrees of freedom the induced metric depends, up to a conformal scaling, on the physical coordinates q^α by virtue of (3.2). In our case, due to the dependence of the configuration variables on the radial coordinate r , the above property is not automatically satisfied; e.g. the functional derivative $\frac{\delta}{\delta\psi(r)}$ acting on a functional whose integrand contains $\Psi^{(n)}$ will produce, upon partial integration of the n^{th} derivative of the Dirac delta function, a term proportional to $\Psi^{(2n)}$. Therefore, since L in (3.4a) contains derivatives of $\psi(r)$ up to $\Psi^{(n)}$, the above mentioned property must be *enforced*. The need for this can also be traced to the substantially different first Poisson bracket in (2.13), which signals a non trivial mixing between the dynamical evolution generator \mathcal{H}_o and the linear generator \mathcal{H}_1 . Thus, according to the above reasoning, in order to proceed with the generalization of Kuchař's method, we have to demand that:

Requirement: $L(\Psi^{(0)}, \dots, \Psi^{(n)})$ must be such that $g^{\Phi\Phi}$ becomes a general function, say $F(\gamma(r) L(\Psi^{(0)}, \dots, \Psi^{(n)}))$ of the integrand of Φ , so that it can be considered a function of this state: $g^{\Phi\Phi} \stackrel{\text{Assumption}}{\equiv} F(\int \gamma(\tilde{r}) L(\Psi^{(0)}, \dots, \Psi^{(n)}) d\tilde{r}) = F(\Phi)$.

At this point, we must emphasize that the application of the **Requirement** in the subsequent development of our quantum theory will result in very severe restrictions on the form of (3.4a). Essentially, all higher derivatives of $\psi(r)$ (i.e. $\Psi^{(2)} \dots \Psi^{(n)}$) are eliminated from Φ (see (3.10), (3.20) bellow). This might, at first sight, strike as odd; indeed, the common belief is that all the derivatives of the configuration variables should enter the physical states. However, before the imposition of both the linear *and* the quadratic constrains there are no truly physical states. Thus, no physical states are lost by the imposition of the **Requirement**; ultimately the only true physical states are the solutions to (3.24). Of course, if one insists, one can keep higher derivatives, say k , in the functional. But then, in order to enforce the **Requirement** one would have to eliminate by hand all derivatives higher than k from the components of the induced metric. However, to our view, such an action would seem very *un-natural*. In addition, although we do not have a concrete rigorous proof, we believe that this *ad-hoc* elimination would—in the present approach—break the space-time covariance of the quantum theory we are trying to built.

Having clarified the way in which we view the **Assumption** and **Requirement** above, we now proceed to the restrictions implied by their use.

Let us now turn to the degree of derivatives (n) of $\psi(r)$. As we argued before, the functional derivatives $\frac{\delta}{\delta\psi(r)}$ and $\frac{\delta}{\delta\gamma(r)}$ acting on a functional containing in its integrand $\Psi^{(n)}$ will, upon partial integration of the n^{th} derivative of the Dirac delta function,

produce a term proportional to $\Psi^{(2n)}$ and $\Psi^{(2n-1)}$ respectively. More precisely

$$g^{\Phi\Phi} = \dots + 2G^{12} \frac{\delta\Phi}{\delta\gamma(r)} \frac{\delta\Phi}{\delta\psi(r)}.$$

Where the functional derivatives are:

$$\begin{aligned} \frac{\delta\Phi}{\delta\psi} &= \dots + \int \gamma \frac{\partial L}{\partial \Psi^{(n)}} \frac{\delta\Psi^{(n)}}{\delta\psi} d\tilde{r} = \dots + \int \gamma \frac{\partial L}{\partial \Psi^{(n)}} \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \delta(r, \tilde{r}) \right) d\tilde{r} = \\ &= \dots - \int \frac{d}{d\tilde{r}} \left(\frac{\partial L}{\partial \Psi^{(n)}} \right) \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} \delta(r, \tilde{r}) \right) d\tilde{r} = \\ &= \dots - \int \gamma \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(n+1)} \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} \delta(r, \tilde{r}) \right) d\tilde{r} = \\ &\quad \vdots \\ &= \dots + (-1)^n \int \gamma(\tilde{r}) \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(2n)} \delta(r, \tilde{r}) d\tilde{r} = \\ &= \dots + (-1)^n \gamma \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(2n)} \end{aligned}$$

and

$$\begin{aligned} \frac{\delta\Phi}{\delta\gamma} &= \dots + \int \gamma \frac{\partial L}{\partial \Psi^{(n)}} \frac{\delta\Psi^{(n)}}{\delta\gamma} d\tilde{r} = \dots + \int \gamma \frac{\partial L}{\partial \Psi^{(n)}} \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} - \frac{\delta(r, \tilde{r})}{\gamma(\tilde{r})^2} \psi'(\tilde{r}) \right) d\tilde{r} = \\ &= \dots + \int \gamma \frac{\partial L}{\partial \Psi^{(n)}} \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} - \frac{\delta(r, \tilde{r})}{\gamma(\tilde{r})} \Psi^{(1)} \right) d\tilde{r} \\ &= \dots - \int \frac{d}{d\tilde{r}} \left(\frac{\partial L}{\partial \Psi^{(n)}} \right) \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-3} - \frac{\delta(r, \tilde{r})}{\gamma(\tilde{r})} \Psi^{(1)} \right) d\tilde{r} = \\ &= \dots - \int \gamma \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(n+1)} \frac{1}{\gamma} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-3} - \frac{\delta(r, \tilde{r})}{\gamma(\tilde{r})} \Psi^{(1)} \right) d\tilde{r} = \\ &\quad \vdots \\ &= \dots + (-1)^{n-1} \int \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(2n-1)} \Psi^{(1)} \delta(r, \tilde{r}) d\tilde{r} = \\ &= \dots + (-1)^{n-1} \frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \Psi^{(2n-1)} \Psi^{(1)}. \end{aligned}$$

Therefore

$$g^{\Phi\Phi} = \dots - \frac{\gamma}{2\psi} (-1)^{2n-1} \left(\frac{\partial^2 L}{\partial (\Psi^{(n)})^2} \right)^2 \Psi^{(1)} \Psi^{(2n-1)} \Psi^{(2n)},$$

where the ... stand for all other terms, not involving $\Psi^{(2n)}$. Now, according to the aforementioned **Requirement** we need this to be a general function, say $F(\gamma L)$, and for this to happen the coefficient of $\Psi^{(2n)}$ must vanish, i.e.

$$\frac{\partial^2 L}{\partial (\Psi^{(n)})^2} = 0 \Leftrightarrow L = L_1 (\Psi^{(0)}, \dots, \Psi^{(n-1)}) \Psi^{(n)} + L_2 (\Psi^{(0)}, \dots, \Psi^{(n-1)}).$$

Now, the term in Φ corresponding to L_1 is, up to a surface term, equivalent to a general term depending on $\Psi^{(0)}, \dots, \Psi^{(n-1)}$ only: indeed,

$$\Phi_1 = \int \gamma(\tilde{r}) L_1 \frac{1}{\gamma(\tilde{r})} \frac{d}{d\tilde{r}} \Psi^{(n-1)} d\tilde{r},$$

which upon subtraction of the surface term

$$A = \int d\tilde{r} \frac{d}{d\tilde{r}} \left(\int d\Psi^{(n-1)} L_1 \right)$$

produces a smooth functional with arguments up to $\Psi^{(n-1)}$ only. Since a surface term in Φ does not affect the outcome of the variational derivatives $\frac{\delta \Phi}{\delta \psi(r)}$ and $\frac{\delta \Phi}{\delta \gamma(r)}$, we conclude that only L_2 is important for the local part of Φ . The entire argument can be repeated successively for $n-1, n-2, \dots, 2$; therefore all $\Psi^{(n)}$'s are suppressed from L except when $n=1$. The case $n=1$ needs separate consideration since, upon elimination of the linear in $\Psi^{(2)}$ term we are left with a local function of $\Psi^{(1)}$, and thus the possibility arises to meet the **Requirement** by solving a differential equation for L . In more detail, if

$$\Phi \equiv \int \gamma(\tilde{r}) L(\psi, \Psi^{(1)}) d\tilde{r}, \quad (3.10)$$

