Particles approximations of Vlasov equations with singular forces : Part. 2

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Maxime Hauray* and Pierre-Emmanuel Jabin †

Abstract. We obtain the mean field limit and the propagation of chaos for a system of particles interacting with a singular interaction force of the type $1/|x|^\alpha$, with $\alpha < 1$ in dimension $d \geq 3$. We also recover previous results, with a sharper propagation of chaos property, for forces with singularity up to $\alpha < d-1$ but with large enough cut-off. This last result thus almost includes the most interesting case of Coulombian or gravitationnal interaction.
Key words. Derivation of kinetic equations. Particle methods. Vlasov equation. Propagation of chaos.

1 Introduction

The N particle system. We study the evolution of system of N particles in interaction on the whole space $\mathbb{R}^d$, which is described by the system of ODEs below. The vectors $(X_1, \dots, X_N)$ denote the position of the particles in $\mathbb{R}^d$, $(V_1, \dots, V_N)$ denote their velocities in $\mathbb{R}^d$ and $F(x)$ the interaction kernel :

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E(X_i) = \displaystyle\sum_{j \neq i} \frac{1}{N} F(X_i - X_j), \end{cases} \tag{1.1}$$

We us the so-called mean-field scaling which consist in keeping the total mass (or charge) of order 1, in order to recover in the limit a "mean-field" equation. This actually implies corresponding rescaling in position, velocity and time. When we will use a cut-off, the force-field F will also depend on N and will be denoted F_N (some N dependent mollification of F).
We also use the notation $Z_i = (X_i, V_i)$ for the solution and $Z^0 = (X_1^0, V_1^0, \dots, X_n^0, V_n^0)$ for the given initial conditions.
A case of particular interest is the Coulombian force $F(x) = C\, x/|x|^{d-1}$, which is used to describe a plasma, or for $C < 0$ gravitational interactions, in which case the system under study may be a galaxy, a cloud of star or galaxies (and thus particles are "stars" or even "galaxies"). Of course other forces are also of physical interest.

The Jeans-Vlasov equation. As a plasma or a galaxy usually contains a very large number of "particles", the system (1.1) is not very convenient. Instead one usually uses

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the distribution function in time, position and speed. The evolution of that distribution fonction $f(t,x,v)$ is given by the Jeans-Vlasov equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0\,, \\ E(x) = \int_{\mathbb{R}^d} \rho(t,y)\, F(x-y)\, dy, \\ \rho(t,x) = \int_v f(t,x,v)\, dv. \end{cases} \tag{1.2}$$

Here ρ is the spatial density and the initial density f^0 is given.

The question of convergence. The question of the convergence of the N particles system towards the (mean field) Vlasov equation is important for theoretical reasons, to justify the validity of the Vlasov equation. It also plays a role for numerical simulation, and especially Particles in Cells methods which introduce a large number (roughly around 10^6, to compare with the order 10^{10} to 10^{23} of the number of particles in physical systems) of "virtual" particles in order to obtain a particles system solvable on a computer.
One way of precisely formulating the question is the following: Choosing the intial positions and speeds of the particles so that the empirical distribution

$$\mu_N^z(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t),V_i(t)},$$

converges (weakly) towards f^0 at time 0, does the empirical distribution $\mu_N(t)$ still converges towards $f(t)$ (the solution of the Vlasov equation with initial condition f^0) at time t? In other words, is the following diagram commutative?

$$\begin{array}{ccc} \mu_N^z(0) & \overset{\text{cvg}}{\rightsquigarrow} & f(0) \\ {\scriptstyle Npart}\downarrow & & \downarrow{\scriptstyle VP} \\ \mu_N^z(t) & \overset{\text{cvg}}{\rightsquigarrow} & f(t) \end{array}$$

Propagation of chaos. As it may be difficult to prove convergence for a particular sequence of initial condition, the problem may be restated in term of propagation of chaos. If the initial positions and speeds of the particles are choosen randomly with the law $(f^0)^{\otimes N}$ (i.e. randomly and independently with profile f^0), then the empirical measure at time 0 is close to f^0 with large probability. But can we say that for any time $t > 0$, $\mu_N^z(t)$ is close to $f(t)$ with large probability?
The convergence and the propagation of chaos are known to hold for smooth interaction forces (of class C^1) since the end of the seventies and the works of Braun and Hepp [BH77], Neunzert and Wick [NW80] and Dobrushin [Dob79].

Previous results with cut-off or for Euler system. The convergence was proved to hold in the case of Coulombian or gravitational force *with cut-off*, for particles initially on a mesh. The cut-off parameter $\varepsilon(N)$ depends of the number of particles and goes to zero as the number of particles increases. But it cannot be too small, in the sense that it should be larger than the average distance between particles in position. Precisely, it should satisfy $\lim_{N\to\infty} \varepsilon(N)/N^{-1/d} = +\infty$. Reference on that subject are the work of Ganguly and Victory [GV89], Wollman [Wol00] and Batt [Bat01] (the later gives a simpler proof, but valid only for larger cut-off). We also mention a result of Ganguly, Lee and Victory [GLV91] where the initial conditions are not equally distributed. However in that work the cut-off is not the same and converges very slowly to zero.
The vortices system is a numerical approximation of the 2D Euler equation (which is a mean-field equation when written in vorticity formulation), and is also similar to our

problem since the kernel as a singularity of the type $1/|x|$. Two results of convergence without regularization are already known for that system. The work of Goodman, Hou and Lowengrub, [GHL90] and [GH91], has a numerical point of view but use the true singular kernel in a interesting way. The work of Schochet [Sch96] uses the weak formulation of Delort of the Euler equation and prove the propagation of chaos for the true kernel. Unfortunately, both use the symmetry of the forces in the vortex case, a symmetry which does not exist in our kinetic problem. The force is still symmetric with respect to the space variable, but there is now a velocity variable which break the argument used in the vortices case. For a more complete description of the vortices system, we refer to the references already quoted or to [Hau09].

Our result without cut-off. Without regularization, there were not (at least to our knowledge) any results known before our previous article [HJ07] for singular force terms. This article gave a positive answer to the question of convergence in the case of forces with singularity of order $1/|x|^\alpha$, with $\alpha < 1$, but with a very restrictive assumption on the minimal distance (in phase space) on the particles at time 0. More precisely that minimal distance had of the order the average distance between a particle and its closest neighboor.
That restrictive condition did not allow a result of propagation of chaos, because it was not generic for empirical measures chosen with law $(f^0)^{\otimes N}$ (*i.e.* it was satisfied with probability 0).
In the present article, we require only a much weaker assumption on the minimal distance between particles. This allows us to prove the propagation of chaos, for forces satisfying a (S^α)-condition :

$$(S^\alpha) \qquad \forall x \in \mathbb{R}^d, \quad |F(x)| \leq \frac{C}{|x|^\alpha}\,, \quad |\nabla F(x)| \leq \frac{C}{|x|^{\alpha+1}}. \tag{1.3}$$

Our precise result without cut-off is the following

Theorem 1. *Assume that $d \geq 3$ and that F satisfies a (S^α)-condition with $\alpha < 1$. Choose any initial condition $f^0 \in L^\infty$ with compact support for the Vlasov equation* (1.2)*. For each $N \in \mathbb{N}^*$, look at the particles system* (1.1) *with initial positions $(X_i, V_i)_{i\leq N}$ chosen randomly according to the probability $(f^0)^{\otimes N}$. Then for all $T \geq 0$, all*

$$\frac{2+2\alpha}{d+\alpha} < \gamma < 1 \text{ and } \quad 0 < s < \frac{\gamma d - (2-\gamma)\alpha - 2}{2(1+\alpha)},$$

there exists positive constants $C_0(f, F)$, $C_s(\gamma, s, f, F)$ such that for N large enough

$$\mathbb{P}\left(\exists t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq \frac{3e^{C_0 t}}{N^{\gamma/(2d)}}\right) \leq \frac{C_s}{N^s},$$

where $f(t)$ is the unique strong solution of the Vlasov equation (1.2) *with initial condition f^0, (the constant C_s blows up when s approaches its maximum value) and W_1 denotes the* 1 *Monge-Kantorovitch-Wasserstein distance.*
A similar result is also true for $d = 2$, but only for forces satisfying a (S^α)-condition, with $\alpha < 1/2$.

The conditions on γ and s are not completely obvious, but it can be checked that if $\alpha < 1$ and $d \geq 3$, $\frac{2+2\alpha}{d+\alpha} < 1$ so that admissible γ exists. And for an admissible γ, the quantity $\frac{\gamma d-(2-\gamma)\alpha-2}{2(1+\alpha)}$ is also positive, so that admissible s also exists.
Roughly speaking, under the assumption of Theorem 1, the probability of finding a deviation strictly larger than the average inter-particle distance $N^{-1/2d}$ is small.

The notation $C(f)$ means that the constant depends on the function f (essentially via conserved quantities like $\|f\|_\infty$ and also the size of its support) on the whole interval of time under consideration, here $[0,T]$.
Unfortunately, we cannot precise the rank N after which the bound is satisfied. In fact, as we shall see later, the proof uses somewhere a large deviation results which is only asymptotic.

Remark 1. *This is the first result (to our knowledge) that provides propagation of chaos for a particle system with a singular force without cut-off. Unfortunately, the singularity is still not very large, because with $\alpha < 1$ the potential is continuous at the origin. But it may be a first step for the understanding of more singular potential. We refer to [BHJ10] for some ideas in that direction.*

Even if the theorem is stated probabilistically in terms of propagation of chaos, the core of the proof is a deterministic theorem (See the Theorem 3 stated in the second section) which has generic assumptions with respect to the law $(f^0)^{\otimes N}$. Thanks to the deterministic result we can also construct explicit sequences of initial conditions for which the convergence towards the Vlasov equation will holds (for instance, particles well choosen on a mesh, but not only).

The result with cut-off. The result with cut-off presented here is in one sense weaker than the previously known result [GV89], since we do not cover the critical case $\alpha = d-1$. But it has also some advantages, especially if we are not interested by numerical simulation. First of all, it is valid for random initial configurations and not only for well distributed initial positions and speeds (on a mesh). Secondly, for α larger but close to one it is valid for smaller cut-offs, much smaller than average (and also minimal) distance between particles.
The result is stated for forces depending on N and satisfying a (S_m^α)-condition

$$(S_m^\alpha) \qquad \begin{array}{ll} i) & F \text{ satisfy a } (S^\alpha) - \text{condition} \\ ii) & \forall\, |x| \geq N^{-m},\ F_N(x) = F(x) \\ iii) & \forall\, |x| \leq N^{-m},\ |F_N(x)| \leq N^{-m\alpha}, \end{array} \tag{1.4}$$

Note that in fact we would not need any estimate on the gradient of F_N for very small x. The result would still be true if F_N only converges to F for large enough x, with an error satisfying $\|F_N - F\|_1 \leq N^{-1/2d}$. The proof could be adapted to that case, but for simplicity we choose this simplified presentation. We also point out that one would like to take m as large as possible if we want to be close to the dynamics without cut-off.

