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# RECORD PROCESS ON THE CONTINUUM RANDOM TREE

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

ABSTRACT. We consider the number of cuts  $X_n^*$  needed to isolate the root of the sub-tree spanned by  $n$  leaves uniformly chosen at random in Aldous's continuum random tree  $\mathcal{T}$ . We prove the almost sure convergence of  $X_n^*/\sqrt{2n}$  to a Rayleigh random variable  $Z$ . We get from the a.s. convergence a representation of  $Z$  as the integral on the leaves of  $\mathcal{T}$  of a record process indexed by the tree  $\mathcal{T}$ . The proof relies on a Brownian Snake approach. This result was motivated by Janson's convergence in distribution of the renormalized number of cuts in a discrete random tree.

## 1. INTRODUCTION

The problem of random cutting down of a rooted tree arises first in MEIR and MOON [18]. The problem is the following: consider a rooted tree  $\mathcal{T}_n$  with  $n$  vertices, select an edge uniformly at random, delete that edge and keep the part of the tree that contains the root. Continue recursively until only the root is left. We let  $X_n$  denote the number of cuts that are needed to isolate the root. What is the asymptotic behavior of  $X_n$  as  $n$  tends to  $\infty$ ?

This problem has been in particular tackled in [18] and in BERTOIN [9] for Cayley trees (uniform labeled tree with  $n$  vertices), in PANHOLZER [19] for simply generated trees or in JANSON [15] for conditioned Galton-Watson trees. The main result in [15] states that, if the offspring distribution of the Galton-Watson process is critical (that is with mean equal to 1) with finite variance, which we take equal to 1 for simplicity, then the following convergence in distribution holds:

$$(1) \quad (\mathcal{T}_n/\sqrt{n}, X_n/\sqrt{n}) \xrightarrow[n \rightarrow +\infty]{(d)} (\mathcal{T}, Z_{\mathcal{T}})$$

where  $\mathcal{T}$  is Aldous's continuum random tree (CRT) introduced in ALDOUS [6, 7], the conditioned law of  $Z_{\mathcal{T}}$  given  $\mathcal{T}$  is specified by its moments, and  $Z_{\mathcal{T}}$  has a Rayleigh distribution with density  $x e^{-x^2/2} \mathbf{1}_{\{x>0\}}$ . However, there is no constructive description of  $Z_{\mathcal{T}}$  conditionally on  $\mathcal{T}$ .

The goal of this paper is to consider a continuous cutting of the CRT and to give an a.s. convergence for a quantity similar to  $X_n$ . More precisely, we consider a CRT  $\mathcal{T}$ , associated with a branching mechanism  $\psi(u) = \alpha u^2$  under the excursion measure  $\mathbb{N}$ . This tree is coded by the height process  $\sqrt{2/\alpha} B_{\text{ex}}$ , where  $B_{\text{ex}}$  is a positive Brownian excursion. Let  $m^{\mathcal{T}}$  be the corresponding measure on the leaves of  $\mathcal{T}$  (see Section 3.1 for a precise definition of  $m^{\mathcal{T}}$ ). We set  $\sigma = m^{\mathcal{T}}(\mathcal{T})$  the total mass of the tree  $\mathcal{T}$ . Notice that Aldous's CRT is distributed as  $\mathcal{T}$  under  $\mathbb{N}[d\mathcal{T} \mid \sigma = 1]$  with  $\alpha = 1/2$ .

We throw points uniformly on the CRT at rate  $2\alpha$  in the same spirit as in ALDOUS and PITMAN [8] (see also ABRAHAM and SERLET [4] for a direct construction and ABRAHAM,

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DELMAS and VOISIN [3] for the general Lévy tree). When a mark appears, we cut the tree on this mark and discard the sub-tree that does not contain the root. For  $x \in \mathcal{T}$ , we denote by  $\theta(x)$  the time at which  $x$  is separated from the root. Then we define:

$$\Theta = \int_{\mathcal{T}} \theta(x) m^{\mathcal{T}}(dx) \quad \text{and} \quad Z = \sqrt{\frac{2\alpha}{\sigma}} \Theta.$$

Conditionally on  $\sigma$ , the random variable  $Z$  is distributed according to the Rayleigh distribution, see Proposition 4.4. Let us denote by  $T_n$  the reduced tree spanned from the CRT  $\mathcal{T}$  by  $n$  leaves uniformly chosen at random and the root. The tree  $T_n$  is distributed as a uniform binary tree with  $n$  leaves and hence  $2n - 1$  vertices, but with edges with random lengths. Let  $X_n^*$  be the number of cuts (generated by the continuous cutting) that are needed to isolate the root of this reduced tree. In this setting the root is isolated as soon as the remaining tree containing the root is reduced to one edge. Notice that several cuts may appear on the same edge of  $T_n$ , so  $X_n^*$  looks like  $X_{2n-1}$  for binary trees but is not exactly the same object. The main result of this paper, see Theorem 6.1, states that  $\mathbb{N}$ -a.e.:

$$\lim_{n \rightarrow +\infty} \frac{X_n^*}{\sqrt{2n}} = \sqrt{\frac{2\alpha}{\sigma}} \Theta = Z.$$

In fact, we also prove the result conditionally on  $\{\sigma = 1\}$ , that is for Aldous's CRT. This strongly suggests that  $Z$  and  $Z_{\mathcal{T}}$  are equal. To get this equality, it is enough to check the equality of all their moments conditionally on  $\mathcal{T}$ . In fact, we were able to compute only the 3 first moments of  $Z$  conditionally on  $\mathcal{T}$ , and we present the two first in Proposition 4.8 and Remark 4.9. The 3 moments indeed coincide when  $\alpha = 1/2$ . So, at this stage the question is still open.

We provide also another representation of  $\Theta$  in terms of the mass of the pruned tree (a similar result also appears in ADDARIO-BERRY, BROUTIN and HOLMGREN [5]). More precisely, we set for  $q \geq 0$ :

$$\sigma_q = \int_{\mathcal{T}} \mathbf{1}_{\{\theta(x) \geq q\}} m^{\mathcal{T}}(dx)$$

the mass of the remaining tree at time  $q$ . It is also the mass of a tagged fragment in Aldous-Pitman's fragmentation. The process  $(\sigma_q, q \geq 0)$  is distributed conditionally on  $\{\sigma = 1\}$ , as  $(1/(1 + \tau_q), q \geq 0)$  where  $\tau$  is a stable subordinator of index  $1/2$ , see ALDOUS and PITMAN [8] or ABRAHAM and DELMAS [1]. Then we prove that:

$$\Theta = \int_0^{+\infty} \sigma_q dq.$$

Using results from ABRAHAM, DELMAS and HOSCHEIT [2] on pruning of Lévy trees, we also derive asymptotics about  $(\sigma^i, i \in \mathcal{I})$  the sizes of the removed sub-trees during the cutting procedure. According to Propositions 8.2 and 8.3, we have  $\mathbb{N}$ -a.e.:

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} \mathbf{1}_{\{\sigma^i \geq 1/n\}} = \lim_{n \rightarrow +\infty} \sqrt{n} \sum_{i \in \mathcal{I}} \sigma^i \mathbf{1}_{\{\sigma^i \leq 1/n\}} = 2\sqrt{\frac{\alpha}{\pi}} \Theta = \sqrt{\frac{2\sigma}{\pi}} Z.$$

The paper is organized as follows. In Section 2, we define the record process constructed from the continuous cutting procedure on the real line, then we construct the cutting procedure on the CRT in Section 3. In Section 4 we compute the law of  $\Theta$  and some moments under various probability laws. We then introduce the reduced tree with  $n$  leaves in Section 5 and state and prove the a.s convergence of the number of cuts in Section 6. Section 7 is devoted to the proof of a key result needed in the proof of Theorem 6.1. We give the other

representation of  $\Theta$  in Section 8 and we end the paper with an Appendix that gather technical proofs of several lemmas used along the paper.

## 2. THE RECORD PROCESS

Let  $\alpha > 0$ . We consider  $\theta = (\theta(t), t \geq 0)$  the record process and  $X = (X(t), t \geq 0)$  the record counting process defined as follows. Let  $N(dt, dq) = \sum_{i \in I} \delta_{t_i, q_i}(dt, dq)$  be a Poisson point measure with intensity  $2\alpha \mathbf{1}_{\{t \geq 0, q \geq 0\}} dt dq$ . For  $\theta(0) \in [0, +\infty]$  and  $X(0) \in \mathbb{N}$ , we set for all  $t \geq 0$ :

$$\theta(t) = \min(\theta(0), \inf\{q_i; t_i \leq t\}) \quad \text{and} \quad X(t) = X(0) + \sum_{0 < s \leq t} \mathbf{1}_{\{\theta_s > \theta_s\}}.$$

By construction  $\theta$  and  $(\theta, X)$  are Markov processes. We shall denote by  $\mathbb{P}_q$  with  $q \in [0, +\infty]$  (resp.  $\mathbb{P}_{(q,k)}$  with  $q \in [0, +\infty]$  and  $k \in \mathbb{N}$ ) the law of the process  $\theta$  (resp.  $(\theta, X)$ ) starting at  $q$  (resp.  $(q, k)$ ), and we denote by  $\mathbb{E}_q$  (resp.  $\mathbb{E}_{(q,k)}$ ) the corresponding expectations.

Notice that  $\theta$  and  $X$  are non-increasing, and a.s.  $X(t) = +\infty$  for every  $t > 0$  if  $\theta(0) = +\infty$ .

*Remark 2.1.* Let us denote by  $1 \geq t_1 > t_2 > \dots$  the jumping times of the process  $(\theta(t), 0 \leq t \leq 1)$  under  $\mathbb{P}_\infty$ . By standard arguments on Poisson point measure, the random variable  $t_1$  is uniformly distributed on  $[0, 1]$ . Conditionally given  $t_1$ , the random variable  $t_2$  is uniformly distributed on  $[0, t_1]$  and so on. We are thus considering the standard stick breaking scheme and the random vector  $(1 - t_1, t_1 - t_2, \dots)$  is distributed according to the Poisson-Dirichlet distribution with parameter  $(0, 1)$ .

*Remark 2.2.* The coefficient  $2\alpha$  in the intensity is added to have the same intensity as in the pruning procedures of [4, 3, 2] but, as we can see from the previous remark, it does not appear in the law of the number of records.

Let  $Y_t$  be an exponential random variable with parameter  $2\alpha t$ . Notice that  $\inf\{q_i; t_i \leq t\}$  is distributed as  $Y_t$ . Let  $g$  be a bounded measurable function defined on  $[0, +\infty]$ . For every  $q \in [0, +\infty]$  and  $t > 0$ , we have

$$\mathbb{E}_q[g(\theta(t))] = \mathbb{E}[g(\min(q, Y_t))] = e^{-2\alpha q t} g(q) + \int_0^q g(x) 2\alpha t e^{-2\alpha t x} dx.$$

Notice that if  $g$  belongs to  $\mathcal{C}^1(\mathbb{R}^+)$  with  $g'$  bounded on  $\mathbb{R}_+$ , we have by an obvious integration by parts that, for  $q \in [0, +\infty]$  and  $t > 0$ ,

$$\mathbb{E}_q[g(\theta(t))] = g(0) + \int_0^q g'(x) e^{-2\alpha t x} dx.$$

We can then compute the infinitesimal generator of  $\theta$  denoted by  $\mathcal{L}$ . Let  $g$  be a bounded measurable function defined on  $[0, +\infty]$  such that  $g - g(+\infty)$  is integrable with respect to the Lebesgue measure on  $\mathbb{R}^+$ . For  $q \in [0, +\infty]$ , we have:

$$\begin{aligned} \mathcal{L}(g)(q) &= \lim_{t \rightarrow 0} \frac{\mathbb{E}_q[g(\theta(t))] - g(q)}{t} \\ &= \lim_{t \rightarrow 0} -g(q) \frac{1 - e^{-2\alpha q t}}{t} + \int_0^q 2\alpha g(x) e^{-2\alpha t x} dx \\ &= 2\alpha \int_0^q (g(x) - g(q)) dx. \end{aligned}$$

Let  $q_0 > 0$ . This result also holds for  $q < q_0$  and  $g$  a bounded measurable function defined on  $[0, q_0]$ . In that case, we get that the process  $M^g = (M_t^g, t \geq 0)$  is a martingale, where  $M^g$  defined by:

$$(2) \quad M_t^g = g(\theta(t)) + 2\alpha \int_0^t ds \int_0^{\theta(s)} \left( g(\theta(s)) - g(x) \right) dx.$$

*Remark 2.3.* If furthermore  $g$  belongs to  $\mathcal{C}^1(\mathbb{R}^+)$  and if  $x \mapsto xg'(x)$  is integrable with respect to the Lebesgue measure on  $\mathbb{R}^+$ , then we have for  $q \in [0, +\infty]$ :

$$\mathcal{L}(g)(q) = -2\alpha \int_0^q xg'(x) dx.$$

Similarly, we can also compute the infinitesimal generator of  $(\theta, X)$ , which we still denote by  $\mathcal{L}$ . This quantity is of interest only for  $\theta(0)$  finite. Let  $g$  be a bounded measurable function defined on  $\mathbb{R}^+ \times \mathbb{N}$ . Standard computations on birth and death processes yield that for  $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$ :

$$\begin{aligned} \mathcal{L}(g)(q, k) &= \lim_{t \rightarrow 0} \frac{\mathbb{E}_{(q, k)}[g(\theta(t), X(t))] - g(q, k)}{t} \\ &= \lim_{t \rightarrow 0} -g(q, k) \frac{1 - e^{-2\alpha qt}}{t} + \int_0^q 2\alpha g(x, k+1) e^{-2\alpha tx} dx + o(1) \\ &= 2\alpha \int_0^q (g(x, k+1) - g(q, k)) dx. \end{aligned}$$

In that case, we get that the process  $M^g = (M_t^g, t \geq 0)$  defined by:

$$(3) \quad M_t^g = g(\theta(t), X(t)) - 2\alpha \int_0^t ds \int_0^{\theta(s)} \left( g(x, X(s)+1) - g(\theta(s), X(s)) \right) dx,$$

is a bounded martingale.

