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Statistical Energy Analysis and the second principle of thermodynamics

Alain Le Bot

Abstract Statistical Energy Analysis is a statistical method in vibroacoustics entirely based on the application of energy balance that is the first principle of thermodynamics. In this study, the definition of vibrational entropy is introduced for sub-systems containing energy and modes. The rate of entropy production at interfaces between sub-systems is also derived. Finally, in steady-state condition, an entropy equilibrium is reached. The meaning of entropy and some implications of this entropy balance are also discussed.

1 Introduction

Statistical Energy Analysis [1, 2] is born from the application of statistical physics concepts to vibroacoustical systems. The idea is quite simple. When the number of modes of vibroacoustical systems is so large that the solving of governing equations becomes unpracticable, it is preferable to give up a deterministic description of the system and to adopt statistical methods. This standpoint is reasonable. It has been followed in several other fields in physics, kinetic theory of gases instead of point mechanics, statistical theory of turbulence instead of Navier-Stoke's equation, statistical behaviour of granular material and some others. In all these cases, the number of entities is very large so that a complete description of all of them would lead to a set of equations untractable in practice and whose solution would be very sensitive to errors of modelling. In addition, the amount of produced information would be huge, the major part being useless.

The application of concepts and methods of statistical physics to the audio frequency range instead of thermal vibration raises some difficulties. First of all, the number of entities in vibroacoustics (the modes) is not so large than in thermics

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(atoms or molecules). Statistical methods applied to poor populations are fragile. This explains why results of SEA are sometimes disappointing. SEA is too often used outside its validity domain. Secondly, small populations lead to large fluctuations. While it is almost impossible (or at least very difficult) to observe fluctuations in thermodynamical systems, large fluctuations are common in SEA and it is sometimes necessary to deal with large population of similar systems, the so-called Gibb's ensemble, to increase the quality of SEA results. Finally, the equipartition of energy among modes is sometimes violated. Poor population and low modal overlap are conditions favourable to a deviation from the equilibrium state. This is why it is so urgent to extend SEA to non-equilibrium states [3, 4, 5] by relaxing the diffuse field assumption following the example of non-equilibrium thermodynamics.

Up to now, SEA is entirely based on the application of energy balance that is the first principle of thermodynamics. But nothing is told on entropy in SEA [6, 7]. Conceptually, both quantities energy and entropy are necessary for a complete derivation of statistical physics and therefore SEA. Energy is indeed necessary to describe the state of the system. Entropy is necessary to measure the loss of information induced by the renouncement to a full description of the system. In this regard, different levels of approximations would lead to different values of entropy. In particular, starting from a complete theory (null entropy), a sequence of theories each of them being an approximation of the preceeding one, leads to an increasing sequence of entropy values. In SEA, the level of approximation has been stated, but the entropy has not yet been defined. This is an important shortcoming.

Entropy is also necessary to define the temperature. The classical definition of temperature T with [8],

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad (1)$$

where S is the entropy and E the internal energy, shows that the temperature measures the rate of increasing of entropy (and therefore the loss of information on the system) with energy. A positive temperature means that entropy is a non-decreasing function of internal energy. The loss of information during the approximation process is therefore more important for energetic systems. It is rather common in SEA literature to define the vibrational temperature as being the modal energy. This result is true but a formal proof requires to introduce the entropy [9]. This is the purpose of this paper.

2 First principle of thermodynamics in SEA

SEA is a simple method to assess the vibrational energy of complex systems. SEA is entirely based on some statistical considerations and the application of energy balance (first principle) and, as we shall see, the application of entropy balance (second principle).

In SEA, a complex system is sub-divided into n components called sub-systems. Sub-systems are considered as tanks containing vibroacoustical energy E_i with $i = 1, \dots, n$ whatever their nature, structural component or acoustical cavity (Fig. 1).