Theorem 2. *Assume that $d \geq 3$, $\gamma \in (0,1)$ and that F_N satisfies a (S_m^α)-condition for some $1 \leq \alpha < d-1$ with a cut-off order m such that*

$$m < m^* := \frac{\gamma}{2d} \min\left(\frac{d-2}{\alpha - 1}, \frac{2d-1}{\alpha} \right).$$

Choose any initial condition $f^0 \in L^\infty$ with compact support for the Vlasov equation (1.2). For each $N \in \mathbb{N}^*$, look at the particles system (1.1) with initial positions $(X_i, V_i)_{n \leq N}$ chosen randomly according to the probability $(f^0)^{\otimes N}$. Then for all T there exists positive constants $C_0(f,F)$, and $C_1(\gamma, m, f, F)$ such that

$$\limsup_{N \to +\infty} \frac{1}{N^\lambda} \ln \mathbb{P}\left(\exists t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq \frac{4e^{C_0 t}}{N^{\gamma/2d}} \right) \leq -C_1,$$

where $\lambda = \min\left(1-\gamma, \frac{d-\alpha}{2d}\right)$ and $f(t)$ is the unique strong solution of the Vlasov equation. (1.2) with initial condition f^0.

In dimension $d = 3$, the minimal cut-off is given by the order of $m^* = \frac{\gamma}{6}\min((\alpha - 1)^{-1}, 5\alpha^{-1})$. As γ can be chosen very close to one, for α larger but close to one, the previous bound tells us that we can choose cut-off of order almost $N^{-5/6}$, i.e. much smaller than the likely minimal inter-particles distance in position space (of order $N^{-2/3}$, see the third section).
With such a small cut-off, one could hope that it is almost never used when we calculate the interaction forces between particles. Only a negligible number of particles will become so close to each others during the time T. This suggests that there should be some way to extend the result of convergence without cut-off at least to some $\alpha > 1$.
Unfortunately, we do not know how to make rigorous the previous probabilistic argument on the close encounters. First it is highly difficult to translate for particles system that are highly correlated. To state it properly we need infinite bounds on the 2 particles marginal. But obtaining such a bound for singular interaction seems difficult. Moreover, it remains to neglect the influence of particles that have had a close encounters (its trajectory after a encounter is not well controlled) on the other particles.
Let us also mention that astro-physicists doing gravitational simulations ($\alpha = d-1$) with tree codes usually use small cut-off parameters, lower than $N^{-1/d}$ by some order. But their scaling is usually different from ours, since they mainly work close to the vacuum. See [Deh00] for a physical oriented discussion about the optimal length of this parameter.

About the proof. The Vlasov equation (1.2) is satisfied by the empirical distribution μ_N of the interacting particle system provided that $F(0)$ is set to 0 (in the case of singular forces, it imposes a discontinuity at 0 but we nevertheless can do this hypothesis that simplify the presentation). Hence the problem of convergence can be reformulated into a problem of stability of the empirical measures - seen initially as perturbation of the smooth profile f^0 - around the solution $f(t)$ of the Vlasov equation.
The stability result is proved thanks to Grönwall estimates involving Monge-Kantorovitch-Wasserstein distances (precisely W_1 and W_∞). The estimates use as pivot a distribution $f_N(t)$ which is the solution of the Vlasov equation with a small enlargement of μ_N (Dirac masses are replaced by "blobs") as initial conditions. The distance between $f(t)$ and $f_N(t)$ is controlled in W_1 distance using a standard stability result for Vlasov equation. The distance between $\mu_N(t)$ and $f_N(t)$ is controlled in W_∞ distance thanks to careful estimates : we separate particles that are far away, particles close from each other but with large relative velocity, and finally close particles with small relative speed. We remark that the use of the infinite MKW distance is important. We were not able to perform it with other MKW distance of order $p < +\infty$. It may seem strange to propagate a stronger norm for a problem with low regularity but in fact it turns out to be the only MKW distance with which we can handle a localized singularity in the force and Dirac masses in the distribution. The "strong" distance help us to localize the singularity.
To conclude the propagation of chaos, we need some hypothesis on the initial conditions that are quite common, and also one more which is uncommon and as no physical interest : the minimal inter-particle distance in phase space.

Organization of the paper. In the next section, we introduce the notations, and state the deterministic results on which the propagation of chaos relies. In the third section, we explain how to obtain the propagation of chaos from the deterministic results. The fourth section is devoted to the proof of the two deterministic theorems.

2 Notations and other important theorems

2.1 Notations and useful results

We first need to introduce some notations and to define different quantities in order to state the result.

• **Empirical distribution μ_N and minimal inter-particle distance d_N**
Given a configuration $(X_i, V_I)_{i\leq N}$ of the particles in the phase space $\mathbb{R}^{2dN}$, the associated empirical distribution is the measure

$$\mu_N = \frac{1}{N}\sum \delta_{X_i,V_i}.$$

An important remark is that if $(X_i(t,), V_i(t))_{i\leq N}$ is a solution of the system of ODE (1.1), then the measure $\mu_N(t)$ is a solution of the Vlasov equation (1.2), provided that the interaction force satisfies $F(0) = 0$. This condition is necessary to avoid self-interaction of Dirac masses. It means that the interaction force is defined everywhere, but discontinuous and has a singularity at 0. In that conditions, the previously known results [BH77], [NW80] cannot be applied.
For every empirical measure, we define the minimal distance d_N between particles in phase-space:

$$d_N(\mu_N) = \min_{i\neq j}(|X_i - X_j| + |V_i - V_j|). \tag{2.1}$$

This is a non physical quantity, but it is crucial to control the possible concentrations of particles and we will need to bound that quantity from below.

• **Infinite MKW distance**
First, we use many times the Monge-Kantorovitch-Wasserstein distance of order one and infinite. The order one distance, denoted by W_1, is classical and we refer to the very clear book of Villani for definition and properties [Vil03]. The second one denoted W_∞ is not widely used, so we recall its definition :

Definition 1. *For two probability measures μ and ν on X, a polish space, with $\Pi(\mu,\nu)$ the set of transference plan from μ to ν:*

$$W_\infty(\mu,\nu) = \inf\{\lambda - esssup\,|x-y|\ |\lambda \in \Pi\}.$$

In one of the few works on the subject [CDPJ08] Champion, De Pascale and Juutineen prove that if μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$, then at least one optimal transference plane is given by a optimal transport map. In other words there exists a measurable map $T : X \to X$ such that $(Id, T)_\#\mathcal{L} \in \Pi$ (it implies in particular that $T_\#\mu = \nu$) and

$$W_\infty(\mu,\nu) = \text{esssup}_x\,|Tx - x|.$$

Although that is not mandatory, we will use this result that will greatly simplify the proof.
Optimal transport is useful to compare the discrete sum of the N particles dynamics to the integrals of the continuous Vlasov system. For instance, if f is a continuous distribution and μ_N an empirical distribution we may rewrite the interaction force of μ_N using a transport map $T = (T_x, T_v)$ of f onto μ_N

$$\frac{1}{N}\sum_{i\neq j} F(X_i^0 - X_j^0) = \int F(X_i^0 - T_x(y,w))f(y,w)\,dydw.$$

Note that in the equality above, the function F is singular at $x = 0$. Using infinite MKW distance, the singularity is still localized "in a ball" after the transport. The term under the integral in the right-hand-side has no singularity out of a ball of radius $W_\infty(f, \nu_N)$ in x. Others MKV distance of order p destroys that simple localization after the transport, which is why it seems more difficult to use them.

- **The scale ε.** We also introduce a scale

$$\varepsilon(N) = N^{-\gamma/2d}, \tag{2.2}$$

for some $\gamma \in (0, 1)$ to be fixed later but close enough from 1. Remark that this scale is larger than the average distance between a particle and its closest neighbor, which is of order $N^{-1/2d}$. We shall do a wide use of that scale in the sequel, and will often define quantities directly in term of ε rather than N. For instance, the cut-off order m used in the (S^α_m)-condition may be rewritten in term of ε, with $\bar{m} := \frac{2d}{\gamma} m$.

$$(S^\alpha_m) \qquad \begin{array}{ll} i) & F \text{ satisfy a } (S^\alpha) - \text{condition} \\ ii) & \forall\, |x| \geq \varepsilon^{\bar{m}},\ F_N(x) = F(x) \\ iii) & \forall\, |x| \leq \varepsilon^{\bar{m}},\ |F_N(x)| \leq \varepsilon^{-\bar{m}\alpha}, \end{array} \tag{2.3}$$

- **The solution f_N of Vlasov equation with blob initial condition.**
Now we defined a smoothing of μ_N at the scale $\varepsilon(N)$. For this, we choose a radial and decreasing kernel $\phi : \mathbb{R}^{2d} \to \mathbb{R}$ with compact support in $B_1 \times B_1$ (B_1 denoting the ball of center 0 and radius 1 of $\mathbb{R}^d$), and denote $\phi_\varepsilon(\cdot) = \varepsilon^{-2d}\phi(\cdot/\varepsilon)$. We use this to smooth μ_N and define

$$f^0_N = \mu^0_N * \phi_{\varepsilon(N)}, \tag{2.4}$$

and denote by $f_N(t, x, v)$ the solution to the Vlasov Eq. (1.2) for the initial condition f^0_N. The interest of f_N is that we may assume that it belongs to L^∞ (see the density kernel estimates of Gao [Gao03] introduced in the next subsection). It allows to use standard stability estimates to control its W_1 distance to another solution of the Vlasov equation (See Loeper result [Loe06]).

2.2 Statement of the deterministic result without cut-off

As mentioned in the introduction, the dynamic is entirely deterministic. In theorem 1 the randomness comes only from the choice of the starting initial data. Precisely, the probability on the initial conditions is used to ensure that some conditions on minimal inter-particle distances and MKV distances are satisfied with large probability. But, once that conditions are fulfilled, we are able to propagate them with deterministic estimates. The following theorem shows that the particles system may be approximated by the solution of the Vlasov equation with the "blob" distribution f^0_N as initial conditions, provided that two conditions on the minimal inter-particle distance $d_N(0)$ and the infinite norm of f^0_N are satisfied.