Let  $n \in \mathbb{N}$ . Taking  $g(q, k) = k \wedge n$ , we deduce that the process  $N^{(n)} = (N_t^{(n)}, t \geq 0)$  defined for  $t \geq 0$  by:

$$N_t^{(n)} = X(t) \wedge n - 2\alpha \int_0^t \theta(s) \mathbf{1}_{\{X(s) < n\}} ds$$

is a bounded martingale under  $\mathbb{P}_{(q, k)}$  (for  $q < +\infty$ ). Notice that for  $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$ , we have:

$$\begin{aligned} \mathbb{E}_{(q, k)}[|N_t^{(n)}|] &\leq \mathbb{E}_{(q, k)}[X(t) \wedge n] + 2\alpha \int_0^t \mathbb{E}_{(q, k)}[\theta(s)] ds \\ &= k \wedge n + 2\alpha \int_0^t \mathbb{E}_{(q, k)}[\theta(s) \mathbf{1}_{\{X(s) < n\}}] ds + 2\alpha \int_0^t \mathbb{E}_{(q, k)}[\theta(s)] ds \\ &\leq k + 4\alpha tq, \end{aligned}$$

where we used that  $X$  is non-negative in the first equality, that  $N^{(n)}$  is a martingale in the second one, and that  $\theta$  is non-increasing in the last one. As  $(N^{(n)}, n \in \mathbb{N})$  converges a.s. to the process  $N = (N_t, t \geq 0)$  defined for  $t \in \mathbb{R}^+$  by:

$$(4) \quad N_t = X(t) - 2\alpha \int_0^t \theta(s) ds,$$

we deduce that  $N$  is a martingale under  $\mathbb{P}_{(q, k)}$  for every  $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$ .

By taking  $g(q, k) = k^2$  and using elementary stochastic calculus and similar arguments as above, we also get that the process  $M = (M_t, t \geq 0)$  defined for  $t \geq 0$  by:

$$(5) \quad M_t = N_t^2 - 2\alpha \int_0^t \theta(s) ds$$

is a martingale under  $\mathbb{P}_{(q,k)}$  for every  $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$ .

### 3. THE BROWNIAN SNAKE OF RECORDS

**3.1. Real trees.** We recall here the definition and basic properties of real trees. We refer to Evans's Saint Flour lectures [14] for more details on the subject.

**Definition 3.1.** *A real tree is a metric space  $(\mathcal{T}, d)$  satisfying the following two properties for every  $x, y \in \mathcal{T}$ :*

- *(unique geodesic) There is a unique isometric map  $f_{x,y}$  from  $[0, d(x, y)]$  into  $\mathcal{T}$  such that  $f_{x,y}(0) = x$  and  $f_{x,y}(d(x, y)) = y$ .*
- *(no loop) If  $\varphi$  is a continuous injective map from  $[0, 1]$  into  $\mathcal{T}$  such that  $\varphi(0) = x$  and  $\varphi(1) = y$ , then*

$$\varphi([0, 1]) = f_{x,y}([0, d(x, y)]).$$

*A rooted real tree is a real tree with a distinguished vertex denoted  $\emptyset$  and called the root.*

We denote by  $\llbracket x, y \rrbracket$  the range of the mapping  $f_{x,y}$ , which is the unique injective path between  $x$  and  $y$  in the tree. We also define a length measure denoted by  $\ell(dx)$  on a real tree by:

$$\ell(\llbracket x, y \rrbracket) = d(x, y).$$

We will consider here only compact real trees and these trees can be coded by some continuous function which is very useful for constructing real trees, in particular random trees. We consider a continuous function  $\zeta : [0, +\infty) \rightarrow [0, +\infty)$  with compact support  $[0, \sigma]$  and such that  $\zeta(0) = \zeta(\sigma) = 0$ . This function  $\zeta$  will be called in the following the height function. For every  $s, t \geq 0$ , we set

$$m_\zeta(s, t) = \inf_{r \in [s \wedge t, s \vee t]} \zeta(r),$$

and

$$d(s, t) = \zeta(s) + \zeta(t) - m_\zeta(s, t).$$

We then define the equivalence relation  $s \sim t$  iff  $d(s, t) = 0$ . We set  $\mathcal{T}_\zeta$  the quotient space

$$\mathcal{T}_\zeta = [0, +\infty) / \sim.$$

The distance  $d$  induces a distance on  $\mathcal{T}_\zeta$  and we keep notation  $d$  for this distance. For simplicity, for  $s \geq 0$ , we shall denote by  $s$  the equivalence class which contains  $s$ . The metric space  $(\mathcal{T}_\zeta, d)$  is a compact real tree, see [13] Theorem 2.1. It can be viewed as a rooted real tree by setting  $\emptyset = 0$ . We shall write  $m^\mathcal{T}$  for the measure on (the leaves of)  $\mathcal{T}$  defined as the Lebesgue measure on  $[0, \sigma]$ . In particular, we have  $m^\mathcal{T}(\mathcal{T}) = \sigma$ .

We denote by  $\mathbb{N}$  the  $\sigma$ -finite measure on the Polish set  $\mathbb{T}$  of real trees (with the Gromov-Hausdorff distance) of the real tree  $\mathcal{T}_\zeta$  when  $\zeta$  is an excursion away from 0 of  $\sqrt{\frac{2}{\alpha}}|B|$  where  $|B|$  is a standard reflected Brownian motion. The tree  $\mathcal{T}_\zeta$  is then the genealogical tree of a continuous state branching process with branching mechanism  $\psi(u) = \alpha u^2$  under its canonical measure.

The density, with respect to Lebesgue measure, of the length  $\sigma$  of the excursion  $\zeta$  under  $\mathbb{N}$  is given by:

$$(6) \quad \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} \mathbf{1}_{\{r>0\}}.$$

In particular, we have for  $\mu \geq 0$ :

$$\mathbb{N} [1 - e^{-\mu\sigma}] = \sqrt{\mu/\alpha}.$$

Using the scaling property of the Brownian motion, there exists a regular version of the measure  $\mathbb{N}$  conditioned on the length of the height process  $\zeta$ . We write  $\mathbb{N}^{(r)}$  for the probability measure  $\mathbb{N}[\cdot | \sigma = r]$ .

**3.2. The spatial process.** We now consider a snake with lifetime process the excursion  $\zeta$  and with spatial motion the record process  $\theta$  (see [17] for the definition and the existence of a snake and [10] for the case of a discontinuous spatial motion). A snake is a path-valued Markov process  $((\zeta_s, W_s), s \geq 0)$  such that, conditionally given  $(\zeta_s, s \geq 0)$ , the process is still a (inhomogeneous) Markov process such that

- For every  $s \geq 0$ , the process  $(W_s(t), t \geq 0)$  is distributed as the record process  $\theta$  killed at time  $\zeta_s$ .
- For every  $0 \leq s \leq s'$ , we have
  - $W_s(t) = W_{s'}(t)$  for every  $t \in [0, m_\zeta(s, s')]$ .
  - The process  $(W_s(t) - W_s(m_\zeta(s, s')), t \in [m_\zeta(s, s'), \zeta_s])$  and the process  $(W_{s'}(t) - W_{s'}(m_\zeta(s, s')), t \in [m_\zeta(s, s'), \zeta_{s'}])$  are independent.

Let us remark that, by definition, the starting points  $W_s(0)$  are the same for all  $s$ . We denote by  $\mathbb{N}_q$  (resp.  $\mathbb{N}_q^{(r)}$ ) the law of the snake when  $\zeta$  is distributed according to  $\mathbb{N}$  (resp.  $\mathbb{N}^{(r)}$ ) and when  $W_s(0) = q$ .

Equivalently, we can define directly the record process on the tree using a Poisson point measure: conditionally given  $\zeta$ , we consider a Poisson point measure  $\sum_{i \in I} \delta_{(q_i, x_i)}(dq, dx)$  on  $\mathbb{R}_+ \times \mathcal{T}_\zeta$  with intensity  $2\alpha dq \ell(dx)$ . For every  $x \in \mathcal{T}_\zeta$ , we set

$$\theta(x) = \min(q, \inf\{q_i, x_i \in \llbracket \emptyset, x \rrbracket\}).$$

Then it is not difficult to see that the law of the process

$$((\zeta_s, (\theta(x), x \in \llbracket \emptyset, s \rrbracket)), s \geq 0)$$

when  $\zeta$  is distributed according to  $\mathbb{N}$  (resp.  $\mathbb{N}^{(r)}$ ) is  $\mathbb{N}_q$  (resp.  $\mathbb{N}_q^{(r)}$ ).

We will write, for every  $s \geq 0$ ,

$$\hat{\theta}_s = W_s(\zeta_s) = \theta(s).$$

#### 4. DISTRIBUTION OF $\Theta$ AND RELATED COMPUTATIONS

Recall that  $\sigma = m^\mathcal{T}(\mathcal{T})$ . We set:

$$\Theta = \int_0^\sigma \hat{\theta}_s ds = \int_{\mathcal{T}} \theta(x) m^\mathcal{T}(dx) \quad \text{and} \quad Z = \sqrt{\frac{2\alpha}{\sigma}} \Theta.$$

In order to stress that  $\Theta$  is defined from  $\mathcal{T}$ , we may write  $\Theta(\mathcal{T})$  for  $\Theta$ .

Let  $\lambda > 0$ ,  $\mu \geq 0$  and  $f(x) = \lambda x + \mu$ . We set for  $q \in [0, +\infty]$ :

$$(7) \quad F(q) = \mathbb{N}_q \left[ 1 - e^{-\int_0^\sigma f(\hat{\theta}_s) ds} \right].$$

We define the function:

$$(8) \quad G(x) = \left( \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \right) e^{\frac{2\alpha}{\lambda}(x - \sqrt{\mu/\alpha})} - x - \frac{\lambda}{2\alpha}.$$

The function  $G$  is one-to-one from  $[\sqrt{\mu/\alpha}, +\infty)$  to  $[0, +\infty)$ , is increasing and is of class  $\mathcal{C}^\infty$ .

**Lemma 4.1.** *Let  $\lambda > 0$ ,  $\mu \geq 0$ . The function  $F$  is of class  $\mathcal{C}^1$  on  $[0, +\infty)$  and solves the following equation on  $[0, +\infty)$ :*

$$(9) \quad \alpha F(q)^2 + 2\alpha \int_0^q x F'(x) dx = f(q).$$

Furthermore, we have  $F = G^{-1}$ .

The proof of this Lemma is postponed to the Appendix, Section 9.

