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Joaquin Fontbona, Benjamin Jourdain

The Annals of Probability
2016, Vol. 44, No. 1, 131–170
DOI: 10.1214/14-AOP969

A TRAJECTORIAL INTERPRETATION OF THE DISSIPATIONS OF ENTROPY AND FISHER INFORMATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS

BY JOAQUIN FONTBONA[1] AND BENJAMIN JOURDAIN[2]

Universidad de Chile and Université Paris-Est

The dissipation of general convex entropies for continuous time Markov processes can be described in terms of backward martingales with respect to the tail filtration. The relative entropy is the expected value of a backward submartingale. In the case of (not necessarily reversible) Markov diffusion processes, we use Girsanov theory to explicit the Doob–Meyer decomposition of this submartingale. We deduce a stochastic analogue of the well-known entropy dissipation formula, which is valid for general convex entropies, including the total variation distance. Under additional regularity assumptions, and using Itô's calculus and ideas of Arnold, Carlen and Ju, we obtain moreover a new Bakry–Emery criterion which ensures exponential convergence of the entropy to 0. This criterion is nonintrinsic since it depends on the square root of the diffusion matrix and cannot be written only in terms of the diffusion matrix itself. We provide examples where the classic Bakry–Emery criterion fails, but our nonintrinsic criterion applies without modifying the law of the diffusion process.

Introduction. We are interested in the long-time behavior of solutions to the stochastic differential equation

$$dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt, \tag{0.1}$$

where $b:\mathbb{R}^d \to \mathbb{R}^d$, $\sigma:\mathbb{R}^d \to \mathbb{R}^{d\otimes d'}$ and $W = (W_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}^{d'}$.

If (0.1) admits a reversible probability measure, the celebrated Bakry–Emery curvature dimension criterion, which involves the generator, the carré du champs and the iterated carré du champs, is a sufficient condition for this reversible measure to satisfy a Poincaré inequality and a logarithmic Sobolev inequality. From these inequalities, one can, respectively, deduce exponential convergence to 0 as $t \to \infty$ of the chi-square distance or the relative entropy between the marginal at

Received July 2013; revised February 2014.

[1]Supported in part by Fondecyt Grant 1110923, BASAL-Conicyt and Millenium Nucleus NC120062.

[2]Supported by French National Research Agency Grants ANR-09-BLAN-0216-01 (MEGAS) and ANR-12-BS01-0019 (STAB).

MSC2010 subject classifications. 60H10, 60H30, 37A35, 26D10, 35B40.

Key words and phrases. Long-time behavior, stochastic differential equations, Bakry–Emery criterion, convex Sobolev inequalities, time reversal, Girsanov theory.

time t of the process and its reversible measure. These results have been extended to more general entropy functionals; see, for instance, [2].

In general, even when the stochastic differential equation (0.1) admits an invariant probability measure, this measure might be not reversible. It is well known from both a probabilistic point of view [13] and the point of view of partial differential equations [1] that a contribution in the drift term, antisymmetric with respect to the invariant measure, may accelerate convergence to this invariant measure as $t \to \infty$.

Throughout this paper, we assume

$(H0)$ $U : [0, \infty) \to \mathbb{R}$ is a convex function such that $\inf U > -\infty$,

and we consider the U-entropy of a probability measure p on a measurable space $(E, \mathcal{E})$, with respect to another probability measure q on $(E, \mathcal{E})$, defined by

$$H_U(p|q) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} U\left(\frac{dp}{dq}(x)\right) dq(x), & \text{if } p \ll q, \\ +\infty, & \text{otherwise.} \end{cases}$$

The particular cases $U(x) = \mathbf{1}_{x>0} x \ln(x)$ and $U(x) = (x-1)^2$, respectively, correspond to the usual entropy and the χ^2-distance. For $U(x) = |x-1|$, $H_U(p|q)$ coincides with the total variation distance when $p \ll q$. Notice that U is continuous on $(0, +\infty)$ and that $U(0) \geq \lim_{x\to 0^+} U(x)$.

The primal goal of this work is to recover, by arguments using Itô's stochastic calculus, the results of [1] and [2] about the long-time behavior of the U-entropy of the law of X_t with respect to the invariant measure. Our approach is based on the following simple remark, valid for an arbitrary (possibly nonhomogeneous) continuous-time Markov process $(X_t : t \geq 0)$ with values in a measurable space $(E, \mathcal{E})$.

If we denote:

- by P_t and Q_t the time marginal laws of X_t when the initial laws are P_0 and Q_0, respectively, and
- by $(X_t^P)_{t\geq 0}$ and $(X_t^Q)_{t\geq 0}$ realizations of the process (X_t) with X_0, respectively, distributed according to P_0 and Q_0,

then, as soon as $H_U(P_t|Q_t) < +\infty$ for some $t \geq 0$, one has $P_s \ll Q_s$ for all $s \geq t$, and the process

$$\left(U\left(\frac{dP_s}{dQ_s}(X_s^Q)\right)\right)_{s\geq t}$$

is a backward $\mathcal{F}_s$-submartingale with respect to the filtration $\mathcal{F}_s := \sigma(X_r^Q, r \geq s)$. In fact, it is easily deduced from the Markov property that if $P_t \ll Q_t$ for some $t \geq 0$, then the law of $(X_r^P)_{r\geq t}$ is absolutely continuous with respect to the one of $(X_r^Q)_{r\geq t}$ and moreover, $P_s \ll Q_s$ for all $s \geq t$ with $(\frac{dP_s}{dQ_s}(X_s^Q))_{s\geq t}$ a backward martingale with respect to the filtration $\mathcal{F}_s$. Jensen's inequality ensures that $t \mapsto H_U(P_t|Q_t)$ is nonincreasing and implies the remark.

The convergence of the U-entropy

$$
\begin{aligned}
H_U(P_s|Q_s) &= \mathbb{E}\left(\left(U\left(\frac{dP_s}{dQ_s}(X_s^Q)\right)\right)\right) \\
&\underset{s\to\infty}{\longrightarrow} \mathbb{E}\left(U\left(\mathbb{E}\left(\frac{dP_t}{dQ_t}(X_t^Q)\Big| \bigcap_{s\geq 0} \mathcal{F}_s\right)\right)\right) < \infty
\end{aligned}
\tag{0.2}
$$

is then deduced from the a.s. convergence of $\frac{dP_s}{dQ_s}(X_s^Q)$ to $\mathbb{E}(\frac{dP_t}{dQ_t}(X_t^Q)|\bigcap_{s\geq 0}\mathcal{F}_s)$ [the fact that for $r \geq t$, $\frac{dP_r}{dQ_r}(X_r^Q) = 0$ a.s. on the set $\{\mathbb{E}(\frac{dP_t}{dQ_t}(X_t^Q)|\bigcap_{s\geq 0}\mathcal{F}_s) = 0\}$ permits to cope with the possible discontinuity of U at 0].

The first section of the paper is dedicated to time-inhomogeneous Markov diffusions given by the stochastic differential equation

$$
dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt, \tag{0.3}
$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\otimes d'}$. Under assumptions that guarantee that for both initial laws, the time-reversed processes are still diffusions, we use Girsanov theory to make explicit the Doob–Meyer decomposition of the submartingale $(U(\frac{dP_s}{dQ_s}(X_s^Q)))_{s\geq t}$. In this way, we obtain a stochastic analogue of the well-known entropy dissipation formula, valid for general convex entropies (including total variation). Taking expectations in this formula, we recover the well-known fact that the U-entropy dissipation is equal to the U-Fisher information. The proofs of the main results of this section are given in Appendix A.

It should be noticed that the idea of considering a trajectorial interpretation of entropy to obtain functional inequalities is not new, at least for reversible diffusions; see, for example, the work of Cattiaux [5] whose results are nevertheless of quite a different nature. However, even in the reversible case, time reversal of a diffusion starting out of equilibrium modifies the dynamics of the diffusion. The backward martingale approach takes this fact into account and moreover permits the use of Itô's calculus under less regularity than is a priori needed when working in the forward time direction. Its interest thus goes beyond the treatment of nonreversible situations.

In the second section, we further suppose that the stochastic differential equation is time-homogeneous [i.e., of the form (0.1)] and that it admits an invariant probability distribution that is chosen as the initial law Q_0. Under additional regularity assumptions, and using Itô's calculus and some ideas similar to those of Arnold, Carlen and Ju [1], we obtain a new Bakry–Emery criterion which ensures exponential convergence of the U-Fischer information to 0 and therefore exponential convergence of the U-entropy to 0. In addition, under this criterion, the invariant measure satisfies a U-convex Sobolev inequality. This criterion is nonintrinsic: it depends on the square root σ of the diffusion matrix $a = \sigma\sigma^*$ and cannot be written only in terms of the diffusion matrix itself, whereas, under mild regularity assumptions on b and a, the law of $(X_t)_{t\geq 0}$ solving (0.1) is characterized by the associated martingale problem only written in terms of a and b. The main results of this section are proved in Appendix B. In Appendix C, we point out that our approach allows us to recover the results and criterion provided in [1]. We also highlight the difference between the arguments leading to each of the two criteria. Additionally, we provide a combined criterion.

Finally, we provide in the third section two examples where the classic Bakry–Emery criterion fails, but our nonintrinsic criterion ensures exponential convergence to equilibrium without modifying the law of the diffusion process.

As future work, we plan to investigate how to choose the square root σ of the diffusion matrix in order to maximize the rate of exponential convergence to equilibrium given by our nonintrinsic Bakry–Emery criterion.

Throughout this work, we use the convention of summation over repeated indexes.

1. Entropy dissipation for diffusion processes.

From now on we assume that $(X_t, t \geq 0)$ is a Markov diffusion process which satisfies the stochastic differential equation

$$dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt, \tag{1.1}$$

where $W = (W_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}^{d'}$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\otimes d'}$ are mesurable coefficients satisfying conditions that will be specified below.

For P_0 and Q_0 two probability measures on $\mathbb{R}^d$, we now denote by $(X_t^P)_{t\geq 0}$ and $(X_t^Q)_{t\geq 0}$ two solutions of (1.1) with X_0, respectively, distributed according to P_0 and Q_0. For $t \geq 0$, the law of X_t^P (resp., X_t^Q) is denoted by P_t (resp., Q_t).

Our first goal is to explicitly describe the backward submartingale $U(\frac{dP_s}{dQ_s}(X_s^Q))$ when $P_0 \ll Q_0$ and, as a byproduct, the decrease of its expectation $H_U(P_s|Q_s)$. In a way, this backward-in-time approach to entropy is converse to Föllmer's approach to the study of time reversal of diffusion processes [8] (see [9] for the infinite-dimensional case) based on the stability under time reversal of the usual pathwise entropy. The latter corresponds to $U(r) = r \ln r$ in Remark 1.1 below.

We fix a finite time-horizon $T \in (0, +\infty)$ in order to work with standard (forward) filtrations by time reversal in $[0, T]$. Let us introduce some notation:

- $\mathbb{Q}^T$ (resp., $\mathbb{P}^T$) will denote the law of the time reversed processes $(X_{T-t}^Q)_{t\leq T}$ [resp., $(X_{T-t}^P)_{t\leq T}$] on the canonical space $C([0, T], \mathbb{R}^d)$.
- Throughout the sequel, $\mathbb{E}^T$ will denote the expectation under the law $\mathbb{Q}^T$.
- $(Y_t)_{t\leq T}$ stands from now on for the canonical process on $C([0, T], \mathbb{R}^d)$, and $\mathcal{G}_t = \sigma(Y_s, 0 \leq s \leq t)$ denotes its natural (complete, right continuous) filtration.

Whenever $P_0 \ll Q_0$, by the Markov property we have $\mathbb{P}^T \ll \mathbb{Q}^T$ with $\frac{d\mathbb{P}^T}{d\mathbb{Q}^T} = \frac{dP_0}{dQ_0}(Y_T)$, and

$$D_t \stackrel{\text{def}}{=} \frac{d\mathbb{P}^T}{d\mathbb{Q}^T}\bigg|_{\mathcal{G}_t} = \frac{dP_{T-t}}{dQ_{T-t}}(Y_t), \qquad 0 \le t \le T \tag{1.2}$$

is a $\mathbb{Q}^T - \mathcal{G}_t$ martingale. Moreover, $H_U(P_s|Q_s) < +\infty$ for $s \in [0,T]$ if and only if $(U(D_t))_{0\le t\le T-s}$ is a uniformly integrable $\mathbb{Q}^T - \mathcal{G}_t$ submartingale, in which case one has

$$H_U(P_t|Q_t) = \mathbb{E}^T\big(U(D_{T-t})\big) \qquad \text{for all } t \in [s,T].$$

REMARK 1.1. If $\mathbb{H}_U(\mathbb{P}_1|\mathbb{P}_2)$ denotes the pathwise U-entropy of a probability measure $\mathbb{P}_1$ on $C([0,T],\mathbb{R}^d)$ with respect to a second probability measure $\mathbb{P}_2$,

$$\mathbb{H}_U(\mathbb{P}_1|\mathbb{P}_2) := \begin{cases} \displaystyle\int_{C([0,T],\mathbb{R}^d)} U\bigg(\frac{d\mathbb{P}_1}{d\mathbb{P}_2}(w)\bigg)\, d\mathbb{P}_2(w), & \text{if } \mathbb{P}_1 \ll \mathbb{P}_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

we easily deduce that

$$\begin{aligned} H_U(P_0|Q_0) &= \mathbb{H}_U\big(\mathrm{law}\big(X_t^P : 0 \le t \le T\big) \,|\, \mathrm{law}\big(X_t^Q : 0 \le t \le T\big)\big) \\ &= \mathbb{H}_U\big(\mathbb{P}^T|\mathbb{Q}^T\big). \end{aligned}$$

In order to use Itô calculus to obtain the explicit form of the Girsanov density D_t as a $\mathbb{Q}^T - \mathcal{G}_t$ martingale, and then deduce the Doob–Meyer decomposition of the submartingale $U(D_t)$, we will assume that the Markov processes $(X_{T-t}^Q, t \le T)$ and $(X_{T-t}^P, t \le T)$ are diffusion processes as well. Conditions ensuring this fact have been studied in Föllmer [8], in Hausmann and Pardoux [11], in Pardoux [18] and in Millet, Nualart and Sanz [17] among others, who in particular provide the semimartingale decomposition of $(X_{T-t}^Q, t \le T)$ in its filtration. We recall in Theorem 1.2 below the general results in [17] in a slightly more restrictive setting. The following conditions are needed:

$(H1)$ For each $T > 0$, $\sup_{t\in[0,T]}(|b(t,0)| + |\sigma(t,0)|) < +\infty$ and for every $A > 0$ there is a constant $K_{T,A} > 0$ such that

$$\big|b(t,x) - b(t,y)\big| + \sum_{i=1}^{d'} \big|\sigma_{\bullet i}(t,x) - \sigma_{\bullet i}(t,y)\big| \le K_{T,A}|x-y| \qquad \forall t \in [0,T], \forall x,y \in B(0,A),$$

where $\sigma_{\bullet i}$ denotes the ith column of the matrix σ and $B(0,A)$ is the ball of radius $A > 0$ centered at the origin in $\mathbb{R}^d$. Moreover:

$(H1)'$ the constants $K_{T,A}$ do not depend on A, or

$(H1)''$ for each $s \geq 0$, equation (1.1) starting at time s is strictly conservative, and for any bounded open set $D \subset \mathbb{R}^d$,

$$\sup_{x\in D}\sup_{s\in[0,T]} \mathbb{E}\left\{\exp\left[\int_s^T \left[4B_{s,t}(x) + 8\sum_j \left|A_{s,t}^j(x)\right|^2\right] dt\right]\right\} < \infty,$$

where

$$B_{s,t}(x) = \left[\sum_{i,k=1}^d \partial_i b_k\big(t, X_{s,t}(x)\big)^2\right]^{1/2},$$

$$A_{s,t}^j(x) = \left[\sum_{i,k=1}^d \partial_i \sigma_{kj}\big(t, X_{s,t}(x)\big)^2\right]^{1/2}$$

and $X_{s,t}(x)$ denotes the solution to (1.1) starting from x at time $s < t$.