Theorem 3. *Assume that the interaction force F satisfies a (S^α) condition, for some $\alpha < 1$ and let $0 < \gamma < 1$. Assume also that the empirical distribution μ_N of the particles and its ε-enlargement f_N satisfy :*

i) $d^0_N := d_N(\mu_N(0)) \geq \varepsilon^{1+r} = N^{-\gamma(1+r)/2d}$ for some $r \in (1, r^*)$ where $r^* := \frac{d-1}{1+\alpha}$,

ii) $\|f^0_N\|_\infty \leq C_\infty$, a constant independent of N,

iii) For some $R > 0$, $\forall N \in \mathbb{N}$, $\mathrm{Supp}\mu_N^0 \subset B(0,R)$, the ball of radius R and center 0 of R^{2d}.

Then for any $T > 0$, there exists two constants $C_0(R, C_\infty, F, T)$ and $C_1(R, C_\infty, F, \gamma, r, T)$ such that for $N \geq e^{C_1 T}$ the following estimate is true

$$\forall t \in [0,\ T], \quad W_\infty(\mu_N(t), f_N(t)) \leq \frac{e^{C_0 t}}{N^{\gamma/2d}}. \tag{2.5}$$

Remark. This is a inequality of the type $W_\infty(t) \leq W_\infty(0)e^{Ct}$, where the value of $W_\infty(0)$ has been bounded by $N^{-\gamma/2d}$. But that last bound is true since f_N^0 is a blob approximation of μ_N, with blob contained in balls of radius $N^{-\gamma/2d}$ around the Dirac of μ_N^0.

2.3 Statement of the deterministic result with cut-off

As in the case without cut-off, the probabilistic result 2 relies on a deterministic result, much simpler with cut-off since it does not need any control on the minimal inter-particles distance. The result is the following

Theorem 4. *Assume that $d \geq 3$ and that the interaction force $F = F_N$ satisfies a (S_m^α), for some $1 < \alpha < d-1$, with a cut-off order satisfying*

$$m < m^* := \frac{\gamma}{2d} \min\left(\frac{d-2}{\alpha - 1}, \frac{2d-1}{\alpha}\right).$$

Assume also that the initial empirical distribution of the particles μ_N^0 and its ε enlargement f_N satisfy :

i) $\|f_N^0\|_\infty \leq C_\infty$, a constant independent of N,

ii) For some $R > 0$, $\forall N \in \mathbb{N}$, $\mathrm{Supp}\mu_N^0 \subset B(0,R)$, the ball of radius R and center 0 of R^{2d}.

Then for any $T > 0$, there exists two constants $C_0(R, C_\infty, F, T)$ and $C_1(R, C_\infty, F, \gamma, r, T)$ such that for $N \geq e^{C_1 T}$ the following estimate is true

$$\forall t \in [0,\ T], \quad W_\infty(\mu_N(t), f_N(t)) \leq \frac{e^{C_0 t}}{N^{\gamma/2d}}. \tag{2.6}$$

Theorem 4 result has also an interest for numerical simulation because one obvious way to fulfill the hypothesis on the infinite norm of f_N^0 is is to put particles initially on a mesh (with a grid length of $N^{-1/2d}$ in $\mathbb{R}^{2d}$). In that case, the result is even valid with $\gamma = 1$.

3 From deterministic results (Theorem 3 and 4) to propagation of chaos.

The assumptions made in Theorem 3 may seem a little bit strange, but they are in some sense generic, when the initial positions and speed are choosen with the law $(f^0)^{\otimes N}$. Therefore, to prove Theorem 1 form Theorem 3, we need to

- Obtain a bound on the W_1 distance between $f(t)$ and $f_N(t)$, which are two solutions of the Vlasov equation.

- Estimate the probability that empirical measure chosen with the law $(f^0)^{\otimes N}$, do not satisfy the conditions i) and ii) of the deterministic theorem 3, and are far away from f^0 in W_1 distance (the last conditions is important for the previous point on the distance between f and f_N).

For these two points, we will use known results detailed in the next two sections. After that, a good choice of the parameter γ and r will allow us to conclude the proof.

3.1 Stability around solution of the Vlasov equation.

The following result is proved in [Loe06] for $\alpha = d-1$, but its proof may be adapted to our less singular case (The adaptation is done in [Hau09] in the Vortex case)

Proposition 1 (From Loeper)**.** *If f_1 and f_2 are two solutions of Vlasov Poisson equations with different kernel K_1 and K_2 both satisfying a (S^α)-condition, with $\alpha < d-1$, then*

$$\frac{d}{dt}W_1(f_1(t), f_2(t)) \leq C\max(\|\rho_1\|_\infty, \|\rho_1\|_\infty)\, W_1(f_1(t), f_2(t)) + C\|\rho_1\|_\infty \|K_1 - K_2\|_1$$

The bound on the density may be obtained in our case with the argument of Pfaffelmoser for solution with compact support [Pfa92] (It is even simpler for $\alpha < 1$ as it is explained in the appendix of [HJ07]).
Using that theorem in the case without cut-off ($K_1 = K_2 = F$), with $\alpha < 1$ (for $d \geq 3$) and a $\|f_N^0\|$ compactly supported, we obtain that there exists a constant C_0 depending on F, an uniform bounds on the infinite norms of the f_N and the size of their supports (denoted C_∞ and R in Theorem 3), such that

$$W_1(f(t), f_N(t)) \leq e^{C_0 t} W_1(f^0, f_N^0) \leq e^{C_0 t}\left(W_1(f^0, \mu_N^0) + N^{-\gamma/2d}\right), \tag{3.1}$$

.

3.2 Estimates in probability on the initial distribution.

The bound ii) of the deterministic theorem 3. A result in the theory of density kernel estimates by Gao [Gao03] shows that this bound is satisfied almost surely in the limit of large N, if the positions and velocities of the initial particles distribution μ_N^0 are chosen randomly and independently according to the law f^0. As we shall only use a small part of the Gao's results, we present in the the following proposition only what we need (with our notations) and explain in the appendix how to obtain it from Gao's results.

Proposition 2 (From Gao)**.** *Assume that ϕ is bounded, radial, decreasing. Then, with the previous notations*

$$\lim_{N\to\infty} \frac{1}{N^{1-\gamma}} \ln \mathbb{P}\left(\|f_N^0\|_\infty \geq L\|f^0\|_\infty\right) = -\|f^0\|_\infty I_\phi(L)$$

with $I_\phi(L) = \sup_{t\in\mathbb{R}}\left\{Lt + \int_{\mathbb{R}^d}[1 - e^{t\phi(x)}]\,dx\right\} > 0$.

Remark 2. *To get a quantitative version of Theorem 1 (in which we can at least precise the rank N after which the inequality is true), one would need a quantitative version of the previous proposition 2. Unfortunately, this is not available (at least to our knowledge), and seems more difficult than the asymptotic result. This is why we cannot precise the rank N after which the estimate of Theorem 1 becomes true. However, under the additionnal assumption that f^0 and ϕ are Lipschitz, Bolley, Guillin and Villani obtained in [BGV07] quantitative concentration inequality for f^N in infinite norm. But unfortunately, they can be used in our setting because they requires to large smoothing paramater in order to give precise results.*

Deviations for the minimal inter-particle distance. It may be proved with simple arguments that the scale η_m is almost surely larger than $N^{-1/d}$ when $f^0 \in L^\infty$. A precise result is stated in the Proposition below, proved in [Hau09]:

Proposition 3. *There exists a constant c_{2d} depending only on the dimension such that if $f^0 \in L^\infty(\mathbb{R}^{2d})$, then*

$$\mathbb{P}\left(d_N(Z) \geq \frac{l}{N^{1/d}}\right) \geq e^{-c_{2d}\|f^0\|_\infty l^d}.$$

Be careful that the inequalities are in the bad sense and that this is not a large deviation result. It is that condition that prevent us to obtain a "large deviation" result in Theorem 1 (contrarily to the cut-off case of Theorem 2). In fact, the only bound it provides on the "bad" set is

$$\mathbb{P}\left(d_N(Z) \leq \frac{l}{N^{1/d}}\right) \leq 1 - e^{-c_{2d}\|f^0\|_\infty l^d} \leq c_{2d}\|f^0\|_\infty l^d.$$

With the notation of Theorem 3 it comes that if $s = \gamma\frac{1+r}{2} - 1 > 0$ then

$$\mathbb{P}\left(d_N(Z) \leq \varepsilon^{1+r}\right) = \mathbb{P}\left(d_N(Z) \leq \frac{N^{-s/d}}{N^{1/d}}\right) \leq c_{2d}\|f^0\|_\infty N^{-s}. \tag{3.2}$$

Deviations for the W_1 MKW distance. Peyre has obtained in [Pey] the following result

Proposition 4 (Peyre)**.** *If $d \geq 2$, and the empirical measures μ_N^0 are chosen according to the law $(f^0)^{\otimes N}$, then there exists an explicit constant L_d such that*

$$\mathbb{P}\left(W_1(\mu_N^0, f^0) \geq \frac{L}{N^{1/(2d)}}\right) \leq e^{L_d - LN^{\frac{d-1}{2d}}}. \tag{3.3}$$

It will particularly interests us when $L = N^{\frac{1-\gamma}{2d}}$, in which case it maybe rewritten

$$\mathbb{P}\left(W_1(\mu_N^0, f^0) \geq \varepsilon\right) \leq Ce^{-N^{\frac{d-\gamma}{2d}}}, \quad \text{with} \quad C = e^{L_d}. \tag{3.4}$$

3.3 Conclusion

Now take the assumptions of Theorem 1. It means that we assume that F satisfies a (S^α) condition for $\alpha < 1$ and $f^0 \in L^\infty$ for $d > 3$. We chose

$$\gamma \in \left(\frac{2+2\alpha}{d+\alpha}, 1\right), \text{ and } \quad r \in \left(\frac{2}{\gamma} - 1, r^* = \frac{d-1}{1+\alpha}\right).$$

(The condition on γ ensures that the second interval is non empty). We also define

$$s := \gamma\,\frac{1+r}{2} - 1 > 0, \quad \lambda = \min\left(1-\gamma, \frac{d-\gamma}{2d}\right)$$

Denote by ω_1, ω_2 the sets of initial conditions s.t. respectively (i), and (ii) (with $C_\infty = 2\|f^0\|_\infty$) of Theorem 3 hold and ω_3 s.t. $W_1(\mu_N, f^0) \leq \frac{1}{N^{\gamma/(2d)}}$.