Within sub-system i , the vibrational energy E_i is repartited among N_i modes. Modes are therefore the sites which carry vibrational energy, exactly in the same way that molecules carry kinetic energy in kinetic theory of gases and that atoms carry vibrational energy in solids. In the two latter cases, this energy (kinetic or vibrational) is called heat. In SEA, the equivalent of heat is therefore the vibrational energy itself. The only difference between a true heat and the vibrational heat introduced in SEA is the frequency of the underlying vibration, thermal range for the former and audio frequency range for the latter.

The heat in SEA is thus defined as the vibrational energy E_i in broadband. The analysis is confined to a frequency band $\Delta\omega$ about the central frequency ω (rad/s). No strict definition is given for the width of the frequency band, but it is commonly admitted that octave bands are well suited.

The first principle of thermodynamics can now be introduced. All sub-systems can receive energy from sources, driving forces in the structural case or noise sources in acoustics. The power being injected into sub-system i is noted P_i^{inj} . But they also dissipate vibrational energy by natural mechanisms such as damping of vibration, absorption of sound by walls, attenuation of sound... The power being dissipated is noted P_i^{diss} . Finally, the vibrational energy can be exchanged with adjacent sub-systems. The net exchanged power between sub-system i and j is noted P_{ij} . In steady-state condition, the energy balance for sub-system i reads,

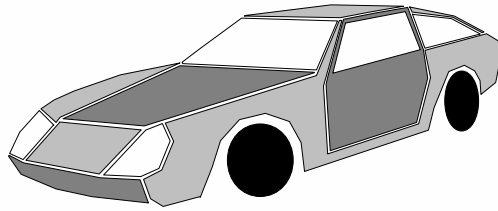
$$P_i^{\text{diss}} + \sum_{j \neq i} P_{ij} = P_i^{\text{inj}}. \quad (2)$$

The power being injected P_i^{inj} is assumed to be known, but the powers being dissipated P_i^{diss} and being exchanged P_{ij} must be expressed in terms of vibrational energies (Fig. 2).

The power being dissipated by internal losses is,

$$P_i^{\text{diss}} = \omega \eta_i E_i, \quad (3)$$

Fig. 1 Decomposition of complex structure in SEA. The vibroacoustical problem is solved by splitting the complex structure into n elements. In all these sub-systems, the vibration field is diffuse *i.e.* homogeneous and isotropic. The SEA approach consists in writing the exchange of energy between these sub-systems.



where η_i is the damping loss factor usually determined by a direct measurement.

The power supplied by the sub-system i to the sub-system j is,

$$P_{i \rightarrow j} = \omega \eta_{ij} E_i, \quad (4)$$

where η_{ij} is the coupling loss factor. The net exchanged power between sub-systems i and j is $P_{ij} = P_{i \rightarrow j} - P_{j \rightarrow i}$ and therefore [10],

$$P_{ij} = \omega (\eta_{ij} E_i - \eta_{ji} E_j), \quad (5)$$

Coupling loss factors are phenomenological constants attached to the junction between two sub-systems. Their values must be determined in each case either by a direct measurement or by using some predictive relationships. These theoretical relationships have been derived for a large number of cases, coupling between adjacent plates, acoustical cavities, connection between plate and beam, sound radiation, vibrational response and so on. We do not enter into the discussion of the validity of a particular method to derive such relationships. A large literature is devoted to this problem and many discussions on the efficiency of these relationships can be found. For the purpose of the present discussion, it is enough to admit that for every junctions, two coupling loss factors η_{ij} and η_{ji} exist.

Coupling loss factors verify the reciprocity relationship,

$$N_i \eta_{ij} = N_j \eta_{ji}. \quad (6)$$

The reciprocity relationship highlights the importance of the modal energies E_i/N_i . Substituting Eq. (6) into Eq. (5) leads to,

$$P_{ij} = \omega \eta_{ij} N_i \left(\frac{E_i}{N_i} - \frac{E_j}{N_j} \right), \quad (7)$$

showing that the net exchanged power is proportional to the difference of modal energies.