$g^{\Phi\Phi}$ reads

$$\begin{aligned} g^{\Phi\Phi} &= \frac{\gamma}{4\psi^2} \left(L - \Psi^{(1)} \frac{\partial L}{\partial \Psi^{(1)}} \right) \left[L - \Psi^{(1)} \frac{\partial L}{\partial \Psi^{(1)}} - 2\psi \left(\frac{\partial L}{\partial \psi} - \Psi^{(1)} \frac{\partial^2 L}{\partial \psi \partial \Psi^{(1)}} \right) \right] + \\ &+ \frac{\gamma}{2\psi} \left(L - \Psi^{(1)} \frac{\partial L}{\partial \Psi^{(1)}} \right) \frac{\partial^2 L}{\partial (\Psi^{(1)})^2} \Psi^{(2)}. \end{aligned} \quad (3.11)$$

Through the definition

$$H \equiv L - \Psi^{(1)} \frac{\partial L}{\partial \Psi^{(1)}} \quad (3.12)$$

we obtain

$$\frac{\partial H}{\partial \psi} = \frac{\partial L}{\partial \psi} - \Psi^{(1)} \frac{\partial^2 L}{\partial \psi \partial \Psi^{(1)}}, \quad \frac{\partial H}{\partial \Psi^{(1)}} = -\Psi^{(1)} \frac{\partial^2 L}{\partial (\Psi^{(1)})^2}.$$

Thus (3.11) assumes the form

$$g^{\Phi\Phi} = \frac{\gamma}{4\psi^2} \left(H^2 - 2\psi H \frac{\partial H}{\partial \psi} - \frac{2\psi}{\Psi^{(1)}} H \frac{\partial H}{\partial \Psi^{(1)}} \Psi^{(2)} \right),$$

which upon addition, by virtue of the **Assumption**, of the surface term

$$A = \frac{d}{dr} \left(\int \frac{1}{2\psi\Psi^{(1)}} H \frac{\partial H}{\partial\Psi^{(1)}} d\Psi^{(1)} \right)$$

gives

$$g^{\Phi\Phi} = \frac{\gamma}{4\psi^2} \left(H^2 - 2\psi H \frac{\partial H}{\partial\psi} + 4\psi^2\Psi^{(1)} \frac{\partial}{\partial\psi} \int \frac{1}{2\psi\Psi^{(1)}} H \frac{\partial H}{\partial\Psi^{(1)}} d\Psi^{(1)} \right). \quad (3.13)$$

Since in the last expression we have only a multiplicative $\gamma(r)$, it is obvious that the **Requirement**

$$g^{\Phi\Phi} = F(\gamma L)$$

can be satisfied only by

$$g^{\Phi\Phi} = \kappa \gamma L, \quad (3.14)$$

with $g^{\Phi\Phi}$ given by (3.13) and κ any constant. Upon differentiation of this equation with respect to $\Psi^{(1)}$ we get

$$\frac{\partial}{\partial\psi} \int \frac{1}{2\psi\Psi^{(1)}} H \frac{\partial H}{\partial\Psi^{(1)}} d\Psi^{(1)} = \kappa \frac{\partial L}{\partial\Psi^{(1)}}.$$

Multiplying the last expression by $\Psi^{(1)}$ and subtracting it from (3.14) (with $g^{\Phi\Phi}$ given by (3.13)) we end up with the autonomous necessary condition for $H(\psi, \Psi^{(1)})$:

$$H \left(\frac{1}{4\psi^2} H - \frac{1}{2\psi} \frac{\partial H}{\partial\psi} - \kappa \right) = 0,$$

where (3.12) was also used. The above equation can be readily integrated giving

$$\begin{aligned} H &= 0, \\ H &= -\frac{4\kappa\psi^2}{3} + \sqrt{\psi} a(\Psi^{(1)}), \end{aligned}$$

where $a(\Psi^{(1)})$ is an arbitrary function of its argument. The first possibility gives according to (3.12) $L = \lambda \Psi^{(1)}$ which, however, contributes to Φ a surface term, and can thus be ignored. Inserting the second solution into (3.12) we construct a partial differential equation for L , namely

$$L - \Psi^{(1)} \frac{\partial L}{\partial\Psi^{(1)}} = -\frac{4\kappa\psi^2}{3} + \sqrt{\psi} a(\Psi^{(1)}),$$

which upon integration gives

$$L = -\frac{4\kappa\psi^2}{3} - \sqrt{\psi} \Psi^{(1)} \int \frac{a(\Psi^{(1)})}{\Psi^{(1)2}} d\Psi^{(1)} + c_1(\psi) \Psi^{(1)}.$$

Since this form of L emerged as a necessary condition, it must be inserted (along with H) in (3.14). The result is that $c_1(\psi) = 0$. Thus L reads

$$L = -\frac{4\kappa\psi^2}{3} - \sqrt{\psi} \Psi^{(1)} \int \frac{a(\Psi^{(1)})}{\Psi^{(1)2}} d\Psi^{(1)}. \quad (3.15)$$

By assuming that the $\Psi^{(1)}$ -dependent part of L equals $b(\Psi^{(1)})$, i.e.

$$-\Psi^{(1)} \int \frac{a(\Psi^{(1)})}{\Psi^{(1)2}} d\Psi^{(1)} = b(\Psi^{(1)}),$$

we get, upon a double differentiation with respect to $\Psi^{(1)}$, the ordinary differential equation

$$-\frac{a'(\Psi^{(1)})}{\Psi^{(1)}} = b''(\Psi^{(1)})$$

with solution

$$a(\Psi^{(1)}) = b(\Psi^{(1)}) + \kappa_1 - \Psi^{(1)} b'(\Psi^{(1)}),$$

where κ_1 is a constant. Substituting this equation into (3.15) and performing a partial integration we end up with

$$L = -\frac{4\kappa\psi^2}{3} + \kappa_1\sqrt{\psi} + \sqrt{\psi} b(\Psi^{(1)}). \quad (3.16)$$

κ , κ_1 and $b(\Psi^{(1)})$ being completely arbitrary and to our disposal; the two simplest choices $\kappa = 0$, $b(\Psi^{(1)}) = 0$ and $\kappa_1 = 0$, $b(\Psi^{(1)}) = 0$ lead respectively to the following two basic ultra-local smooth functionals:

$$q^1 = \int d\tilde{r} \gamma(\tilde{r}) \sqrt{\psi(\tilde{r})}, \quad q^2 = \int d\tilde{r} \gamma(\tilde{r}) \psi(\tilde{r})^2.$$