$(H2)_Q$ For each $t > 0$, the law $Q_t(dx)$ of X_t^Q has a density $q_t(x)$ with respect to Lebesgue measure.

$(H3)_Q$ Setting $a_{ij} = (\sigma\sigma^*)_{ij}$, for each $i = 1, \ldots, d$ the distributional derivative $\partial_j(a_{ij}(t,x)q_t(x))$ is a locally integrable function on $[0,T] \times \mathbb{R}^d$:

$$\int_0^T \int_D \left|\partial_j\big(a_{ij}(t,x)q_t(x)\big)\right| dx\, dt < \infty$$

for any bounded open set $D \subset \mathbb{R}^d$.

For $(t,x) \in [0,T] \times \mathbb{R}^d$ we write:

- $\bar{a}_{ij}(t,x) := a_{ij}(T-t,x), i, j = 1, \ldots, d,$
- $\bar{b}_i^Q(t,x) := -b_i(T-t,x) + \frac{\partial_j(a_{ij}(T-t,x)q_{T-t}(x))}{q_{T-t}(x)}$ [with the convention that the term involving $\frac{1}{q_{T-t}(x)}$ is zero when $q_{T-t}(x)$ is zero]

and notice that $\bar{b}^Q(t,x)$ is defined $dt \otimes dx$ a.e. on $[0,T] \times \mathbb{R}^d$ under assumption $(H3)_Q$.

THEOREM 1.2. *Assume that $(H1)$ and $(H2)_Q$ hold.*

(a) *Suppose moreover $(H3)_Q$. Then $\mathbb{Q}^T$ is a solution to the martingale problem*

$$(MP)_Q : M_t^f := f(Y_t) - f(Y_0) - \int_0^t \frac{1}{2}\bar{a}_{ij}(s,Y_s)\partial_{ij} f(Y_s) + \bar{b}_i^Q(s,Y_s)\partial_i f(Y_s)\, ds, \qquad t \in [0,T],$$

is a continuous martingale with respect to the filtration $(\mathcal{G}_t)$ for all $f \in C_0^\infty(\mathbb{R}^d)$.

(b) *Let* $\tilde{b}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ *and* $\tilde{\sigma}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^{d\otimes d'}$ *be measurable functions such that* $\int_0^T\int_D|\tilde{a}_{ij}(t,x)|+|\tilde{b}_i(t,x)|q_{T-t}(x)\,dx\,dt<\infty$ *for any bounded open set* $D\subset\mathbb{R}^d$. *Assume moreover that* $\mathbb{Q}^T$ *is a solution to the martingale problem with respect to* $(\mathcal{G}_t)$ *for the generator* $\mathcal{L}_t f(x)=\frac{1}{2}\tilde{a}_{ij}(t,x)\partial_{ij}f(x)+\tilde{b}_i(t,x)\partial_i f(x)$. *Then* $(H3)_Q$ *holds,* $\tilde{b}=\bar{b}$ *and* $\tilde{a}=\bar{a}$.

PROOF. According to Theorem 3.3 in [17], under $(H1)$, $(H2)_Q$ and $(H3)_Q$, $(M_t^f)_{t\in[0,T)}$ is a continuous $\mathcal{G}_t$-martingale under $\mathbb{Q}^T$. When f is C^∞ on $\mathbb{R}^d$ and vanishes outside $B(0,A)$, we have

$$\begin{aligned}
&\mathbb{E}^T\left(\int_0^T\left|\bar{b}_i^Q(s,Y_s)\right|\left|\partial_i f(Y_s)\right|ds\right)\\
&\quad\le\sup_{B(0,A)}|\nabla f|\left(T\sup_{[0,T]\times B(0,A)}\left|b(s,x)\right|\right.\\
&\qquad\left.+\int_{[0,T]\times B(0,A)}\sum_{i=1}^d\left|\partial_j\big(a_{ij}(s,x)q_s(x)\big)\right|ds\,dx\right),
\end{aligned}\tag{1.3}$$

where the right-hand side is finite under $(H1)$ and $(H3)_Q$. This implies that $\mathbb{E}^T(|M_T^f|)<+\infty$, and together with $(H1)$, that $(M_t^f)_{t\in[0,T]}$ is a continuous $\mathcal{G}_t$-martingale under $\mathbb{Q}^T$. Part (b) follows from Theorem 3.3 in [17]. $\square$

Assume $(H1)$, $(H2)_P$, $(H2)_Q$, $(H3)_P$ and $(H3)_Q$. Then, under $(MP)_Q$ and $(MP)_P$, the process Y_t is, respectively, a weak solution to the SDEs

$$dX_t=\bar{\sigma}(t,X_t)\,d\overline{W}_t+\bar{b}^Q(t,X_t)\,dt,\qquad t\in[0,T]\tag{1.4}$$

and

$$dX_t=\bar{\sigma}(t,X_t)\,d\widetilde{W}_t+\bar{b}^P(t,X_t)\,dt,\qquad t\in[0,T],$$

where $\bar{\sigma}(t,x)=\sigma(T-t,x)$ and $\overline{W}$ and $\widetilde{W}$ are d'-dimensional Brownian motions in possibly enlarged probability spaces. If for all $t>0$, $x\mapsto p_t(x)$ and $x\mapsto q_t(x)$ are strictly positive and differentiable, then the difference between the drift terms of the two equations is given by

$$\begin{aligned}
\bar{b}_i^P(t,x)-\bar{b}_i^Q(t,x)&=\bar{a}_{ij}(t,\cdot)\partial_j\ln p_{T-t}(x)-\bar{a}_{ij}(t,\cdot)\partial_j\ln q_{T-t}(x)\\
&=\bar{a}_{ij}(t,x)\partial_j\left[\ln\frac{p_{T-t}}{q_{T-t}}(x)\right].
\end{aligned}$$

If uniqueness in law holds for the second stochastic differential equation, then the simplest form of Girsanov theorem allows us to deduce that

$$\begin{aligned}
D_t=\frac{p_T}{q_T}(Y_0)\exp\biggl\{&\int_0^t\nabla^*\left[\ln\frac{p_{T-t}}{q_{T-t}}(Y_t)\right]\bar{\sigma}(t,Y_t)\,d\overline{W}_t\\
&-\frac{1}{2}\int_0^t\nabla^*\left[\ln\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\bar{a}(s,Y_s)\nabla\left[\ln\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\,ds\biggr\}
\end{aligned}$$

(in the above equation and from now on, we denote by ∇^* the transpose of the gradient). However, in the general case when $q_t(x)$ or $p_t(x)$ may vanish and are possibly not differentiable, it is not clear what sense should be given to the derivatives above. If the diffusion matrix is singular, neither is it clear that the difference of drift terms $\bar{b}^Q$ and $\bar{b}^P$ (defined by means of distributional derivatives) is in the range of the diffusion matrix, which is required in order to use Girsanov theorem.

The problem of finding D_t in the general case is reminiscent of and, somehow, reciprocal to the stochastic construction of Nelson processes, where $\mathbb{Q}^T$ and the possibly singular difference of drift terms are given, and one aims to construct $\mathbb{P}^T$; see, for instance, [6]. The following technical lemma answers the question in the most general situations covered by Theorem 1.2. Its proof, not hard but lengthy, relies on Girsanov theory in the absolutely continuous setting and is given in the Appendix A.1. Recall that an element $\mathbb{P}_0 \in \mathcal{M}$ of a given set $\mathcal{M}$ of probability measures in $C([0,T],\mathbb{R}^d)$ is said to be *extremal* if $\mathbb{P}_0 = \alpha\mathbb{P}_1 + (1-\alpha)\mathbb{P}_2$ for some $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}$ and $\alpha \in (0,1)$ implies $\mathbb{P}_0 = \mathbb{P}_1 = \mathbb{P}_2$.

LEMMA 1.3. *Assume that* $(H1)$, $(H2)_Q$, $(H3)_Q$ *and* $(H3)_P$ *hold, with* $P_0 \ll Q_0$, *and let* $\frac{p_t}{q_t}(x)$ *be the Radon–Nikodym derivative of* $p_t(x)\,dx\,dt$ *w.r.t.* $q_t(x)\,dx\,dt$ *on* $[0,T]\times\mathbb{R}^d$. *Then*:

(a) *there exists a measurable function in* $[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ *denoted* $(t,x)\mapsto \nabla\ln[\frac{p_t}{q_t}](x)$ *such that*

$$\bar{b}^P(t,x) - \bar{b}^Q(t,x) = \bar{a}(t,x)\nabla\left[\ln\frac{p_{T-t}}{q_{T-t}}(x)\right], \qquad p_{T-t}(x)\,dx\,dt \text{ a.e.}$$

(b) *Define* $q_t(x)\,dx\,dt$ *a.e. in* $[0,T]\times\mathbb{R}^d$ *the function* $(t,x)\mapsto\nabla[\frac{p_t}{q_t}](x)$ *by*

$$\nabla\left[\frac{p_t}{q_t}\right](x) := \frac{p_t}{q_t}(x)\nabla\left[\ln\frac{p_t}{q_t}\right](x),$$

and assume moreover that $\mathbb{Q}^T$ *is an extremal solution to the martingale problem* $(MP)_Q$. *Then the* $\mathbb{Q}^T$-$(\mathcal{G}_t)$ *martingale* $(D_t)_{t\in[0,T]}$ *introduced in* (1.2) *has a continuous version* (*denoted in the same way*) *satisfying*

$$\begin{aligned}
D_t &= \frac{p_T}{q_T}(Y_0) + \int_0^t D_s\nabla\ln\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s \\
&= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{\{(p_{T-s}/q_{T-s})(Y_s)>0\}}\cdot dM_s,
\end{aligned}$$

where $M_t = (M_t^i)_{i=1}^d$ *are the continuous local martingales w.r.t.* $\mathbb{Q}^T$ *and* $(\mathcal{G}_t)$ *defined by*

$$M_t^i := Y_t^i - Y_0^i - \int_0^t \bar{b}_i^Q(s,Y_s)\,ds, \qquad t\in[0,T],$$

and R is the $(\mathcal{G}_t)$-stopping time $R := \inf\{s \in [0,T] : D_s = 0\}$. Moreover, $\mathbb{Q}^T$ a.s., one has

$$\langle D\rangle_t = \int_0^t \nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\bar{a}(s,Y_s)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\,ds \qquad \forall t \in [0,T].$$

From the proof of Lemma 1.3 it will be clear that if p_t and q_t are everywhere strictly positive and of class C^1, $(t,x) \mapsto \nabla[\frac{p_t}{q_t}](x)$ and $(t,x) \mapsto \nabla \ln[\frac{p_t}{q_t}](x)$ can be, respectively, taken to be the usual gradient and gradient of the logarithm of $\frac{p_t}{q_t}$.

We now introduce the notation U'_- and $U''(dy)$ for the left-hand derivative of the restriction of the convex function $U : [0,\infty) \to \mathbb{R}$ to $(0,+\infty)$ and the nonnegative measure on $(0,+\infty)$ equal to the second order distribution derivative of this restriction.

We are ready to state the main result of this section:

THEOREM 1.4 (Stochastic U-entropy dissipation). *Let Q_0 and P_0 be probability measures on $\mathbb{R}^d$ such that*

$$H_U(P_0|Q_0) < \infty,$$

and assume that $(H1)$, $(H2)_Q$, $(H3)_Q$ and $(H3)_P$ hold. Suppose moreover that $\mathbb{Q}^T$ is an extremal solution to the martingale problem $(MP)_Q$.

Then the submartingale $(U(D_t))_{t\in[0,T]}$ has the Doob–Meyer decomposition

$$(1.5)\qquad \begin{aligned} U(D_t) &= U(D_0) + \int_0^t U'_-(D_s)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s \\ &\quad + \frac{1}{2}\int_{(0,+\infty)} L_t^r(D)U''(dr) - \mathbf{1}_{\{0<R\le t\}}\Delta U(0) \qquad \forall t \in [0,T] \end{aligned}$$

where $R := \inf\{s \in [0,T] : D_s = 0\}$, $\Delta U(0) = \lim_{x\to 0^+} U(x) - U(0) \le 0$ and $L_t^r(D)$ denotes the local time at level $r \ge 0$ and time t of the continuous version of the martingale $(D_s)_{s\in[0,T]}$.

In particular, if U is continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$, one has

$$(1.6)\qquad \begin{aligned} U(D_t) &= U(D_0) + \int_0^t U'(D_s)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s \\ &\quad + \frac{1}{2}\int_0^t U''\left(\frac{p_{T-s}}{q_{T-s}}(Y_s)\right)\left(\nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right]\bar{a}(s,\cdot)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right]\right)(Y_s)\mathbf{1}_{s<R}\,ds \\ &\qquad \forall t \in [0,T]. \end{aligned}$$

Theorem 1.4 is proved in Appendix A. We next briefly discuss some of its assumptions and then state some consequences.

REMARK 1.5. (a) By Theorem 3.1 in [11], conditions $(H2)_Q$ and $(H3)_Q$ hold under condition $(H1)'$ if Q_0 has a density q_0 w.r.t. the Lebesgue measure s.t. $\int_{\mathbb{R}^d} \frac{q_0^2(x)\,dx}{1+|x|^k} < +\infty$ for some $k > 0$ and either

$$\forall T > 0, \exists \varepsilon > 0, \forall (t,x) \in [0,T] \times \mathbb{R}^d \qquad a(t,x) = \sigma\sigma^*(t,x) \geq \varepsilon I_d$$

or the second-order distribution derivatives $\frac{\partial^2 a^{ij}}{\partial x_i\,\partial x_j}(t,x)$ are bounded on $[0,T] \times \mathbb{R}^d$ for each $T > 0$ (by Theorem 3.1, page 1199 of [11], the latter conditions imply that (A)(ii) on page 1189 and thus Theorem 2.1 therein hold). In particular, under $(H1)'$ and the previous conditions, $(H2)_P$ and $(H3)_P$ also hold if, for instance, $P_0 \ll Q_0$ and $\frac{dP_0}{dQ_0}$ has polynomial growth.