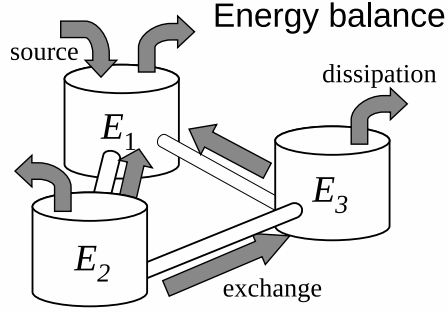


Fig. 2 Energy balance in SEA. Sources supply energy to sub-systems. Part of this energy is dissipated by natural processes. Energy is finally exchanged between adjacent sub-systems. Vibrational energy E_i of sub-systems results from the balance of sources, dissipation and exchanges.

The SEA equation is simply obtained by substituting Eqs.(3, 5) in Eq. (2),

$$\omega \begin{pmatrix} N_1 \sum_j \eta_{1j} & & -N_1 \eta_{1k} \\ & \ddots & \\ -N_k \eta_{kl} & & N_n \sum_j \eta_{nj} \end{pmatrix} \begin{pmatrix} E_1/N_1 \\ \vdots \\ E_n/N_n \end{pmatrix} = \begin{pmatrix} P_1^{\text{inj}} \\ \vdots \\ P_n^{\text{inj}} \end{pmatrix} \quad (8)$$

This is a linear system on vibrational temperatures whose matrix is symmetric.

3 Vibrational entropy, vibrational temperature

In order to rigorously define the entropy of vibrational systems within the context of SEA, it is important to well understand the approximation process applied to equations of motion to derive SEA equations. The equations of motion are the reference theory which gives an access to every physical quantities. There is no loss of information, the entropy is null. SEA is a simplified theory which does not give access to the detail of all physical quantities. There is a loss of information. In SEA, the vibrational energy is repartited among modes and the exact repartition of energy is not known. This exact repartition is refered as microstate of the system. A microstate of the system is a list of modes with a specification of their energy level such as:

Mode 1 Energy e_1
 Mode 2 Energy e_2
 ...
 Mode N Energy e_N

The sum of all modal energies is the energy of the overall system $E = \sum_{i=1}^N e_i$. Indeed, the solving of the equations of motion provides this list and therefore the knowledge of the microstate. Conversely, one can admit that the knowledge of this list gives of the total information on the system. Actually, a complete description of the state of system in steady condition is given by a list of modal amplitudes (complex-valued numbers) rather than modal energies (real-valued numbers). But in this text, we will not discuss the question of entropy induced by the loss of information when one neglects phase shifts between modal amplitudes.

The problem is therefore to quantify the loss of information between the knowledge of a microstate, that is the repartition e_i with $i = 1 \dots N$ and the knowledge of a macrostate, that is the total energy E and the number N of modes. The question is to count the number of microstates that correspond to a given macrostate.

With this idea in mind, a SEA system can be viewed as a set of N linear oscillators whose natural frequencies is an increasing sequence $\omega_1, \omega_2 \dots \omega_N$ (Fig. 3). And the number W of microstates attached to a macrostate E , N is given by the structure function [9],

$$\Omega(E) = \left(\frac{2\pi}{\omega} \right)^N \frac{E^{N-1}}{N-1!} \quad (9)$$

and,

$$W = \frac{\Omega(E)}{h^N} \delta E \quad (10)$$

where δE is the uncertainty on E and h is a constant introduced during the discretization of phase space into equiprobable cells. The final step is to apply Boltzmann's definition of entropy,

$$S = k \log W \quad (11)$$

with the result,

$$S(E, N) = kN \left[1 + \log \left(\frac{2\pi E}{h\omega N} \right) \right] \quad (12)$$

This is the microcanonical entropy of a SEA sub-system. Eq.(12) gives the complete expression of the "vibrational entropy" of any SEA sub-system.