The next simpler choice $\kappa = 0$, $\kappa_1 = 0$ and $b(\Psi^{(1)})$ arbitrary leads to a generic $q^3 = \int d\tilde{r} \gamma(\tilde{r}) \sqrt{\psi} b(\Psi^{(1)})$. However, it can be proven that, for any choice of $b(\Psi^{(1)})$, the corresponding renormalized induced metric

$$g^{AB} = G^{\alpha\beta} \frac{\delta q^A}{\delta x^\alpha} \frac{\delta q^B}{\delta x^\beta} \quad \text{where } A, B = 1, 2, 3$$

is singular. The calculation of g^{AB} gives:

$$\begin{aligned} g^{11} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^1}{\delta x^\beta} = 0 \stackrel{\text{Assumption}}{\iff} g_{ren}^{11} = 0, \\ g^{12} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = -\frac{3}{8} \gamma \sqrt{\psi} \stackrel{\text{Assumption}}{\iff} g_{ren}^{12} = -\frac{3}{8} q^1, \\ g^{22} &= G^{\alpha\beta} \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = -\frac{3}{4} \gamma \psi^2 \stackrel{\text{Assumption}}{\iff} g_{ren}^{22} = -\frac{3}{4} q^2, \\ g^{13} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \frac{1}{4} \gamma \Psi^{(2)} b'' = \frac{d}{dr} \left(\frac{1}{4} b' \right) \stackrel{\text{Assumption}}{\iff} g_{ren}^{13} = 0, \end{aligned}$$

$$\begin{aligned}
g^{23} &= G^{\alpha\beta} \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \frac{1}{8} \gamma \sqrt{\psi} (-3b + 3\Psi^{(1)} b' + 2\psi \Psi^{(2)} b'') \stackrel{Assumption}{\iff} \\
g_{ren}^{23} &= \frac{1}{8} \int dr \gamma \sqrt{\psi} (-3b + 3\Psi^{(1)} b' + 2\psi \Psi^{(2)} b'') - \int dr \frac{d}{dr} \left(\frac{1}{4} \int d\Psi^{(1)} \psi^{3/2} b'' \right) = \\
&= -\frac{3}{8} \int dr \gamma \sqrt{\psi} b = -\frac{3q^3}{8}, \\
g^{33} &= G^{\alpha\beta} \frac{\delta q^3}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \frac{1}{2} \gamma (b - \Psi^{(1)} b') \Psi^{(2)} b'' \stackrel{Assumption}{\iff} \\
g_{ren}^{33} &= \frac{1}{2} \int dr \gamma (b - \Psi^{(1)} b') \Psi^{(2)} b'' - \int dr \frac{d}{dr} \left[\frac{1}{2} \int d\Psi^{(1)} (b - \Psi^{(1)} b') b'' \right] = 0,
\end{aligned}$$

where by ' we denote differentiation with respect to $\Psi^{(1)}$. Thus the re-normalized induced metric reads

$$g_{ren}^{AB}(q^1, q^2, q^3) = -\frac{3}{8} \begin{pmatrix} 0 & q^1 & 0 \\ q^1 & 2q^2 & q^3 \\ 0 & q^3 & 0 \end{pmatrix}.$$

Effecting the transformation $(\tilde{q}^1, \tilde{q}^2, \tilde{q}^3) = (q^1, q^2, f(\frac{q^3}{q^1}))$ we bring g_{ren}^{AB} into a manifestly degenerate form:

$$g_{ren}^{AB}(q^1, q^2) = -\frac{3}{8} \begin{pmatrix} 0 & q^1 & 0 \\ q^1 & 2q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, it seems that, as far as the ultra local part of the functionals is concerned, the re-normalized metric is given by the upper left 2×2 block of the above expression. It is interesting to observe that the integrands of q^1, q^2 form a base in the space spanned by γ, ψ . It is convenient to change these two integrands (namely $\gamma\sqrt{\psi}$ and $\gamma\psi^2$) to γ and $\gamma\psi^2$ respectively, since the latter choice complies with the ultra local parts of the potential (2.11). One might wonder if this action is permitted, namely if the new re-normalized metric, resulting from the choice of the new functionals,

$$y^1 = \int \gamma(\tilde{r}) d\tilde{r}, \quad y^2 = \int \gamma(\tilde{r}) \psi(\tilde{r})^2 d\tilde{r}$$

is equivalent to the previous. Interestingly enough, the answer is yes. Indeed, following the line of thoughts leading to $g_{ren}^{AB}(q^1, q^2)$ one arrives at

$$g_{ren}^{AB}(y^1, y^2) = -\frac{1}{4} \begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 \\ y^1 & 3y^2 \end{pmatrix},$$

which is related to the upper left 2×2 block of $g_{ren}^{AB}(q^1, q^2)$ through the transformation

$$(y^1, y^2) = \left(\frac{(q^1)^{4/3}}{(q^2)^{1/3}}, q^2 \right).$$

Quite unexpectedly, this transformation is identical to the transformation connecting the integrands of the two pairs of functionals $(\gamma\sqrt{\psi}, \gamma\psi^2)$, $(\gamma, \gamma\psi^2)$. This is a strong indication that the use of the **Assumption** preserves the geometry of the re-normalized manifold. We thus adopt, without any loss of generality the ultra local functionals:

$$y^1 = \int \gamma(\tilde{r}) d\tilde{r}, \quad y^2 = \int \gamma(\tilde{r}) \psi(\tilde{r})^2 d\tilde{r} \quad (3.17)$$

One might think that this preservation of the geometry of the re-normalized manifold is due to the ultra local nature of the integrands but, as we shall subsequently see, this state of affairs continuous to hold even when functionals with integrands which contain derivatives of the configuration variables are considered.

Indeed it is quite essential to have a functional that contains first derivative of $\psi(r)$, since a term of this kind (namely $\frac{\psi'^2}{\gamma}$) does appear in the potential (2.11). Thus, it is clear that this is not the end of our investigation for a suitable space of state vectors: the caveat is that the argument leading to y^1, y^2 crucially depends upon the original choice of one initial candidate smooth scalar functional (3.10). Therefore, to complete the search we must close the circle by starting with the two already secured smooth functionals (y^1, y^2) , and a third of the general form

$$y^3 = \int d\tilde{r} \gamma(\tilde{r}) L(\Psi^{(1)})$$

(since the ψ dependence has already been fixed to either 1 or ψ^2). The calculation of

the, related to y^3 , components of the induced metric g^{AB} gives:

$$\begin{aligned}
g^{13} &= \frac{\gamma}{4\psi^2} (L - \Psi^{(1)} L' + \psi \Psi^{(2)} L'') \stackrel{\text{Assumption}}{\iff} \\
g_{ren}^{13} &= \int dr \frac{\gamma}{4\psi^2} (L - \Psi^{(1)} L' + \psi \Psi^{(2)} L'') - \int dr \frac{d}{dr} \left(\frac{1}{4} \int d\Psi^{(1)} \frac{L''}{\psi} \right) = \\
&= \frac{1}{4} \int dr \frac{\gamma}{\psi^2} L \stackrel{\text{Assumption}}{=} \frac{1}{4} \frac{\gamma}{\gamma \psi^2} \gamma L \stackrel{\text{Assumption}}{=} \frac{1}{4} \frac{\int dr \gamma}{\int dr \gamma \psi^2} \int dr \gamma L = \frac{y^1 y^3}{4 y^2}, \\
g^{23} &= \frac{\gamma}{4} (-L + \Psi^{(1)} L' + \psi \Psi^{(2)} L'') \stackrel{\text{Assumption}}{\iff} \\
g_{ren}^{23} &= \int dr \frac{\gamma}{4} (-L + \Psi^{(1)} L' + \psi \Psi^{(2)} L'') - \int dr \frac{d}{dr} \left(\frac{1}{4} \int d\Psi^{(1)} \psi L'' \right) = \\
&= -\frac{1}{4} \int dr \gamma L = -\frac{y^3}{4}, \\
g^{33} &= \frac{\gamma}{4\psi^2} (L - \Psi^{(1)} L')^2 + \frac{\gamma}{2\psi} (L - \Psi^{(1)} L') \Psi^{(2)} L'' \stackrel{\text{Assumption}}{\iff} \\
g_{ren}^{33} &= \int dr \left[\frac{\gamma}{4\psi^2} (L - \Psi^{(1)} L')^2 + \frac{\gamma}{2\psi} (L - \Psi^{(1)} L') \Psi^{(2)} L'' \right] - \\
&\quad - \int dr \frac{d}{dr} \int d\Psi^{(1)} \left(\frac{(L - \Psi^{(1)} L') L''}{2\psi} \right) \\
g_{ren}^{33} &= \int \frac{\gamma}{4\psi^2} \left[(L - \Psi^{(1)} L')^2 - \Psi^{(1)} \int \frac{d\Psi^{(1)}}{\Psi^{(1)}} \frac{\partial}{\partial \Psi^{(1)}} (L - \Psi^{(1)} L')^2 \right] dr. \quad (3.18)
\end{aligned}$$