Notice that  $F(+\infty) = +\infty$  which doesn't allow us to compute the Laplace transform of  $\int_0^\sigma f(\hat{\theta}_s) ds$ . However, we have the following result.

**Corollary 4.2.** *Let  $\lambda > 0$ ,  $\mu \geq 0$ . We have:*

$$(10) \quad \mathbb{N}_\infty \left[ \sigma e^{-\mu\sigma - \lambda\Theta} \right] = \frac{1}{2\sqrt{\alpha\mu} + \lambda}.$$

*Proof.* We have for  $q \in [0, +\infty)$ :

$$(11) \quad \partial_\mu F(q) = \mathbb{N}_q \left[ \sigma e^{-\int_0^\sigma f(\hat{\theta}_s) ds} \right].$$

Since  $G(F(q)) = q$  we get:

$$(\partial_\mu G)(F(q)) + G'(F(q)) \partial_\mu F(q) = 0.$$

We have:

$$\partial_\mu G(x) = -\frac{1}{\lambda} e^{\frac{2\alpha}{\lambda}(x - \sqrt{\mu/\alpha})} = -\frac{1}{\lambda} \frac{1}{2\sqrt{\alpha\mu} + \lambda} (2\alpha G(x) + 2\alpha x + \lambda).$$

Notice that  $G'(F(q)) = 1/F'(q)$ . We deduce from (37) that:

$$\begin{aligned} \partial_\mu F(q) &= \frac{1}{2\alpha(F(q) + q)} \frac{1}{2\sqrt{\alpha\mu} + \lambda} (2\alpha q + 2\alpha F(q) + \lambda) \\ &= \frac{1}{2\sqrt{\alpha\mu} + \lambda} \left( 1 + \frac{\lambda}{2\alpha(F(q) + q)} \right). \end{aligned}$$

Letting  $q$  go to infinity gives the result.  $\square$

Using (6), we get that for every non-negative measurable random variable  $V$ , for  $q \in [0, +\infty]$ :

$$(12) \quad \mathbb{N}_q[V] = \int_0^\infty \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} \mathbb{N}_q^{(r)}[V].$$

In particular, we deduce from Corollary 4.2 that:

$$(13) \quad \frac{1}{2\sqrt{\alpha\pi}} \int_0^\infty \frac{dr}{\sqrt{r}} e^{-\mu r} \mathbb{N}_\infty^{(r)} \left[ e^{-\lambda\Theta} \right] = \frac{1}{2\sqrt{\alpha\mu} + \lambda}.$$

The proof of the next Lemma is postponed to the Appendix, Section 9.



**Lemma 4.3.** *Let  $Z'$  be a Rayleigh random variable with density  $x e^{-x^2/2} \mathbf{1}_{\{x>0\}}$ . Let  $\mu > 0$ ,  $c \geq 0$ . We have:*

$$(14) \quad \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{dr}{\sqrt{r}} e^{-\mu r} \mathbb{E} \left[ e^{-\sqrt{2r} c Z'} \right] = \frac{1}{c + \sqrt{\mu}}.$$

We then deduce the following result.

**Proposition 4.4.** *Let  $Z'$  be a Rayleigh random variable with density  $x e^{-x^2/2} \mathbf{1}_{\{x>0\}}$ . We have that for all  $r > 0$ , the random variable  $\Theta$  is distributed under  $\mathbb{N}_\infty^{(r)}$  as  $\sqrt{\frac{r}{2\alpha}} Z'$ . In other words, for all  $r > 0$ , under  $\mathbb{N}_\infty^{(r)}$ , we have  $Z \stackrel{(d)}{=} Z'$ .*

*Proof.* By comparing Equations (13) and (14), we obtain that:

$$(15) \quad \frac{1}{2\sqrt{\alpha\pi}} \int_0^\infty \frac{dr}{\sqrt{r}} e^{-\mu r} \mathbb{N}_\infty^{(r)} \left[ e^{-\lambda\Theta} \right] = \frac{1}{2\sqrt{\alpha\pi}} \int_0^{+\infty} \frac{dr}{\sqrt{r}} e^{-\mu r} \mathbb{E} \left[ e^{-\lambda\sqrt{\frac{r}{2\alpha}} Z'} \right]$$

for every  $\mu \geq 0$ . This implies that  $dr$ -a.e.:

$$\mathbb{N}_\infty^{(r)} \left[ e^{-\lambda\Theta} \right] = \mathbb{E} \left[ e^{-\lambda\sqrt{\frac{r}{2\alpha}} Z'} \right],$$

but, thanks to the scaling property for Brownian motion, the function

$$r \mapsto \mathbb{N}_\infty^{(r)} \left[ e^{-\lambda\Theta} \right]$$

is continuous, and so is the right-hand side of Equation (15). Thus the equality holds for every  $r$ , which ends the proof.  $\square$

We deduce the following Corollary.

**Corollary 4.5.** *We have:*

$$(16) \quad \mathbb{N}_\infty^{(r)} [\Theta] = \frac{1}{2} \sqrt{\frac{\pi r}{\alpha}} \quad \text{and} \quad \mathbb{N}_\infty^{(r)} [\Theta^2] = \frac{r}{\alpha}.$$

*Proof.* This is a direct consequence of Proposition 4.4 and the following fact:

$$(17) \quad \mathbb{E}[Z'^n] = 2^{n/2} \Gamma((n+2)/2) \quad \text{for } n > -2,$$

which implies  $\mathbb{E}[Z'] = \sqrt{\pi/2}$  and  $\mathbb{E}[Z'^2] = 2$ .  $\square$

The last part of the Section is devoted to the computation of the first moment of  $\Theta$  under  $\mathbb{N}_q^{(r)}$ , with  $q < +\infty$ . We first give the asymptotic expansion of  $F$  with respect to small  $\lambda$ . We write  $O(\lambda^k)$  for any function  $g$  of  $q, \mu$  and  $\lambda$  such that for any  $q > 0$ ,  $\mu > 0$  and  $\varepsilon > 0$  there exists a finite constant  $C$  (depending on  $q, \mu$  and  $\varepsilon$ ) such that for all  $\lambda \in [0, \varepsilon]$ ,  $|g(q, \mu, \lambda)| \leq C\lambda^k$ . Notice that  $O(\lambda^k)$  is not uniform in  $q$  or  $\mu$ .

**Lemma 4.6.** *Let  $q \in (0, +\infty)$ . We set  $z = q\sqrt{\frac{\alpha}{\mu}}$ . We have:*

$$(18) \quad F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log(1+z) - \frac{\lambda^2}{4\alpha^{3/2}\mu^{1/2}} \frac{z - \log(1+z)}{1+z} + O(\lambda^3).$$

*In particular, we deduce that:*

$$(19) \quad \partial_\lambda F(q)|_{\lambda=0} = \frac{1}{2\alpha} \log(1+z) \quad \text{and} \quad \partial_\lambda^2 F(q)|_{\lambda=0} = -\frac{1}{2\alpha^{3/2}\mu^{1/2}} \frac{z - \log(1+z)}{1+z}.$$

*Proof.* Using the second part of Lemma 4.1 and (8), we get:

$$(20) \quad F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log \left( \frac{2\alpha q + 2\alpha F(q) + \lambda}{2\sqrt{\alpha\mu} + \lambda} \right).$$

Using (7), we get that  $F(q)$  decreases to  $\sqrt{\mu/\alpha}$  when  $\lambda$  goes down to 0, that is  $F(q) = \sqrt{\mu/\alpha} + O(1)$ . Plugging this in the right-hand side of (20), we get:

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + O(\lambda).$$

Plugging this in the right-hand side of (20), we get:

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log(1 + z) + O(\lambda^2).$$

Plugging this again in the right-hand side of (20), we get (18). This readily implies (19).  $\square$

We can then compute the first moment of  $\Theta$  under  $\mathbb{N}_q^{(r)}$ .

**Proposition 4.7.** *Let  $Z'$  be a Rayleigh random variable with density  $x e^{-x^2/2} \mathbf{1}_{\{x>0\}}$ . We set*

$$(21) \quad H_q(r) = \sqrt{\frac{r}{2\alpha}} \int_0^{q\sqrt{2\alpha r}} dy \mathbb{E} \left[ e^{-yZ'} \right].$$

For  $r > 0$  and  $q \in [0, +\infty)$ , we have:

$$\mathbb{N}_q^{(r)}[\Theta] = H_q(r).$$

Since  $0 \leq 1 - e^{-z} \leq z$  for  $z \geq 0$ , we get:

$$(22) \quad 0 \leq qr - H_q(r) \leq \frac{1}{2} \sqrt{\pi\alpha} q^2 r^{3/2}.$$

We also deduce from the previous Proposition that:

$$\mathbb{N}_\infty^{(r)}[\Theta] = H_\infty(r) = \sqrt{\frac{r}{2\alpha}} \int_0^{+\infty} dy \mathbb{E} \left[ e^{-yZ'} \right] = \sqrt{\frac{r}{2\alpha}} \mathbb{E}[1/Z'] = \frac{\sqrt{\pi r}}{2\sqrt{\alpha}},$$

where we used (17) with  $n = -1$ . Thus we recover the first part of (16) (notice that this relies on the following identity for the Rayleigh random variable:  $\mathbb{E}[Z'] = \mathbb{E}[1/Z']$ ).

*Proof.* By the change of variable  $y = q\sqrt{2\alpha z}$ , we have

$$H_q(r) = \frac{q\sqrt{r}}{2} \int_0^r \frac{dz}{\sqrt{z}} \int_0^{+\infty} dx x \exp \left( -\frac{x^2}{2} - q\sqrt{2\alpha z} x \right).$$

Then we compute for  $\mu > 0$ ,

$$\begin{aligned} \int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi r}} e^{-\mu r} H_q(r) &= \frac{q}{4\sqrt{\pi\alpha}} \int_0^{+\infty} dr e^{-\mu r} \int_0^r \frac{dz}{\sqrt{z}} \int_0^{+\infty} dx x \exp \left( -\frac{x^2}{2} - q\sqrt{2\alpha z} x \right) \\ &= \frac{q}{4\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dz}{\sqrt{z}} \int_z^{+\infty} dr e^{-\mu r} \int_0^{+\infty} dx x \exp \left( -\frac{x^2}{2} - q\sqrt{2\alpha z} x \right) \\ &= \frac{q}{4\sqrt{\pi\alpha}} \frac{1}{\mu} \int_0^{+\infty} \frac{dz}{\sqrt{z}} e^{-\mu z} \int_0^{+\infty} dx x \exp \left( -\frac{x^2}{2} - q\sqrt{2\alpha z} x \right) \\ &= \frac{q}{4\sqrt{\alpha}} \frac{1}{\mu} \frac{1}{\sqrt{\mu} + q\sqrt{\alpha}}, \end{aligned}$$

where we used equality (14) for the last equality.

On the other hand, we have:

$$\begin{aligned}
\int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi r}} e^{-\mu r} \mathbb{N}_q^{(r)}[\Theta] &= -\partial_\mu \int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} e^{-\mu r} \mathbb{N}_q^{(r)}[\Theta] \\
&= -\partial_\mu \mathbb{N}_q[e^{-\mu\sigma} \Theta] \\
&= -\partial_\mu [\partial_\lambda F(q)|_{\lambda=0}] \\
&= -\frac{1}{2\alpha} \partial_\mu \log \left( 1 + q\sqrt{\frac{\alpha}{\mu}} \right) \\
&= \frac{q}{4\sqrt{\alpha}} \frac{1}{\mu} \frac{1}{\sqrt{\mu} + q\sqrt{\alpha}},
\end{aligned}$$

where we used (12) for the second equality, the definition (7) of  $F$  for the third one and (19) for the fourth one. Therefore, we have that  $dr$ -a.e.  $\mathbb{N}_q^{(r)}[\Theta] = H_q(r)$ . Then the equality holds for all  $r > 0$  by continuity (using again a scaling argument).  $\square$

We end that section with the computation of the first moments of  $\Theta$  conditionally given the tree  $\mathcal{T}_\zeta$  or equivalently conditionally on the height process  $\zeta$ .