(b) Condition $(H1)''$ introduced in [17] allows us to include in our study the fundamental examples of Langevin diffusions with $a(x) = I_d$ and $b(x) = -\nabla V(x)$ for a nonnegative C^2 potential V, possibly superquadratic, but satisfying

$$\begin{cases} \limsup\limits_{|x|\to\infty} \dfrac{-x^*\nabla V(x)}{|x|^2} < +\infty, \qquad \limsup\limits_{|x|\to\infty} \dfrac{\Delta V}{|\nabla V|^2}(x) < 2, \\ \limsup\limits_{|x|\to\infty} \dfrac{\sqrt{\partial_{ik} V \partial_{ik} V}}{V}(x) = 0. \end{cases} \tag{1.7}$$

See Section A.5 for a proof of this fact.

(c) Extremality of the solution $\mathbb{Q}^T$ to the martingale problem $(MP)_Q$ is implied by pathwise uniqueness for the stochastic differential equation (1.4). In the relevant case when σ and b in (1.1) are time-homogeneous and (0.1) admits an invariant density $p_\infty(x) > 0$, for the choice $Q_0(dx) = p_\infty(x)\,dx$, equation (1.4) takes the form

$$dX_t = \sigma(X_t)\,dW_t + \left(\frac{\partial_j(a_{\bullet j}p_\infty)}{p_\infty}(X_t) - b(X_t)\right)dt, \qquad t \in [0,T].$$

Pathwise uniqueness for this SDE can be proved under $(H1)$ by a standard argument using localization, Itô's formula and Gronwall's lemma, whenever the function $-\frac{\partial_j(a_{\bullet j}p_\infty)}{p_\infty}$ is the sum of a locally Lipschitz function and a monotone function. This is, for instance, the case when $a = I_d$ and $p_\infty(x) = Ce^{-2V(x)}$ for some convex function $V : \mathbb{R}^d \to \mathbb{R}$, or when the strictly positive density p_∞ and a have locally Lipschitz derivatives.

The proof of Theorem 1.4 will justify that expectations can be taken in (1.5) and (1.6), yielding

COROLLARY 1.6 (U-entropy dissipation). *Under the assumptions of Theorem 1.4,*

$$\begin{aligned} H_U(P_t|Q_t) = {} & H_U(P_T|Q_T) - \Delta U(0)\mathbb{Q}^T(0 < R \leq T-t) \\ & + \frac{1}{2}\int_{(0,+\infty)} \mathbb{E}^T\big(L^r_{T-t}(D)\big)U''(dr) \qquad \forall t \in [0,T]. \end{aligned} \tag{1.8}$$

If U is moreover continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$, we get the well-known expression for the entropy dissipation, $\forall t \in [0,T]$,

$$\begin{aligned} H_U(P_t|Q_t) &= -H_U(P_0|Q_0) \\ &\quad - \frac{1}{2}\int_0^t \int_{\{(p_s/q_s)(x)>0\}} U''\Big(\frac{p_s}{q_s}(x)\Big) \\ &\qquad \times \Big(\nabla^*\Big[\frac{p_s}{q_s}\Big] a(s,\cdot)\nabla\Big[\frac{p_s}{q_s}\Big]\Big)(x) q_s(x)\,dx\,ds, \end{aligned} \tag{1.9}$$

with $U''(r)$ now standing for the second order derivative of U at $r>0$.

The particular case $U(x)=|x-1|$ of the total variation distance is more intricate, but we are still able to derive an analogous dissipation formula. To our knowledge this formula is new:

COROLLARY 1.7 (Dissipation of total variation). *Under the assumptions of Theorem 1.4, suppose moreover that for a.e. $t\in[0,T]$, the functions $x\mapsto q_t(x)$ and $x\mapsto \frac{p_t}{q_t}(x)$ are, respectively, of class C^1 and C^2, and there exists a sequence $(r_n)_n$ of positive numbers tending to $+\infty$ as $n\to\infty$, such that* $\lim_{n\to\infty}\frac{1}{r_n}\int_{\{r_n\le|x|<2r_n\}}|a(t,x)\nabla[\frac{p_t}{q_t}](x)|q_t(x)\,dx = 0$. *Furthermore, assume that* $\int_0^T\int_{\mathbb{R}^d}|\nabla\cdot[\bar a(s,x)\nabla[\frac{p_{T-s}}{q_{T-s}}](x)q_{T-s}(x)]|\,dx\,ds<\infty$. *Then*, $\forall t\in[0,T]$,

$$\begin{aligned} &\|P_t-Q_t\|_{\mathrm{TV}} \\ &\quad = \|P_0-Q_0\|_{\mathrm{TV}} \\ &\qquad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \widetilde{\operatorname{sign}}\Big(\frac{p_s}{q_s}-1\Big)(x)\nabla\cdot\Big[a(s,x)\nabla\Big[\frac{p_s}{q_s}\Big](x)q_s(x)\Big]\,dx\,ds, \end{aligned} \tag{1.10}$$

where $\widetilde{\operatorname{sign}}(r) = -\mathbf{1}_{(-\infty,0)}(r)+\mathbf{1}_{(0,\infty)}(r)$ *and the integral is nonpositive for all* $t\in[0,T]$.

The proof is given in Appendix A.3.

REMARK 1.8. (a) Denote by $\mathbb{Q}$ the law of $(X_t^Q, t\le T)$ and by $\mathbb{E}$ the corresponding expectation. The following "forward" version of formula (1.8) holds under the assumptions of Theorem 1.4 if moreover $\frac{p_t}{q_t}(Y_t)$ is a continuous $(\mathcal{G}_t)$ semimartingale under $\mathbb{Q}$ [in particular if $(t,x)\mapsto \frac{dP_t}{dQ_t}(x)$ has a version of class $C^{1,2}$]:

$$\begin{aligned} H_U(P_t|Q_t) &= H_U(P_0|Q_0) \\ &\quad + \Delta U(0)\mathbb{Q}(0<S\le t) - \frac{1}{2}\int_{(0,+\infty)} \mathbb{E}\Big(L_t^r\Big(\frac{p_\cdot}{q_\cdot}(Y_\cdot)\Big)\Big)U''(dr) \\ &\qquad \forall t\in[0,T], \end{aligned}$$

where $S := \inf\{s \in [0,T] : \frac{p_s}{q_s}(Y_s) > 0\}$. This follows from the pathwise relation

$$L_T^r\left(\frac{p_{T-\cdot}}{q_{T-\cdot}}(X^Q_{T-\cdot})\right) - L_{T-t}^r\left(\frac{p_{T-\cdot}}{q_{T-\cdot}}(X^Q_{T-\cdot})\right) = L_t^r\left(\frac{p_\cdot}{q_\cdot}(X^Q_\cdot)\right)$$

and the fact that $(\frac{p_{T-t}}{q_{T-t}}(X^Q_{T-t}))_{t\in[0,T]}$ is a.s. stopped upon hitting 0, by Lemma 1.3.

(b) The limit type assumption in Corollary 1.7 is not too stringent. Thanks to (1.9) and the Cauchy–Schwarz inequality, it holds true, for instance, if the matrix a is uniformly bounded and $H_U(P_0|Q_0) < \infty$ for $U(r) = (r-1)^2$, since $|a\nabla[\frac{p_t}{q_t}]| = \sup_{|v|\le 1}(\sigma v)^*(\sigma\nabla\frac{p_t}{q_t}) \le \sqrt{|a|}\sqrt{\nabla^*[\frac{p_t}{q_t}]a\nabla[\frac{p_t}{q_t}]}$.

We end this section providing sufficient conditions in order that $\lim_{t\to\infty} H_U(P_t|Q_t) = 0$. The proof of the following result is differed to Appendix A.4.

PROPOSITION 1.9. *Let us assume that the coefficients σ and b are time-homogeneous and globally Lipschitz continuous. Then the semigroup associated with the SDE* (0.1) *is Feller. Let us also suppose that* (0.1) *admits an invariant density p_∞, locally Lipschitz and bounded away from* 0 *and* $+\infty$, *and such that* $\int_{\mathbb{R}^d} \frac{p_\infty^2(x)\,dx}{1+|x|^k} < +\infty$ *for some* $k > 0$ *and that* $-\frac{\partial_j(a_{\bullet j}p_\infty)}{p_\infty}$ *is the sum of a locally Lipschitz function and a monotone function. Finally, we suppose that*

$$\forall A > 0, \exists \varepsilon_A > 0, \forall |x| \le A \qquad a(x) \ge \varepsilon_A I_d \tag{1.11}$$

with either ε_A not depending on A or the second-order distribution derivatives $\frac{\partial a^{ij}}{\partial x_i \partial x_j}$ *bounded on* $\mathbb{R}^d$. *Then the tail sigma-field* $\bigcap_{t\ge 0}\sigma(X_r, r \ge t)$ *is trivial a.s. w.r.t. the law of* $(X_t^Q)_{t\ge 0}$. *In particular, if* $U(1) = 0$, *then as soon as* $H_U(P_s|Q_s) < +\infty$ *for some* $s < +\infty$, *one has* $\lim_{t\to\infty} H_U(P_t|Q_t) = 0$.

REMARK 1.10. The triviality of the tail sigma-field still holds when $(X_t)_{t\ge 0}$ is Feller, has an invariant distribution and a strictly positive transition density $\varphi_t(x,y)$ with respect to the Lebesgue measure which is continuous in (x,y) for each $t > 0$. (The continuity implies the strong Feller property, the positivity implies the ergodicity of the invariant measure and combining both, one checks that $(X_t)_{t\ge 0}$ is Harris recurrent. Then one concludes by Theorem 1.3.9 in [15].) Notice that conditions ensuring the positivity and joint continuity in (x,y) of $\varphi_t(x,y)$ can be found in [10], Chapter 9, under uniform ellipticity, and in [16], Theorem 4.5, under hypoellipticity.

2. Dissipation of the Fisher information and nonintrinsic Bakry–Emery criterion. From this point forward, we will focus on the case when $Q_0(dx) =$

$p_\infty(x)\,dx$ is a stationary probability law for the time-homogeneous Markov diffusion (0.1). We denote

$$I_U(p_s|p_\infty) = \frac{1}{2}\int_{\{p_s/p_\infty>0\}} U''\left(\frac{p_s}{p_\infty}\right)\nabla^*\left[\frac{p_s}{p_\infty}\right]a\nabla\left[\frac{p_s}{p_\infty}\right]p_\infty\,dx$$

the integral that appears in the right-hand side of (1.9), and we refer to it as the U-Fisher information.

Inspired by the famous Bakry–Emery approach, we want to compute the derivative of $I_U(p_s|p_\infty)$ with respect to the time variable.

Throughout the sequel, we make the following assumptions:

$(H4)$ The drift function b and the matrix σ are time-homogeneous and such that $(H1)$ holds. Moreover, b (resp., σ) admits first (resp., second) order derivatives which are locally α-Hölder-continuous on $\mathbb{R}^d$ for some $\alpha > 0$.

$(H5)_{p_\infty}$ The Markov process defined by (0.1) has an invariant density $p_\infty(x)$ and $Q_0(dx) = p_\infty(x)\,dx$. Moreover, p_∞ admits derivatives up to the second order which are locally α-Hölder-continuous on $\mathbb{R}^d$ for some $\alpha > 0$ and $p_\infty(x) > 0$ for all $x \in \mathbb{R}^d$.

$(H6)^T_{p_0}$ The initial distribution P_0 admits a probability density p_0 with respect to the Lebesgue measure. Moreover, we assume that $(H2)_{p_0}$ holds and that $p_t(x)$ has spatial derivatives up to the second order for each $t > 0$, which are continuous in $(t,x) \in (0,T] \times \mathbb{R}^d$ and bounded and Hölder continuous in $x \in \mathbb{R}^d$ uniformly on $[\delta,T] \times \mathbb{R}^d$ for each $\delta \in (0,T]$.

Let us also introduce some notation:

- We write $\mathbb{P}^T_\infty := \mathbb{Q}^T$ and $\bar{b}_i := \bar{b}^Q_i$, $i = 1,\dots,d$.
- By possibly enlarging the probability space $\mathcal{G}_t - \mathbb{P}^T_\infty$, we introduce a Brownian motion $\overline{W}$ such that $(Y_t)_{t\in[0,T]}$ solves the stochastic differential equation

$$dY_t = \sigma(Y_t)\,d\overline{W}_t + \bar{b}(Y_t)\,dt \tag{2.1}$$
$$\text{where } \bar{b}_i(y) = -b_i(y) + \frac{\partial_j(a_{ij}(y)p_\infty(y))}{p_\infty(y)}.$$

  By assumptions $(H4)$ and $(H5)_\infty$, the coefficients σ and $\bar{b}$ are locally Lipschitz so that trajectorial uniqueness holds for this SDE. By the Yamada–Watanabe theorem, one deduces that uniqueness holds for the martingale problem $(MP)_Q$.
- We write $\rho_t(x) := \frac{p_{T-t}}{p_\infty}(x)$, $t \in [0,T]$.

Notice that $(H5)_{p_\infty}$ implies $(H2)_Q$ for $Q_0(dx) = p_\infty(x)\,dx$ and combined with $(H4)$, it implies $(H3)_Q$. Moreover $(H6)^T_{p_0}$ implies $(H2)_P$ and $(H3)_P$. Therefore the hypotheses of Theorem 1.4 hold within the present section. Notice also that, under $(H5)_\infty$ and $(H6)^T_{p_0}$, the first-order spatial derivatives of $\frac{p_t}{p_\infty}$ are defined everywhere. Thus, we may and will assume in the sequel that Lemma 1.3(b) and equation (1.9) hold with the standard gradient $\nabla\frac{p_t}{p_\infty}$. Under $(H4)$, if moreover a and b are bounded with a uniformly elliptic, then $(H6)^T_{p_0}$ holds for any compactly supported probability density p_0, by [10] Chapter 9. We refer to [16] for conditions ensuring that $(H6)^T_{p_0}$ holds under hypoellipticity.

To compute the dissipation of the U-Fischer information, throughout the sequel we make the following regularity assumption on U:

$(H7)$ The convex function $U:[0,\infty)\to\mathbb{R}$ is of class C^4 on $(0,+\infty)$, continuous on $[0,+\infty)$ and satisfies $U(1)=U'(1)=0$.

The assumption that $U'(1)=0$ is inspired in the analysis on admissible entropies developed in Arnold et al. [2] and is granted without modifying the functions $p\mapsto H_U(p|p_\infty)$ and $p\mapsto I_U(p|p_\infty)$ by replacing $U(r)$ by $U(r)-U'(1)(r-1)$ if needed. Notice that if $(H7)$ holds, $U(r)$ attains the minimum 0 at $r=1$ and therefore $U\geq 0$ by convexity. Following [3], page 202 (see also [2, 7]), we introduce an additional assumption on U,

$(H7')$ $\forall r\in(0,\infty)$, $(U^{(3)}(r))^2\leq \frac{1}{2}U''(r)U^{(4)}(r)$,

which is satisfied, for instance, by $U(r)=r\ln r-(r-1)$ and by $U(r)=(r-1)^2$. Let us recall consequences of $(H7)'$ pointed out in [2] (see Remark 2.3 therein) which will be used in proving the following results.