$$\omega_1 := \{Z \text{ s.t. } d_N(Z) \geq \varepsilon^{1+r}\}, \quad \omega_2 := \{Z \text{ s.t. } \|f_N^0\|_\infty \leq 2\|f^0\|_\infty\}$$
$$\omega_3 := \{Z \text{ s.t. } W_1(\mu_N^0, f^0) \leq \varepsilon\}$$

By the results stated in the previous section, one knows that

$$\mathbb{P}(\omega_1^c) \leq C\, N^{-s}, \quad \mathbb{P}(\omega_3^c) \leq \quad Ce^{-N^{\frac{d-\gamma}{2d}}}, \quad \lim_{N\to\infty} \frac{1}{N^{1-\gamma}} \ln \mathbb{P}(\omega_2^c) = -C_1 := I_\phi(2) < 0.$$

Denote $\omega = \omega_1 \cap \omega_2 \cap \omega_3$. Hence $|\omega^c| \leq |\omega_1^c| + |\omega_2^c| + |\omega_3^c|$ and for N large enough

$$\mathbb{P}(\omega^c) \leq C\, N^{-s} + C\, e^{-\min(1,C_1)\, N^{-\lambda}} \leq C\, N^{-s} \tag{3.5}$$

If the initial conditions belong to ω then one may apply Theorem 3 and get on $[0,\ T]$

$$W_1(f_N, \mu_N) \leq W_\infty(f_N, \mu_N) \leq \frac{e^{C_0 t}}{N^{\gamma/(2d)}}.$$

Now apply the stability around solution of Vlasov equation given by (3.1) and get

$$W_1(f, f_N) \leq W_1(f^0, f_N^0)\, e^{C_0 t} \leq \frac{2}{N^{\gamma/(2d)}}\, e^{C_0 t}.$$

The factor 2 comes from the fact that $W_1(f^0, f_N^0) \leq W_1(f^0, \mu_N^0) + W_1(\mu_N^0, f_N^0)$. We conclude that

$$W_1(f, \mu_N) \leq \frac{3}{N^{\gamma/2d}} e^{C_0 t},$$

which proves that

$$\mathbb{P}(\omega) \leq \mathbb{P}\left(\forall t \in [0,T],\ W_1(f, f_N) \leq \frac{3e^{C_0 t}}{N^{\gamma/d}}\right).$$

The bound 3.5 then gives Theorem 1.

3.4 From Theorem 4 to Theorem 2

In the cut-off case, one can derive Theorem 2 from Theorem 4 in the same manner. As we do not use the minimal distance in that case, the proof is simpler in the case $\alpha < d-1$ and we get a stronger convergence result. The only difference is that we shall use the Theorem 1 with $K_1 = F$ and $K_2 = F_N$, so that an error term appears. But that error term is bounded by

$$C\|\rho_f\|_\infty \|F - F_N\|_1 \leq C\varepsilon^{d-\alpha} \leq C W_\infty(t)$$

for any t so that the proof is unchanged. In fact, with the same λ, we obtain that

$$\lim_{N\to+\infty} \frac{1}{N^\lambda} \ln \mathbb{P}\left(\exists t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq \frac{4e^{C_0 t}}{N^{\gamma/(2d)}}\right) = -C_1.$$

4 Proof of Theorem 3

4.1 Definition of the transport

We try now to compare the the dynamics of μ_N and f_N, two distributions which have a compact support. For that, we choose an optimal transport T^0 from f_N^0 to μ_N^0 for the infinite MKW distance (See the remark after Definition 1). The existence of such a transport is ensured by [CDPJ08]. Since both f_N^0 and μ_N^0 are compactly supported, note that $T^0(z)$ is defined for $|z| \leq R^0$ (the size of the support). Since f_N^0 is an ε-enlargement of μ_N^0, it is clear that $W_\infty(f_N^0, \mu_N^0) \leq \varepsilon$.

We also denote by $Z^f = (X^f, V^f)$ the smooth flow associated to f_N and by $Z^N = (X^N, V^N)$ the flow of the N particles system (with the convention $Z(t,s)$ transport from time s to time t). A simple way to get a transport of $f_N(t)$ on $\mu_N(t)$ is to transport along the flows the map T^0, i.e. to define

$$T^t = Z^N(t,0) \circ T^0 \circ Z^f(0,t), \qquad \text{and} \quad T^t = (T^t_x, T^t_v)$$

We use the following notation, for a test-"particle" of the continuous system at the position $z_t = (x_t, v_t)$ at time t, $z_s = (x_s, v_s)$ will be its position at time s for $s \in [t-\tau, t]$. Precisely

$$z_s = Z^f(s,t,z_t)$$

Since f_N is the solution of a transport equation, we have $f_N(t,z_t) = f_N(s,z_s)$. And since the vector-field of that transport equation is divergence free

$$\int \Phi(z)\, f_N(s,z)\, dz = \int \Phi(Z^f(s,t,z))\, f_N(t,z)\, dz = \int \Phi(z_s)\, f_N(t,z_t)\, dz_t.$$

Finally let us remark that f_N is a solution to the continuous Vlasov equations with an initial L^∞ norm and support that are uniformly bounded in N. Therefore this remains true uniformly in N for any finite time. In particular there exists a constant C independent of N such that for any $t \in [0,\ T]$

$$\begin{aligned}
&\|f_N(t,.,.)\|_\infty \le C,\ \|f_N(t,.,.)\|_{L^1} = 1,\\
&|E(t,x)| \le \int |F(x-y)|\, f_N(t,y,w)\, dy\, dw \le C\\
&|\nabla E(t,x)| \le \int |\nabla F(x-y)|\, f_N(t,y,w)\, dy\, dw| \le C\\
&\text{supp}\, f_N(t,.,.) \in B(0,R(t)), \quad R(t) \le C,
\end{aligned} \tag{4.1}$$

as of course $R(t) \le R^0 + \int_0^t \|E(s,.)\|_\infty\, ds$. This is always true for $\alpha < 1$. In dimension $d \le 3$ it remains true for $\alpha < d-1$ and even $\alpha = d-1$. In fact, all that estimates where central in the work of see Pfaffelmöser [Pfa92] about existence and uniqueness of compactly supported solution of Vlasov-Poisson equation (See also [Hor93] for a result with improved bounds). The proofs can be adapted to our simpler cases (See the Appendix of [HJ07] for the case $\alpha < 1$).
In dimension $d > 3$ and for attractive forces with $1 < \alpha < d-1$, there can be a blow-up in finite time (for α larger than a critical value depending on the dimension). In that case, we simply restrict ourselves to a time interval on which this does not occur.

In what follows, the final time T is fixed and independent of N. For simplicity, C will denote a generic universal constant, which may actually depend on T, the size of the initial support, the infinite norms of the f_N... But those constants are always independent of N as in (4.1).

4.2 The quantities to control

We will not be able to control the infinite norm of the field (and its derivative) created by the empirical distribution, but only a small temporal average of this norm. For this, we introduce in the case without cut-off a small time step $\tau = \varepsilon^{r'}$ for some $r' > r$ and close to r (the precise condition will appear later).
In the case with cut-off where r and r' are useless, the time step will by $\tau = \varepsilon$.

- **The MKW infinite distance between $\mu_N(t)$ and $f(t)$.**

  We of course wish to bound $W_\infty(t) := \sup_{0\le s\le t} W_\infty(\mu_N(s), f_N(s))$ (note that W_∞ is hence automatically non decreasing). For the transport introduced before, one has

$$W_\infty(t) \le \sup_{s\le t} \sup_{(x_s,v_s)\in \mathrm{supp}\, f_N(s,.,.)} |T^s(x_s,v_s) - (x_s,v_s)|.$$

  In fact, we will provide bound for the quantity of the right hand side. Our result maybe stated for that quantity, rather than the infinite MKW distance. It is a little stronger, since it means that rearrangement in the transport are not necessary to keep the MKW distance bounded. The transport chosen at time $t = 0$ is preserved during the time.

- **The support of μ_N**

  We shall also need a uniform control on the support in position and velocity of the empirical distributions :

$$R^N(t) = \max_i |(X_i(t),\ V_i(t))|.$$

  But using the infinite MKW distance, it is clear that $R^N(t) \le R(t) + W_\infty(t)$. We will do a wide use of this control in the following.

- **The infinite norm $|E^N|_\infty$ of the time averaged discrete force field.**

  We also define the average of the discrete force field over small time intervals of length τ (the dependence on t is implicit)

$$|E_N|_\infty = \sup_i \frac{1}{\tau} \int_{t-\tau}^{t} |E_N(X_i(s))|\, ds.$$

- **The infinite norm $|\nabla^N E|_\infty$ of the time averaged discrete derivative of the force field.**

  We also define a version of the infinite norm of its averaged derivative

$$|\nabla^N E|_\infty = \sup_{i\neq j} \frac{1}{\tau} \int_{t-\tau}^{t} \frac{|E_N(X_i(s)) - E_N(X_j(s))|\, ds}{|X_i(s) - X_j(s)| + \varepsilon^{(1+r')}}\, ds.$$

  For both E_N and $\nabla^N E$, we use the convention that when the interval of integration contain 0 (for $t < \tau$), the integrand is null on the left side.

- **The minimal distance in phase space d_N**

  which has already be defined by the equation (2.1) in the Section 2.

- **Two useful integrals $I_\alpha(t, z_t)$ and $J_{\alpha+1}(t, z_t)$**

  Finally the technical computations involve

$$I_\alpha(t, \bar z_t, z_t) = \frac{1}{\tau} \int_{t-\tau}^{t} |F(T_x^s(\bar z_s) - T_x^s(z_s)) - F(\bar x_s - x_s)|\, ds.$$

  Defining a second kernel as

$$K_\varepsilon = \min\left(\frac{1}{|x|^{1+\alpha}},\ \frac{1}{\varepsilon^{1+r'}\,|x|^\alpha}\right),$$

we define a second useful quantity

$$J_{\alpha+1}(t,\bar z_t,z_t)=\frac{1}{\tau}\int_{t-\tau}^{t}K_\varepsilon(|T_x^s(\bar z_s)-T_x^s(z_s)|\,ds \\ =\frac{1}{\tau}\int_{t-\tau}^{t}K_\varepsilon(|X_i(s)-X_j(s)|\,ds,$$

if i and j is the indices such that $Z_i(t)=T^t(\bar z_t)$ and $Z_j(t)=T^t(z_t)$.