**Proposition 4.8.** *We have:*

$$\mathbb{N}_\infty[\Theta \mid \zeta] = \frac{1}{2\alpha} \int_0^\sigma \frac{dt_1}{\zeta_{t_1}} \quad \mathbb{N}_\infty[\Theta^2 \mid \zeta] = \frac{2}{(2\alpha)^2} \int_0^\sigma \int_0^\sigma \frac{dt_1 dt_2}{L_1 L_2},$$

where  $L_1 = \zeta_{t_1}$  and  $L_2$  is the length of the tree spanned by  $t_1$  and  $t_2$  i.e.  $L_2 = \zeta_{t_1} + \zeta_{t_2} - m_\zeta(t_1, t_2)$ .

*Remark 4.9.* We deduce from Proposition 4.8, that:

$$\mathbb{N}_\infty[Z \mid \zeta] = \frac{1}{\sqrt{2\alpha\sigma}} \int_0^\sigma \frac{dt_1}{\zeta_{t_1}} \quad \mathbb{N}_\infty[Z^2 \mid \zeta] = \frac{1}{\alpha\sigma} \int_0^\sigma \int_0^\sigma \frac{dt_1 dt_2}{L_1 L_2}.$$

Recall the rate at which marks are thrown is  $2\alpha$ . Using notations (1), we deduce from Theorems 1.9 and 1.10 of [15] that, for  $\alpha = 1/2$ ,  $Z$  and  $Z_{\mathcal{T}}$  have, conditionally on  $\mathcal{T}$  the same first two moments. We also check this is true for the third moment. But we were not able to perform computations of  $\mathbb{N}_\infty[Z^k \mid \zeta]$  in closed form for general  $k$  in order to check that  $Z$  and  $Z_{\mathcal{T}}$  have the same distribution conditionally on  $\mathcal{T}$ .

*Proof.* We have:

$$\mathbb{N}_\infty[\Theta \mid \zeta] = \int_0^\sigma \mathbb{N}_\infty[\hat{\theta}_s \mid \zeta] ds.$$

But, conditionally on  $\zeta_s$ ,  $\hat{\theta}_s$  is distributed according to an exponential random variable with parameter  $2\alpha\zeta_s$  by standard results on Poisson point measures. Therefore, we have:

$$\mathbb{N}_\infty[\Theta \mid \zeta] = \frac{1}{2\alpha} \int_0^\sigma \frac{ds}{\zeta_s}.$$

For the second moment, we have:

$$\mathbb{N}_\infty[\Theta^2 \mid \zeta] = \int_0^\sigma \int_0^\sigma dt_1 dt_2 \mathbb{N}_\infty[\hat{\theta}_{t_1} \hat{\theta}_{t_2} \mid \zeta].$$

Let us set  $h_0 = 2\alpha m_\zeta(t_1, t_2)$ ,  $h_1 = 2\alpha\zeta_{t_1} - h_0$ , and  $h_2 = 2\alpha\zeta_{t_2} - h_0$ . Remark that  $L_1 = (h_0 + h_1)/2\alpha$  and  $L_2 = (h_0 + h_1 + h_2)/2\alpha$ . Moreover, under  $\mathbb{N}_\infty$  conditionally given  $\zeta$ , we have

$$(\hat{\theta}_{t_1}, \hat{\theta}_{t_2}) \stackrel{(d)}{=} (Y_0 \wedge Y_1, Y_0 \wedge Y_2)$$

where  $Y_0, Y_1, Y_2$  are independent exponentially distributed random variables with respective parameter  $h_0, h_1$  and  $h_2$ . Consequently, we have

$$\begin{aligned} \mathbb{N}_\infty[\hat{\theta}_{t_1} \hat{\theta}_{t_2} \mid \zeta] &= \int_0^{+\infty} du h_0 e^{-h_0 u} \mathbb{E}[u \wedge Y_1] \mathbb{E}[u \wedge Y_2] \\ &= \int_0^{+\infty} du h_0 e^{-h_0 u} \frac{1}{h_1 h_2} (1 - e^{-h_1 u}) (1 - e^{-h_2 u}) \\ &= \frac{1}{h_1 h_2} \left( 1 - \frac{h_0}{h_0 + h_1} - \frac{h_0}{h_0 + h_2} + \frac{h_0}{h_0 + h_1 + h_2} \right) \\ &= \frac{1}{(h_0 + h_1)(h_0 + h_1 + h_2)} + \frac{1}{(h_0 + h_2)(h_0 + h_1 + h_2)}. \end{aligned}$$

When integrating with respect to  $dt_1 dt_2$ , these two terms give the same contribution by symmetry and we get the second moment of the Proposition.  $\square$

## 5. SUB-TREE WITH $n$ LEAVES

**5.1. Definition.** Let  $r \geq 0$  and let  $\mathcal{T}$  be a tree distributed according to  $\mathbb{N}^{(r)}$ . Let  $(t_1, \dots, t_n)$  be  $n$  points uniformly chosen at random on  $[0, r]$  and let  $T_n$  be the sub-tree of  $\mathcal{T}$  spanned from these  $n$  points and the root:

$$T_n = \bigcup_{k=1}^n \llbracket \emptyset, t_k \rrbracket.$$

Notice that  $T_n$  has  $2n - 1$  edges. Let  $(h_1, \dots, h_{2n-1})$  be the lengths of the edges given in lexicographic order. We shall consider the total length of  $T_n$ :

$$L_n = \sum_{k=1}^{2n-1} h_k.$$

We first recall the density of  $(h_1, \dots, h_{2n-1})$ , see also [13]. The proof of this Lemma is given in the Appendix, Section 9.

**Lemma 5.1.** *Under  $\mathbb{N}^{(r)}$ ,  $(h_1, \dots, h_{2n-1})$  has density:*

$$f_n^{(r)}(h_1, \dots, h_{2n-1}) = 2 \frac{(2n-2)!}{(n-1)!} \frac{\alpha^n}{r^n} L_n e^{-\alpha L_n^2/r} \mathbf{1}_{\{h_1 > 0, \dots, h_{2n-1} > 0\}}.$$

We set  $h_{\emptyset, n} = h_1$  for the length of the edge of  $T_n$  originating from the root, that is  $h_{\emptyset, n} = d(\emptyset, m_n) = \ell(\llbracket \emptyset, m_n \rrbracket)$  with  $m_n$  defined by:

$$(23) \quad \bigcap_{k=1}^n \llbracket \emptyset, t_k \rrbracket = \llbracket \emptyset, m_n \rrbracket.$$

**5.2. The total length of the sub-tree.** The main result of this Section is the following.

**Proposition 5.2.** *Let  $L_n$  be the total length of  $T_n$ . Then  $L_n^2$  is distributed under  $\mathbb{N}^{(r)}$  as  $r\Gamma_n/\alpha$  where  $\Gamma_n = E_1 + \dots + E_n$ , with  $(E_k, k \in \mathbb{N}^*)$  independent exponential random variables with mean 1. In particular, we have that  $\mathbb{N}^{(r)}$ -a.s.*

$$\lim_{n \rightarrow +\infty} L_n / \sqrt{n} = \sqrt{r/\alpha}.$$

*Proof.* Use the density  $f_n^{(r)}$  to get:

$$\begin{aligned} \mathbb{N}^{(r)}[g(L_n^2)] &= \int g \left( \left( \sum_{k=1}^{2n-1} h_k \right)^2 \right) f_n^{(r)}(h_1, \dots, h_{2n-1}) dh_1 \cdots dh_{2n-1} \\ &= \frac{(2n-2)!}{(n-1)!} \int g(L_n^2) \frac{\alpha^n}{n^n} 2L_n e^{-\alpha L_n^2/r} \mathbf{1}_{\{h_1 \geq 0, \dots, h_{2n-1} \geq 0\}} dh_1 \cdots dh_{2n-1}. \end{aligned}$$

Consider now the change of variables:

$$u_1 = \sqrt{\frac{\alpha}{r}} h_1, \dots, u_{2n-2} = \sqrt{\frac{\alpha}{r}} h_{2n-2}, x = \sqrt{\frac{\alpha}{r}} L_n,$$

with Jacobian equal to  $(\frac{\alpha}{r})^{n-\frac{1}{2}}$ . We have:

$$\begin{aligned} \mathbb{N}^{(r)}[g(L_n^2)] &= \frac{(2n-2)!}{(n-1)!} \int g \left( \frac{rx^2}{\alpha} \right) \frac{\alpha^n}{r^n} 2\sqrt{\frac{r}{\alpha}} x e^{-x^2} \left( \frac{\alpha}{r} \right)^{n-\frac{1}{2}} \\ &\quad \mathbf{1}_{\{u_1 \geq 0, \dots, u_{2n-2} \geq 0\}} \mathbf{1}_{\{u_1 + \dots + u_{2n-2} \leq x\}} du_1 \cdots du_{2n-2} dx \\ &= \frac{1}{(n-1)!} \int g \left( \frac{rx^2}{\alpha} \right) 2x e^{-x^2} x^{2n-2} \mathbf{1}_{x \geq 0} dx \\ &= \frac{1}{(n-1)!} \int g \left( \frac{rz}{\alpha} \right) e^{-z} z^{n-1} \mathbf{1}_{z \geq 0} dz, \end{aligned}$$

where we used that:

$$\int \mathbf{1}_{\{u_1 \geq 0, \dots, u_{2n-2} \geq 0\}} \mathbf{1}_{\{u_1 + \dots + u_{2n-2} \leq x\}} du_1 \cdots du_{2n-2} = \frac{x^{2n-2}}{(2n-2)!}$$

for the second equality and considering the obvious change of variables  $z = x^2$  for the last equality. Then use that  $\Gamma_n$  has distribution gamma with parameter  $(n, 1)$  to get the first part of the Proposition.

For the second part, we compute

$$\mathbb{N}^{(r)} \left[ \sum_{n=1}^{+\infty} \left( \frac{L_n^2}{n} - \frac{r}{\alpha} \right)^4 \right] = \frac{r}{\alpha} \sum_{n=1}^{+\infty} \mathbb{E} \left[ \left( \frac{\Gamma_n}{n} - 1 \right)^4 \right] = \frac{r}{\alpha} \sum_{n=1}^{+\infty} \frac{1}{n^2} \left( 3 + \frac{1}{n} \right) < +\infty.$$

This implies that  $\mathbb{N}^{(r)}$ -a.s.  $\sum_{n=1}^{+\infty} \left( \frac{L_n^2}{n} - \frac{r}{\alpha} \right)^4$  is finite, which proves the last part of the Proposition.  $\square$

### 5.3. The length of the vertex originating at the root.

**Proposition 5.3.** *The sequence  $(\sqrt{n}h_{\emptyset, n}, n \geq 1)$  converges in distribution to  $\sqrt{r/\alpha} E_1/2$ , where  $E_1$  is an exponential random variable with mean 1.*

*Proof.* Let  $k \in (-1, +\infty)$ . We set  $H_k = (\alpha/r)^{k/2} \mathbb{N}^{(r)}[h_{\emptyset, n}^k]$ . We have:

$$H_k = 2 \frac{(2n-2)!}{(n-1)!} \frac{\alpha^{n+k/2}}{r^{n+k/2}} \int_{\mathbb{R}_+^{2n-1}} dh_1 \cdots dh_{2n-1} h_1^k L_n e^{-\alpha L_n^2/r}.$$

Consider the same change of variables as in the proof of Proposition 5.2 to get:

$$\begin{aligned}
H_k &= 2 \frac{(2n-2)!}{(n-1)!} \frac{\alpha^{n+k/2}}{r^{n+k/2}} \int_{\mathbb{R}_+^{2n-1}} \left(\frac{r}{\alpha}\right)^{k/2} u_1^k \left(\frac{r}{\alpha}\right)^{1/2} x e^{-\alpha x^2/r} \mathbf{1}_{\{u_1+\dots+u_{2n-2}\leq x\}} \left(\frac{r}{\alpha}\right)^{n-\frac{1}{2}} \\
&\quad du_1 \cdots du_{2n-2} dx \\
&= 2 \frac{(2n-2)!}{(n-1)!} \int \int_{\mathbb{R}_+^2} du_1 dx \mathbf{1}_{\{u_1\leq x\}} u_1^k x e^{-x^2} \int \int_{\mathbb{R}_+^{2n-3}} du_2 \dots du_{2n-2} \mathbf{1}_{\{u_2+\dots+u_{2n-2}\leq x-u_1\}} \\
&= 2 \frac{(2n-2)!}{(n-1)!} \frac{1}{(2n-3)!} \int_0^{+\infty} dx x e^{-x^2} \int_0^x dh h^k (x-h)^{2n-3}.
\end{aligned}$$

Set  $y = x^2$ , to get:

$$\begin{aligned}
H_r &= 2 \frac{(2n-2)!}{(n-1)!} \frac{1}{(2n-3)!} \beta(k+1, 2n-2) \int_0^{+\infty} dx x^{2n+k-1} e^{-x^2} \\
&= \frac{(2n-2)!}{(n-1)!} \frac{1}{(2n-3)!} \beta(k+1, 2n-2) \int_0^{+\infty} dy y^{n+\frac{k}{2}-1} e^{-y} \\
&= \frac{(2n-2)!}{(n-1)!} \frac{1}{(2n-3)!} \frac{\Gamma(k+1)(2n-3)!}{\Gamma(2n+k-1)} \Gamma\left(n+\frac{k}{2}\right) \\
&= \frac{\Gamma(k+1)}{2^k} \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(n+\frac{k}{2}-\frac{1}{2}\right)},
\end{aligned}$$

where, for the last equality, we used twice the duplication formula:

$$(24) \quad \frac{\Gamma(2n-1)}{\Gamma(n)} = \frac{2^{2n-2} \Gamma(n-1/2)}{\sqrt{\pi}}.$$

We observe that  $\lim_{n \rightarrow +\infty} \mathbb{N}^{(r)}[n^{k/2} h_{\emptyset, n}^k] = \frac{k!}{2^k} \left(\frac{r}{\alpha}\right)^{k/2} = \mathbb{E}[(\sqrt{r} E_1 / (2\sqrt{\alpha}))^k]$ . This gives the result, as the exponential distribution is characterized by its moments.  $\square$

From the proof of Proposition 5.3, we also get the following result.