The expression inside the square brackets of g_{ren}^{33} above, being a generic function of $\Psi^{(1)}$, can also be considered as a function of L , say $W(L(\Psi^{(1)}))$. It is thus clear that the **Requirement** is satisfied for any $L(\Psi^{(1)})$:

$$\stackrel{\text{Assumption}}{\iff} g_{ren}^{33} = \frac{(\int \gamma dr)^2}{4 \int \gamma \psi^2 dr} W\left(\frac{\int \gamma L dr}{\int \gamma dr}\right) = \frac{(y^1)^2}{4y^2} W\left(\frac{y^3}{y^1}\right).$$

Let this expression $W(L(\Psi^{(1)}))$ be parameterized as

$$L(\Psi^{(1)})^2 - \frac{4 F[L(\Psi^{(1)})]^2}{3 F'[F[L(\Psi^{(1)})]]^2}. \quad (3.19)$$

This ‘‘peculiar’’ parametrization of the arbitrariness in $L(\Psi^{(1)})$ has been chosen in order to facilitate the subsequent proof that the freedom in the choice of L (left by the imposition of the **Requirement**) is a pure general coordinate transformation (gct) of the induced re-normalized metric.

The reduced re-normalized manifold is thus parameterized by the following three smooth scalar functionals:

$$y^1 = \int \gamma(\tilde{r}) d\tilde{r}, \quad y^2 = \int \gamma(\tilde{r}) \psi(\tilde{r})^2 d\tilde{r}, \quad y^3 = \int \gamma(\tilde{r}) L(\Psi^{(1)}) d\tilde{r}. \quad (3.20)$$

Any other functional, say $y^4 = \int d\tilde{r} \gamma(\tilde{r}) K [\psi(\tilde{r}), \Psi^{(1)}(\tilde{r})]$, can be considered as a function of y^1, y^2, y^3 ; indeed, since the scalar functions appearing in the integrands of y^2, y^3 form a base in the space spanned by $\psi, \Psi^{(1)}$, we can express the generic K in y^4 as $K \left[\sqrt{\frac{\gamma\psi^2}{\gamma}}, \Psi^{(1)} \right]$, which (through the **Assumption**) gives $y^4 = y^1 K \left[\sqrt{\frac{y^2}{y^1}}, L^{-1} \left(\frac{y^3}{y^1} \right) \right]$.

The geometry of this space is described by the induced re-normalized metric

$$g_{ren}^{AB}(y^1, y^2, y^3) = -\frac{1}{4} \begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 & -\frac{y^1 y^3}{y^2} \\ y^1 & 3y^2 & y^3 \\ -\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} + \frac{4(y^1)^2 F \left(\frac{y^3}{y^1} \right)^2}{3y^2 F' \left[F \left(\frac{y^3}{y^1} \right) \right]^2} \end{pmatrix},$$

$$g_{ABren}(y^1, y^2, y^3) = \begin{pmatrix} \frac{3y^2}{(y^1)^4} \left((y^1)^2 - \frac{(y^3)^2 F' \left[F \left(\frac{y^3}{y^1} \right) \right]^2}{F \left(\frac{y^3}{y^1} \right)^2} \right) & -\frac{1}{y^1} & \frac{3y^2 y^3 F' \left[F \left(\frac{y^3}{y^1} \right) \right]^2}{(y^1)^3 F \left(\frac{y^3}{y^1} \right)^2} \\ -\frac{1}{y^1} & -\frac{1}{y^2} & 0 \\ \frac{3y^2 y^3 F' \left[F \left(\frac{y^3}{y^1} \right) \right]^2}{(y^1)^3 F \left(\frac{y^3}{y^1} \right)^2} & 0 & -\frac{3y^2 F' \left[F \left(\frac{y^3}{y^1} \right) \right]^2}{(y^1)^2 F \left(\frac{y^3}{y^1} \right)^2} \end{pmatrix}. \quad (3.21)$$

Any function $\Psi(y^1, y^2, y^3)$ on this manifold is of course annihilated by the quantum linear constraint, i.e.

$$\hat{\mathcal{H}}_1 \Psi(y^1, y^2, y^3) = \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^1} \hat{\mathcal{H}}_1 y^1 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^2} \hat{\mathcal{H}}_1 y^2 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^3} \hat{\mathcal{H}}_1 y^3 = 0$$

since the derivatives with respect to r are transparent to the partial derivatives of Ψ (which are, just like the y^A 's, r-numbers).

The covariant metric (3.21) describes a three dimensional conformally flat geometry, since the corresponding Cotton-York tensor vanishes. The Ricci scalar is $R = \frac{3}{8y^2}$, indicating that the arbitrariness in F (and thus also in L) is a *pure gauge*. The change of coordinates

$$(y^1, y^2, y^3) = (e^{-\frac{1}{8}(5Y^1+3Y^3)}, e^{Y^1+Y^2+Y^3}, e^{-\frac{1}{8}(5Y^1+3Y^3)} F^{-1}(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)})) \quad (3.22)$$

(where F^{-1} denotes the function inverse to F , i.e. $F^{-1}(F(x)) = x$) brings the metric to

the manifestly conformally flat form:

$$g_{ABren}(Y^1, Y^2, Y^3) = \begin{pmatrix} e^{Y^1+Y^2+Y^3} & 0 & 0 \\ 0 & -\frac{4}{3}e^{Y^1+Y^2+Y^3} & 0 \\ 0 & 0 & -e^{Y^1+Y^2+Y^3} \end{pmatrix}, \quad (3.23)$$

in which all the F dependence has indeed disappeared.