The temperature is obtained as for any thermodynamic system with,

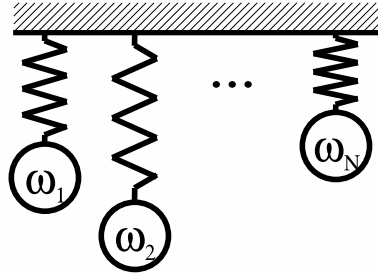
$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_N \quad (13)$$

with the result,

$$T = \frac{E}{kN} \quad (14)$$

As it was expected from Eq. (7) giving the power flow between two SEA sub-systems, the "vibrational temperature" is well defined by the modal energy. In the literature, this result is generally obtained from an analogy between SEA and thermodynamics. This is now a logical consequence of the expression of the entropy obtained from Boltzmann's definition.

Fig. 3 Principle of a SEA system. The vibrational energy E is repartited among N linear oscillators whose natural frequencies is an increasing sequence $\omega_1, \omega_2 \dots \omega_N$. The detailed repartition of the energy $e_1, e_2 \dots e_N$ is a microstate whereas the knowledge of the total energy E and the number N of oscillators is a macrostate.



4 Second principle of thermodynamics in SEA

The second principle of thermodynamics states that entropy of isolated system cannot decrease. This is also true within the context of SEA. The problem that must be considered is the mixing of vibrational energy of two adjacent sub-systems.

Let us consider two sub-systems with energies E_1 , E_2 and mode counts N_1 and N_2 . If these sub-systems are isolated, their entropies are respectively $S(E_1, N_1)$ and $S(E_2, N_2)$ given by Eq. (12). Now, if these two sub-systems are connected together, the final state has energy $E_1 + E_2$ and the total number of modes is $N_1 + N_2$. When equilibrium is reached, the final entropy of the overall system is $S(E_1 + E_2, N_1 + N_2)$, once again given by Eq. (12).

The difference between the final entropy and the sum of initial entropy is therefore the entropy created during the mixing process.

$$\Delta S = S(E, N) - [S(E_1, N_1) + S(E_2, N_2)] \quad (15)$$

By substituting Eq. (12), it yields,

$$\Delta S = k(N_1 + N_2) \log \frac{E_1 + E_2}{N_1 + N_2} - kN_1 \log \frac{E_1}{N_1} - kN_2 \log \frac{E_2}{N_2} \quad (16)$$

This is the entropy created by mixing the energy of the two sub-systems. This entropy production is non-negative $\Delta S > 0$. This result stems from the convexity of the function $f(x, y) = -y \log(x/y)$.

The fact that ΔS is non-negative can be interpreted in terms of loss of information. The loss of information for sub-system i is the log of the number of microstates corresponding to E_i , N_i . And the loss of information for the overall system is the log of the number of microstates corresponding to $E = E_1 + E_2$, $N = N_1 + N_2$. But there is many more possibilities to share the energy $E = E_1 + E_2$ over $N = N_1 + N_2$ modes than the sum of possibilities to share on the one hand, E_1 over N_1 modes and, on the second hand, E_2 over N_2 modes. The difference between the two is exactly the loss of information during the mixing process.

5 Entropy balance in SEA

Coming back to the initial situation of n sub-systems in interaction, we are now in position to state an entropy balance of the overall system in SEA. Three questions must be examined: sources, dissipation and mixing.

The variation of entropy is driven by the variation of vibrational energy around a vibrational temperature. Since the number of modes N is always fixed, Eq. (13) gives the infinitesimal variation of entropy dS for an infinitesimal variation δE ,

$$dS = \frac{\delta E}{T} \quad (17)$$

For sources, the injected power is P_i^{inj} . The vibrational energy δE supplied to the sub-system during time dt is $\delta E = P_i^{\text{inj}} dt$. It follows that the increase rate of entropy by sources is,

$$\frac{dS_i^{\text{inj}}}{dt} = \frac{P_i^{\text{inj}}}{T_i} = k \frac{P_i^{\text{inj}} N_i}{E_i} \quad (18)$$

The last equality stems from Eq. (14).