**Lemma 5.4.** *For all  $k \in (-1, +\infty)$ , we have, when  $n$  goes to infinity:*

$$\mathbb{N}^{(r)}[h_{\emptyset, n}^k] = \left(\frac{r}{\alpha}\right)^{k/2} \frac{\Gamma(k+1)}{2^k} \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(n+\frac{k}{2}-\frac{1}{2}\right)} \sim (r/\alpha)^{k/2} n^{-k/2} 2^{-k} \Gamma(k+1).$$

## 6. NUMBER OF RECORDS ON SUB-TREES

Recall the vertex originating from the root of  $T_n$  is  $[\emptyset, m_n]$ , with  $m_n$  defined by (23). Let  $T_n^*$  be the sub-tree of  $T_n$  where we remove the edge  $[\emptyset, m_n]$ :

$$T_n^* = T_n \setminus [\emptyset, m_n],$$

and  $L_n^*$  its total length *i.e.*  $L_n^* = L_n - h_{\emptyset, n}$ . We also set  $\theta_{\emptyset, n} = \theta(m_n)$ , and conditionally on  $h_{\emptyset, n}$ ,  $\theta_{\emptyset, n}$  is an exponential random variable with mean  $1/h_{\emptyset, n} < +\infty$ . Thus, we have:

$$\mathbb{N}_{\infty}^{(r)} \left[ \int_{T_n^*} \theta(x) \ell(dx) \mid T_n^*, \theta_{\emptyset, n} \right] \leq L_n^* \theta_{\emptyset, n} < +\infty.$$

Let  $X_n^*$  be the number of records on the tree  $T_n^*$ :

$$X_n^* = \sum_{x \in T_n^*} \mathbf{1}_{\{\theta(x-) > \theta(x)\}},$$

where  $\theta(x-) = \lim_{\substack{y \rightarrow x \\ y \in [\emptyset, x] \setminus \{x\}}} \theta(y)$ . We can then state the main result.

**Theorem 6.1.** *We have that, for all  $r > 0$ ,  $\mathbb{N}_\infty^{(r)}$ -a.s.:*

$$\lim_{n \rightarrow +\infty} \frac{X_n^*}{\sqrt{n}} = 2\sqrt{\frac{\alpha}{r}}\Theta.$$

*Remark 6.2.* We have:

$$(25) \quad \frac{X_n^*}{\sqrt{2n-1}} \xrightarrow[n \rightarrow \infty]{\mathbb{N}_\infty^{(r)\text{-a.s.}}} \sqrt{\frac{2\alpha}{r}}\Theta = Z.$$

Notice that the binary tree  $T_n$  has  $2n - 1$  vertices; and it corresponds to a critical Galton-Watson tree with reproduction law taking values in  $\{0, 2\}$  and with variance  $\sigma^2 = 1$  conditionally on its number of edges being  $2n - 1$ . Therefore, we get a similar result as Theorem 1.6 in [15].

The proof of Theorem 6.1 relies on the following technical result, which will be proved in Section 7.

**Proposition 6.3.** *We have that, for all  $r > 0$ ,  $\mathbb{N}_\infty^{(r)}$ -a.s.:*

$$(26) \quad \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x) \ell(dx) = \frac{1}{\sqrt{r\alpha}}\Theta.$$

and  $\mathbb{N}^{(r)}$ -a.s.:

$$(27) \quad r\mathbb{N}_\infty^{(r)} \left[ \frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x) \ell(dx) \mid T_n \right] \leq \frac{L_n}{\sqrt{n}} \mathbb{N}_\infty^{(r)} [\Theta \mid T_n] + R'_n,$$

where  $R'_n$  is non-negative and  $\sigma(T_n)$ -measurable and the sequence  $(R'_n, n \geq 1)$  converges  $\mathbb{N}_\infty^{(r)}$ -a.s. to 0.

*Proof of Theorem 6.1.* We set:

$$\Delta_n = \frac{X_n^*}{\sqrt{n}} - \frac{2\alpha}{\sqrt{n}} \int_{T_n^*} \theta(x) \ell(dx).$$

Using the martingale of Equation (5), we have that:

$$(28) \quad \mathbb{N}_\infty^{(r)} [\Delta_n^2 \mid T_n] = \frac{2\alpha}{\sqrt{n}} \mathbb{N}_\infty^{(r)} \left[ \frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x) \ell(dx) \mid T_n \right].$$

We have:

$$\begin{aligned} \mathbb{N}_\infty^{(r)} \left[ \sum_{n \geq 1} \Delta_{n^4}^2 \mathbf{1}_{\{R'_{n^4} \leq 1\}} \right] &= \sum_{n \geq 1} \mathbb{N}_\infty^{(r)} \left[ \mathbb{N}_\infty^{(r)} [\Delta_{n^4}^2 \mid T_{n^4}] \mathbf{1}_{\{R'_{n^4} \leq 1\}} \right] \\ &\leq \sum_{n \geq 1} \frac{2\alpha}{n^2 r^2} \mathbb{N}_\infty^{(r)} \left[ \left( \frac{L_{n^4}}{n^2} \mathbb{N}_\infty^{(r)} [\Theta \mid T_{n^4}] + R'_{n^4} \right) \mathbf{1}_{\{R'_{n^4} \leq 1\}} \right] \\ &\leq \sum_{n \geq 1} \frac{2\alpha}{n^2 r^2} \left( \frac{1}{n^2} \mathbb{N}_\infty^{(r)} [L_{n^4}^2]^{1/2} \mathbb{N}_\infty^{(r)} [\Theta^2]^{1/2} + 1 \right) \\ &< +\infty, \end{aligned}$$

where we used (28) and (27) for the first inequality, Cauchy-Schwarz inequality for the second one, and Proposition 5.2 as well as (16) for the last one. This result implies that  $\mathbb{N}_\infty^{(r)}$ -a.s.  $\lim_{n \rightarrow +\infty} \Delta_{n^4} \mathbf{1}_{\{R'_{n^4} \leq 1\}} = 0$  and thus  $\mathbb{N}_\infty^{(r)}$ -a.s.  $\lim_{n \rightarrow +\infty} \Delta_{n^4} = 0$  as the sequence  $(R'_n, n \geq 1)$  converges  $\mathbb{N}^{(r)}$ -a.s. to 0. We deduce from (26), that  $\mathbb{N}_\infty^{(r)}$ -a.s. the sequence  $(X_{n^4}^*/n^2, n \geq 1)$  converges to  $2\sqrt{\frac{\pi}{\alpha}}\Theta$ . Then using that  $(X_n^*, n \geq 1)$  is increasing, we get for  $k \in \mathbb{N}$ , such that  $n^4 < k \leq (n+1)^4$ , that:

$$\frac{n^2}{(n+1)^2} \frac{X_{n^4}^*}{n^2} \leq \frac{X_k^*}{\sqrt{k}} \leq \frac{(n+1)^2}{n^2} \frac{X_{(n+1)^4}^*}{(n+1)^2}.$$

Thus, we get that  $\mathbb{N}_\infty^{(r)}$ -a.s. the sequence  $(X_k^*/\sqrt{k}, k \geq 1)$  converges to  $2\sqrt{\frac{\pi}{\alpha}}\Theta$ .  $\square$

## 7. PROOF OF PROPOSITION 6.3

Let  $\mathcal{F}_n$  be the  $\sigma$ -field generated by  $T_n$  and  $(\theta(x), x \in T_n)$ . The filtration  $(\mathcal{F}_n, n \geq 1)$  is increasing towards  $\bigvee_{n \geq 1} \mathcal{F}_n = \mathcal{F}$ , the  $\sigma$ -field generated by  $\mathcal{T}$  and  $(\hat{\theta}_s, s \in [0, \sigma]) = (\theta(x), x \in \mathcal{T})$ .

In order to first give a description of the Brownian snake conditionally on  $\mathcal{F}_n$ , we consider the sub-trees that are grafted on  $T_n$ . For  $x, y \in \mathcal{T}$ , we define an equivalence relation by setting

$$x \sim_{T_n} y \iff [\emptyset, x] \cap T_n = [\emptyset, y] \cap T_n$$

and we set  $(\mathcal{T}_i, i \in I_n)$  for the different equivalent classes. The set  $\mathcal{T}_i$  can be viewed as a rooted real tree with root  $x_i = \mathcal{T}_i \cap T_n$ . Notice that  $x_i$  represents the point of  $T_n$  at which the tree  $\mathcal{T}_i$  is grafted on  $T_n$ . Finally, we set  $\theta_i = \theta(x_i)$  and  $\sigma_i = m^{\mathcal{T}}(\mathcal{T}_i)$  which corresponds to the length of the height process of  $\mathcal{T}_i$ .

Using Theorem 3 of [16] (combined with the spatial motion  $\theta$ ), we get the following result.

**Lemma 7.1.** *Under  $\mathbb{N}_q$  conditionally on  $\mathcal{F}_n$ , the point measure*

$$\sum_{i \in I_n} \delta_{(\mathcal{T}_i, \theta_i, x_i)}(d\mathcal{T}, dq, dx)$$

*is a Poisson point measure with intensity*

$$2\alpha \mathbf{1}_{T_n}(x) \ell(dx) \mathbb{N}[d\mathcal{T}] \delta_{\theta(x)}(dq).$$

We deduce from that Lemma the next result.

**Lemma 7.2.** *Under  $\mathbb{N}_q^{(r)}$  and conditionally on  $\mathcal{F}_n$ , the point measure*

$$\mathcal{N}_n(d\sigma, dq, dx) = \sum_{i \in I_n} \delta_{(\sigma_i, \theta_i, x_i)}(d\sigma, dq, dx)$$

*is distributed as a Poisson point measure:*

$$\tilde{\mathcal{N}}(d\sigma, dq, dx) = \sum_{j \in J} \delta_{(\tilde{\sigma}_j, \theta_j, x_j)}(d\sigma, dq, dx)$$

*with intensity  $2\alpha \mathbf{1}_{T_n}(x) \ell(dx) \frac{d\sigma}{2\sqrt{\alpha\pi} \sigma^{3/2}} \mathbf{1}_{\sigma > 0} \delta_{\theta(x)}(dq)$  conditioned on  $\{\sum_{j \in J} \tilde{\sigma}_j = r\}$ .*

We can compute some elementary functionals of  $\mathcal{N}_n$ .