The final restriction on the form of Ψ will be obtained by the imposition of the quantum analog of the quadratic constraint \mathcal{H}_o . According to the above exposition we postulate that the quantum gravity of the geometries given by (2.5) will be described by the following partial differential equation (in terms of the Y^A 's)

$$\hat{\mathcal{H}}_o \Psi \equiv \left[-\frac{1}{2} \square_c + V_{ren}\right] \Psi(Y^1, Y^2, Y^3) = 0 \quad (3.24)$$

with

$$\square_c = \square + \frac{d-2}{4(d-1)} R \quad (3.25)$$

being the conformal Laplacian based on $g_{ABren}(Y^1, Y^2, Y^3)$, R the Ricci scalar, and d the dimensions of g_{ABren} . The metric (3.23) is conformally flat with Ricci scalar $R = \frac{3}{8} e^{-Y^1-Y^2-Y^3}$, and its dimension is $d = 3$. The re-normalized form of the potential (2.11) offers us the possibility to introduce, in a dynamical way, topological effects into our wave functional: Indeed, under our **Assumption**, the first two terms become $-2\epsilon y^1$ and $2\Lambda y^2$, respectively, while the last, being a total derivative, becomes $A_T \equiv 4 \frac{\psi \psi'}{\gamma} \Big|_{\alpha}^{\beta}$ (*if* $\alpha < r < \beta$). In the spirit previously explained we should drop this term, however one could also keep it. The re-normalized form of the remaining, third, term of the potential can be obtained as follows

$$y^3 = \gamma L(\Psi^{(1)}) \Leftrightarrow L(\Psi^{(1)}) = \frac{y^3}{\gamma} \stackrel{\text{Assumption}}{\Leftrightarrow} L(\Psi^{(1)}) = \frac{y^3}{y^1} \Leftrightarrow \Psi^{(1)} = L^{-1} \left(\frac{y^3}{y^1} \right),$$

thus finally

$$\frac{\psi'}{\gamma} = L^{-1} \left(\frac{y^3}{y^1} \right)$$

and the third term becomes $-2y^1 \left[L^{-1} \left(\frac{y^3}{y^1} \right) \right]^2$. Finally, effecting the transformation (3.22) the form of the re-normalized potential is

$$V_{ren} = -2\epsilon e^{-\frac{1}{8}(5Y^1+3Y^3)} - 2e^{-\frac{1}{8}(5Y^1+3Y^3)} \left[L^{-1} \left(F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) \right) \right]^2 + 2\Lambda e^{Y^1+Y^2+Y^3} + A_T \quad (3.26)$$

and the Wheeler-deWitt equation is given as

$$\begin{aligned}
& -2\epsilon e^{\frac{1}{8}(3Y^1+5Y^3)+Y^2} \Psi(Y^1, Y^2, Y^3) + 2\Lambda e^{2(Y^1+Y^2+Y^3)} \Psi(Y^1, Y^2, Y^3) - \\
& 2e^{\frac{1}{8}(3Y^1+5Y^3)+Y^2} \left[L^{-1} \left(F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) \right) \right]^2 \Psi(Y^1, Y^2, Y^3) + \\
& A_T e^{Y^1+Y^2+Y^3} \Psi(Y^1, Y^2, Y^3) - \frac{3}{128} \Psi(Y^1, Y^2, Y^3) - \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^1} + \\
& \frac{3}{16} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^2} + \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^3} - \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^1)^2} + \frac{3}{8} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^2)^2} + \\
& \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^3)^2} = 0.
\end{aligned}$$

Since F^{-1} is an arbitrary function of its arguments, we may contemplate the choice:

$$F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) = L \left(\sqrt{e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} - \epsilon} \right). \quad (3.27)$$

Of course there is a question of existence for such a choice: since F which appears in (3.19) is nothing but a convenient parametrization of the expression inside the square brackets of (3.18), any demand that F has a specified form (much more in terms of L) constitutes an implicit restriction on the form of L itself. Subsequently, at least the existence of such an L must be secured. Indeed, in the Appendix B it is shown that an appropriate L exists; its form is given by (B.4):

$$L(\Psi^{(1)}) = m + \int \frac{(\Psi^{(1)})^{3/2}}{((\Psi^{(1)})^2 - \epsilon)^{13/16}} e^{k - \frac{3\epsilon}{16((\Psi^{(1)})^2 - \epsilon)}} d\Psi^{(1)} \quad \text{where } c_1 m + c_2 + c_3 e^k = 0.$$

This choice for F reduces the Wheeler-deWitt equation to the final almost-separable form

$$\begin{aligned}
& 2\Lambda e^{2(Y^1+Y^2+Y^3)} \Psi(Y^1, Y^2, Y^3) - 2\epsilon e^{\frac{4}{3}Y^2} \Psi(Y^1, Y^2, Y^3) + A_T e^{Y^1+Y^2+Y^3} \Psi(Y^1, Y^2, Y^3) - \\
& \frac{3}{128} \Psi(Y^1, Y^2, Y^3) - \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^1} + \frac{3}{16} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^2} + \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^3} - \\
& \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^1)^2} + \frac{3}{8} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^2)^2} + \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^3)^2} = 0. \quad (3.28)
\end{aligned}$$

This equation is separable for $\Lambda = 0$ and $A_T = 0$. In this case it can readily be solved: assuming $\Psi(Y^1, Y^2, Y^3) = \Psi^1(Y^1) \Psi^2(Y^2) \Psi^3(Y^3)$ and dividing (3.28) by Ψ we get the three ordinary differential equations:

$$\begin{aligned}
& \frac{1}{4\Psi^1(Y^1)} \frac{d\Psi^1(Y^1)}{dY^1} + \frac{1}{2\Psi^1(Y^1)} \frac{d^2\Psi^1(Y^1)}{d(Y^1)^2} = m, \\
& \frac{3}{16\Psi^2(Y^2)} \frac{d\Psi^2(Y^2)}{dY^2} + \frac{3}{8\Psi^2(Y^2)} \frac{d^2\Psi^2(Y^2)}{d(Y^2)^2} - 2\epsilon e^{\frac{4}{3}Y^2} = n, \\
& \frac{1}{4\Psi^3(Y^3)} \frac{d\Psi^3(Y^3)}{dY^3} + \frac{1}{2\Psi^3(Y^3)} \frac{d^2\Psi^3(Y^3)}{d(Y^3)^2} - \frac{3}{128} = m - n,
\end{aligned}$$

where m and n are separation constants. Their solutions are:

$$\begin{aligned}\Psi^1(Y^1) &= c_1 e^{\frac{1}{4}(-1-\sqrt{1+32m})Y^1} + c_2 e^{\frac{1}{4}(-1+\sqrt{1+32m})Y^1}, \\ \Psi^2(Y^2) &= c_3 e^{-Y^2/4} I_{-\frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\epsilon\sqrt{3} e^{2Y^2/3} \right) + c_4 e^{-Y^2/4} I_{\frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\epsilon\sqrt{3} e^{2Y^2/3} \right) \\ \Psi^3(Y^3) &= c_5 e^{\frac{1}{8}(-2-\sqrt{7+128m-128n})Y^3} + c_6 e^{\frac{1}{8}(-2+\sqrt{7+128m-128n})Y^3},\end{aligned}$$

where $I_{\pm\frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\sqrt{3} e^{2Y^2/3} \right)$ are modified Bessel functions of the first kind and non-integer order.

4 Discussion

We have considered the canonical analysis and subsequent quantization of the (3+1)-dimensional action of pure gravity plus a cosmological constant term, under the assumption of the existence of two-dimensional (spacelike) surfaces of maximal symmetry. At the classical level, the application of the Dirac algorithm results in one linear and one quadratic first class constraints. The linear constraint is shown to correspond to arbitrary changes of the radial coordinate. The quadratic constraint is the generator of the time evolution. Adopting the Schrödinger picture for the quantum momentum operators, we are led to choose as our initial collection of state vectors all smooth (integrals over the radial coordinate r) functionals, in order to avoid an ill-defined action of these operators. The quantum linear constraint entails a reduction of this collection to all smooth scalar functionals. At this stage the need emerges to somehow obtain an induced metric on the so far “physical” states, which is composed out of these states. This leads us to firstly adopt a particular (formal) re-normalization prescription and secondly impose the **Requirement**. As a result, the final collection of state vectors is reduced to the three (essentially unique) smooth scalar functionals (y^1, y^2, y^3) . The quantum analogue of the kinetic part of the quadratic constraint is then realized as the conformal Laplace-Beltrami operator based on the induced re-normalized metric. After the interpretation (through the **Assumption**) of the potential part of \mathcal{H}_o a Wheeler-deWitt equation emerges. In order to analytically solve this equation we exploit the freedom in the choice of L appearing in y^3 , which is left by the imposition of **Requirement** and which is shown to be a *pure* general coordinate transformation on the re-normalized manifold. Effecting an appropriate change of variables the metric is put in conformally flat form. Then, the aforementioned freedom is used to make the equation separable.