REMARK 2.1. Condition $(H7')$ implies that $(\frac{1}{U''})''\leq 0$ at points where $U''\neq 0$. Since $U''\geq 0$, and excluding the uninteresting case where U'' identically vanishes, the previous implies that $\frac{1}{U''}$ is finite in $[0,\infty)$, and therefore that U is strictly convex. We then deduce from $(H7')$ that $U^{(4)}\geq 0$ in $(0,\infty)$. By concavity and positivity of $\frac{1}{U''}$ this function is moreover nondecreasing, and we deduce that $U^{(3)}\leq 0$ in $(0,\infty)$.

We do not assume that the entropy function U is C^4 on the closed interval $[0,+\infty)$, since we want to deal with $U(r)=r\ln(r)-(r-1)$. This is why we introduce some regularization U_δ indexed by a positive parameter δ: we choose U_δ such that $U_\delta(r)=U(r+\delta)$ for $r\geq 0$, and U_δ is extended to a C^4 function on $\mathbb{R}$. In the next proposition as well as in the remaining of the paper, we will omit the argument (t,Y_t) in order to obtain more compact formulas.

PROPOSITION 2.2. *Under $(H4)$, $(H5)_{p_\infty}$, $(H6)^T_{p_0}$ and $(H7)$, one has on the time-interval* $[0,T]$,

$$d\big[U''_\delta(\rho)\nabla^*\rho a\nabla\rho\big]=\mathrm{tr}(\Lambda_\delta\Gamma)\,dt+U''_\delta(\rho)\bar{\theta}\,dt+d\widehat{M}^{(\delta)}$$

$$\textit{with } \mathrm{tr}(\Lambda_\delta\Gamma)\geq 0 \textit{ under } (H7)'$$

and where $\widehat{M}^{(\delta)}_t=\int_0^t\partial_k[U''_\delta(\rho)\nabla^*\rho a\nabla\rho]\sigma_{kr}\,d\overline{W}^r_s$ *is a* $\mathcal{G}_t-\mathbb{P}^T_\infty$*-local martingale,*

$$\begin{aligned}\bar{\theta}=2\big\{\partial_{l'}\rho\partial_l\rho\big[&\tfrac{1}{4}(\partial_k\sigma_{lj}a_{km}\partial_m\sigma_{l'j}-\sigma_{ki}\partial_k\sigma_{lj}\sigma_{mj}\partial_m\sigma_{l'i})\\&+\tfrac{1}{2}\bar{b}_m\partial_m a_{ll'}+\tfrac{1}{2}\sigma_{l'i}a_{mk}\partial_{mk}\sigma_{li}-a_{ml'}\partial_m\bar{b}_l\big]\\&+[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho\big\},\end{aligned}$$

and Λ_δ *and* Γ *are the square matrices defined by*

$$\Lambda_\delta := \begin{bmatrix} U''_\delta(\rho) & U^{(3)}_\delta(\rho) \\ U^{(3)}_\delta(\rho) & \frac{1}{2}U^{(4)}_\delta(\rho) \end{bmatrix},$$

$$\Gamma := \begin{bmatrix} \Gamma_{11} & (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) \\ (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) & |\nabla^*\rho a\nabla\rho|^2 \end{bmatrix}$$

with $\Gamma_{11} = \sum_{i,j=1}^d(\sigma_{kj}\sigma_{li}\partial_{kl}\rho + \frac{1}{2}(\sigma_{kj}\partial_k\sigma_{li} + \sigma_{ki}\partial_k\sigma_{lj})\partial_l\rho)^2$.

The computation of $d[U''_\delta(\rho)\nabla^*\rho a\nabla\rho]$ is postponed to Appendix B. Let us nevertheless discuss the sign of the term $\mathrm{tr}(\Lambda_\delta\Gamma)$ which is inspired from [3], page 202, and also from the term $\mathrm{tr}(\mathbf{XY})$ in [1], pages 163–164; see Appendix C for a detailed comparison with the computations in that paper. Since, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\big((\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho a\nabla(\sigma_{\bullet i}\cdot\nabla\rho)\big)^2 \\
&\quad = \bigg((\sigma^*\nabla\rho)_i(\sigma^*\nabla\rho)_j\bigg(\sigma_{kj}\sigma_{li}\partial_{kl}\rho + \frac{1}{2}(\sigma_{kj}\partial_k\sigma_{li} + \sigma_{ki}\partial_k\sigma_{lj})\partial_l\rho\bigg)\bigg)^2 \\
&\quad \le \Gamma_{11}\sum_{i,j=1}^d(\sigma^*\nabla\rho)_i^2(\sigma^*\nabla\rho)_j^2 = \Gamma_{11}|\nabla^*\rho a\nabla\rho|^2,
\end{aligned}$$

the determinant of the matrix Γ is nonnegative, and this matrix is positive semidefinite. Under $(H7')$, Λ_δ is also positive semidefinite and $\mathrm{tr}(\Lambda_\delta\Gamma)\ge 0$.

REMARK 2.3. In a previous version of this paper, the coefficient Γ_{11} was chosen equal to

$$\begin{aligned}
\sum_{i,j=1}^d(\sigma_{kj}\sigma_{li}\partial_{kl}\rho + \sigma_{kj}\partial_k\sigma_{li}\partial_l\rho)^2 &= \sum_{i,j=1}^d\big(\sigma^*\nabla(\sigma^*\nabla\rho)_i\big)_j^2 \\
&= \nabla^*(\sigma\nabla\rho)_i a\nabla(\sigma\nabla\rho)_i.
\end{aligned}$$

We thank Anton Arnold for pointing out to us that the positive semidefiniteness of the matrix Γ is preserved under the new choice of this coefficient. Notice that, by symmetry of $\sigma_{kj}\sigma_{li}\partial_{kl}\rho$ in i and j,

$$\begin{aligned}
&\sum_{i,j=1}^d(\sigma_{kj}\sigma_{li}\partial_{kl}\rho + \sigma_{kj}\partial_k\sigma_{li}\partial_l\rho)^2 - \Gamma_{11} \\
&\qquad = \frac{1}{4}\sum_{i,j=1}^d\big((\sigma_{kj}\partial_k\sigma_{li} - \sigma_{ki}\partial_k\sigma_{lj})\partial_l\rho\big)^2 \\
&\qquad = \frac{1}{2}(\partial_k\sigma_{lj}a_{km}\partial_m\sigma_{l'j} - \sigma_{ki}\partial_k\sigma_{lj}\sigma_{mj}\partial_m\sigma_{l'i})\partial_l\rho\partial_{l'}\rho
\end{aligned}$$

is a nonnegative quadratic form applied to $\nabla\rho$ which implies that the Bakry–Emery criterion below improves upon the one of the previous version.

We introduce one last assumption on the density flow $\rho_t = \frac{p_{T-t}}{p_\infty}$:

$(H6')^T_{p_0}$ For each $T' \in (0, T)$ the following integrals are finite:

- $\int_0^{T'} |U^{(3)}(\rho) \vee -1|^2 |\nabla^*\rho a \nabla\rho|^3 p_\infty(x)\, dx\, dt$;
- $\int_0^{T'} (U''(\rho) \wedge 1)^2 \nabla^*(\nabla^*\rho a\nabla\rho) a \nabla(\nabla^*\rho a \nabla\rho) p_\infty(x)\, dx\, dt$;
- $$\int_0^{T'} (U''(\rho)\wedge 1)\big[\big|(\sigma_{l'i}a_{m\bullet} - \sigma_{\bullet i}a_{ml'})\partial_m\sigma_{li}\big| + \big|\partial_k\big([\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li}\big)\big|\big]|\partial_{l'}\rho||\partial_l\rho| p_\infty(x)\, dx\, dt;$$
- $$\int_0^{T'} (U''(\rho)\wedge 1)\big|(\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}) \times \partial_m\sigma_{li}(\partial_l\rho\partial_k \ln p_\infty + \partial_{lk}\rho)\big||\partial_{l'}\rho| p_\infty(x)\, dx\, dt.$$

We also denote by $(H6)^\infty_{p_0}$ [resp., $(H6')^\infty_{p_0}$] the assumption that $(H6)^T_{p_0}$ [resp., $(H6')^T_{p_0}$] holds for each $T > 0$.

THEOREM 2.4. *Let Θ denote the $d \times d$ symmetric matrix defined by*

$$\begin{aligned}\Theta_{ll'} = {}& -\tfrac{1}{2}b_m\partial_m a_{ll'} + \tfrac{1}{2}(a_{kl'}\partial_k b_l + a_{kl}\partial_k b_{l'}) - \tfrac{1}{4}a_{mk}\partial_{mk}a_{ll'}\\ & - \tfrac{1}{2}(a_{kl'}\partial_{kj}a_{lj} + a_{kl}\partial_{kj}a_{l'j}) - a_{kl}a_{jl'}\partial_{kj}\ln(p_\infty)\\ & - \tfrac{1}{2}(a_{kl}\partial_k a_{l'j} + a_{kl'}\partial_k a_{lj})\partial_j \ln(p_\infty)\\ & - \tfrac{1}{4}(a_{mk}\partial_m\sigma_{li}\partial_k\sigma_{l'i} + \sigma_{ki}\partial_k\sigma_{lj}\sigma_{mj}\partial_m\sigma_{l'i})\\ & + \tfrac{1}{2}\sigma_{ki}(\partial_m\sigma_{li}a_{ml'} + \partial_m\sigma_{l'i}a_{ml})\partial_k\ln(p_\infty)\\ & + \tfrac{1}{2}\partial_k\big[\sigma_{ki}(\partial_m\sigma_{li}a_{ml'} + \partial_m\sigma_{l'i}a_{ml})\big],\end{aligned}$$

and assume that $\Theta(x)$ is $p_\infty(x)\,dx$-a.e. positive semidefinite. Then, under $(H4)$, $(H5)_{p_\infty}$, $(H6)^T_{p_0}$ $(H6')^T_{p_0}$, $(H7)$ and $(H7')$, for a.e. $t \in [0, T]$ one has

$$\frac{d}{dt}\int_{\rho_t>0} U''(\rho_t)\big[\nabla^*\rho_t a \nabla\rho_t\big] p_\infty\, dx \geq 2\int_{\rho_t>0} U''(\rho_t)\nabla^*\rho_t\Theta\nabla\rho_t p_\infty\, dx. \tag{2.2}$$

If moreover $I_U(p_0|p_\infty) < +\infty$, $(H6)^\infty_{p_0}$ and $(H6')^\infty_{p_0}$ hold and the matrix Θ satisfies the nonintrinsic Bakry–Emery criterion

(NIBEC) $\exists\lambda > 0, \forall x \in \mathbb{R}^d, \Theta(x) \geq \lambda a(x)$,

then $\forall t \geq 0$, $I_U(p_t|p_\infty) \leq e^{-2\lambda t} I_U(p_0|p_\infty)$ *and the nonincreasing function* $t \mapsto H_U(p_t|p_\infty)$ *also converges at exponential rate* 2λ *to its limit as* $t \to \infty$.

REMARK 2.5.

- The matrix Θ and therefore our Bakry–Emery criterion are nonintrinsic in the sense that they cannot in general be written in terms of the diffusion matrix a only without making explicit use of σ. This is because we got rid of the nonnegative term $\operatorname{tr}(\Lambda_\delta \Gamma)$ which appears in the first equation in Proposition 2.2 and involves the nonintrinsic term Γ_{11}.
- In case $a = 2\nu I_d$ and $b = -(\nabla V + F)$ with F such that $\nabla \cdot (e^{-V/\nu} F) = 0$, then $p_\infty \propto e^{-V/\nu}$, $\bar{b} = -b + 2\nu \nabla \ln p_\infty = -\nabla V + F$ and $\Theta = \nu(2\nabla^2 V - \nabla F - \nabla F^*)$. For the canonical choice $\sigma = \sqrt{2\nu} I_d$, condition (NIBEC) therefore writes $\exists\lambda > 0, \forall x \in \mathbb{R}^d, \nabla^2 V(x) - \frac{\nabla F + \nabla F^*}{2}(x) \geq \lambda I_d$ which is exactly condition (A2) in the Introduction of [1], page 158.

The proof of (2.2) is postponed to Appendix B.2. Let us deduce the last assertion of Theorem 2.4. Reverting time in (2.2) and using (NIBEC), one obtains that for $r \geq 0$,

$$\frac{d}{dr} I_U(p_r|p_\infty) \leq -2\lambda I_U(p_r|p_\infty).$$

Hence $\forall r \geq t \geq 0$, $I_U(p_r|p_\infty) \leq e^{-2\lambda(r-t)} I_U(p_t|p_\infty)$. Since by Theorem 1.4, one has $\frac{d}{dr} H_U(p_r|p_\infty) = -I_U(p_r|p_\infty)$, we deduce that

$$\begin{aligned}
0 &\leq H_U(p_t|p_\infty) - \lim_{r\to\infty} H_U(p_r|p_\infty) \\
&= \int_t^\infty I_U(p_r|p_\infty)\,dt \qquad (2.3)\\
&\leq \frac{I_U(p_t|p_\infty)}{2\lambda} \leq \frac{e^{-2\lambda t} I_U(p_0|p_\infty)}{2\lambda}.
\end{aligned}$$

We deduce the following theorem.

THEOREM 2.6. *Assume* $(H4)$, $(H5)_{p_\infty}$, $(H6)_{p_0}^\infty$ $(H6')_{p_0}^\infty$, $(H7)$ *and* $(H7')$, *that the matrix* $\Theta(x)$ *is* $p_\infty(x)\,dx$*-a.e. positive semidefinite, that the diffusion matrix* a *is locally uniformly strictly positive definite and that* $H_U(p_s|p_\infty) < +\infty$ *for some* $s \geq 0$. *Then* $H_U(p_t|p_\infty)$ *converges to* 0 *as* $t \to \infty$. *Moreover, under* (NIBEC), *for* $t > s$, *one has the convex Sobolev inequality*

$$H_U(p_t|p_\infty) \leq \frac{1}{2\lambda} I_U(p_t|p_\infty) \quad \text{and} \tag{2.4}$$

$$\forall t \geq s \qquad H_U(p_t|p_\infty) \leq e^{-2\lambda(t-s)} H_U(p_s|p_\infty). \tag{2.5}$$

PROOF. Reverting time in (2.2), we obtain that $t \mapsto I_U(p_t|p_\infty)$ is nonincreasing. When $H_U(p_s|p_\infty)$ is finite for some $s \geq 0$, writing (1.9) on the interval $[s, T]$ in place of $[0, T]$ with arbitrarily large T, we deduce that $I_U(p_t|p_\infty)$ is finite on $(s, +\infty)$ and tends to 0 as $t \to \infty$. When a is locally uniformly strictly positive definite, the beginning of the proof of Theorem 2.5 [1] [before part (a)], ensures that p_t tends to p_∞ in $L^1(\mathbb{R}^d)$. As a consequence, in the notation of the Introduction, $\mathbb{E}|\frac{dP_t}{dQ_t}(X_t^Q) - 1|$ tends to 0 as $t \to \infty$, and therefore the a.s. limit $\mathbb{E}(\frac{dP_t}{dQ_t}(X_t^Q)|\bigcap_{s\geq 0}\mathcal{F}_s)$ of $\frac{dP_t}{dQ_t}(X_t^Q)$ is equal to 1. By (0.2), one concludes that $H_U(p_t|p_\infty)$ tends to $U(1) = 0$.