All previous quantities are relatively easily bounded by I_α and $J_{\alpha+1}$. Those last two will not be bounded by direct calculation on the discrete system, but we compare them to similar ones for the continuous system, paying for that in terms of the distance between $\mu_N(t)$ and $f(t)$. That strategy is interesting because the integrals are easier to manipulate than the discrete sums.
We summarize the first easy bounds in the following

Proposition 5. *Under the assumptions of Theorem 3, one has for some constant C uniform in N*

$$(i)\quad R_N(t)\le W_\infty(t)+R(t)\le W_\infty(t)+C,$$
$$(ii)\quad W_\infty(t)\le W_\infty(t-\tau)+C\,\tau\sup_{\bar z_t}\int_{|z_t|\le R(t)}I_\alpha(t,\bar z_t,z_t)\,dz_t,$$
$$(iii)\quad |\nabla^N E|_\infty\le C\sup_{\bar z_t}\int_{|z_t|\le R(t)}J_{\alpha+1}(t,\bar z_t,z_t)\,dz_t.$$
$$(iv)\quad d_N(t)+\varepsilon^{1+r'}\ge[d_N(t-\tau)+\varepsilon^{1+r'}]e^{-\tau(1+|\nabla^N E|_\infty(t))}.$$

Note that the control on $R_N(t)$ is simple enough that it will actually be used implicitly in the rest many times, and that the iv) is a simple consequence of the iii). In fact, in that proposition the crucial estimates are the ii) and iii).
Remark also that in the case of very singular interaction force ($\alpha\ge 1$) with cut-off - in short (S_m^α) conditions - the control on minimal distance d_N and therefore the control on $|\nabla^N E|_\infty$ are useless, so that the only interesting inequality is the second one.

4.3 Proof of Prop. 5

Let us start with (i). Simply write

$$R^N(t)=\sup_{z_t\in\operatorname{supp} f_N(t,\cdot)}|T^t(z_t)|\le\sup_{z_t\in\operatorname{supp} f_N(t,\cdot)}|T^t(z_t)-z_t|+\sup_{z_t\in\operatorname{supp} f_N(t,\cdot)}|z_t|,$$

So indeed by (4.1)

$$R^N(t)\le W_\infty(t)+R(t)\le W_\infty(t)+C.$$

As for (ii), simply differentiate in time W_∞ to find

$$\frac{W_\infty(t)-W_\infty(t-\tau)}{\tau}\le\sup_{\bar z_t}\int\left|\frac{1}{\tau}\int_{t-\tau}^{t}[F(T_x^s(\bar z_s)-T_x^s(z_s))-F(\bar x_s-x_s)]\,ds\right|f_N(z_t)\,dz_t.$$

Since f_N is uniformly bounded in L^∞ and compactly supported in $B(0,R(t))$, one gets by the definition of I_α

$$\frac{W_\infty(t)-W_\infty(t-\tau)}{\tau}\le\|f^0\|_\infty\sup_{\bar z_t}\int_{|z_t|\le R(t)}I_\alpha(t,\bar z_t,z_t)\,dz_t,$$

which is exactly (ii).

Concerning $|\nabla^N E|_\infty$ in (iii), noting that

$$\int_{t-\tau}^{t} \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} = \frac{1}{N} \sum_{k \neq i,j} \int_{t-\tau}^{t} \frac{|F(X_i(s) - X_k(s)) - F(X_j(s) - X_k(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \, ds. \\ + \frac{1}{N} \int_{t-\tau}^{t} \frac{|F(X_i(s) - X_j(s)) - F(X_j(s) - X_i(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \, ds$$

By the assumption (1.3), one has that

$$|F(x) - F(y)| \leq C \left(\frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}} \right) |x - y|.$$

So

$$\frac{|F(X_i(s) - X_k(s)) - F(X_j(s) - X_k(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \leq \frac{C}{|X_i(s) - X_k(s)|^{1+\alpha}} + \frac{C}{|X_j(s) - X_k(s)|^{1+\alpha}},$$

and that bound is also true for the remaining term where $k = i$ or j. One also obviously has, still by (1.3)

$$\frac{|F(X_i(s) - X_k(s)) - F(X_j(s) - X_k(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \leq \frac{C}{\varepsilon^{1+r'} |X_i(s) - X_k(s)|^{\alpha}} \\ + \frac{C}{\varepsilon^{1+r'} |X_j(s) - X_k(s)|^{\alpha}}.$$

Therefore by the definition of K_ε

$$\frac{|F(X_i(s) - X_k(s)) - F(X_j(s) - X_k(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \leq K_\varepsilon(X_i(s) - X_k(s)) + K_\varepsilon(X_j(s) - X_k(s)).$$

Summing up, this implies that

$$|\nabla^N E|_\infty \leq C \max_{i \neq j} \Big(\frac{1}{\tau} \int_{t-\tau}^{t} \frac{1}{N} \sum_{k \neq i} K_\varepsilon(X_i(s) - X_k(s)) \, ds \\ + \frac{1}{\tau} \int_{t-\tau}^{t} \frac{1}{N} \sum_{k \neq j} K_\varepsilon(X_j(s) - X_k(s)) ds \Big).$$

Transforming the sum into integral thank to the transport, we get exactly the bound (iii) involving $J_{\alpha+1}$.

Finally for $d_N(t)$, consider any $i \neq j$, then obviously

$$\frac{d}{ds} |(X_i(s) - X_j(s),\ V_i(s) - V_j(s))| \geq -|V_i(s) - V_j(s)| - |E_N(X_i(s)) - E_N(X_j(s))|.$$

Simply write

$$|E_N(X_i(s)) - E_N(X_j(s))| \leq \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \left(|X_i(s) - X_j(s)| + \varepsilon^{1+r'} \right)$$

to obtain that

$$\frac{d}{ds} |(X_i(s) - X_j(s),\ V_i(s) - V_j(s))| \geq - \left(1 + \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} \right) \\ \left(|(X_i(s) - X_j(s),\ V_i(s) - V_j(s))| + \varepsilon^{1+r'} \right).$$

Integrating this inequality and taking the minimum, we get

$$d_N(t)+\varepsilon^{1+r'} \geq (d_N(t-\tau)+\varepsilon^{1+r'}) \inf_{i\neq j} \exp\left(-\tau - \int_{t-\tau}^{t} \frac{|E_N(X_i(s))-E_N(X_j(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}}\,ds\right)$$
$$\geq [d_N(t-\tau)+\varepsilon^{1+r'}] \exp^{-\tau(1+|\nabla^N E|_\infty(t))}.$$

4.4 The bound for I_α and $J_{\alpha+1}$

To close the the system of inequality of the Proposition 5, it remains to bound the two integrals involving I_α and J_α. It is done with the following lemmas

Lemma 1. *Assume that F satisfies an (S^α)-condition with $\alpha < 1$, and that τ is small enough such that*

$$C\,\tau\,(1+|\nabla^N E|_\infty(t))\,(W_\infty(t)+\tau) \leq d_N(t).$$

Then one has the following bounds, uniform in $\bar{z}_t$

$$\int_{|z_t|\leq R(t)} I_\alpha(t,\bar{z}_t,z_t)\,dz_t \leq C\left[W_\infty(t)+(W_\infty(t)+\tau)^d\tau^{-\alpha}+(W_\infty(t)+\tau)^{2d}(d_N(t))^{-\alpha}\tau^{-\alpha}\right].$$

$$\int_{|z_t|\leq R(t)} J_{\alpha+1}(t,\bar{z}_t,z_t)\,dz_t \leq C\left(1+(W_\infty(t)+\tau)^d\varepsilon^{-(1+r')}\,\tau^{-\alpha}\right.$$
$$\left.+\,(W_\infty(t)+\tau)^{2d}\varepsilon^{-(1+r')}\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}\right).$$

In the cut-off case where the interaction force satisfy a (S^α_m) condition (we recall that it means that the cut-off is of size $N^{-m}=\varepsilon^{\bar{m}}$), we only need to bound the integral of I_α, with the result

Lemma 2. *Assume that $1\leq\alpha<d-1$, and that F satisfies a (S^α_m) condition, one as the following bound, uniform in $\bar{z}_t$*

$$\int_{|z_t|\leq R(t)} I_\alpha(t,\bar{z}_t,z_t)\,dz_t \leq C\left(W_\infty(t)+(W_\infty(t)+\tau)^d\tau^{-1}\varepsilon^{\bar{m}(1-\alpha)}+(W_\infty(t)+\tau)^{2d}\varepsilon^{-\bar{m}\alpha}\right). \quad (4.2)$$

with the convention (if $\alpha=1$) that $(\varepsilon^{\bar{m}})^0=|\ln(\varepsilon^{\bar{m}})|$ [1].

The proofs with or without cut-off follow the same line and we will prove the above lemmas at the same time. We begin by an explanation of the sketch of the proof, and then perform the technical calculation.

4.4.1 Rough sketch of the proof

The point $\bar{z}_t=(\bar{x}_t,\bar{v}_t)$ is considered fixed through all this subsection (as the integration is carried over $z_t=(x_t,v_t)$). Accordingly we decompose the integration in z_t over several domains. First

$$A_t=\{z_t\,|\,|\bar{x}_t-x_t|\geq 4W_\infty(t)+2\tau(|\bar{v}_t-v_t|+\tau|E|_\infty(t))\,\}.$$

[1] That convention may be justified by the fact that it implies a very simple algebra $(x^{1-\alpha})'\approx x^{-\alpha}$ even if $\alpha=1$. It allows us to give an unique formula rather than three different cases.

This set consist of points z_t such that x_s and $T_x^s(z_s)$ are sufficiently far away from $\bar{x}_s$ on the whole interval $[t-\tau, t]$, so that they will not see the singularity of the force. The bound over this domain will be obtained using traditional estimates for convolutions.
One part of the integral can be estimated easily on A_t^c (the part corresponding to the flow of the regular solution f_N to the Vlasov equation). For the other part it is necessary to decompose further. The next domain is

$$B_t = A_t^c \bigcap \{z_t \,|\, |\bar{v}_t - v_t| \geq 4\, W_\infty(t) + 4\,\tau |E|_\infty|(t)\}.$$

This contains all particles z_t that are close to $\bar{z}_t$ in position (i.e. x_t close to $\bar{x}_t$), but with enough relative velocity not to interact too much. The small average in time will be useful in that part, as the two particles remains close only a small amount of time.
The last part is of course the remainder

$$C_t = (A_t \cup B_t)^c.$$

This is a small set, but where the particles remains close together a relatively long time. Here, we are forced to deal with the corresponding term at the discrete level of the particles. This is the only term which requires the minimal distance in phase space; and the only term for which we need a time step τ small enough as per the assumption in Lemma 1.