Generally (and somewhat loosely) speaking, our goal is, at a first stage, to assign a unique number between 0 and 1 to each and every geometry of the families considered, in a way that is independent of the coordinate system used to represent the metric. Of course, at the present status of things we cannot do this, since the following two problems remain to be solved: i) render finite the three smooth functionals and ii) select an appropriate inner product.

The first will need a final regularization of y^1, y^2, y^3 , but most probably, the detailed way to do this will depend upon the particular geometry under consideration.

For the second, a natural choice would be the determinant of the induced re-normalized metric, although the problem with the positive definiteness may dictate another choice.

At this point it is instructive to reflect upon the fate of the classical "open" Poisson bracket algebra (2.13) of the constraints $\mathcal{H}_0, \mathcal{H}_1$ within the context of the present quantization scheme. It is true that it is very difficult to investigate the quantum realization of this Poisson algebra in the space spanned by the configuration functions and their derivatives of arbitrary order. The way we have chosen to overcome this difficulty is to generalize Kuchar's construction; an essential element of this construction is the reduction of the initial configuration space. This reduction consists in restricting to the space spanned by the "physical" degrees of freedom, annihilated by the linear constraints. Thus, even in the case of finite degrees of freedom, the classical algebra is not, at the quantum level, realized in the original configuration space, and therefore its realization can not be faithful in this space. As a second step in this construction, the reduced space of "physical" states is endowed with a metric (induced by the super-metric) and the kinetic part of the quadratic constraint is realized as the Laplace-Beltrami operator based on this metric. The consistency of the scheme is guaranteed by the existence of the reduced space of "physical" states, the fact that, by virtue of the first class algebra (3.2), the induced metric is composed out of the "physical" degrees of freedom q^α , and standard existence theorems concerning second order partial differential equations. Of course, the ultimate finding of solutions to this equation can also serve as proof of consistency.

In the case we address, the first step of the generalization of Kuchar's construction is smoothly carried out, resulting in the "physical" states (3.4a),(3.4b). It is the generalization of the second step that can not be implemented without the use of the central idea of our work, i.e. the **Assumption** and the **Requirement**. It is only after their acceptance that it is possible to construct the finally reduced physical space spanned by the three functionals (3.20) and endowed with the physical metric (3.21). All the requirements for consistency of our quantum theory are thus met: The reduced space of "physical" states exists, and the postulated operator equation (3.24) admits solutions. Furthermore, the entire solution space (for $\Lambda = 0, A_T = 0$) is given.

There is also a similarity with string theory which we think is worth pointing out: Indeed, in string theory the Virasoro algebra is made consistent by adjusting the dimensions of the target space. In our quantum theory, the initial degrees of freedom are infinite and any possible realization of the open Poisson algebra (2.13) is bound to be plagued by infinities (anomalies). In this respect, the function of the **Assumption** and the **Requirement** leads to a unique choice of the dimensions of the physical space i.e. three. Thus the well known anomaly cancelation in string theory is appearing in a natural way in our scheme, hence the parallelism.

Finally, we would like to comment upon the relation of the results here obtained to the results presented in our previous work [31]. There, the 2+1 action of pure gravity plus a cosmological constant term was quantized in a similar manner, under the assumption of existence of *one* space-like Killing vector field. We would like to point out the quite interesting fact that, although the systems considered are different, the resulting re-

normalized manifolds, their geometry and the corresponding Wheeler–deWitt equations are strikingly similar. To our view, this constitutes a very strong indication that the imposition of the **Assumption** and the **Requirement** is not simply an elegant way to reduce the number of spatial derivatives of the configuration fields involved in the scalar functionals, but is rather a tool for unraveling the underlying geometrical structure of Quantum Gravity (in the approximation considered, of course).

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A Appendix: The Classical Solution Space

In this Appendix we outline the investigation of the solution space to (2.14) for the family of geometries described by (2.5). We first observe that the block-diagonal form of this metric is preserved under the (restricted) coordinate transformations $(t, r) \rightarrow (T(t, r), R(t, r))$. Under any such change the 2×2 (t, r) part of (2.5) transforms as a second rank covariant tensor and can thus serve as a two-dimensional metric, say $g_{\mu\nu}^{(2)}$ (of signature $(-, +)$), while $\psi(t, r)$ transforms as a scalar. Therefore, the doublet $F \equiv (\frac{\partial\psi(t,r)}{\partial t}, \frac{\partial\psi(t,r)}{\partial r})$ is an one-form under the above mentioned restricted transformations and its measure $K = g^{(2)\mu\nu} F_\mu F_\nu$ is invariant. We can thus distinguish three cases corresponding to K being positive, negative or zero respectively. When these cases are combined with the three types of symmetry, they produce nine possibilities which have to be considered.

Let us take the case $K > 0$. The one-form F is space-like and therefore can, by virtue of a standard theorem of differential geometry, be reduced to the form $F = (0, 1)$ by a change of coordinates in the (t, r) subspace. Because of the scalar character of $\psi(t, r)$ we conclude that it has to be identified to the new radial coordinate $R \equiv \psi(t, r)$. At this stage, a further transformation of the time coordinate, keeping R unchanged, can be employed to eliminate the off-diagonal component of $g_{\mu\nu}^{(2)}$. The line-element (2.5) is thus, without loss of generality, reduced to the form

$$ds^2 = -A(T, R) dT^2 + B(T, R) dr^2 + R^2 d\theta^2 + R^2 f(\theta)^2 d\phi^2. \quad (\text{A.1})$$

When this line-element is inserted in (2.14) the constraint equation $E_1^0 \equiv -\frac{\partial_T B(T, R)}{RA(T, R)B(T, R)} = 0$ is obtained. Insertion of the solution to this equation $B(T, R) = \beta(R)$ into (2.14) makes the combination $R\beta(R)(E_1^1 - E_0^0) \equiv \frac{\partial_R \beta(R)}{\beta(R)} + \frac{\partial_R A(T, R)}{A(T, R)} = 0$ easy to integrate, resulting in $A(T, R) = \frac{\alpha(T)}{\beta(R)}$. At this stage, the line-element (A.1) reads:

$$ds^2 = -\frac{\alpha(T)}{\beta(R)} dT^2 + \beta(R) dr^2 + R^2 d\theta^2 + R^2 f(\theta)^2 d\phi^2. \quad (\text{A.2})$$

The form of this metric reveals that it is static, since $\alpha(T)$ can be absorbed by a time redefinition $d\tau \equiv \alpha(T)^{1/2} dT$. For $\Lambda = 0$ and spherical symmetry this result is known as Birkhoff's theorem. Using these standard coordinates we can easily integrate the only functionally independent equation E_0^0 obtaining the following three "point-like" solutions for the cases of spherical, *GBL* and plane symmetry respectively:

$$ds^2 = -\left(1 - \frac{A}{R} - \frac{\Lambda R^2}{3}\right) d\tau^2 + \left(1 - \frac{A}{R} - \frac{\Lambda R^2}{3}\right)^{-1} dR^2 + R^2 d\theta^2 + R^2 (\sin \theta)^2 d\phi^2 \quad (\text{A.3})$$

$$ds^2 = -\left(-1 + \frac{A}{R} - \frac{\Lambda R^2}{3}\right) d\tau^2 + \left(-1 + \frac{A}{R} - \frac{\Lambda R^2}{3}\right)^{-1} dR^2 + R^2 d\theta^2 + R^2 (\sinh \theta)^2 d\phi^2 \quad (\text{A.4})$$

$$ds^2 = -\left(\frac{\kappa}{R} - \frac{\Lambda R^2}{3}\right) d\tau^2 + \left(\frac{\kappa}{R} - \frac{\Lambda R^2}{3}\right)^{-1} dR^2 + R^2 d\theta^2 + R^2 \theta^2 d\phi^2 \quad (\text{A.5})$$