Similarly, dissipation processes of vibration within sub-systems leads to a modification of vibrational level. This energy is dissipated that is transformed into heat (in the sense of thermics). In classical thermodynamics, dissipation processes induce an increase of entropy by creating heat inside the system. But, in SEA, that we have called vibrational heat is the vibrational energy itself. And dissipation leads to a decreasing of vibrational energy. In other words, dissipation tends to cool the system and therefore induces a decreasing of vibrational entropy.

$$\frac{dS_i^{\text{diss}}}{dt} = -\frac{P_i^{\text{diss}}}{T_i} = -k\omega\eta_i N_i \quad (19)$$

where the last equality is deduced from Eq. (3).

The mixing of energy at interfaces between sub-systems also induces a variation of entropy. As we have seen in previous section, this variation is always non-negative. Let us develop this point. The power being exchanged between sub-systems i and j is P_{ij} . Therefore, the vibrational entropy introduced in or extracted from sub-system i is $-P_{ij}/T_i$. Following the same reasoning, the vibrational entropy introduced in sub-system j is $-P_{ji}/T_j$. Since $P_{ij} = -P_{ji}$, the net vibrational entropy introduced in the entire system by the mixing process is,

$$\frac{dS_{ij}}{dt} = P_{ij} \left(\frac{1}{T_j} - \frac{1}{T_i} \right) = k\omega(\eta_{ij}E_i - \eta_{ji}E_j) \left(\frac{N_j}{E_j} - \frac{N_i}{E_i} \right) \quad (20)$$

The last equality stems from Eq. (5).

For the entire system, the vibrational entropy introduced in the system is,

$$\frac{dS}{dt} = \sum_{i=1}^n \frac{dS_i^{\text{inj}}}{dt} + \frac{dS_i^{\text{diss}}}{dt} + \sum_{i>j} \frac{dS_{ij}}{dt} \quad (21)$$

It is easy to deduce from Eq. (2) that this entropy is null,

$$\frac{dS}{dt} = 0 \quad (22)$$

It means that there is no production of entropy for the entire SEA system. But there is an exchange of entropy with exterior which exactly balances the production of entropy by mixing processes (Fig. 4).

6 Conclusion

In this paper, it has been shown that the second principle of thermodynamics can be stated in the context of SEA. Explicit relationships have been derived for the vibrational entropy of sub-systems and the production of entropy at interfaces of sub-systems. SEA is up to now only based on the application of the energy balance. An entropy balance also applies.

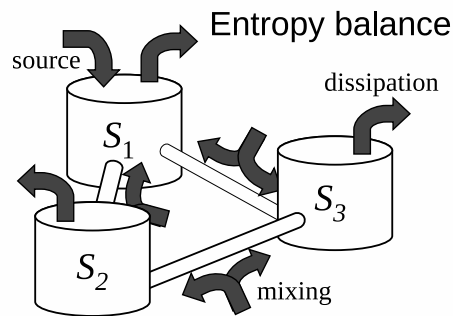
Acknowledgments

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Fig. 4 Entropy balance in SEA. Sources supply entropy to sub-systems. Dissipation processes extract entropy since they are responsible of a cooling of the sub-system. Entropy is also produced by mixing process at interfaces between adjacent sub-systems. Vibrational entropy S_i of sub-systems results from the balance of sources, dissipation and mixing.



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7 Questions

G. Tanner:

What is the meaning of the number of modes N if one considers the continuum limit which is natural in acoustics?

A. Le Bot:

This problem is also encountered in classical mechanical statistics. The usual response of physicists is to say that basically nothing is continuous, everything is quantum that is discrete. I think that this response is not satisfactory for vibroacoustics. In physical statistics, we start from systems which have a huge but finite number of states and we derive continuum mechanics. But in statistical vibroacoustics we start from a continuous equation and our problem is not to know if this equation is an approximation of a more fundamental equation but rather to degrade it with further statistical assumptions. What was discussed in this talk is how to quantify the amount of information that has been lost during this second level statistical process.