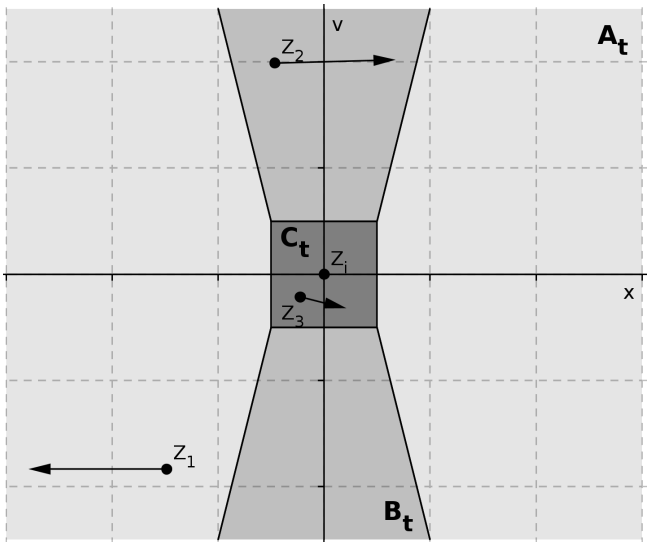

Figure 1: The partition of the phase space.

4.4.2 Step 1: Estimate over A_t

If $z_t \in A_t$, we have for $s \in [t-\tau, t]$

$$|\bar{x}_s - x_s| \geq |\bar{x}_t - x_t| - (t-s)|\bar{v}_t - v_t| - (t-s)^2 |E|_\infty(t) \geq \frac{|\bar{x}_t - x_t|}{2} \quad (4.3)$$

$$|T_x^s(\bar{z}_s) - T_x^s(z_s)| \geq |\bar{x}_s - x_s| - 2W_\infty(s)| \geq \frac{|\bar{x}_t - x_t|}{2}. \quad (4.4)$$

For I_α, we use the direct bound for $\bar{z}_t \in A_t$

$$\begin{aligned} |F(T_x^s(\bar{z}_s) - T_x^s(z_s)) - F(\bar{x}_s - x_s)| &\leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} (|T_x^s(\bar{z}_s) - \bar{x}_s| + |T_x^s(z_s) - x_s|) \\ &\leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(s) \leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(t), \end{aligned}$$

and obtain by integration on $[t-\tau, t]$

$$I(t, \bar{z}_t, z_t) \leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(t).$$

Then integrating in z_t we may get since $\alpha + 1 < 2 \leq d$

$$\begin{aligned} \int_{A_t} I_\alpha(t, \bar{z}_t, z_t)\, dz_t &\leq C W_\infty(t) \int_{A_t} \frac{dz_t}{|\bar{x}_t - x_t|^{1+\alpha}} \\ &\leq C\, R(t)^{2d-1-\alpha}\, W_\infty(t) \leq C\, W_\infty(t). \end{aligned} \tag{4.5}$$

For $J_{\alpha+1}$, we have using (4.4) on the set A_t the bound

$$|K_\varepsilon(T_x^s(\bar{z}_t) - T_x^s(z_t))| \leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}}.$$

Integrating with respect to time and z_t we get since $1 + \alpha < d$.

$$\begin{aligned} \int_{A_t} J_{\alpha+1}(t, \bar{z}_t, z_t)\, dz_t &\leq C \int_{A_t} \frac{dz_t}{|\bar{x}_t - x_t|^{1+\alpha}} \\ &\leq C\, R(t)^{d-1-\alpha} \leq C. \end{aligned} \tag{4.6}$$

For the cut-off case, the estimation on I_α for this step is unchanged.

4.4.3 Step 1' : Estimate over A_t^c for the continuous part of I_α.

For the remaining term in I_α, we use the rude bound

$$|F(T_x^s(\bar{z}_s) - T_x^s(z_s)) - F(\bar{x}_s - x_s)| \leq |F(T_x^s(\bar{z}_s) - T_x^s(z_s))| + |F(\bar{x}_s - x_s)|.$$

The term involving T^s is complicated and requires the additional decompositions. It will be treated in the next sections. The other term is simply bounded by

$$\begin{aligned} \int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^t |F(\bar{x}_s - x_s)| ds\; dz_t &\leq \frac{1}{\tau} \int_{t-\tau}^t \int_{z_t \in A_t^c} \frac{C\, dz_t}{|\bar{x}_s - x_s|^\alpha}\, ds \\ &\leq \frac{1}{\tau} \int_{t-\tau}^t \int_{z_s \in Z^f(s,t,A_t^c)} \frac{C\, dz_s}{|\bar{x}_s - x_s|^\alpha}\, ds. \end{aligned}$$

¿From the bound $R(t)$ and $|E|_\infty(t)$ we see that

$$|A_t^c| \leq C R(t)^d (W_\infty(t) + \tau)^d \leq C (W_\infty(t) + \tau)^d.$$

Since the flow Z_f is measure preserving, the measure of the set $Z^f(s, t, A_t^c)$ satisfies the same bound. This set is also included in $[-R(t), R(t)]^{2d}$ (if R is increasing, a property that we may assume). We use the above lemma which implies that above all the set $Z(s, t, A_t^c)$, the integral reaches is maximum when the set is a cylinder

Lemma 3. *Let $\Omega \subset B(0, R) \subset \mathbb{R}^n$. Let P be a projection from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m \leq n$. Then for any $a < m$*

$$\int_\Omega \frac{dx}{|Px|^a} \leq C_a R^{a(n-m)/m}\, |\Omega|^{1-a/m}.$$

Proof of Lemma 3. We can freely assume that $Px = (x_1, \dots, x_m)$. Now maximize the integral

$$\int_\omega |Px|^{-a} dx$$

over all sets $\omega \subset \mathbb{R}^n$ satisfying $\omega \subset B(0,R)$ and $|\omega| = |\Omega|$. It is clear that the maximum is obtained by concentrating as much as possible ω near $Px = 0$, *i.e.* with a cylinder of the form $B_m(0,r) \times B_{n-m}(0,R)$ where B_k denotes the k-dimensional ball. Since $|\omega| = |\Omega|$ we have $r = |\Omega|^{1/m} R^{1-n/m}$. The integral can now be computed explicitly and gives the lemma. □

Applying the lemma, we get

$$\int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^t |F(\bar{x}_s - x_s)| \, dz_t ds \leq C(W_\infty(t) + \tau)^{d-\alpha}. \tag{4.7}$$

That term do not appear in Lemma 1 since it is strictly smaller than the bound of the remaining term (involving T), as we shall see in the next section.
For the cut-off case, we do not need the estimate on $J_{\alpha+1}$ and the bound in this case is similar since $\alpha \leq d-1 < d$. The cut-off cannot in fact help to provide a better bound for this term.

At this point, the remaining term to bound in I_α is only

$$\int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^t |F(T_x^s(\bar{z}_s) - T_x^s(z_s))| \, ds \leq C \int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^t \frac{dz_t}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha} \, ds. \tag{4.8}$$

For $J_{\alpha+1}$ one may bound the integral of the continuous part on A_t^c in a similar manner. As $K_\varepsilon \leq \frac{1}{\varepsilon^{1+r'}|x|^\alpha}$, the remainder can be controlled by (4.8)

$$\int_{A_t^c} J_{\alpha+1}(t, \bar{z}_t, z_t) \, dz_t \leq C \, (W_\infty(t) + \tau)^{d-\alpha} + \frac{C}{\varepsilon^{1+r'}} \int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^t \frac{dz_t}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha} \, ds. \tag{4.9}$$

Therefore in the next sections we focus on giving a bound for (4.8).

4.4.4 Step 2: Estimate over B_t

If $z_t \in B_t$, we have, as for A_t, for $s \in [t-\tau, t]$

$$|\bar{v}_s - v_s - \bar{v}_t + v_t| \leq 2\tau |E_\infty|(t) \leq \frac{|\bar{v}_t - v_t|}{2}, \tag{4.10}$$

$$|T_v^s(\bar{z}_s) - T_v^s(z_s) - \bar{v}_t + v_t| \leq |\bar{v}_s - v_s - \bar{v}_t + v_t| + 2W_\infty(s)| \leq \frac{|\bar{v}_t - v_t|}{2}. \tag{4.11}$$

This means that the particles involved are close to each others (in the positions variables), but with a sufficiently large relative velocity, so that they do not interact a lot on the interval $[t-\tau, t]$.
First we introduce a notation for the term of (4.8)

$$\int_{z_t \in B_t} I_{bc}(t, \bar{z}_t, z_t) \, dz_t, \quad \text{with } I_{bc}(t, \bar{z}_t, z_t) = I_{bc}(t, i, j) := \frac{1}{\tau} \int_{t-\tau}^t \frac{1}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha} \, ds, \tag{4.12}$$

where (i,j) are s.t. $T_x^s(\bar{z}_s) = X_i(s)$, $T_x^s(z_s) = X_j(s)$.