Under (NIBEC), for $t > s$, $I_U(p_t|p_\infty) < +\infty$ and reasoning like in the derivation of (2.3), one obtains (2.4). This implies that

$$\frac{d}{dt}H_U(p_t|p_\infty) = -I_U(p_t|p_\infty) \leq -2\lambda H_U(p_t|p_\infty)$$

from which the last assertion follows readily. $\square$

REMARK 2.7. In view of (0.2) and Remark 1.10, the local uniform strict positive definiteness assumption on the diffusion matrix a may be replaced by some hypoellipticity assumption, in order to ensure that $H_U(p_t|p_\infty)$ tends to 0 as $t \to \infty$ at exponential rate 2λ as soon as $H_U(p_s|p_\infty) < \infty$ for some $s \geq 0$. By the last step of the proof of Theorem 2.6, this implies (2.4) and (2.5) under (NIBEC).

3. Examples. Consider the reversible diffusion process in $\mathbb{R}^2$ with coefficients given for each $(x_1, x_2) \in \mathbb{R}^2$ by

$$a(x_1, x_2) = I_2 \quad \text{and} \quad b(x_1, x_2) = -\nabla V(x_1, x_2),$$

where V is the globally C^2 convex potential

$$V(x_1, x_2) := |x_1|^2 + |x_1 - x_2|^{2+\alpha} + |x_2|^{2+\alpha}$$

for some $\alpha \in (0, 1)$. The invariant measure is $p_\infty \propto e^{-2V}$, and we have

$$\begin{aligned}
\partial_1 V &= 2x_1 + (2+\alpha)\,\mathrm{sign}(x_1 - x_2)|x_1 - x_2|^{1+\alpha},\\
\partial_2 V &= (2+\alpha)\,\mathrm{sign}(x_2)|x_2|^{1+\alpha} + (2+\alpha)\,\mathrm{sign}(x_2 - x_1)|x_2 - x_1|^{1+\alpha}
\end{aligned}$$

and

$$\nabla^2 V = \begin{pmatrix} 2 & 0 \\ 0 & (2+\alpha)(1+\alpha)|x_2|^\alpha \end{pmatrix} + (2+\alpha)(1+\alpha)|x_1 - x_2|^\alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The drift $b = -\nabla V$ is locally Lipschitz continuous. Moreover,

$$(x_1, x_2)\nabla V(x_1, x_2) \geq 0$$

and $\sqrt{\partial_{ik} V \partial_{ik} V(x_1, x_2)} \le C\sqrt{1 + |x_2|^{2\alpha} + |x_1 - x_2|^{2\alpha}}$ so that

$$\limsup_{|(x_1,x_2)|\to+\infty} \frac{\sqrt{\partial_{ik} V \partial_{ik} V(x_1, x_2)}}{V(x_1, x_2)} = 0.$$

Finally $\Delta V(x_1, x_2) \le C(1 + |x_2|^\alpha + |x_1 - x_2|^\alpha)$ whereas

$$\begin{aligned}|\nabla V|^2(x_1, x_2) &\ge \big(2|x_2| + (2+\alpha)|x_1 - x_2|^{1+\alpha}\big)^2 \mathbf{1}_{\mathrm{sign}(x_2)\neq\mathrm{sign}(x_2-x_1)} \\ &\quad + (2+\alpha)^2\big(|x_2|^{1+\alpha} + |x_1 - x_2|^{1+\alpha}\big)^2 \mathbf{1}_{\mathrm{sign}(x_2)=\mathrm{sign}(x_2-x_1)}\end{aligned}$$

since $\mathrm{sign}(x_2) \neq \mathrm{sign}(x_2 - x_1)$ if and only if $x_1 \ge x_2 \ge 0$ or $x_1 \le x_2 \le 0$. Therefore $\limsup_{|(x_1,x_2)|\to+\infty} \frac{\Delta V}{|\nabla V|^2}(x_1, x_2) = 0$ and, by Remark 1.5(b), $(H1)''$ is satisfied.

The classic Bakry–Emery criterion fails since $\nabla^2 V(0, 0)$ is singular, but a logarithmic Sobolev inequality can be obtained by the perturbative argument of Holley and Stroock [12]. The potential V is also a particular case of the examples considered by Arnold, Carlen and Ju in Section 3 of [1]. We notice that in order to check that p_∞ satisfies the convex Sobolev inequality (2.4), they first modify the Fokker–Planck equation by adding a nonsymmetric drift term F, as described in Remark 2.5(ii), above. Exponential convergence to 0 of $H_U(p_t|p_\infty)$ for the solution p_t of the original Fokker–Planck equation is only deduced in a second step.

It is nevertheless of interest to see how our nonintrinsic Bakry–Emery criterion allows us to prove directly that p_∞ satisfies the convex Sobolev inequality (2.4) and that $H_U(p_t|p_\infty)$ converges exponentially to 0. In contrast to [1] we modify the stochastic differential equation associated with the diffusion processes, by changing the square root σ of the diffusion matrix, but not the law of its solution or the associated Fokker–Planck equation. We consider

$$\sigma = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

for a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^2 to be chosen later. We obtain, after some computation,

$$\begin{aligned}\Theta = \nabla^2 V &- \frac{1}{4}|\nabla\phi|^2 I_2 - \frac{1}{4}\begin{pmatrix} (\partial_2\phi)^2 & -\partial_1\phi\partial_2\phi \\ -\partial_1\phi\partial_2\phi & (\partial_1\phi)^2 \end{pmatrix} \\ &+ \begin{pmatrix} \partial_{12}\phi & \dfrac{\partial_{22}\phi - \partial_{11}\phi}{2} \\ \dfrac{\partial_{22}\phi - \partial_{11}\phi}{2} & -\partial_{12}\phi \end{pmatrix} \\ &+ \begin{pmatrix} -2\partial_1\phi\partial_2 V & \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V \\ \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V & 2\partial_2\phi\partial_1 V \end{pmatrix}.\end{aligned}$$

We now consider a parameter $\varepsilon > 0$ which will be chosen small and a C^2 function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(s) = s$ if $|s| \le 1$ and $\varphi(s) = 0$ if $|s| \ge 2$. Then we define

$$\phi(x_1,x_2) = -\varepsilon\varphi_\varepsilon(x_1)\varphi_\varepsilon(x_2), \qquad (x_1,x_2)\in\mathbb{R}^2,$$

where $\varphi_\varepsilon(s) = \varepsilon\varphi(s/\varepsilon)$. Notice that

$$\varphi_\varepsilon = O(\varepsilon), \qquad \varphi_\varepsilon'' = O(1/\varepsilon) \quad \text{and} \quad \varphi_\varepsilon' = \begin{cases} 1, & \text{if } |s|\le\varepsilon, \\ O(1), & \text{if } \varepsilon<|s|<2\varepsilon, \\ 0, & \text{if } |s|\ge 2\varepsilon. \end{cases}$$

Then, defining $B_\varepsilon := \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } |x_1|\vee|x_2|\le\varepsilon\}$ and $C_\varepsilon := B_{2\varepsilon}\setminus B_\varepsilon$, we have

$$\partial_1\phi(x_1,x_2),\, \partial_2\phi(x_1,x_2) = \begin{cases} O(\varepsilon^2), & \text{if } (x_1,x_2)\in B_{2\varepsilon}, \\ 0, & \text{if } (x_1,x_2)\in B_{2\varepsilon}^c, \end{cases}$$

$$\partial_{12}\phi(x_1,x_2) = \begin{cases} -\varepsilon, & \text{if } (x_1,x_2)\in B_\varepsilon, \\ O(\varepsilon), & \text{if } (x_1,x_2)\in C_\varepsilon, \\ 0, & \text{if } (x_1,x_2)\in B_{2\varepsilon}^c, \end{cases}$$

$$\frac{1}{2}\big(\partial_{11}\phi(x_1,x_2) - \partial_{22}\phi(x_1,x_2)\big) = \begin{cases} 0, & \text{if } (x_1,x_2)\in B_\varepsilon, \\ O(\varepsilon), & \text{if } (x_1,x_2)\in C_\varepsilon, \\ 0, & \text{if } (x_1,x_2)\in B_{2\varepsilon}^c \end{cases}$$

and $\partial_1 V = O(\varepsilon)$, $\partial_2 V = O(\varepsilon^{1+\alpha})$ on $B_{2\varepsilon}$. It follows that

$$\Theta = \nabla^2 V + \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) \ge \begin{pmatrix} 2-\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) \qquad \text{on } B_\varepsilon.$$

Next, the smallest eigenvalue of $\nabla^2 V(x_1,x_2)$ is given by

$$\begin{aligned} \gamma_- &:= 1+\kappa_1+\kappa_2/2 - \sqrt{1+\kappa_1^2-\kappa_2+\kappa_2^2/4} \\ &\ge 1+\kappa_2/2 - \sqrt{(\kappa_2/2-1)^2} = \kappa_2\wedge 2 \end{aligned} \tag{3.1}$$

with $\kappa_1 = \kappa_1(x_1,x_2) := (2+\alpha)(1+\alpha)|x_1-x_2|^\alpha$ and $\kappa_2 = \kappa_2(x_1,x_2) := (2+\alpha)(1+\alpha)|x_2|^\alpha$. Since $\gamma_- = \kappa_1+\kappa_2+O(\kappa_1^2+\kappa_2^2)$ as $\kappa_1^2+\kappa_2^2\to 0$ and $|x_2|^\alpha + |x_1-x_2|^\alpha \ge (|x_2|+|x_1-x_2|)^\alpha \ge |x_1|^\alpha$, we deduce that on C_ε,

$$\Theta = \nabla^2 V + O(\varepsilon) \ge (2+\alpha)(1+\alpha)\varepsilon^\alpha I_2 + o(\varepsilon^\alpha).$$

Finally, by (3.1), $\inf_{(x_1,x_2)\in B_{2\varepsilon}^c}\gamma_- \ge ((2+\alpha)(1+\alpha)(2\varepsilon)^\alpha)\wedge 2 > 0$. We conclude that for ε small enough, (NIBEC) holds.

We next study a related second example of application of our criterion, where $\nabla^2 V$ is singular on a ball with positive radius. Once again, the perturbative argument of Holley and Stroock [12] also ensures that a logarithmic Sobolev inequality holds for this choice of potential.

Let v be a convex C^2 function which vanishes on $[-\frac{1}{4}, \frac{1}{4}]$ and such that $v'' = 2$ on $(-\infty, \frac{1}{2}] \cup [\frac{1}{2}, +\infty)$. We set $v_\varepsilon(s) = \varepsilon^2 v(\frac{s}{\varepsilon})$ and $V_\varepsilon(x_1, x_2) = x_1^2 + v_\varepsilon(x_2) + v_\varepsilon(x_1 - x_2)$. For $\varepsilon < \frac{1}{3}$, let φ_ε be a C^2 function such that

$$\varphi_\varepsilon(s) = \begin{cases} s, & \text{when } |s| \le \varepsilon, \\ 0, & \text{when } |s| \ge 1 \end{cases}$$

and such that $\frac{-2\varepsilon}{1-\varepsilon} \le \varphi'_\varepsilon \le 1$, $|\varphi_\varepsilon| \le 2\varepsilon$ and $|\varphi''_\varepsilon| \le C$ where C is a constant not depending on ε. We set $\phi(x_1, x_2) = -\varphi_\varepsilon(x_1)\varphi_\varepsilon(x_2)$ so that $-1 \le \partial_{12}\phi(x_1, x_2) \le \frac{2\varepsilon}{1-\varepsilon}$ with the first inequality being an equality on B_ε. We have $|\partial_{22}\phi - \partial_{11}\phi| \le 4C\varepsilon$ and $|\nabla\phi| = O(\varepsilon)$. As a consequence, $\Theta = \widehat{\Theta} + O(\varepsilon)$ where

$$\widehat{\Theta} = \begin{pmatrix} 2 + v''_\varepsilon(x_1 - x_2) + \partial_{12}\phi(x_1, x_2) & -v''_\varepsilon(x_1 - x_2) \\ -v''_\varepsilon(x_1 - x_2) & v''_\varepsilon(x_2) + v''_\varepsilon(x_1 - x_2) - \partial_{12}\phi(x_1, x_2) \end{pmatrix}.$$

On B_ε, we have $\partial_{12}\phi(x_1, x_2) = -1$ and $\widehat{\Theta} \ge I_2$. If $|x_2| \ge \frac{\varepsilon}{2}$, then $v''_\varepsilon(x_2) = 2$ so that $\widehat{\Theta} \ge (2-1) \wedge (2 - \frac{2\varepsilon}{1-\varepsilon})I_2$. When $|x_2| \le \frac{\varepsilon}{2}$ and $|x_1| > \varepsilon$, $|x_1 - x_2| \ge \frac{\varepsilon}{2}$ holds so that $v''_\varepsilon(x_1 - x_2) = 2$ and

$$\begin{aligned} \widehat{\Theta} &\ge \begin{pmatrix} 4 + \partial_{12}\phi & -2 \\ -2 & 2 - \partial_{12}\phi \end{pmatrix} \ge \big(3 - \sqrt{5 + 2\partial_{12}\phi - (\partial_{12}\phi)^2}\big) I_2 \\ &\ge \left(3 - \sqrt{5 + \frac{4\varepsilon}{1-\varepsilon}}\right) I_2. \end{aligned}$$

We conclude that

$$\forall \lambda \in (0, 3 - \sqrt{5}) \qquad \text{for } \varepsilon > 0 \text{ and small enough, } \forall x \in \mathbb{R}^d, \Theta(x) \ge \lambda I_2.$$

APPENDIX A

In the present Appendix section we give the proofs of the main results of Section 1.