Y. Ben-Haim:

Could you extend this formalism to open systems? For instance, when vibroacoustic systems radiate and they have a loss of energy and they also exchange entropy because energy is leaving the system and going into vacuum or air.

A. Le Bot:

I just said that this approach is limited to the case where the surface of constant energy is closed but not that the system itself is closed. I don't know if this a possibility to apply this formalism to open systems, but I feel that this formalism is not simply well-suited for non-finite systems which, by nature, cannot be in thermal equilibrium.

P. Shorter:

Let's me make a comment on the number of configurations. It's not only a story of number of modes. If you consider a plate with only two modes but in a very large ensemble of similar plates, they may have varying damping, material properties or boundary conditions. So the variation of entropy is taken over this space of configurations. But, if you look at a single system, a single system has a very smooth response, while it has a huge number of modes. But that doesn't definitely suppose it has only one value, the one given by a SEA model. It's value can be different from the one you've got from an ensemble average.

A. Le Bot:

In the expression of the vibrational entropy I propose to use, it is just included the missing information relative to fact that we don't know the exact repartition of

energy over modes. But if we want to measure the effect of particular statistical properties such as the boundary conditions are not known, we must modify this entropy expression. And we can even imagine to add all these individual entropies for boundary conditions, size of system, properties of material and so on, to get a more accurate expression of vibrational entropy.

F. Ziegler:

You mentioned that when you define the "vibrational temperature" as modal energy divided by Boltzmann constant and when you calculate the vibrational temperature of a plate for usual acoustical levels, you get very a high temperature about ten power eleven Kelvins! So, what are the physical consequences of this fact? Can we will not use this theory of thermodynamics only because of the improper scale of these numbers?

A. Le Bot:

I think that there is absolutely no physical consequences of this fact. In astronomy, for instance, measurements are so accurate and signals so tiny that they must aware of noise level induced by mechanical vibration. They have very large structures which are in thermal equilibrium with surrounding air and therefore, the "vibrational temperature" is equal to the true thermal temperature of hundred Kelvins. But in our case, the sources of vibration have a mechanical origin and therefore we deals with very large vibration, which are not in thermal equilibrium with usual thermal frequencies. This is why we get so high vibrational temperatures. But this very hot vibration is confined into a relatively low and narrow frequency band which is disconnected from thermal frequencies. The value of Boltzmann constant is not well-suited in SEA for usual vibrational systems and therefore, the question which raises is the definition of an appropriate scale for vibrational temperature.

A. Belayev:

Antonio was the first to introduce the vibrational entropy in SEA.

A. Carcaterra:

I just worked on that subject some years ago. And it is a very interesting theoretical point of view. But my concern was that it is not really obvious how we can use this additional concept of vibrational entropy in terms of improving the SEA models. How is it possible to use an entropy balance equation in order to get more information on SEA models or in order to get an advantage in the description of vibroacoustical systems?

A. Le Bot:

This is indeed an important question and may be the only question of interest in fact. Is it useful to use the entropy concept in vibroacoustics? I now work on this problem for a couple of years and I sincerely hope that it is useful! But let me read some lines from Ingo Muller who recently wrote an interesting paper in Entropy journal on historical aspects of thermodynamics: "In the nineteenth century - after the formulation of the second law - there was a noisy controversy between *energetics*, represented by Oswald, and thermodynamics favoured by Boltzmann. Energeticists maintained that the entropy was not needed. They were wrong, but they did have a point, albeit

only at small temperatures. Planck was involved - in a minor role - in the discussion as an unappreciated supporter of Boltzmann's thermodynamic view. It was this controversy which prompted Planck to issue his oft-quoted *dictum*: *The only way to get revolutionary advances in science accepted is to wait for all scientists to die.*"