For $z_t \in B_t$, define for $s \in [t-\tau,\ t]$

$$\phi(s) = (T_x^s(\bar{z}_s) - T_x^s(z_s)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|}.$$

Note that $|\phi(s)| \leq |T_x^s(\bar{z}_s) - T_x^s(z_s)|$ and that

$$\begin{aligned}\phi'(s) &= (T_v^s(\bar{z}_s) - T_v^s(z_s)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|} \\ &= |\bar{v}_t - v_t| + (T_v^s(\bar{z}_s) - T_v^s(z_s) - (\bar{v}_t - v_t)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|} \geq \frac{|\bar{v}_t - v_t|}{2},\end{aligned}$$

where we have used (4.10). Therefore if ϕ attains its minimum on $[t-\tau,\ t]$ at s_0, then

$$|T_x^s(\bar{z}_s) - T_x^s(z_s)| \geq |t - s_0| \, \frac{|\bar{v}_t - v_t|}{2}.$$

Using this directly gives, as $\alpha < 1$

$$|I_{bc}(t, \bar{z}_t, z_t)| \leq \frac{C}{\tau} \, |\bar{v}_t - v_t|^{-\alpha} \int_{t-\tau}^{t} \frac{ds}{|s - s_0|^{\alpha}} \leq C \, \tau^{-\alpha} \, |\bar{v}_t - v_t|^{-\alpha}. \tag{4.13}$$

Now integrating

$$\begin{aligned}\int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)| \, dz_t &\leq C \, \tau^{-\alpha} \int_{A_t^c} \frac{dz_t}{|\bar{v}_t - v_t|^{-\alpha}} \\ &\leq C \, \tau^{-\alpha} \, (W_\infty(t) + \tau)^d \, (R(t))^{d-\alpha},\end{aligned}$$

by using again Lemma 3. In conclusion

$$\int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)| \, dz_t \leq C \, \tau^{-\alpha} \, (W_\infty(t) + \tau)^d. \tag{4.14}$$

With the cut-off where $\alpha \geq 1$, the reasoning follows the same line up to the bound (4.13) which relies on the hypothesis $\alpha < 1$. (4.13) is replaced by

$$\begin{aligned}|I_{bc}(t, \bar{z}_t, z_t)| \leq & \frac{C}{\tau} \int_{t-\tau}^{t} \frac{ds}{(|s - s_0||\bar{v}_t - v_t| + 4\varepsilon^{\bar{m}})^{\alpha}} \\ \leq & \frac{C}{\tau} \int_0^{\frac{\tau}{2|\bar{v}_t - v_t|}} \frac{ds}{(s + 4\varepsilon^{\bar{m}})^{\alpha}} \leq \frac{C\varepsilon^{\bar{m}(1-\alpha)}}{\tau},\end{aligned} \tag{4.15}$$

with the convention $(\varepsilon^{\bar{m}})^0 = |\ln(\varepsilon^{\bar{m}})|$ for $\alpha = 1$. Integrating that bound (which do not depend on v_t) over B_t, we get the estimate

$$\int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)| \, dz_t \leq C \, (W_\infty(t) + \tau)^d \, \tau^{-1} \, \varepsilon^{\bar{m}(1-\alpha)}. \tag{4.16}$$

4.4.5 Step 3: Estimate over C_t

First remark that $C_t \subset \{|z_t| \leq C(W_\infty(t) + \tau)\}$, so that its volume is bounded by $C(W_\infty(t) + \tau)^{2d}$. ¿From the previous steps, it only remains to bound

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t) \, dz_t.$$

We begin by the cut-off case, which is the simpler one. In that case, one simply bounded $I_{bc} \leq C\,\varepsilon^{-\bar{m}\alpha}$ which implies

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t \leq C(W_\infty(t) + \tau)^{2d} \varepsilon^{-\bar{m}\alpha}. \tag{4.17}$$

It remains the case without cut-off. Denote $\tilde{C}_t = \{j \mid \exists z_t \in C_t,\ s.t.\ Z_j(t) = T^t(z_t)\}$, and bound

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t \leq \sum_{j \in \tilde{C}_t} \frac{a_{ij}}{N} I_{Nc}(t, i, j) \quad \text{with } I_{Nc}(t, i, j) = \frac{1}{\tau} \int_{t-\tau}^{t} \frac{dz_t}{|X_i(s) - X_j(s)|^\alpha}\, ds,$$

where i is the number of the particle associated to $\bar{z}_t$ ($T^t(\bar{z}_t) = Z_i(t)$) and

$$a_{ij} = N|\{z_t \in C_t,\ T^t(z_t) = Z_j(t)\}|, \quad \text{so that } \frac{1}{N} \sum_{j \in \tilde{C}_t} a_{ij} = |C_t|.$$

To bound I_{Nc} over $\tilde{C}_t$, we do another decomposition in j. Define

$$JX_t = \left\{ j \in \tilde{C}_t\,,\ |X_j(t) - X_i(t)| \geq \frac{d_N(t)}{2} \right\},$$

and

$$JV_t = \left\{ j \in \tilde{C}_t\,,\ |X_j(t) - X_i(t)| \leq |V_j(t) - V_i(t)| \text{ and } |V_j(t) - V_i(t)| \geq \frac{d_N(t)}{2} \right\}.$$

By the definition of the minimal distance in phase space $d_N(t)$, one has that $\tilde{C}_t = JX_t \cup JV_t$. Since

$$|T^t(z_t) - z_t| \leq W_\infty(t),$$

one has by the definition of $\tilde{C}_t$ and of C_t that for all $j \in \tilde{C}_t$, $|Z_j(t) - Z_i(t)| \leq C\,(W_\infty(t) + \tau)$.

Let us start with the bound over JX_t. If $j \in JX_t$, one has that

$$|X_j(s) - X_i(s)| \geq |X_j(t) - X_i(t)| - \int_s^t |V_j(u) - V_i(u)|\, du.$$

On the other hand

$$|V_j(u) - V_i(u)| \leq 2W_\infty(t) + |\bar{v}_u - v_u| \leq 2(W_\infty(t) + \tau|E|_\infty) + |\bar{v}_t - v_t| \leq C(W_\infty(t) + \tau).$$

Therefore assuming that

$$C\,\tau(W_\infty(t) + \tau) \leq d_N(t)/4, \tag{4.18}$$

we have that for any $s \in [t - \tau,\ t]$, $|X_j(s) - X_i(s)| \geq d_N(t)/4$. Consequently for any $j \in JX_t$

$$I_{Nc}(t, i, j) \leq C\,(d_N(t))^{-\alpha}. \tag{4.19}$$

For $j \in JV_t$, we write

$$|(V_j(s) - V_i(s)) - (V_j(t) - V_i(t))| \geq -\int_s^t |E_N(X_j(u)) - E_N(X_i(u))|\, du.$$

Note that

$$\begin{aligned} |X_j(s) - X_i(s)| &\le |X_j(t) - X_i(t)| + \int_s^t |V_j(u) - V_i(u)|\, du \\ &\le C(W_\infty(t) + \tau) + 2\int_s^t (W_\infty(u) + R(u))\, du \\ &\le C(W_\infty(t) + \tau). \end{aligned} \tag{4.20}$$

Hence we get for $s \in [t-\tau, t]$

$$\int_s^t |E_N(X_j(u)) - E_N(X_i(u))|\, du \le C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t) + \tau + \varepsilon^{1+r'}).$$

Note that the constant C still does not depend on $\tau = \varepsilon^{1+r'}$. Therefore provided that

$$C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t) + \tau) \le d_N(t)/4, \tag{4.21}$$

one has that

$$|V_j(s) - V_i(s) - (V_j(t) - V_i(t))| \ge d_N(t)/4 \quad \text{and also} \quad |V_i(s) - V_j(s)| \ge \frac{d_N(t)}{4}.$$

As in the step for B_t this implies the dispersion estimate $|X_j(s) - X_i(s)| \ge |s - s_0|\, d_N(t)/4$ for some $s_0 \in [t-\tau,\ t]$. As a consequence for $j \in JV_t$,

$$I_{Nc}(t,i,j) \le \frac{C}{\tau}\,(d_N(t))^{-\alpha}\int_{t-\tau}^t \frac{ds}{|s - s_0|^\alpha} \le C\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}. \tag{4.22}$$

Summing (4.19) and (4.22), one gets

$$\sum_{j\in\tilde{C}_t} \frac{a_{ij}}{N} I_{Nc}(t,i,j) \le C\,|C_t|\ \left((d_N(t))^{-\alpha} + \tau^{-\alpha}\ (d_N(t))^{-\alpha}\right).$$

Coming back to I_{bc} and keeping only the largest term of the sum

$$\int_{C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t \le C\,(W_\infty(t) + \tau)^{2d}\tau^{-\alpha}\ (d_N(t))^{-\alpha}. \tag{4.23}$$

4.4.6 Conclusion of the proof of Lemmas 1, 2

Assumptions (4.18) and (4.21) are ensured by the hypothesis of the lemma. Summing up (4.5) for I_α or (4.6) for $J_{\alpha+1}$, with (4.7), (4.14) and (4.23), we indeed find the conclusion of the first lemma.
In the S_m^α case, no assumption is needed, and summing up the bounds (4.5), (4.7), (4.16), (4.17), we obtain the second lemma.

4.5 Conclusion of the proof of Theorem 3 (without cut-off)

In this subsection, in order to make the argument clearer, we number explicitly the constants. Let us summarize the important information of Prop. 5 and Lemma 1.
Note that $W_\infty(t) \ge W_\infty(0) \ge \tau$ so we freely replace $C(W_\infty + \tau)$ by $2CW_\infty$ in the inequalities of lemma 1. Let us also rescale the interested quantities s.t. all may be of order 1

$$\varepsilon\,\tilde{W}_\infty(t) = W_\infty(t), \quad \varepsilon^{1+r}\,\tilde{d}_N(t) = d_N(t).$$

We also assume that $\tilde{W}_\infty$ and $|\nabla^N E|_\infty$ are non decreasing functions of t, and that $\tilde{d}_N$ is a non increasing function of t. In fact the bound proved before are also valid for $\sup_{s\le t}\tilde{W}_\infty$, $\sup_{s\le t}|\nabla^N E|_\infty$, and $\inf_{s\le t}\tilde{d}_N(s)$. With that convention $\tilde{W}_\infty(t)\ge\tilde{W}_\infty(0)\ge 1$. By assumption (i) in Theorem 3, also note that $\tilde{d}_N(0)\ge 1$.
Recalling $\tau=\varepsilon^{r'}$, the condition of Lemma 1 after rescaling reads

$$C_1\,\varepsilon^{r'-r}\,(1+|\nabla^N E|_\infty(t))\,\tilde{W}_\infty(t)\le\ \tilde{d}_N(t),\quad\forall\ t\in[0,\ t_0]. \tag{4.24}$$

We have that for some constants C_i independent of N (and hence ε), such that if (4.24) is satisfied, then for any $t\in[0,\ t_0]$

$$\begin{aligned}
\tilde{W}_\infty(t) &\le \tilde{W}_\infty(t-\tau)+C_0\,\varepsilon^{r'}\,\left(\tilde{W}_\infty(t)+\varepsilon^{\lambda_1}\,\tilde{W}_\infty^d(t)+\varepsilon^{\lambda_2}\,\tilde{W}_\infty^{2d}(t)\,\tilde{d}_N^{-\alpha}(t)\right),\\
|\nabla^N E|_\infty(t) &\le C_2\left(1+\varepsilon^{\lambda_3}\,\tilde{W}_\infty^d(t)+\varepsilon^{\lambda_4}\,\tilde{W}_\infty^{2d}(t)\,\tilde{d}_N^{-\alpha}(t))\right)\\
\tilde{d}_N(t)+\varepsilon^{r'-r} &\ge [\tilde{d}_N(t-\tau)+\varepsilon^{r'-r}]e^{-\tau(1+|\nabla^N E|_\infty(t))},
\end{aligned}$$

where ε appear four times with four different exponents $\lambda_i, i=1,\dots,4$ defined by

$$\begin{aligned}
\lambda_1&=d-1-\alpha\,r', &\qquad \lambda_2&=2d-1-\ \alpha(1+r'+r),\\
\lambda_3&=d-1-r'-\alpha\,r', &\qquad \lambda_4&=2d-1-r'-\ \alpha(1+r'+r).
\end{aligned}$$