A.1. Proof of Lemma 1.3. The proof of part (a) relies on the following technical result:

LEMMA A.1. *Assume that $(H1)$, $(H2)_P$ and $(H3)_P$ hold.*

(i) *For each $i = 1, \dots, d$ and a.e. $t \in (0, T]$, the distribution $[a_{ij}(t, \cdot)\partial_j p_t] := \partial_j(a_{ij}(t, \cdot)p_t) - p_t \partial_j a_{ij}(t, \cdot)$ is a function in $L^1_{\text{loc}}(dx)$ and, as a Radon measure in $[0, T] \times \mathbb{R}^d$, one has $[a_{ij}(t, \cdot)\partial_j p_t](x)\, dx\, dt \ll p_t(x)\, dx\, dt$. A measurable in (t, x) version of the Radon–Nikodym density is given by $[a_{ij}(t, \cdot)\partial_j p_t](x)/p_t(x)$. Moreover, there exists a measurable function $(t, x) \mapsto K^p(t, x) \in \mathbb{R}^d$ such that for each $i = 1, \dots, d$,*

$$\big[a_{ij}(t, \cdot)\partial_j p_t\big](x)/p_t(x) = a_{i\bullet}(t, x)K^p(t, x), \qquad p_t(x)\, dx\, dt \text{ a.e.},$$

where $a_{i\bullet}$ denotes the row vector $(a_{i1}, \dots, a_{id})$.

(ii) *If, moreover, $(H2)_Q$, $(H3)_Q$ and $P_0 \ll Q_0$ hold, one has $[a_{ij}(t,\cdot)\partial_j p_t](x)\,dx\,dt \ll q_t(x)\,dx\,dt$, and $[a_{ij}(t,\cdot)\partial_j p_t](x)/q_t(x)$ is a measurable in (t,x) version of the Radon–Nikodym derivative. Furthermore, it holds $p_{T-t}(x)\,dx\,dt$ [but not necessarily $q_{T-t}(x)\,dx\,dt$] a.e. that*

$$\begin{aligned}\bar b_i^P(t,x) - \bar b_i^Q(t,x) &= [\bar a_{ij}(t,\cdot)\partial_j p_{T-t}](x)/p_{T-t}(x)\\ &\quad - [\bar a_{ij}(t,\cdot)\partial_j q_{T-t}](x)/q_{T-t}(x)\\ &= \bar a_{i\bullet}(t,x)\big(K^p(T-t,x) - K^q(T-t,x)\big),\end{aligned}$$

and $q_{T-t}(x)\,dx\,dt$ [and thus $p_{T-t}(x)\,dx\,dt$] a.e. that

$$\begin{aligned}&\frac{p_{T-t}(x)}{q_{T-t}(x)}\big(\bar b_i^P(t,x) - \bar b_i^Q(t,x)\big)\\ &\qquad = \frac{p_{T-t}(x)}{q_{T-t}(x)}\bar a_{i\bullet}(t,x)\big(K^p(T-t,x) - K^q(T-t,x)\big).\end{aligned}$$

PROOF. The Lipschitz character of a [following from $(H1)$] ensures that a has a.e. defined spatial derivatives of order 1 in $L^\infty_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$. Thus, the distribution $a_{ij}(t,\cdot)\partial_j p_t$ is a function in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ under $(H3)_P$. This implies, by Lemma A.2 in [17] (see also Lemma A.2 in [11]), that $a_{ij}(t,x)\partial_j p_t(x)$ vanishes a.e. on $\{x : p_t(x) = 0\}$. This fact easily yields the remaining assertions, except the existence of the functions K^p or K^q, which we establish in what follows.

We will, on one hand, use the fact asserted in the proof of Lemma A.2 in [17] that, for a.e. $t > 0$ and each bounded open set O, $a_{ij}(t,x)\partial_j p_t(x)$ is the $\sigma(L^1(O), L^\infty(O))$-weak limit of some subsequence of $a_{ij}(t,x)\partial_j[\rho_n * p_t](x)$, for compactly supported regularizing kernels $\rho_n(x) = n^d\rho(nx)$. It is indeed shown in Lemma A.1 in [11] that for a suitable bounded sequence $\alpha_n > 0$, $\alpha_n^{-1}|x||\nabla\rho_n(x)|$ is again a regularizing kernel. The local Lipschitz character of a then yields the domination $\forall x \in O$, $|a_{ij}(t,x)\partial_j[\rho_n * p_t](x)| \le |\rho_n * \partial_j(a_{ij}(t,\cdot)p_t)(x)| + C\alpha_n^{-1}\int |x-y||\nabla\rho_n(x-y)|p_t(y)\,dy$, the right-hand side being, by the previous, an $L^1(O)$-converging sequence. Weak compactness is then provided by the Dunford–Pettis criterion, and the limit is identified integrating by parts against smooth test functions compactly supported in O. On the other hand, diagonalizing the symmetric positive semidefinite matrix $(a_{ij}(t,x)) = [u_1(t,x),\dots,u_d(t,x)]\Lambda(t,x)[u_1(t,x),\dots,u_d(t,x)]^*$ provides orthonormal vectors $(u_i(t,x))_{i=1}^d$ and the corresponding eigenvalues and diagonal components $(\lambda_i(t,x))_{i=1}^d$ of $\Lambda(t,x)$, that are measurable as functions of (t,x).

We take O as before and $a_{ij}(t,x)\partial_j[\rho_n * p_t](x)$ to be the subsequence described above. Defining the vectorial functions $w^{(n)} := [u_1,\dots,u_d]^*\nabla[\rho_n * p_t]$ and $v_k = \mathrm{sign}(u_k^*[a\nabla p])u_k$, $k = 1,\dots,d$, we have

$$\begin{aligned}\int_{O\cap\{\lambda_k=0\}} \big|v_k^*[a\nabla p_t]\big| &= \lim_{n\to\infty}\int_{O\cap\{\lambda_k=0\}} v_k^*[a\nabla[\rho_n * p_t]]\\ &= \lim_{n\to\infty}\int_{O\cap\{\lambda_k=0\}} \lambda_k w_k^{(n)}\,\mathrm{sign}\big(u_k^*[a\nabla p_t]\big) = 0,\end{aligned}$$

since $a\nabla[\rho_n * p_t] = \sum_{j=1}^d \lambda_j w_j^{(n)} u_j$ by the spectral decomposition of a. Consequently, for each t and a.e. $x \in \mathbb{R}^d$, the vector $[a(t,x)\nabla p_t(x)]$ belongs to the linear space $\langle (u_i(t,x))_{i=1,\dots,d;\lambda_i(t,x)\neq 0}\rangle$. Denote now by $w = (w_j)_{j=1}^d := (u_j^* a \nabla p_t)_{j=1}^d$ the coordinates of $a\nabla p_t$ w.r.t. the orthogonal basis $(u_j(t,x))_{j=1,\dots,d}$, so that w is a measurable function of (t,x). If we moreover denote by $\overline{\Lambda}$ the diagonal matrix with diagonal coefficients $\lambda_j^{-1}\mathbf{1}_{\lambda_j\neq 0}$, $j=1,\dots,d$, and set $v := [u_1,\dots,u_d]\overline{\Lambda}w$, then

$$
\begin{aligned}
av &= [u_1,\dots,u_d]\Lambda[u_1,\dots,u_d]^*[u_1,\dots,u_d]\overline{\Lambda}w = [u_1,\dots,u_d]\Lambda\overline{\Lambda}w \\
&= [u_1,\dots,u_d]w
\end{aligned}
$$

since $w = (w_j \mathbf{1}_{\lambda_j\neq 0})_{j=1}^d$. That is, $(t,x)\mapsto v(t,x)\in\mathbb{R}^d$ is a measurable function such that for almost every $t\in[0,T]$ and each i, $a_{i\bullet}(t,x)v(t,x) = [a_{ij}\partial_j p_t(x)]$, dx a.e. Finally, $K^p(t,x) := v(t,x)/p_t(x)\mathbf{1}_{p_t(x)>0}$ has the required properties. $\square$

We can now take $\nabla \ln \frac{p_t}{q_t}(x)$ to be an arbitrary representant of the equivalence class of the function $K^p(t,x) - K^q(t,x)$ under the relation $f(t,x) - g(t,x) \in \mathrm{Ker}(a(t,x))$, $p_t(x)\,dx\,dt$ a.e. The identity in Lemma 1.3(a) is then satisfied by construction.

The proof of part (b) of Lemma 1.3 first relies on the following martingale representation property ensured by the extremality assumption, according to Theorem 12.21 in [14]:

LEMMA A.2. *Assume that $(H1)$, $(H2)_Q$ and $(H3)_Q$ hold. For each $i = 1,\dots,d$,*

$$
M_t^i := Y_t^i - Y_0^i - \int_0^t \bar{b}_i^Q(s,Y_s)\,ds, \qquad t\in[0,T]
$$

is a continuous local martingale with respect to $\mathbb{Q}^T$ and $(\mathcal{G}_t)$, and $\langle M^i, M^j\rangle_t = \int_0^t \bar{a}^{ij}(s,Y_s)\,ds$ for all $i,j=1,\dots,d$. Moreover, if $\mathbb{Q}^T$ is an extremal solution to the martingale problem $(MP)_Q$, then for any martingale $(N_t)_{t\in[0,T]}$ with respect to $\mathbb{Q}^T$ and $(\mathcal{G}_t)$ such that $N_0 = 0$, there exist predictable processes $(h_t^j)_{t\in[0,T],j=1,\dots,d}$ with $\sum_{i,j=1}^d \int_0^T h_s^i \bar{a}_{ij}(s,Y_s)h_s^j\,ds < \infty$, $\mathbb{Q}^T$ a.s., and such that $(\int_0^t h_s\cdot dM_s = \sum_{j=1}^d \int_0^t h_s^j\,dM_s^j)_{t\in[0,T]}$ is a modification of $(N_t)_{t\in[0,T]}$. In particular, $(N_t)_{t\in[0,T]}$ has a continuous modification.

The main assertions in part (b) of Lemma 1.3 are then consequences of the next result.

LEMMA A.3. *Assume that $(H1)$, $(H2)_Q$, $(H3)_Q$ and $(H3)_P$ hold together. Suppose moreover that $P_0 \ll Q_0$ and that $\mathbb{Q}^T$ is an extremal solution to the martingale problem $(MP)_Q$. Recall that $(t,x)\mapsto \nabla[\frac{p_t}{q_t}](x)$ is $q_t(x)\,dx\,dt$ a.e. defined in $[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ by $\nabla[\frac{p_t}{q_t}](x) := \frac{p_t}{q_t}(x)\nabla[\ln\frac{p_t}{q_t}](x)$.*

(i) *With* R *the* $(\mathcal{G}_t)$*-stopping time* $R := \inf\{s \in [0,T] : D_s = 0\}$, *we have* $\mathbb{Q}^T$*-a.s. that*

$$\forall t \in [0,T] \qquad \int_0^t \nabla^* \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\bar{a}(s,Y_s)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\,ds < \infty,$$

and on $\{R > 0\}$,

$$\forall t \in [0,R) \qquad \int_0^t \nabla^* \left[\ln\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\bar{a}(s,Y_s)\nabla\left[\ln\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\,ds < \infty.$$

(ii) *The process* $(D_t)_{t\in[0,T]}$ *has a continuous version, denoted in the same way, such that* $\mathbb{Q}^T$ *a.s.*, $\forall t \in [0,T]$,

$$\begin{aligned} D_t &= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\cdot \mathbf{1}_{s<R}\,dM_s \\ &= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\cdot \mathbf{1}_{\{(p_{T-s}/q_{T-s})(Y_s)>0\}}\,dM_s \quad \textit{and} \\ \langle D\rangle_t &= \int_0^t \nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\bar{a}(s,Y_s)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\,ds. \end{aligned}$$

PROOF. By Lemma A.2, the $\mathbb{Q}^T$-martingale $(D_t)_{t\in[0,T]}$ admits the continuous version $D_0 + \sum_{j=1}^d \int_0^t h_s^j\,dM_s^j$ still denoted by D_t for simplicity. The martingale representation property and standard properties of stochastic integrals moreover imply that D_t is determined by the processes $\langle D, M^i\rangle = \int_0^\cdot \bar{a}_{ij}(t,Y_t)h_t^j\,dt, i = 1,\dots,d$. Consequently, h_t can be replaced (leaving D_t unchanged) by any predictable process k_t such that for each i, $\int_0^\cdot \sum_{j=1}^d h_t^j \bar{a}_{ij}(t,Y_t)\,dt = \int_0^\cdot \bar{a}_{ij}(t,Y_t)k_t^j\,dt$ $\mathbb{Q}^T$ a.s. [the fact that $\int_0^T k_s^i \bar{a}_{ij}(s,Y_s)k_s^j\,ds = \int_0^T \sum_{i,j=1}^d h_s^j \bar{a}_{ij}(s,Y_s)h_s^i\,ds < \infty$ $\mathbb{Q}^T$ a.s. then follows immediately]. Furthermore, since $D_t = D_{t\wedge R}$ by standard properties of nonnegative continuous martingales, we may and shall assume that $\mathbb{Q}^T$ a.s. $h_t = h_t\mathbf{1}_{t<R} = h_t\mathbf{1}_{D_t>0}$ for all $t \in [0,T]$. Let us also notice that, by Fubini's theorem, it $\mathbb{Q}^T$-a.s. holds that $D_s = \frac{p_{T-s}}{q_{T-s}}(Y_s)$ (and then $\mathbf{1}_{\{R>s\}} = \mathbf{1}_{\{(p_{T-s}/q_{T-s})(Y_s)>0\}}$) for a.e. $s \in [0,T]$.