**Lemma 7.3.** *The point measure  $\mathcal{N}_n$  has intensity:*

$$2\alpha \mathbf{1}_{T_n}(x) \ell(dx) \mathbb{E}^{(r), L_n}[d\sigma] \delta_{\theta(x)}(dq),$$

where  $\mathbb{E}^{(r), L_n}$  satisfies, for any non-negative measurable function  $F$ :

$$2\alpha \int_{T_n} \ell(dx) \mathbb{E}^{(r), L_n}[F(x, \sigma)] = \mathbb{E} \left[ \sum_{j \in J} F(s_j, \tilde{\sigma}_j) \mid \sum_{j \in J} \tilde{\sigma}_j = r \right].$$

We also have:

$$(29) \quad \mathbb{E}^{(r), L_n}[\sigma] = \frac{r}{2\alpha L_n} \quad \text{and} \quad \mathbb{E}^{(r), L_n}[\sigma^{3/2}] \leq \frac{1}{2\sqrt{\alpha\pi}} \frac{1}{L_n} r^2 e^{-\alpha L_n^2/4r}.$$

*Proof.* The first part of the Lemma is a consequence of the exchangeability of  $(\sigma_i, i \in I_n)$ . With  $F(q, r') = r'$ , we get:

$$2\alpha L_n \mathbb{E}^{(r), L_n}[\sigma] = 2\alpha \int_{T_n} \ell(dx) \mathbb{E}^{(r), L_n}[\sigma] = \mathbb{E} \left[ \sum_{j \in J} \tilde{\sigma}_j \mid \sum_{j \in J} \tilde{\sigma}_j = r \right] = r.$$

This gives the first equality of (29). Recall that:

$$\mathbb{N} [1 - e^{-\mu\sigma}] = \int_0^\infty \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} (1 - e^{-\mu r}) = \sqrt{\mu/\alpha}.$$

We have, using the Palm formula for Poisson point measures, for  $a > 1/2$ :

$$\begin{aligned} \mathbb{E} \left[ \sum_{j \in J} \tilde{\sigma}_j^a e^{-\mu \sum_{i \in J} \tilde{\sigma}_i} \right] &= \mathbb{E} \left[ \sum_{j \in J} \tilde{\sigma}_j^a e^{-\mu \tilde{\sigma}_j} e^{-\mu \sum_{i \in J, i \neq j} \tilde{\sigma}_i} \right] \\ &= 2\alpha L_n \mathbb{N} [\sigma^a e^{-\mu\sigma}] \exp \left( -2\alpha L_n \int_0^{+\infty} \frac{d\sigma}{2\sqrt{\alpha\pi} \sigma^{3/2}} (1 - e^{-\mu\sigma}) \right) \\ &= 2\alpha L_n \mathbb{N} [\sigma^a e^{-\mu\sigma}] e^{-2L_n \sqrt{\alpha\mu}}. \end{aligned}$$

Moreover, we have:

$$\mathbb{N} [\sigma^a e^{-\mu\sigma}] = \int_0^\infty \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} r^a e^{-\mu r} = \frac{1}{2\sqrt{\alpha\pi}} \Gamma(a - 1/2) \mu^{1/2-a}.$$

We deduce that:

$$\begin{aligned} \mathbb{E} \left[ \sum_{j \in J} \left( 2\sqrt{\alpha} L_n \tilde{\sigma}_j^{3/2} + \frac{1}{\Gamma(3/2)} \tilde{\sigma}_j^2 \right) e^{-\mu \sum_{i \in J} \tilde{\sigma}_i} \right] \\ &= 2\alpha L_n e^{-2L_n \sqrt{\alpha\mu}} \left( 2\sqrt{\alpha} L_n \mathbb{N} [\sigma^{3/2} e^{-\mu\sigma}] + \frac{1}{\Gamma(3/2)} \mathbb{N} [\sigma^2 e^{-\mu\sigma}] \right) \\ &= 2\alpha L_n e^{-2L_n \sqrt{\alpha\mu}} \frac{1}{2\sqrt{\alpha\pi}} \left( \frac{2\sqrt{\alpha} L_n}{\mu} + \frac{1}{\mu^{3/2}} \right) \\ &= \frac{2}{\sqrt{\pi}} \frac{\partial^2}{\partial \mu^2} e^{-L_n \sqrt{\mu\alpha}}. \end{aligned}$$

Using (39), we have

$$\begin{aligned}
\frac{\partial^2}{\partial \mu^2} e^{-L_n \sqrt{\mu \alpha}} &= \frac{\partial^2}{\partial \mu^2} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{dx}{x^{3/2}} e^{-1/x} e^{-\frac{1}{4} \alpha L_n^2 \mu x} \\
&= \frac{1}{\sqrt{\pi}} \left( \frac{\alpha}{4} L_n^2 \right)^2 \int_0^{+\infty} dx \sqrt{x} e^{-1/x} e^{-\frac{1}{4} \alpha L_n^2 \mu x} \\
&= \frac{L_n \sqrt{\alpha}}{2\sqrt{\pi}} \int_0^{+\infty} dr \sqrt{r} e^{-\alpha L_n^2 / 4r} e^{-\mu r} \\
&= \alpha L_n \int_0^{+\infty} \frac{dr}{2\sqrt{\alpha \pi} r^{3/2}} r^2 e^{-\alpha L_n^2 / 4r} e^{-\mu r}.
\end{aligned}$$

We deduce that:

$$\mathbb{E} \left[ \sum_{j \in J} \left( 2\sqrt{\alpha} L_n \tilde{\sigma}_j^{3/2} + \frac{1}{\Gamma(3/2)} \tilde{\sigma}_j^2 \right) \mid \sum_{i \in J} \tilde{\sigma}_i = r \right] = \frac{2\alpha L_n}{\sqrt{\pi}} r^2 e^{-\alpha L_n^2 / 4r}.$$

Then, using the first part of Lemma 7.3 with  $F(s, \sigma) = 2\sqrt{\alpha} L_n \sigma^{3/2} + \frac{1}{\Gamma(3/2)} \sigma^2$ , we get the second equality of (29).  $\square$

We consider the process  $(M_n, n \geq 1)$  defined by, for  $q \in [0, +\infty[$ :

$$M_n = \mathbb{N}_q^{(r)} \left[ \Theta \mid \mathcal{F}_n \right].$$

Thanks to Corollary 4.5, we get that:

$$\mathbb{N}_q^{(r)} [M_n^2] \leq \mathbb{N}_q^{(r)} [\Theta^2] \leq \mathbb{N}_\infty^{(r)} [\Theta^2] = \frac{r}{\alpha}.$$

Therefore  $(M_n, n \geq 1)$  is (a well defined) square integrable non-negative martingale. In particular it converges  $\mathbb{N}_q^{(r)}$ -a.s. (and in  $L^2(\mathbb{N}_q^{(r)})$ ) to  $\Theta$  as the increasing  $\sigma$ -fields  $\mathcal{F}_n$  increase to  $\mathcal{F}$ .

**Lemma 7.4.** *We have, for  $n \geq 1$ ,*

$$(30) \quad -R_n \leq M_n - \frac{r}{L_n} \int_{T_n^*} \theta(x) \ell(dx) \leq V_n,$$

where  $(R_n, n \geq 1)$  and  $(V_n, n \geq 1)$  are non-negative sequences which converge  $\mathbb{N}_\infty^{(r)}$ -a.s. to 0. Furthermore the non-negative sequence  $(R'_n, n \geq 1)$ , with  $R'_n = \mathbb{N}_\infty^{(r)} [R_n | T_n] L_n / \sqrt{n}$ , converges  $\mathbb{N}_\infty^{(r)}$ -a.s. to 0.

This Lemma and the  $\mathbb{N}_\infty^{(r)}$ -a.s. convergences of  $(M_n, n \geq 1)$  to  $\Theta$  and of  $(L_n / \sqrt{n}, n \geq 1)$  to  $\sqrt{r/\alpha}$  (see Proposition 5.2) prove (26). Then use that  $\mathbb{N}_\infty^{(r)} [M_n | T_n] = \mathbb{N}_\infty^{(r)} [\Theta | T_n]$  to get (27). This ends the proof of Proposition 6.3.

*Proof of Lemma 7.4.* We consider the set  $I_n^* = \{i \in I_n, x_i \geq m_n\}$  of indexes such that  $\mathcal{T}_i$  is not grafted on the edge of  $T_n$  originating from the root. We set:

$$A_n = \{s \geq 0; [\emptyset, s] \cap T_n^* \neq \emptyset\} = \bigcup_{i \in I_n^*} T^i, \quad M_n^* = \mathbb{N}_q^{(r)} \left[ \int_{A_n} \hat{\theta}_s ds \mid \mathcal{F}_n \right] \quad \text{and} \quad V_n = M_n - M_n^*.$$

Notice that the sequence  $(A_n, n \in \mathbb{N}^*)$  is non-decreasing and that  $\bigcap_{n \in \mathbb{N}^*} A_n^c = \emptyset$ , as there is no tree grafted on the root. By dominated convergence, this implies that  $\mathbb{N}_q^{(r)}$ -a.s.:

$$\lim_{n \rightarrow +\infty} \int_{A_n^c} \hat{\theta}_s ds = 0.$$

As:

$$V_{n+m} = \mathbb{N}_q^{(r)} \left[ \int_{A_{n+m}^c} \hat{\theta}_s ds \mid \mathcal{F}_{n+m} \right] \leq \mathbb{N}_q^{(r)} \left[ \int_{A_n^c} \hat{\theta}_s ds \mid \mathcal{F}_{n+m} \right],$$

and as  $\mathcal{F}_{n+m}$  increases to  $\mathcal{F}$ , we get that  $\limsup_{m \rightarrow +\infty} V_{n+m} \leq \int_{A_n^c} \hat{\theta}_s ds$  and thus  $\mathbb{N}_q^{(r)}$ -a.s.

$$(31) \quad \lim_{n \rightarrow +\infty} V_n = 0.$$

Recall the function  $H_q$  defined by (21). We have, with  $\Theta_i = \Theta(\mathcal{T}_i) = \int_{\mathcal{T}_i} \theta(x) m^{\mathcal{T}}(dx)$ :

$$\begin{aligned} M_n^* &= \mathbb{N}_q^{(r)} \left[ \int_{A_n} \hat{\theta}_s ds \mid \mathcal{F}_n \right] = \mathbb{N}_q^{(r)} \left[ \sum_{i \in I_n^*} \Theta_i \mid \mathcal{F}_n \right] = \mathbb{N}_q^{(r)} \left[ \sum_{i \in I_n^*} \mathbb{N}_{\theta(x_i)}^{(\sigma_i)}[\Theta] \mid \mathcal{F}_n \right] \\ &= \mathbb{N}_q^{(r)} \left[ \sum_{i \in I_n^*} H_{\theta(x_i)}(\sigma_i) \mid \mathcal{F}_n \right]. \end{aligned}$$

Since  $H_q(r) \leq qr$ , see (22), we get using Lemma 7.3:

$$M_n^* = 2\alpha \int_{T_n^*} \ell(dx) \mathbb{E}^{(r), L_n} [H_{\theta(x)}(\sigma)] \leq 2\alpha \int_{T_n^*} \ell(dx) \theta(x) \mathbb{E}^{(r), L_n} [\sigma] = r \frac{1}{L_n} \int_{T_n^*} \ell(dx) \theta(x).$$

This gives the upper bound of (30).