For $\Lambda = 0$ (A.3) is the well-known Schwarzschild metric [32]. The constants A, Λ appearing in (A.3),(A.4),(A.5) are *essential*: They can not be eliminated or even altered by a coordinate transformation, and thus different values of them define distinct geometries. Their character as *essential* can be verified by the use of the Cartan-Karlhede equivalence method [33], or the use of a criterion developed in [34], or by enclosing them in invariant relations satisfied among geometrical objects constructed by the Riemann tensor and its covariant derivatives of an appropriate order. Such relations typically arise when we eliminate the coordinates between the expressions giving a sufficient number of curvature scalars. For metrics (A.3), (A.4) a pair of such relation is given by:

$$\begin{aligned}
R - 4\Lambda &= 0, \\
(3S1 - 8\Lambda^2)^6(27\epsilon A^2 S2 + 64\Lambda^2 + 144\epsilon A^2 \Lambda^3) - 3(3S1 - 8\Lambda^2)^5(27\epsilon A^2 S1 S2 + \\
32\Lambda S2 + 27\epsilon A^2 S3) + 3\epsilon(3S1 - 8\Lambda^2)^4((36A^2 \Lambda + 12\epsilon)S2^2 + 81A^2 S1 S3 + 80\epsilon \Lambda S3) - \\
18\epsilon(3S1 - 8\Lambda^2)^3(A^2 S2^3 + (36A^2 \Lambda + 10\epsilon)S2 S3) + 9\epsilon(3S1 - 8\Lambda^2)^2(18A^2 S2^2 S3 + \\
(108A^2 \Lambda + 25\epsilon)S3^2) - X486\epsilon A^2 S2 S3^2(3S1 - 8\Lambda^2) + 486\epsilon A^2 S3^3 &= 0, \tag{A.6}
\end{aligned}$$

where $\epsilon = -1, 1$ for spherical and *GBL* symmetry respectively, $S1$ is the Kretschmann scalar curvature, $S2$ its Laplacian, and $S3$ the square of the measure of its gradient:

$$S1 \equiv R^{IJKL} R_{IJKL}, \quad S2 \equiv g^{IJ} S1_{;I;J}, \quad S3 \equiv g^{IJ} S1_{;I} S1_{;J}.$$

For metric (A.5) the constant κ is not *essential* as it can be seen by applying the criterion of [34]. It is included only because the value $\kappa = 0$ gives a space different from the family with $\kappa \neq 0$. Thus, in this case, one invariant relation is needed (in order to prove the essentiality of Λ). This could be the common to all metrics $R = 4\Lambda$ or, the alternative relation:

$$(3S1 - 8\Lambda^2)(S2 - 6S1\Lambda + 16\Lambda^3) - 3S3 = 0. \tag{A.7}$$

The case $K > 0$ is completely analogous : F is now time-like and can thus be reduced to the form $F = (1, 0)$, indicating that $\psi(t, r)$ is to be identified to the new time coordinate $T \equiv \psi(t, r)$. Again the freedom in choosing the new radial coordinate can be used to eliminate the off-diagonal element of $g_{\mu\nu}^2$. Therefore, the metric corresponding to (A.1) is

$$ds^2 = -A(T, R) dT^2 + B(T, R) dr^2 + T^2 d\theta^2 + T^2 f(\theta)^2 d\phi^2. \tag{A.8}$$

When this line-element is inserted into (2.14) the constraint equation $E_1^0 \equiv -\frac{\partial_R A(T, R)}{T A(T, R)^2} = 0$ is obtained. Insertion of the solution to this equation $A(T, R) = \alpha(T)$ into (2.14) makes the combination $T\alpha(T)(E_1^1 - E_0^0) \equiv \frac{\partial_T \alpha(T)}{\alpha(T)} + \frac{\partial_T B(T, R)}{B(T, R)} = 0$ easy to integrate, resulting in $B(T, R) = \frac{\beta(R)}{\alpha(T)}$. At this stage, the line-element (A.8) reads:

$$ds^2 = -\alpha(T) dT^2 + \frac{\beta(R)}{\alpha(T)} dr^2 + T^2 d\theta^2 + T^2 f(\theta)^2 d\phi^2 \tag{A.9}$$

revealing that it is spatially homogeneous, since $\beta(R)$ can be absorbed by a redefinition $d\rho \equiv \beta(R)^{1/2}dR$. This result can be considered as the counterpart to Birkhoff's theorem. In these standard coordinates the only remaining functionally independent equation $E_0^0 = 0$ is readily solved, resulting in the following three cosmological solutions

$$ds^2 = -\left(1 - \frac{A}{T} - \frac{\Lambda T^2}{3}\right)^{-1} dT^2 + \left(1 - \frac{A}{T} - \frac{\Lambda T^2}{3}\right) d\rho^2 + T^2 d\theta^2 + T^2 (\sin \theta)^2 d\phi^2 \quad (\text{A.10})$$

$$ds^2 = -\left(-1 + \frac{A}{T} - \frac{\Lambda T^2}{3}\right)^{-1} dT^2 + \left(-1 + \frac{A}{T} - \frac{\Lambda T^2}{3}\right) d\rho^2 + T^2 d\theta^2 + T^2 (\sinh \theta)^2 d\phi^2 \quad (\text{A.11})$$

$$ds^2 = -\left(\frac{\kappa}{T} - \frac{\Lambda T^2}{3}\right)^{-1} dT^2 + \left(\frac{\kappa}{T} - \frac{\Lambda T^2}{3}\right) dR^2 + T^2 d\theta^2 + T^2 \theta^2 d\phi^2 \quad (\text{A.12})$$

For $\Lambda = 0$ (A.10) is the well-known Kantowski-Sachs metric [35]. The character of the constants A, Λ, κ is the same as in the previous case ($K > 0$) and their (locally) defining invariant relations are identical to (A.6), (A.7).

Finally, the case $K = 0$ does not produce any new, non-trivial solution: The investigation is most easily carried out in the conformal gauge and light-cone coordinates for metric (2.5) $ds^2 = 2A(u, v) du dv + \psi(u, v)^2 d\theta^2 + \psi(u, v)^2 f(\theta)^2 d\phi^2$. Since $K \equiv 2 \frac{\partial_u \psi(u, v) \partial_v \psi(u, v)}{A(u, v)} = 0$ we have that $\psi(u, v) = h(u)$ or $k(v)$. In either case, with an appropriate redefinition of u or v the previous line-element can be cast into the form $ds^2 = 2A(U, v) dU dv + U^2 d\theta^2 + u^2 f(\theta)^2 d\phi^2$. The case with v^2 in the spatial part is completely equivalent. The constraint equations now become $E_0^0 \equiv \Lambda + \frac{\epsilon}{U^2} = 0$, $E_1^1 \equiv \frac{2\partial_U A(U, v)}{U A(U, v)^2} = 0$ where $\epsilon = -1, 1, 0$ for the spherical, *GBL* or plane symmetry respectively. The system is compatible only for $\epsilon = 0$ (plane symmetry), its solution being $\Lambda = 0$, $A(U, v) = A_1(v)$ which, however, is nothing but the Minkowski space-time.