To propagate uniform bounds as $\varepsilon\to 0$ and $N\to\infty$, we need all λ_i to be positive. As $r,r'>0$, it is clear that $\lambda_1>\lambda_3$ and $\lambda_2>\lambda_4$. Thus we need only check $\lambda_2>0$ and $\lambda_4>0$. Now, simply note that if

$$r<\frac{d-1}{1+\alpha},\quad\text{and}\quad r<\frac{2d-1-\alpha}{1+2\,\alpha},$$

then for any $r'>r$ close enough to r, one has $\lambda_2>0$ and $\lambda_4>0$.
As $\alpha^2<1<d$, the first inequality is the stronger one. As it is the condition given in Theorem 3, and r' is close to r, we have that all λ are positive and we denote $\lambda=\min_i(\lambda_i)$. Then by a rough estimate,

$$\begin{aligned}
\tilde{W}_\infty(t) &\le \tilde{W}_\infty(t-\tau)+C_0\,\tau\,\left(\tilde{W}_\infty(t)+2\,\varepsilon^{\lambda}\,\tilde{W}_\infty^{2d}(t)\,d_N^{-\alpha}(t)\right),\\
|\nabla^N E|_\infty(t) &\le C_2\left(1\ +2\,\varepsilon^{\lambda}\,\tilde{W}_\infty^{2d}(t)\,\tilde{d}_N^{-\alpha}(t)\right),\\
\tilde{d}_N(t) &\ge [\,\tilde{d}_N(0)+\varepsilon^{r'-r}]e^{-t(1+|\nabla^N E|_\infty(t))}-\varepsilon^{r'-r}.
\end{aligned}\tag{4.25}$$

If one has (4.24) and

$$2\,\varepsilon^{\lambda}\,\tilde{W}_\infty^{2d}(t)\,\tilde{d}_N^{-\alpha}(t)\le 1, \tag{4.26}$$

then using $W_\infty\ge 1$, we get $\tilde{W}_\infty(t)\le\tilde{W}_\infty(t-\tau)+2C_0\tau\tilde{W}_\infty(t)$ so that

$$\begin{aligned}
\tilde{W}_\infty(t) &\le \tilde{W}_\infty(t-\tau)(1-2C_0\tau)^{-1},\\
|\nabla^N E|_\infty(t) &\le 2\,C_2,\\
\tilde{d}_N(t) &\ge\ e^{-(1+2C_2)\,t}-\varepsilon^{r'-r}.
\end{aligned}\tag{4.27}$$

The last inequality implies $\tilde{d}_N(t)\ge\frac{1}{2}\,e^{-(1+2C_2)\,t}$ if $2\varepsilon^{r-r'}e^{(1+2C_2)\,T}<1$. That condition is fulfilled for ε small enough, i.e. N large enough.
The first inequality in (4.27), iterated gives $W_\infty(t)\le W_\infty(0)(1-2C_0\tau)^{-\frac{t}{\tau}}$. If $C_0\tau\le\frac{1}{4}$, then we can use $-\ln(1-x)\ge 2x$ for $x\in[0,\frac{1}{2}]$, and get

$$\tilde{W_\infty}(t)\le\tilde{W}_\infty(0)e^{4C_0t}$$

To summarize

$$\begin{aligned} \tilde{W}_\infty(t) &\le e^{4C_0 t}, \\ |\nabla^N E|_\infty(t) &\le 2\, C_2, \\ \tilde{d}_N(t) &\ge \frac{1}{2}\, e^{-(1+2C_2)\, t}. \end{aligned} \tag{4.28}$$

At the discrete level of the particles, the dynamics is continuous in time, at least for initial conditions not leading to collisions. That set is of full measure for $\alpha < 1$, and $d \ge 2$ (See [Hau04]). So as long as (4.24) and (4.26) are satisfied at $t = 0$, there exists a maximal time $t_0 \in]0,\ T]$ (possibly $t_0 = T$) such that they are satisfied on $[0,\ t_0]$.
We show that for N large enough, *i.e.* ε small enough, then one necessarily has $t_0 = T$. Then we will have (4.28) on $[0,\ T]$ which is the desired result. This is simple enough. By contradiction if $t_0 < T$ then

$$C_1\, \varepsilon^{(r-r')}\, (1 + |\nabla^N E|_\infty(t_0))\, \tilde{W}_\infty(t_0) = \tilde{d}_N(t_0), \quad \text{or } 4\ \varepsilon^\lambda\, \tilde{W}_\infty^{2d}(t_0)\, \tilde{d}_N^{-\alpha}(t_0) = 1.$$

Until t_0, (4.28) holds. Therefore

$$\varepsilon^\lambda\, \tilde{W}_\infty^{2d}(t_0)\, \tilde{d}_N^{-\alpha}(t_0) \le \varepsilon^\lambda\, 2^\alpha\, e^{(\alpha\, +\, (4d+2\alpha)\, \max(C_0, C_2))\, t_0} < 1,$$

for ε small enough with respect to T and the C_i. This is the same for (4.24),

$$C_1\, \varepsilon^{(r-r')}\, (1 + |\nabla^N E|_\infty(t_0))\, \tilde{W}_\infty(t_0) \tilde{d}_N^{-1}(t_0) \le 2\varepsilon^{(r-r')} C_1 (1 + 2C_2) e^{(1+6\, \max(C_0, C_2)) t_0} < 1.$$

Hence we obtain a contradiction and prove Theorem 3.

4.6 Conclusion of the proof of Theorem 4 (cut-off case)

In the cut-off case, using Lemma 2 together with the inequality ii) of the Proposition 5, we may obtain

$$W_\infty(t) \le W_\infty(t-\tau) + C_0 W_\infty(t) \left[1 + (W_\infty(t) + \tau)^{d-1} \tau^{-1} \varepsilon^{\bar{m}(1-\alpha)} + (W_\infty(t) + \tau)^{2d-1} \varepsilon^{-\bar{m}\alpha}\right].$$

We again rescale the quantity $W_\infty(t) = \varepsilon \tilde{W}_\infty(t)$ and replace $\tilde{W}_\infty(t)$ by $\tilde{W}_\infty(t) + 1$. Recall that $\kappa = \varepsilon^m$, $\tau = \varepsilon^{r'}$. Here the optimal estimate is obtained by taking τ as large as possible (because of the τ^{-1} term). This corresponds to $r' = 1$. It comes for $1 \le \alpha < d - 1$,

$$\tilde{W}_\infty(t) \le \tilde{W}_\infty(t-\tau) + C_0 \tilde{W}_\infty(t) \left[1 + \varepsilon^{d-2-\bar{m}(\alpha-1)}\, \tilde{W}_\infty^{d-1}(t) + \varepsilon^{2d-1-\bar{m}\alpha}\, \tilde{W}_\infty^{2d-1}(t)\right].$$

As in the previous section, we will get a good bound provided that the power of ε appearing in parenthesis are positive. The two conditions read

$$\bar{m} < \bar{m}^* := \min\left(\frac{d-2}{\alpha - 1}, \frac{2d-1}{\alpha}\right).$$

In that case, for N large enough (with respect to e^{Ct}), we get a control of the type

$$\frac{d}{dt} \tilde{W}_\infty(t) \le 4 C_0 \tilde{W}_\infty(t)$$

which gives the desired result.

5 Appendix: ¿From Gao's large deviation to Proposition 2

The result proven by Gao that we need is Theorem 1.2 of [Gao03], which is written below with the modifications :

- The sequence a_N of Gao is directly replaced by $a_N = N^\gamma$, $\gamma \in (0,1)$, which satisfies the required assumptions

$$a_N \underset{N\to+\infty}{\longrightarrow} 0, \quad \frac{Na_N^d}{\log a_N} \underset{N\to+\infty}{\longrightarrow} +\infty.$$

- The general but technical assumptions on the kernel ϕ (denoted by K in Gao's article) are replaced by the more simple and less general below. In fact, according to Gao's remarks, a bounded function of the form $\phi(x) = h(\sum_i x_i^2)$ with $h : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing and satisfying $\int \phi = 1$ is admissible.

Theorem 5 (Gao '03)**.** *Assume that f^0 is continuous and goes to zero at infinity, and that ϕ is a non-negative, radial and decreasing function. Then, with the notation introduced in the section 2*

$$\lim_{N\to+\infty} N^{\gamma-1} \ln \mathbb{P}\left(\|f_N - f^0\|_\infty \geq \lambda\right) = -J_\phi(\lambda),$$

where

$$J_\phi(\lambda) = \inf_{x\in\mathbb{R}^d} \sup_{t\in\mathbb{R}} \left\{ t\lambda - f(x) \int_{\mathbb{R}^d} [e^{t\phi(z)} - 1 - t\phi(z)]\, dz \right\}.$$

This is the original formulation of Gao. Remark that in the definition of J, the supremum is decreasing in $f^0(x)$ (the term under the integral is positive), so that the infimum will be reach at the maximum of f^0. Thus J may be rewritten as

$$J_\phi(\lambda) = \sup_{t\in\mathbb{R}} \left\{ t\lambda - \|f^0\|_\infty \int_{\mathbb{R}^d} [e^{t\phi(z)} - 1 - t\phi(z)]\, dz \right\}.$$

Let us explain how to obtain Proposition 2 from that result. As $\|f_N\|_\infty \leq \|f_N - f^0\|_\infty + \|f^0\|_\infty$, it suffices to bound by below $J_\phi((L-1)\|f^0\|_\infty)$ which is equal to

$$J_\phi((L-1)\|f^0\|_\infty) = \|f^0\|_\infty \sup_{t\in\mathbb{R}} \left\{ Lt + \int_{\mathbb{R}^d} [1 - e^{t\phi(z)}]\, dz \right\} =: I_\phi(L).$$

For a fixed ϕ and $L > 1$, we have $I_\phi(L) > 0$, which is enough to provide the large deviation result. Furthermore, in some cases, it is easy to compute explicitly I_ϕ. Consider for instance the uniform $\phi = \frac{1}{|B_1|}\mathbb{I}_{B_1}$ (where $\mathbb{I}$ denotes the characteristic function and $|B_1|$ the volume of B_1). In that case, the supremum is explicit and we get

$$I_\phi(L) = |f|_\infty |B_1| (L \ln L - L + 1).$$

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