We shall now prove the lower bound of (30). Since  $H_q(r) \geq qr - \frac{1}{2} \sqrt{\alpha\pi} q^2 r^{3/2}$ , see (22), we also get using the second equality of (29):

$$\begin{aligned} M_n &\geq M_n^* \geq r \frac{1}{L_n} \int_{T_n^*} \ell(dx) \theta(x) - \frac{1}{2} \sqrt{\alpha\pi} \mathbb{E}^{(r), L_n} [\sigma^{3/2}] \int_{T_n^*} \ell(dx) \theta(x)^2 \\ &\geq r \frac{1}{L_n} \int_{T_n^*} \ell(dx) \theta(x) - \frac{1}{4} r^2 e^{-\alpha L_n^2/4r} \theta_{\emptyset, n}^2. \end{aligned}$$

This proves the lower bound of (30) with:

$$(32) \quad R_n = \frac{1}{4} r^2 e^{-\alpha L_n^2/4r} \theta_{\emptyset, n}^2.$$

It remains to prove that this quantity tends to 0. First, we have:

$$\mathbb{N}_\infty^{(r)} [h_{\emptyset, n}^2 \theta_{\emptyset, n}^2] = \mathbb{N}_\infty^{(r)} [h_{\emptyset, n}^2 \mathbb{N}_\infty^{(r)} [\theta_{\emptyset, n}^2 \mid h_{\emptyset, n}]] = \frac{1}{(2\alpha)^2},$$

where we used that  $\theta_{\emptyset, n}$  is exponentially distributed conditionally given  $h_{\emptyset, n}$  for the second equality. We deduce that:

$$\mathbb{N}_\infty^{(r)} \left[ \sum_{n=1}^{+\infty} \frac{h_{\emptyset, n}^2 \theta_{\emptyset, n}^2}{n^2} \right] < \infty$$

and hence  $\mathbb{N}_\infty^{(r)}$ -a.s.:

$$\sum_{n=1}^{+\infty} \frac{h_{\emptyset, n}^2 \theta_{\emptyset, n}^2}{n^2} < \infty.$$

This implies that for some finite constant  $C_1$ ,  $\mathbb{N}_\infty^{(r)}$ -a.s.:

$$h_{\emptyset,n}^2 \theta_{\emptyset,n}^2 \leq C_1 n^2.$$

Using Lemma 5.4, we have  $\mathbb{N}_\infty^{(r)}[h_{\emptyset,n}^{-1/2}] \sim n^{1/4} \sqrt{\alpha\pi/2r}$ , which implies by similar arguments that for some finite constant  $C_2$ ,  $\mathbb{N}_\infty^{(r)}$ -a.s.:

$$(33) \quad h_{\emptyset,n}^{-1/2} \leq C_2 n^{3/2}.$$

Finally, using Formula (32) for  $R_n$ , we have  $\mathbb{N}_\infty^{(r)}$ -a.s.:

$$R_n \leq C_1 C_2^4 n^8 e^{-\alpha L_n^2/4r}.$$

As  $\mathbb{N}_\infty^{(r)}$ -a.s.  $\lim_{n \rightarrow +\infty} L_n/\sqrt{n} = \sqrt{r/\alpha}$ , we deduce that  $\lim_{n \rightarrow +\infty} R_n = 0$ .

Using (33), we deduce that:

$$R'_n = \frac{L_n}{\sqrt{n}} \mathbb{N}_\infty^{(r)}[R_n | T_n] = \frac{L_n}{\sqrt{n}} \frac{r^2}{4} e^{-\alpha L_n^2/4r} \frac{1}{4\alpha^2} \frac{1}{h_{\emptyset,n}^2} \leq C_2^4 \frac{r^2}{4} n^{11/2} L_n e^{-\alpha L_n^2/4r}.$$

Thus, we get the non-negative sequence  $(R'_n, n \geq 1)$ , converges  $\mathbb{N}_\infty^{(r)}$ -a.s. to 0, which ends the proof.  $\square$

## 8. ANOTHER EXPRESSION FOR $\Theta$

For  $q \geq 0$ , we define the measure on real trees:

$$\mathbb{N}^q[\mathcal{T} \in \bullet] = \mathbb{N}[\mathcal{T} \in \bullet; e^{-\alpha q^2 \sigma}].$$

This measure corresponds to the excursion measure of the genealogical tree of the continuous state branching process with branching mechanism  $\psi_q(u) = \psi(u+q) - \psi(q) = \alpha u^2 + 2\alpha u$ , see [1]. We consider the sub-tree of  $\mathcal{T}$  on which the record process is no smaller than  $q$ :

$$\mathcal{T}^q = \{x \in \mathcal{T}; \theta(x) \geq q\}.$$

We set  $\sigma_q = m^{\mathcal{T}}(\mathcal{T}^q)$ . Notice that  $\sigma_q$  is also the Lebesgue measure of  $\{s \in [0, \sigma], \hat{\theta}_s \geq q\}$ . See [1] for the distribution of the decreasing process  $(\sigma_q, q \geq 0)$ . According to [1], the distribution of  $\mathcal{T}^q$  is given by  $\mathbb{N}^q$ . In particular, we have:

$$\mathbb{N}[\sigma_q] = \mathbb{N}[\sigma e^{-\alpha q^2 \sigma}] = \frac{1}{2\alpha q}.$$

Let  $(\theta_i, i \in \mathcal{I})$  be the set of jumping times of  $(\sigma_q, q \geq 0)$ . We set:

$$\mathcal{T}^i = \{x \in \mathcal{T}; \theta(x) = \theta_i\} \quad \text{and} \quad \sigma^i = m^{\mathcal{T}}(\mathcal{T}^i) = \sigma_{\theta_i^-} - \sigma_{\theta_i}.$$

According to [1], we have that  $\mathbb{N}_\infty^{(r)}$ -a.s.  $\mathcal{T}^i$  is a real tree for all  $i \in \mathcal{I}$ . Then the following result is straightforward as by definition  $\Theta = \sum_{i \in \mathcal{I}} \theta_i \sigma^i$  and  $\sigma_q = \sum_{\theta_i \geq q} \sigma^i$ .

**Proposition 8.1.** *We have  $\mathbb{N}_\infty$ -a.e. or  $\mathbb{N}_\infty^{(r)}$ -a.s.:*

$$\Theta = \int_0^{+\infty} \sigma_q dq.$$

We define the backward filtration  $\mathcal{G} = (\mathcal{G}_q, q \geq 0)$  with  $\mathcal{G}_q = \sigma(\mathcal{T}^r, r \geq q)$ . Following [2], we get that the random measure:

$$\mathcal{N}(d\mathcal{T}', dq) = \sum_{i \in \mathcal{I}} \delta_{\mathcal{T}^i, \theta_i}(d\mathcal{T}', dq)$$

is under  $\mathbb{N}_\infty$  a point measure with intensity:

$$\mathbf{1}_{\{q>0\}} 2\alpha\sigma_q \mathbb{N}^q [d\mathcal{T}'] dq.$$

This means that for every non-negative predictable process  $(Y(\mathcal{T}', q), q \in \mathbb{R}_+, \mathcal{T}' \in \mathbb{T})$  with respect to the backward filtration  $\mathcal{G}$ ,

$$(34) \quad \mathbb{N}_\infty \left[ \int Y(\mathcal{T}', q) \mathcal{N}(d\mathcal{T}, dq) \right] = \mathbb{N}_\infty \left[ \int \mathcal{Y}_q \mathbf{1}_{\{q>0\}} 2\alpha\sigma_q dq \right],$$

where  $(\mathcal{Y}_q = \int Y(\mathcal{T}', q) \mathbb{N}^q [d\mathcal{T}'], q \in \mathbb{R}_+)$  is predictable with respect to the backward filtration  $\mathcal{G}$ . We refer to [11, 12] for the general theory of random point measures.

Recall that, according to Proposition 4.4,  $Z = \sqrt{\frac{2\alpha}{\sigma}} \Theta$  is a Rayleigh random variable with density  $x e^{-x^2/2} \mathbf{1}_{\{x>0\}}$ .

**Proposition 8.2.** *We have  $\mathbb{N}_\infty$ -a.e.:*

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} \mathbf{1}_{\{\sigma^i \geq 1/n\}} = 2\sqrt{\frac{\alpha}{\pi}} \Theta = \sqrt{\frac{2\sigma}{\pi}} Z.$$

*Proof.* Let  $M > 0$  be large. We consider the  $\mathcal{G}$ -stopping time  $\tau_M = \inf\{q; \sigma_q < M/2\alpha\}$ . We define for every  $\theta > 0$  and every positive integer  $n$ ,

$$Q_n(\theta) = \sum_{i \in \mathcal{I}} \mathbf{1}_{\{\sigma^i \geq 1/n\}} \mathbf{1}_{\{\theta_i > \theta\}}.$$

We have  $Q_n(\tau_M) = \sum_{i \in \mathcal{I}} \mathbf{1}_{\{\sigma^i \geq 1/n\}} \mathbf{1}_{\{\sigma_{\theta_i} < M/2\alpha\}}$  so that:

$$\begin{aligned} \mathbb{N}_\infty [Q_n(\tau_M)] &= \mathbb{N}_\infty \left[ \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right] \right] \\ &\leq \mathbb{N}_\infty \left[ \int_0^{+\infty} dq \min \left( \sigma_q, \frac{M}{2\alpha} \right) \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right] \right] \\ &\leq \frac{1}{2\alpha} \int_0^{+\infty} dq \min \left( M, \frac{1}{q} \right) \int_{\frac{1}{n}}^{+\infty} \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} e^{-\alpha q^2 r} \\ &= \frac{\sqrt{n}}{4\alpha^{3/2} \sqrt{\pi}} \int_0^{+\infty} dq \min \left( M, \frac{1}{q} \right) \int_1^{+\infty} \frac{dr}{r^{3/2}} e^{-\alpha q^2 r/n}, \end{aligned}$$

where the first equality is derived from (34). Elementary computations yields there exists a finite constant  $c$  which depends on  $M$  but not on  $n$  such that:

$$(35) \quad \mathbb{N}_\infty [Q_n(\tau_M)] = \mathbb{N}_\infty \left[ \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right] \right] \leq c\sqrt{n} (1 + \log(n) \mathbf{1}_{\{n \geq \alpha M^2\}}).$$

Classical results on random point measures imply that the process  $(N_n(\theta \vee \tau_M), \theta \geq 0)$ , with:

$$N_n(\theta) = Q_n(\theta) - 2\alpha \int_\theta^{+\infty} dq \sigma_q \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right]$$

is a backward martingale with respect to  $\mathcal{G}$ . Moreover, since  $(Q_n(\theta), \theta \geq 0)$  is a pure jump process with jumps of size 1, the process  $(M_n(\theta \vee \tau_M), \theta \geq 0)$ , with:

$$M_n(\theta) = N_n(\theta)^2 - 2\alpha \int_{\theta}^{+\infty} dq \sigma_q \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right]$$

is also a backward martingale with respect to  $\mathcal{G}$ . Using (35), we get that  $\mathbb{N}_{\infty} \left[ (N_{n^4}(\tau_M)/n^2)^2 \right]$  is less than a constant times  $n^{-3/2}$ ; therefore

$$\sum_{n=1}^{+\infty} \left( \frac{N_{n^4}(\tau_M)}{n^2} \right)^2$$

is finite in  $L^1(\mathbb{N}_{\infty})$  and thus is  $\mathbb{N}_{\infty}$ -a.e. finite. This implies that  $\mathbb{N}_{\infty}$ -a.e.:

$$\lim_{n \rightarrow +\infty} \frac{N_{n^4}(\tau_M)}{n^2} = 0.$$

Moreover, we have by monotone convergence:

$$\begin{aligned} \frac{2\alpha}{\sqrt{n}} \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \mathbf{1}_{\{\sigma \geq 1/n\}} e^{-\alpha q^2 \sigma} \right] &= 2\alpha \int_{\tau_M}^{+\infty} dq \sigma_q \int_1^{+\infty} \frac{dr}{2\sqrt{\alpha\pi} r^{3/2}} e^{-\alpha q^2 \frac{r}{n}} \\ &\xrightarrow[n \rightarrow \infty]{\mathbb{N}_{\infty}\text{-a.e.}} 2\sqrt{\frac{\alpha}{\pi}} \int_{\tau_M}^{+\infty} dq \sigma_q. \end{aligned}$$

We get that the sequence  $(Q_n(\tau_M)/n^2, n \geq 1)$  converges  $\mathbb{N}_{\infty}$ -a.e. toward  $2\sqrt{\frac{\alpha}{\pi}} \int_{\tau_M}^{+\infty} dq \sigma_q$ . Since  $(Q_n(\theta), n \geq 1)$  is non-decreasing, we deduce that  $\mathbb{N}_{\infty}$ -a.e.:

$$\lim_{n \rightarrow +\infty} \frac{Q_n(\tau_M)}{\sqrt{n}} = 2\sqrt{\frac{\alpha}{\pi}} \int_{\tau_M}^{+\infty} dq \sigma_q.$$

Since  $\Theta$  is finite  $\mathbb{N}_{\infty}$ -a.e., we get that  $\mathbb{N}_{\infty}$ -a.e.  $\tau_M = 0$  for  $M$  large enough. This gives the result.  $\square$

**Proposition 8.3.** *We have  $\mathbb{N}_{\infty}$ -a.e.:*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \sum_{i \in \mathcal{I}} \sigma^i \mathbf{1}_{\{\sigma^i \leq 1/n\}} = 2\sqrt{\frac{\alpha}{\pi}} \Theta = \sqrt{\frac{2\sigma}{\pi}} Z.$$