Now, by our assumptions and Theorem 1.2(a), $\mathbb{P}^T \ll \mathbb{Q}^T$ are probability measures, respectively, solving the martingale problems $(MP)_P$ and $(MP)_Q$. The processes $\int_0^\cdot \bar{b}_i^P(t,Y_t)\,dt$ and $\int_0^\cdot \bar{b}_i^Q(t,Y_t)\,dt + \int_0^\cdot (D_t)^{-1}h_t^j\,d\langle M^i, M^j\rangle_t$ then are $\mathbb{P}^T$-indistinguishable; see, for example, Proposition 12.18(v) in [14]. Using these facts, the expression for $\langle M^i, M^j\rangle$ in Lemma A.2 and part (ii) of Lemma A.1, we deduce first that, $\mathbb{P}^T$-a.s.,

$$\forall t \in [0,T], \forall i \qquad \bar{b}_i^P(t,Y_t) - \bar{b}_i^Q(t,Y_t)$$

$$\begin{aligned} &= \bar{a}_{ij}(t,Y_t)\left(h_t^j \frac{q_{T-t}}{p_{T-t}}(Y_t)\right) \qquad \text{(A.1)} \\ &= \bar{a}_{i\bullet}(t,Y_t)\big(K^p(T-t,Y_t) - K^q(T-t,Y_t)\big). \end{aligned}$$

By part (ii) of Lemma A.1 we then also get

$$\int_0^{\cdot} \bar{a}_{ij}(t, Y_t) h_t^j \, dt = \int_0^{\cdot} \bar{a}_{i\bullet}(t, Y_t)\big(K^p(T-t, Y_t) - K^q(T-t, Y_t)\big) \frac{p_{T-t}(Y_t)}{q_{T-t}(Y_t)} \, dt,$$

$i = 1, \dots, d$, $\mathbb{P}^T$-a.s., and then $\mathbb{Q}^T$-a.s. because of our assumption on h. From these identities and our previous discussion, we deduce that we can choose $h_t = \nabla \frac{p_{T-t}}{q_{T-t}}(Y_t)\mathbf{1}_{\{(p_{T-t}/q_{T-t})(Y_t)>0\}} = \nabla \frac{p_{T-t}}{q_{T-t}}(Y_t)\mathbf{1}_{\{R>t\}}$. This proves part (ii). The first property of the process $\nabla \frac{p_{T-t}}{q_{T-t}}(Y_t)$ in (i) is thus consequence of the general properties of h in the representation formula for D_t. The second assertion in (i) easily follows from the first one, taking into account the definitions of $\nabla \frac{p_{T-t}}{q_{T-t}}(Y_t)$ and $\nabla \ln \frac{p_{T-t}}{q_{T-t}}(Y_t)$ and the properties of D_t. $\square$

A.2. Proof of Theorem 1.4. Since, by Lemma 1.3, $(D_t)_{t\in[0,T]}$ is a continuous nonnegative $\mathbb{Q}^T$-martingale and U'_- is locally bounded on $(0, +\infty)$, $t \mapsto \int_0^t [U'_-(D_s)]^2 \, d\langle D\rangle_s$ is finite and continuous on $[0, T]$ when $R > T$ and finite and continuous on $[0, R)$ otherwise. In the latter case, $\int_0^R [U'_-(D_s)]^2 \, d\langle D\rangle_s$ makes sense but is possibly infinite. Define for any positive integer n the stopping time

$$R_n := \inf\left\{t \in [0, T \wedge R] : D_t \le \frac{1}{n} \text{ or } \int_0^t \big[U'_-(D_s)\big]^2 \, d\langle D\rangle_s \ge n\right\}.$$

For all $t \in [0, T]$, $\int_0^{t\wedge R_n} [U'_-(D_s)]^2 \, d\langle D\rangle_s \le n$ and $\mathbb{E}(\int_0^{t\wedge R_n} U'_-(D_s)\, dD_s) = 0$. Moreover $R_n \nearrow R$ as $n \to \infty$.

Let $t \in [0, T]$. By Tanaka's formula,

$$U(D_{t\wedge R_n}) = U(D_0) + \int_0^{t\wedge R_n} U'_-(D_s)\, dD_s + \frac{1}{2}\int_{(0,+\infty)} L^r_{t\wedge R_n}(D) U''(dr). \tag{A.2}$$

The finiteness of $H_U(P_0|Q_0)$ implies that $(U(D_s))_{s\in[0,T]}$ is a uniformly integrable $\mathbb{Q}^T$-submartingale. Since the $\mathbb{Q}^T$-expectation of the stochastic integral is zero, one deduces

$$\mathbb{E}^T\big(U(D_{t\wedge R_n})\big) = \mathbb{E}^T\big(U(D_0)\big) + \frac{1}{2}\mathbb{E}^T\left(\int_{(0,+\infty)} L^r_{t\wedge R_n}(D) U''(dr)\right).$$

When $n \to \infty$, since U is continuous on $(0, +\infty)$ by convexity, $U(D_{t\wedge R_n})$ converges to $U(D_{t\wedge R}) + \Delta U(0)\mathbf{1}_{\{0<R\le t\}} = U(D_t) + \Delta U(0)\mathbf{1}_{\{0<R\le t\}}$. Then, by uniform integrability, $\mathbb{E}^T(U(D_{t\wedge R_n}))$ converges to $\mathbb{E}^T(U(D_t)) + \Delta U(0)\mathbb{Q}^T(0 < R \le t)$. Dealing with the expectation of the integral on the right-hand side above by monotone convergence, we obtain

$$\mathbb{E}^T\big(U(D_t)\big) = \mathbb{E}^T\big(U(D_0)\big) - \Delta U(0)\mathbb{Q}^T(0 < R \le t) + \frac{1}{2}\mathbb{E}^T\left(\int_{(0,+\infty)} L^r_{t\wedge R}(D) U''(dr)\right).$$

Since according to Lemma 1.3(b), D is equal to zero on $[R, T]$, one can replace $t \wedge R$ by t in the last expectation. Replacing t by $T - t$ in this equation, one gets (1.8). Moreover $\mathbb{Q}^T$ a.s., $\int_{(0,+\infty)} L_t^r(D)U''(dr)$ is the finite limit of the integral with respect to $U''(dr)$ in the right-hand side of (A.2) as $n \to \infty$. Since the left-hand side converges to $U(D_t) + \Delta U(0)\mathbf{1}_{\{0<R\le t\}}$ we deduce that the stochastic integral on the right-hand side also has a finite limit. Hence $\int_0^{t\wedge R}[U'(D_s)]^2\, d\langle D\rangle_s < +\infty$, $\int_0^{t\wedge R} U'(D_s)\, dD_s$ makes sense, and (1.5) holds. When U is continuous on $[0, +\infty)$ and C^2 on $(0, +\infty)$, (1.6) follows by the occupation times formula. In this case, Lemma 1.3(b) and (1.8) written for $t = 0$, combined with the same arguments, imply that

$$
\begin{aligned}
&H_U(P_0|Q_0)\\
&\quad = H_U(P_T|Q_T)\\
&\qquad + \frac{1}{2}\mathbb{E}^T\biggl(\int_0^T U''(D_s)\mathbf{1}_{\{s<R\}}\nabla^*\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\bar{a}(s, Y_s)\nabla\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\, ds\biggr).
\end{aligned}
$$

Since Y_s admits the density q_{T-s} and for almost all $s \in [0, T)$, $D_s = \frac{p_{T-s}}{q_{T-s}}(Y_s)$ and $\{R > s\} = \{\frac{p_{T-s}}{q_{T-s}}(Y_s) > 0\}$, (1.9) follows by the change of variables $s \mapsto T - s$.

A.3. Proof of Corollary 1.7. We notice first that

$$
\mathbb{E}^T\int_0^T \mathbf{1}_{|D_s-1|<\delta}\nabla^*\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\bar{a}(s, Y_s)\nabla\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\, ds < \infty \qquad \forall\delta \in (0, 1). \tag{A.3}
$$

Indeed, for $\delta \in (0, 1)$, we can easily construct a C^2 convex function $\widehat{U}$ on $\mathbb{R}$ such that $\forall r \in \mathbb{R}$, $0 \le \widehat{U}(r) \le |r - 1|$ and $\forall r \in [1 - \delta, 1 + \delta]$, $\widehat{U}''(r) \ge \alpha$ for some $\alpha > 0$, so that the integral in (A.3) is bounded thanks to (1.9) by $\frac{1}{\alpha}H_{\widehat{U}}(P_0|Q_0) \le \frac{1}{\alpha}\|P_0 - Q_0\|_{\mathrm{TV}}$. For $r \in \mathbb{R}$, since

$$
L_t^r(D) = 2\biggl((D_t - r)^+ - (D_0 - r)^+ - \int_0^t \mathbf{1}_{D_s>r}\, dD_s\biggr),
$$

by Doob's inequality we obtain $|\mathbb{E}^T(L_t^r(D) - L_t^1(D))| \le 4|r - 1| + 2(\mathbb{E}^T\int_0^t \mathbf{1}_{\{1\wedge r<D_s\le r\vee 1\}}\, d\langle D\rangle_s)^{1/2}$. Hence, Lemma 1.3(b) and (A.3) imply that $r \mapsto \mathbb{E}^T(L_t^r(D))$ is continuous (and finite) at $r = 1$. With the occupation times formula, one deduces that

$$
\begin{aligned}
&2\mathbb{E}^T\bigl(L_t^1(D)\bigr)\\
&\quad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1+\varepsilon}\mathbb{E}^T\bigl(L_t^r(D)\bigr)\, dr\\
&\quad = \lim_{\varepsilon\to 0}\mathbb{E}^T\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{|D_s-1|<\varepsilon\}}\nabla^*\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\bar{a}(s, Y_s)\nabla\biggl[\frac{p_{T-s}}{q_{T-s}}\biggr](Y_s)\, ds
\end{aligned}
$$

$$= \lim_{\varepsilon\to 0}\int_0^t \frac{1}{\varepsilon}\int_{\{|(p_{T-s}/q_{T-s})(x)-1|<\varepsilon\}} \nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right](x)\bar{a}(s,x) \nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\,dx\,ds.$$

Define now the function $\varphi_\varepsilon(r) := \mathbf{1}_{[-\varepsilon,\varepsilon]}(r)r\varepsilon^{-1} + \mathbf{1}_{(\varepsilon,\infty)}(r) - \mathbf{1}_{(-\infty,-\varepsilon)}(r)$. Since the function $\varepsilon \mapsto \int_0^t \int_{\{|(p_{T-s}/q_{T-s})(x)-1|\le\varepsilon\}} q_{T-s}(x)\,dx\,ds$ is increasing and right continuous, we can chose $\varepsilon_k \searrow 0$ a sequence with $\int_0^t \int_{\{|(p_{T-s}/q_{T-s})(x)-1|=\varepsilon_k\}} q_{T-s}(x)\,dx\,ds = 0$ so that $2\mathbb{E}^T(L_t^1(D))$ is equal to

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}^d}\nabla^*\left[\varphi_{\varepsilon_k}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)\right](x)\bar{a}(s,x)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\,dx\,ds\\
&\quad= -\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}^d}\varphi_{\varepsilon_k}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)(x)\\
&\qquad\qquad \nabla\cdot\left[\bar{a}(s,x)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\right]dx\,ds\\
&\quad= -\int_0^t\int_{\mathbb{R}^d}\widetilde{\operatorname{sign}}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)(x)\nabla\cdot\left[\bar{a}(s,x)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\right]dx\,ds,
\end{aligned}$$

where the last equality follows from the integrability assumption made on $\nabla\cdot[\bar{a}(s,x)\nabla[\frac{p_{T-s}}{q_{T-s}}](x)q_{T-s}(x)]$. To justify the integration by parts at the second equality, we introduce functions $\phi_n \in C_0^\infty(\mathbb{R}^d)$ such that $\mathbf{1}_{B(0,r_n)} \le \phi_n \le \mathbf{1}_{B(0,2r_n)}$ and $0 \le |\nabla\phi_n| \le 2/r_n$, and functions $\varphi_{\varepsilon_k,m} : \mathbb{R}\to\mathbb{R}$ of class C^1 such that $\varphi_{\varepsilon_k,m} \to \varphi_{\varepsilon_k}$, $|\varphi_{\varepsilon_k,m}| \le |\varphi_{\varepsilon_k}|$ on $\mathbb{R}$ and $\varphi'_{\varepsilon_k,m} \to \varphi'_{\varepsilon_k}$, $|\varphi'_{\varepsilon_k,m}| \le |\varphi'_{\varepsilon_k}|$ on $\mathbb{R}\setminus\{-\varepsilon_k,+\varepsilon_k\}$ as $m\to\infty$. Using the assumptions, (A.3) and the choice of ε_k, we take the limits $n\to\infty$ then $m\to\infty$ by dominated convergence in the equality

$$\begin{aligned}
&\int_{\mathbb{R}^d}\varphi'_{\varepsilon_k,m}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)(x)\nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right](x)a(T-s,x)\\
&\qquad\nabla\cdot\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\phi_n(x)\,dx\\
&\quad= -\int_{\mathbb{R}^d}\varphi_{\varepsilon_k,m}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)(x)\\
&\qquad\qquad\nabla\cdot\left(a(T-s,x)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\right)\phi_n(x)\,dx\\
&\qquad-\int_{\mathbb{R}^d}\varphi_{\varepsilon_k,m}\left(\frac{p_{T-s}}{q_{T-s}}-1\right)(x)\nabla^*\phi_n(x)a(T-s,x)\\
&\qquad\qquad\nabla\cdot\left[\frac{p_{T-s}}{q_{T-s}}\right](x)q_{T-s}(x)\,dx.
\end{aligned}$$

A.4. Proof of Proposition 1.9. To check the Feller property, we introduce a continuous function $f:\mathbb{R}^d \to \mathbb{R}$ going to 0 at infinity. Using Itô's calculus and Gronwall's lemma, we check under the assumptions on the coefficients that the solution X_t^x of (0.1) starting from $x \in \mathbb{R}^d$ satisfies $\mathbb{E}((1+|X_t^x|^2)^{-1}) \le C(1+|x|^2)^{-1}$ for some $C>0$. Then the inequality

$$\bigl|\mathbb{E}\bigl(f(X_t^x)\bigr)\bigr| \le \sup_{|y|\le A}\bigl|f(y)\bigr| C\frac{(1+A^2)}{(1+|x|^2)} + \sup_{|y|>A}\bigl|f(y)\bigr|$$

for all $A>0$ (following from the previous estimate and Markov's inequality) implies that $\mathbb{E}(f(X_t^x)) \to 0$ when $x \to \infty$. Finally, the continuity of $x \to \mathbb{E}(f(X_t^x))$ follows from the bound $\mathbb{E}(|X_t^x - X_t^y|^2) \le C|x-y|^2$ and the uniform continuity and boundedness of f.

By Theorem 1.3.8 [15], since $(X_t)_{t\ge 0}$ is Feller, the tail sigma field is trivial as soon as $\|P_t - Q_t\|_{\mathrm{TV}} \to 0$ as $t \to \infty$ for all pair of initial laws P_0 and Q_0. Since $\|P_t - Q_t\|_{\mathrm{TV}} \le \|P_t - p_\infty\,dx\|_{\mathrm{TV}} + \|p_\infty\,dx - Q_t\|_{\mathrm{TV}}$ and, by Theorem 2.1.3, page 162 of [4], the local uniform ellipticity assumption ensures that P_t admits a density with respect to the Lebesgue measure for all $t>0$, it is enough to show that $\|P_t - p_\infty\,dx\|_{\mathrm{TV}} \to 0$ as $t \to \infty$ when P_0 admits a density p_0 with respect to the Lebesgue measure.