Notice that in the families of metrics (A.3), (A.4), (A.10), (A.11) Minkowski space-time is attained with $A = 0$, $\Lambda = 0$, while we can not set $\kappa = 0$, $\Lambda = 0$ in (A.5), (A.12). Also, setting $A = 0$, $\kappa = 0$ in all metrics results in maximally symmetric space-times. Lastly, for $\Lambda = 0$ all solutions can be concisely summarized, in the conformal light-cone gauge, by (2.6), (2.7).

B Appendix: Existence of L

In this Appendix we show that an $L(\Psi^{(1)})$ exists for which (3.18) is equal to (3.19) for the particular choice of $F[L(\Psi^{(1)})]$ given by (3.27). To begin with let us change coordinates from $\Psi^{(1)}$ to $\omega \equiv \sqrt{\Psi^{(1)2} - \epsilon}$. The term inside the square brackets in the r.h.s of (3.18) becomes

$$\left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right)^2 - 2\sqrt{\epsilon + \omega^2} \int \frac{\left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right) \left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right)'}{\sqrt{\epsilon + \omega^2}} d\omega,$$

where the prime now denotes differentiation with respect to the variable ω . On the other hand (3.19), through the choice $F = L^{-1}(\omega)$ becomes

$$L(\omega)^2 - \frac{4\omega^2}{3} L'(\omega)^2.$$

We thus have to prove the existence of an $L(\omega)$ which secures the equality between the above two expressions, namely that

$$\begin{aligned} I &\equiv \left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right)^2 - 2\sqrt{\epsilon + \omega^2} \int \frac{\left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right) \left(L(\omega) - \frac{\epsilon + \omega^2}{\omega} L'(\omega)\right)'}{\sqrt{\epsilon + \omega^2}} d\omega \\ &- L(\omega)^2 + \frac{4\omega^2}{3} L'(\omega)^2 = 0. \end{aligned} \quad (\text{B.1})$$

Let us assume that (B.1) holds. Then, the expression $-3\omega^2 \left(I - \frac{\epsilon + \omega^2}{\omega} \frac{\partial I}{\partial \omega}\right)$ must also vanish, which leads to

$$-3(\epsilon + \omega^2)^2 L'(\omega)^2 + 4\omega^2(2\epsilon + \omega^2) L'(\omega)^2 + 8\omega^3(\epsilon + \omega^2) L'(\omega) L''(\omega) = 0. \quad (\text{B.2})$$

The case $L(\omega) = \text{const.}$ does not concern us since it corresponds to the functional y^1 . Therefore, dividing (B.2) by $L'(\omega)^2$ and defining $a(\omega) \equiv L'(\omega)/L(\omega)$ we obtain

$$8\omega^3(\epsilon + \omega^2) a'(\omega) + \omega^4 + 2\epsilon\omega^2 - 3\epsilon^2 = 0, \quad (\text{B.3})$$

which is readily integrated, giving

$$a(\omega) = k + \frac{1}{8} \left(-\frac{3\epsilon}{2\omega^2} - 5 \ln \omega + 2 \ln(\epsilon + \omega^2) \right)$$

and thus

$$L(\omega) = m + \int \frac{(\epsilon + \omega^2)^{1/4}}{\omega^{5/8}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega. \quad (\text{B.4})$$

This expression for $L(\omega)$ emerged as an integrability condition for the integro-differential equation (B.1). It is therefore necessary to insert (B.4) into this equation. The result is

the following expression for $A \equiv -4(\epsilon + \omega^2)^{-1/2}I$:

$$A = -\frac{32\omega^{3/4}}{3} e^{2k - \frac{3\epsilon}{8\omega^2}} + \frac{8(\epsilon + \omega^2)^{3/4}}{\omega^{13/8}} e^{k - \frac{3\epsilon}{16\omega^2}} \left(m + \int \frac{(\epsilon + \omega^2)^{1/4}}{\omega^{5/8}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega \right) - \int \left[\frac{(3\epsilon^2 - 10\epsilon\omega^2 - \omega^4)}{\omega^{37/8}(\epsilon + \omega^2)^{1/4}} e^{k - \frac{3\epsilon}{16\omega^2}} \left(m + \int \frac{(\epsilon + \omega^2)^{1/4}}{\omega^{5/8}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega \right) \right] d\omega.$$

Surprisingly enough the above expression is ω -independent, i.e. $\partial A/\partial\omega = 0$. Therefore

$$B \equiv A - F(k, m) = 0. \quad (\text{B.5})$$

We have now to prove that there is a choice of the constants k, m for which $F = 0$, so that $A = 0 \Rightarrow I = 0$. Our strategy is to confine, through integrability conditions for (B.5), as much as possible the form of $F(k, m)$. As a first step, we must dispose off the double integral (in the variable ω) appearing in A . To do this we differentiate B with respect to k and solve the resulting expression for the aforementioned double integral. By inserting the outcome of this operation into (B.5) we get

$$B = \frac{\partial F}{\partial k} - 2F + \frac{8m(\epsilon + \omega^2)^{3/4}}{\omega^{13/8}} e^{k - \frac{3\epsilon}{16\omega^2}} - m \int \frac{(3\epsilon^2 - 10\epsilon\omega^2 - \omega^4)}{\omega^{37/8}(\epsilon + \omega^2)^{1/4}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega = 0.$$

By differentiating this new form of B with respect to k , and repeating the procedure described just above, we can eliminate the remaining integral that appears in B (and in fact all the ω -dependence). Thus, we end up with

$$B = 2F - 3\frac{\partial F}{\partial k} + \frac{\partial^2 F}{\partial k^2} = 0,$$

which has the following solutions:

$$F(k, m) = \lambda_1(m) e^k + \lambda_2(m) e^{2k}. \quad (\text{B.6})$$

Inserting (B.6) into (B.5) and differentiating with respect to m we get

$$\frac{8(\epsilon + \omega^2)^{3/4}}{\omega^{13/8}} e^{k - \frac{3\epsilon}{16\omega^2}} - \int \frac{(3\epsilon^2 - 10\epsilon\omega^2 - \omega^4)}{\omega^{37/8}(\epsilon + \omega^2)^{1/4}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega - \lambda_1'(m) e^k - \lambda_2'(m) e^{2k} = 0,$$

which by differentiation with respect to k gives

$$\frac{8(\epsilon + \omega^2)^{3/4}}{\omega^{13/8}} e^{k - \frac{3\epsilon}{16\omega^2}} - \int \frac{(3\epsilon^2 - 10\epsilon\omega^2 - \omega^4)}{\omega^{37/8}(\epsilon + \omega^2)^{1/4}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega - \lambda_1'(m) e^k - 2\lambda_2'(m) e^{2k} = 0.$$

Subtracting these last two equations we have

$$\lambda_2'(m) = 0 \Rightarrow \lambda_2(m) = c_3.$$

If we insert this result together with (B.6) in (B.5) and we double differentiate with respect to m we will get

$$\lambda_1''(m) = 0 \Rightarrow \lambda_1(m) = c_1 m + c_2.$$

Therefore, (B.6) becomes

$$F(k, m) = c_1 m e^k + c_2 e^k + c_3 e^{2k}$$

and we are finally able to conclude that the choice $m = -\frac{c_2 + c_3 e^k}{c_1}$ satisfies (B.1).

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