*Proof.* The proof is very similar to the proof of Proposition 8.2. We set:

$$Q_n(\theta) = \sum_{i \in \mathcal{I}} \sigma^i \mathbf{1}_{\{\sigma^i \leq 1/n\}} \mathbf{1}_{\{\theta_i \geq \theta\}}.$$

We have:

$$\begin{aligned} \mathbb{N}_{\infty} [Q_n(\tau_M)] &= \mathbb{N}_{\infty} \left[ \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \sigma \mathbf{1}_{\{\sigma \leq 1/n\}} e^{-\alpha q^2 \sigma} \right] \right] \\ &\leq \frac{1}{4\alpha^{3/2} \sqrt{\pi} \sqrt{n}} \int_0^{+\infty} dq \min \left( M, \frac{1}{q} \right) \int_0^1 \frac{dr}{\sqrt{r}} e^{-\alpha q^2 r/n} < +\infty, \end{aligned}$$

as well as for some finite constant  $c$  which depends on  $M$  but not on  $n$ :

$$\begin{aligned} \mathbb{N}_\infty \left[ \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \sigma^2 \mathbf{1}_{\{\sigma \leq 1/n\}} e^{-\alpha q^2 \sigma} \right] \right] \\ \leq \frac{1}{4\alpha^{3/2} \sqrt{\pi} n^{3/2}} \int_0^{+\infty} dq \min \left( M, \frac{1}{q} \right) \int_0^1 dr \sqrt{r} e^{-\alpha q^2 r/n} \\ \leq cn^{-3/2} (1 + \log(n)). \end{aligned}$$

which is finite. Classical results on random point measures imply that the process  $(N_n(\theta \vee \tau_M), \theta \geq 0)$  and  $(M_n(\theta \vee \tau_M), \theta \geq 0)$ , with:

$$\begin{aligned} N_n(\theta) &= Q_n(\theta) - 2\alpha \int_\theta^{+\infty} dq \sigma_q \mathbb{N} \left[ \sigma \mathbf{1}_{\{\sigma \leq 1/n\}} e^{-\alpha q^2 \sigma} \right] \\ M_n(\theta) &= N_n(\theta)^2 - 2\alpha \int_\theta^{+\infty} dq \sigma_q \mathbb{N} \left[ \sigma^2 \mathbf{1}_{\{\sigma \leq 1/n\}} e^{-\alpha q^2 \sigma} \right] \end{aligned}$$

are backward martingales with respect to  $\mathcal{G}$ . We get that  $\mathbb{N}_\infty \left[ (n^2 N_{n^4}(\tau_M))^2 \right]$  is less than a constant times  $n^{-3/2}$ . Following the proof of Proposition 8.2, we deduce that  $\mathbb{N}_\infty$ -a.e.  $\lim_{n \rightarrow +\infty} n^2 N_{n^4}(\tau_M) = 0$ . Furthermore, we have:

$$\begin{aligned} 2\alpha\sqrt{n} \int_{\tau_M}^{+\infty} dq \sigma_q \mathbb{N} \left[ \sigma \mathbf{1}_{\{\sigma \leq 1/n\}} e^{-\alpha q^2 \sigma} \right] &= 2\alpha\sqrt{n} \int_{\tau_M}^{+\infty} dq \sigma_q \int_0^{\frac{1}{n}} \frac{dr}{2\sqrt{\alpha\pi r}} e^{-\alpha q^2 r} \\ &= 2\alpha \int_{\tau_M}^{+\infty} dq \sigma_q \int_0^1 \frac{dr}{2\sqrt{\alpha\pi r}} e^{-\alpha q^2 \frac{r}{n}} \\ &\rightarrow 2\sqrt{\frac{\alpha}{\pi}} \int_{\tau_M}^{+\infty} dq \sigma_q. \end{aligned}$$

We conclude the proof as in the proof of Proposition 8.2.  $\square$

## 9. APPENDIX

*Proof of Lemma 4.1.* The first part of the Lemma is a well known result. We introduce the function  $u_t(q)$  defined for  $t \geq 0$  and  $q \geq 0$  by:

$$u_t(q) = \mathbb{N}_q \left[ 1 - e^{-\int_0^\sigma f(\hat{\theta}_s) \mathbf{1}_{\{\zeta_s \leq t\}} ds} \right].$$

We deduce from Theorem II.5.11 of [20] that  $u$  is the unique non-negative solution of:

$$u_t(q) + \mathbb{E}_q \left[ \int_0^t \alpha u_{t-s}(\theta(s))^2 ds \right] = \mathbb{E}_q \left[ \int_0^t f(\theta(s)) ds \right].$$

Using the Markov property of  $\theta$ , we get that for  $t \geq r \geq 0$ :

$$(36) \quad u_t(q) + \mathbb{E}_q \left[ \int_0^r \alpha u_{t-s}(\theta(s))^2 ds \right] = \mathbb{E}_q \left[ \int_0^r f(\theta(s)) ds \right] + \mathbb{E}_q[u_{t-r}(\theta(r))].$$

Notice that  $\lim_{t \rightarrow +\infty} u_t(q) = F(q)$ . And we have:

$$u_t(q) \leq F(q) \leq \mathbb{N} \left[ 1 - e^{-(q+\mu)\sigma} \right] = \sqrt{(q+\mu)/\alpha}.$$

By monotone convergence, we deduce from (36) that:

$$F(q) + \mathbb{E}_q \left[ \int_0^r \alpha F(\theta(s))^2 ds \right] = \mathbb{E}_q \left[ \int_0^r f(\theta(s)) ds \right] + \mathbb{E}_q[F(\theta(r))].$$

This implies that the process  $N = (N_t, t \geq 0)$  defined by:

$$N_t = F(\theta(t)) + \int_0^t (f(\theta(s)) - \alpha F(\theta(s))^2) ds,$$

is a martingale under  $\mathbb{E}_q$ , for  $q < +\infty$ . We deduce from (2) (with  $g = F$ ) that:

$$\int_0^t \left( f(\theta(s)) - \alpha F(\theta(s))^2 - 2\alpha \int_0^{\theta(s)} (F(x) - F(q)) dx \right) ds$$

is a martingale. Since it is predictable, it is a.s. constant. We get that a.e. for  $q \geq 0$ :

$$f(q) - \alpha F(q)^2 + 2\alpha q F(q) - 2\alpha \int_0^q F(x) dx = 0,$$

that is a.e.:

$$F(q) = \sqrt{q^2 - 2 \int_0^q F(x) dx + (f(q)/\alpha)} + q.$$

Since by construction  $F$  is non-decreasing, we get that  $F$  is continuous and then of class  $\mathcal{C}^1$ . An obvious integration by parts gives (9).

We now prove the second part of the Lemma. Notice that  $F(0) = \mathbb{N}_0 [1 - e^{-\mu\sigma}] = \sqrt{\mu/\alpha}$ . By differentiating (9) we have:

$$(37) \quad 2\alpha F'(q)(F(q) + q) = \lambda.$$

This implies that  $F' > 0$  and thus  $F$  is one-to-one from  $[0, +\infty)$  to  $[\sqrt{\mu/\alpha}, +\infty)$ . Moreover,  $F^{-1}$  solves the differential equation

$$(38) \quad g'(x) = \frac{2\alpha}{\lambda}(g(x) + x).$$

Elementary computations give that the unique solution to (38) with the initial condition  $g(\sqrt{\mu/\alpha}) = 0$  is  $G$ . Thus, we get by uniqueness  $F^{-1} = G$ .  $\square$

*Proof of Lemma 4.3.* We set

$$J = \sqrt{\frac{\mu}{2}} \int_0^\infty \frac{dr}{\sqrt{r}} e^{-\mu r} \int_0^\infty dx x e^{-x^2/2} e^{-c\sqrt{2r}x}.$$

With the change of variable  $t^2 = 2\mu r$  and with  $\rho = c/\sqrt{\mu}$ , we get:

$$\begin{aligned} J &= \int_{[0, +\infty)^2} dt dx x \exp(-(t^2 + x^2 + 2\rho t x)/2) \\ &= \int_{[0, +\infty)^2} dt dx (x + \rho t) e^{-(t^2 + x^2 + 2\rho t x)/2} - \rho \int_{[0, +\infty)^2} dt dx t e^{-(t^2 + x^2 + 2\rho t x)/2} \\ &= \int_0^\infty dt [-\exp(-(t^2 + x^2 + 2\rho t x)/2)]_{x=0}^{x=+\infty} - \rho J \\ &= \int_0^\infty dt e^{-t^2/2} - \rho J \\ &= \sqrt{\pi/2} - \rho J. \end{aligned}$$

This implies that  $J = \frac{\sqrt{\pi/2}}{\rho + 1} = \sqrt{\frac{\mu}{2}} \frac{\sqrt{\pi}}{c + \sqrt{\mu}}$ , which is exactly what we needed.  $\square$



*Proof of Lemma 5.1.* Let  $(h_1, \dots, h_{2n-1})$  be the lengths of the edges of the tree  $T_n$  with leaves  $t_1, \dots, t_n$ . Let  $F$  be a non-negative function defined on  $\mathbb{R}^{2n-1}$ . We set:

$$J_n = \mathbb{N} \left[ \int_{[0, \sigma]^n} dt_1 \dots dt_n e^{-\mu \sigma} F(h_1, \dots, h_{2n-1}) \right].$$

By conditioning with respect to  $\sigma$ , we get:

$$J_n = \frac{1}{2\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dr}{r^{3/2}} r^n e^{-\mu r} \left( \int F(h_1, \dots, h_{2n-1}) f_n^{(r)}(h_1, \dots, h_{2n-1}) dh_1 \dots dh_{2n-1} \right).$$

Using Theorem 3 of [16], we also have, for some constant  $c_n$ ,

$$\begin{aligned} J_n &= c_n \int_{[0, +\infty)^{2n-1}} dh_1 \dots dh_{2n-1} F(h_1, \dots, h_{2n-1}) \mathbb{N} [e^{-\mu \sigma} \mid h_1, \dots, h_{2n-1}] \\ &= c_n \int_{[0, +\infty)^{2n-1}} dh_1 \dots dh_{2n-1} F(h_1, \dots, h_{2n-1}) e^{-2\alpha L_n \mathbb{N}[1 - \exp(-\mu \sigma)]} \\ &= c_n \int_{[0, +\infty)^{2n-1}} dh_1 \dots dh_{2n-1} F(h_1, \dots, h_{2n-1}) e^{-2\sqrt{\alpha\mu} L_n}, \end{aligned}$$

where we used the Poisson decomposition of the continuum random tree (see *e.g.* Lemma 7.1) for the second equality.

Take  $F = 1$  in the previous equality to check that:

$$c_n \frac{1}{(2\sqrt{\alpha\mu})^{2n-1}} = \frac{1}{2\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dr}{r^{3/2}} r^n e^{-\mu r} = \frac{\Gamma(n - \frac{1}{2})}{2\sqrt{\pi\alpha} \mu^{n-1/2}}.$$

This gives, thanks to the duplication formula (24):

$$c_n = \alpha^{n-1} \frac{2^{2n-2} \Gamma(n - \frac{1}{2})}{\sqrt{\pi}} = \alpha^{n-1} \frac{(2n-2)!}{(n-1)!}.$$

We deduce that:

$$c_n e^{-2\sqrt{\alpha\mu} L_n} = \frac{1}{2\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dr}{r^{3/2}} r^n e^{-\mu r} f_n^{(r)}(h_1, \dots, h_{2n-1}).$$

We recall the following Laplace transform for the density of the stable subordinator of index 1/2: for  $a > 0$  and  $\mu \geq 0$ ,

$$(39) \quad a \int_0^{+\infty} \frac{dr}{\sqrt{2\pi r^3}} e^{-\mu r - a^2/(2r)} = e^{-a\sqrt{2\mu}}.$$

We then deduce, with  $a = \sqrt{2\alpha} L_n$  that:

$$\begin{aligned} \frac{1}{2\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dr}{r^{3/2}} r^n e^{-\mu r} \left( \frac{2\alpha}{r^n} c_n L_n e^{-\alpha L_n^2/r} \right) &= c_n \sqrt{2\alpha} L_n \int_0^{+\infty} \frac{dr}{\sqrt{2\pi r^{3/2}}} e^{-\mu r - (\sqrt{2\alpha} L_n)^2/2r} \\ &= c_n e^{-2L_n \sqrt{\alpha\mu}}. \end{aligned}$$

By uniqueness of the Laplace transform, this gives the result.  $\square$

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