For $k \in \mathbb{N}^*$, consider the probability density:

$$p_0^k(x) = \bigl(p_0(x) \wedge kp_\infty(x)\bigr) + p_\infty(x)\int_{p_0>kp_\infty}\bigl(p_0(y) - kp_\infty(y)\bigr)\,dy.$$

Since p_∞ is positive, on one hand we have $\lim_{k\to\infty}\|p_0 - p_0^k\|_1 = 0$ and $p_0^k \le (k+1)p_\infty$. On the other hand, the total variation distance between the marginal laws at time t of the solutions to (0.1) started from the initial densities p_0 and p_0^k is not larger than $\|p_0 - p_0^k\|_1$. Therefore we can moreover restrict ourselves to the case when $\frac{p_0}{p_\infty}$ is bounded. Then

$$\int_{\mathbb{R}^d}\biggl(\frac{p_0}{p_\infty}(x) - 1\biggr)^2 p_\infty(x)\,dx \le \biggl(\int_{\mathbb{R}^d}\biggl(\frac{p_0}{p_\infty}(x) - 1\biggr)^4 p_\infty(x)\,dx\biggr)^{1/2} < +\infty.$$

We set $Q_0 = p_\infty\,dx$. By Remark 1.5(a) and (c), conditions $(H1)$, $(H2)_Q$, $(H3)_Q$ and $(H3)_P$ hold, and for each $T>0$, $\mathbb{Q}^T$ is an extremal solution of the martingale problem (MP). Applying Theorem 1.4, respectively, with $U(r) = (r-1)^4$ and $U(r) = (r-1)^2$, we get that $t \mapsto \int_{\mathbb{R}^d}(\frac{p_t}{p_\infty}(x) - 1)^2 p_\infty(x)\,dx$ is nonincreasing and that

$$\begin{aligned}(\mathrm{A.4})\qquad &\sup_{t\ge 0}\int_{\mathbb{R}^d}\biggl(\frac{p_t}{p_\infty}(x) - 1\biggr)^4 p_\infty(x)\,dx \\ &\quad + \int_0^\infty\int_{\{(p_t/p_\infty)(x)>0\}}\biggl(\nabla^*\biggl[\frac{p_t}{p_\infty}\biggr]a\nabla\biggl[\frac{p_t}{p_\infty}\biggr]\biggr)(x)p_\infty(x)\,dx\,dt < +\infty.\end{aligned}$$

Since a is locally uniformly elliptic, the proof of Lemma A.1 ensures that dt a.e., the gradient ∇p_t (resp., ∇p_∞) of p_t (resp., p_∞) in the sense of distributions is a locally integrable function on $\mathbb{R}^d$ that vanishes a.e. on $\{x : p_t(x) = 0\}$. Moreover, we can choose therein $K^p(t,x) = \mathbf{1}_{\{p_t(x)>0\}} \frac{\nabla p_t}{p_t}(x)$ and $K^q(t,x) = \frac{\nabla p_\infty}{p_\infty}(x)$. Then, in (A.4), $\nabla[\frac{p_t}{p_\infty}] = \frac{\nabla p_t}{p_\infty} - \frac{p_t \nabla p_\infty}{p_\infty^2}$ is a.e. equal to 0 when $\frac{p_t}{p_\infty}$ is equal to 0 so that the restriction of the spatial integration to $\{\frac{p_t}{p_\infty}(x) > 0\}$ can be removed. Since p_∞ is assumed to be locally Lipschitz continuous and bounded away from 0, the function $\frac{1}{p_\infty}$ is locally bounded with a locally bounded distributional gradient equal to $-\frac{\nabla p_\infty}{p_\infty^2}$. We deduce that the gradient $\nabla \frac{p_t}{p_\infty}$ of $\frac{p_t}{p_\infty}$ in the sense of distributions is equal to $\frac{\nabla p_t}{p_\infty}(x) - \frac{p_t \nabla p_\infty}{p_\infty^2}$ and therefore to $\nabla[\frac{p_t}{p_\infty}]$.

From the finiteness of the time-integral in (A.4), we deduce the existence of a sequence $(t_n)_n$ tending to $+\infty$ such that $\lim_{n\to\infty} \int_{\mathbb{R}^d} (\nabla^* \frac{p_{t_n}}{p_\infty} a \nabla \frac{p_{t_n}}{p_\infty})(x) p_\infty(x)\,dx = 0$. For $A > 0$, writing the integral on $\mathbb{R}^d$ as the sum of the integrals on the ball $B(0,A)$ and its complementary $B(0,A)^c$, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left(\frac{p_{t_n}}{p_\infty}(x) - 1\right)^2 p_\infty(x)\,dx \\
&\quad \le \int_{B(0,A)} \left(\frac{p_{t_n}}{p_\infty}(x) - \frac{\int_{B(0,A)} p_{t_n}(y)\,dy}{\int_{B(0,A)} p_\infty(y)\,dy}\right)^2 p_\infty(x)\,dx \\
&\qquad + \frac{(\int_{B(0,A)} (p_{t_n} - p_\infty)(y)\,dy)^2}{\int_{B(0,A)} p_\infty(y)\,dy} \\
&\qquad + \left(\int_{B(0,A)^c} \left(\frac{p_{t_n}}{p_\infty}(x) - 1\right)^4 p_\infty(x)\,dx \int_{B(0,A)^c} p_\infty(x)\,dx\right)^{1/2} \\
&\quad \le \int_{B(0,A)} \left(\frac{p_{t_n}}{p_\infty}(x) - \frac{\int_{B(0,A)} (p_{t_n}/p_\infty)(y)\,dy}{\int_{B(0,A)} dy}\right)^2 p_\infty(x)\,dx \\
&\qquad + \frac{(\int_{B(0,A)^c} ((p_{t_n}/p_\infty)(y) - 1) p_\infty(y)\,dy)^2}{\int_{B(0,A)} p_\infty(y)\,dy} \\
&\qquad + \left(\int_{\mathbb{R}^d} \left(\frac{p_0}{p_\infty}(x) - 1\right)^4 p_\infty(x)\,dx \int_{B(0,A)^c} p_\infty(x)\,dx\right)^{1/2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left(\int_{B(0,A)^c} \left(\frac{p_{t_n}}{p_\infty}(y) - 1\right) p_\infty(y)\,dy\right)^2 \\
&\quad \le \int_{\mathbb{R}^d} \left(\frac{p_0}{p_\infty}(y) - 1\right)^2 p_\infty(y)\,dy \int_{B(0,A)^c} p_\infty(y)\,dy,
\end{aligned}
$$

the sum of the last two terms on the right-hand side tends to 0 uniformly in n as $A \to \infty$. Using (1.11) and denoting by $C_A < +\infty$ the constant of the Poincaré–Wirtinger inequality satisfied by the Lebesgue measure on the ball $B(0, A)$, we check that the first term is smaller than

$$C_A \frac{\sup_{B(0,A)} p_\infty}{\varepsilon_A \inf_{B(0,A)} p_\infty} \int_{\mathbb{R}^d} \left(\nabla^* \frac{p_{t_n}}{p_\infty} a \nabla \frac{p_{t_n}}{p_\infty}\right)(x) p_\infty(x)\, dx,$$

which tends to 0 as $n \to \infty$. Hence $\lim_{n\to\infty} \int_{\mathbb{R}^d} (\frac{p_{t_n}}{p_\infty}(x) - 1)^2 p_\infty(x)\, dx = 0$. Since $\|p_t - p_\infty\|_1^2 \le \int_{\mathbb{R}^d} (\frac{p_t}{p_\infty}(x) - 1)^2 p_\infty(x)\, dx$ where the right-hand side is nonincreasing with t, we conclude that $\lim_{t\to\infty} \|p_t - p_\infty\|_1 = 0$.

A.5. Sufficient conditions for superquadratic potentials to satisfy $(H1)''$.

LEMMA A.4. *Let $b(x) = -\nabla V(x)$ for a nonnegative C^2 potential V in $\mathbb{R}^d$ satisfying* (1.7), *and σ be any globally Lipschitz continuous choice of the square root of the identity I_d. Then condition $(H1)''$ holds for the diffusion process $dX_t = \sigma(X_t)\, dW_t - \nabla V(X_t)\, dt$.*

PROOF. Computing $d|X_t|^2$, we see that the first condition in (1.7) prevents explosion for the SDE which has locally Lipschitz coefficients. Since for $c > 0$,

$$de^{cV(X_t)} = e^{cV(X_t)}\left(c\nabla^* V(X_t)\sigma(X_t)\, dW_t + \frac{c}{2}\left[\Delta V + (c-2)|\nabla V|^2\right](X_t)\, dt\right),$$

the second condition ensures that for c small enough, $\mathbb{E}(e^{cV(X_t)}) \le e^{K(c)t}\mathbb{E}(e^{cV(X_0)})$ for some finite constant $K(c)$ only depending on V and c. The third assumption ensures the existence of a finite constant $\widetilde{K}(\frac{c}{T})$ only depending on $\frac{c}{T}$ and V such that

$$\mathbb{E}\left(\exp\left(4\int_0^T \sqrt{\partial_{ik} V \partial_{ik} V(X_t)}\, dt\right)\right) \le \widetilde{K}\left(\frac{c}{T}\right)\mathbb{E}\left(\exp\left(\frac{c}{T}\int_0^T V(X_t)\, dt\right)\right).$$

By Jensen's inequality, we deduce that

$$\begin{aligned}\mathbb{E}\left(\exp\left(4\int_0^T \sqrt{\partial_{ik} V \partial_{ik} V(X_t)}\, dt\right)\right) &\le \frac{\widetilde{K}(c/T)}{T}\int_0^T \mathbb{E}\left(e^{cV(X_t)}\right) dt \\ &\le \widetilde{K}\left(\frac{c}{T}\right) e^{K(c)T}\mathbb{E}\left(e^{cV(X_0)}\right).\end{aligned}$$
□

APPENDIX B

We next provide the proofs of the main results of Section 2.

B.1. Proof of Proposition 2.2. We will make use of the stochastic flow defined by the two-parameter process $\xi_t(x)$ satisfying

$$
d\xi_t^i(x) = \sigma_{ik}\big(\xi_t(x)\big)\, d\overline{W}_t^k + \bar{b}_i\big(\xi_t(x)\big)\, dt, \qquad (t,x) \in [0,T) \times \mathbb{R}^d, i = 1,\dots,d, \tag{B.1}
$$

and $\xi_0(x) = x$, noting that $\xi_t(Y_0) = Y_t$. We shall also deal with the family of continuous $\mathcal{G}_t - \mathbb{P}_\infty^T$-local martingales $(D_t(x) : t \in [0,T])_{x\in\mathbb{R}^d}$ defined by

$$
dD_t(x) = [\sigma_{ik}\partial_i\rho]\big(t, \xi_t(x)\big)\, d\overline{W}_t^k, \qquad D_0(x) = \frac{p_T}{p_\infty}(x) = \rho_0(x). \tag{B.2}
$$

According to Lemma 1.3, $D_t(Y_0)$ is equal to the process D_t defined in (1.2). Writing $\nabla\rho_t(\xi_t(x)) = (\nabla_x^*\xi_t(x))^{-1}\nabla_x[\rho_t(\xi_t(x))]$ we remark that, thanks to the Itô product rule, $d\nabla\rho_t(\xi_t(x))$ can be obtained with by computing $d(\nabla_x\xi_t(x))^{-1}$ and $d\nabla_x[\rho_t(\xi_t(x))]$. Those computations are part of the contents of the two next lemmas:

LEMMA B.1. *The process $(t,x) \mapsto \xi_t(x)$ has a $\mathbb{P}_\infty^T$ a.s. continuous version such that the mapping $x \mapsto \xi_t(x)$ is a global diffeomorphism of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and every $t \in [0,T]$. Moreover, we have for all $(t,x) \in [0,T) \times \mathbb{R}^d$,*

$$
d\partial_j\xi_t^i(x) = \partial_p\sigma_{ik}\big(t,\xi_t(x)\big)\partial_j\xi_t^p(x)\, d\overline{W}_t^k + \partial_p\bar{b}_i\big(t,\xi_t(x)\big)\partial_j\xi_t^p(x)\, dt, \tag{B.3}
$$

with $\partial_j\xi_0^i(x) = \delta_{ij}$. Finally, writing $\nabla\xi_t(x) = (\partial_j\xi_t^i(x))_{ij}$, it holds that

$$
\begin{aligned}
&d\big(\nabla\xi_t(x)\big)_{kl}^{-1} \\
&\quad = -\big(\nabla\xi_t(x)\big)_{ki}^{-1}[\partial_l\sigma_{ir}]\big(\xi_t(x)\big)\, d\overline{W}_t^r - \big(\nabla\xi_t(x)\big)_{ki}^{-1}[\partial_l\bar{b}_i]\big(\xi_t(x)\big)\, dt \\
&\qquad + \big(\nabla\xi_t(x)\big)_{ki}^{-1}[\partial_m\sigma_{ir}\partial_l\sigma_{mr}]\big(\xi_t(x)\big)\, dt, \qquad (t,x) \in [0,T) \times \mathbb{R}^d.
\end{aligned} \tag{B.4}
$$

PROOF. Under assumptions $(H4)$ and $(H5)_{p_\infty}$, classic results of Kunita [15] (see Theorem 4.7.2) imply the asserted regularity properties of the stochastic flow, as well as the $\mathbb{P}_\infty^T$ a.s. existence of the inverse matrix $(\nabla\xi_t(x))^{-1}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Since the smooth map $A \mapsto A^{-1}$, defined on nonsingular $d \times d$ matrices, has first and second derivatives, respectively, given by the linear and bilinear operators $F \mapsto -A^{-1}FA^{-1}$ and $(F,K) \mapsto A^{-1}FA^{-1}KA^{-1} + A^{-1}KA^{-1}FA^{-1}$ (where F, K are generic square-matrices), we deduce that for $A = (A_{ij})_{i,j=1,\dots,d}$,

$$
\frac{\partial(A^{-1})_{kl}}{\partial A_{ij}} = -A_{ki}^{-1}A_{jl}^{-1} \quad \text{and} \quad \frac{\partial^2(A^{-1})_{kl}}{\partial A_{ij}\,\partial A_{mn}} = A_{ki}^{-1}A_{jm}^{-1}A_{nl}^{-1} + A_{km}^{-1}A_{ni}^{-1}A_{jl}^{-1}
$$

for all $k,l,i,j,m,n \in \{1,\dots,d\}$. Equation (B.4) follows by applying Itô's formula to each of the functions $A \mapsto (A^{-1})_{kl}$ and the semimartingales $(\partial_j\xi_t^i(x))$, $i,j = 1,\dots,d$. $\square$

LEMMA B.2. *The process $D_t(x)$ has a modification still denoted by $D_t(x)$ such that $\mathbb{P}^T_\infty$ a.s. the function $(t,x)\mapsto D_t(x)$ is continuous and $x\mapsto D_t(x)$ is of class C^1 for each t. This modification is indistinguishable from $(\rho_t(\xi_t(x)):(t,x)\in[0,T)\times\mathbb{R}^d)$, and we have*

$$\begin{aligned} d\partial_k D_t(x) &= \partial_m[\sigma_{ir}\partial_i\rho]\big(t,\xi_t(x)\big)\partial_k\xi_t^m(x)\,d\overline{W}_t^r \\ &= d\big[\partial_m\rho\big(t,\xi_t(x)\big)\partial_k\xi_t^m(x)\big] \end{aligned} \tag{B.5}$$

for all $(t,x)\in[0,T)\times\mathbb{R}^d$.

PROOF. Thanks to the regularity of $x\mapsto\xi_t(x)$ established in Lemma B.1 and assumptions $(H5)_{p_\infty}$ and $(H6)^T_{p_0}$, the statements follow from Theorem 3.3.3 of Kunita [15]; see also Exercise 3.1.5 therein. □

We can now proceed to prove Proposition 2.2. Evaluating expressions (B.4) and (B.5) in $x=Y_0$, we obtain using Itô's product rule that

$$\begin{aligned} d\partial_l\rho_t(Y_t) &= [\sigma_{kr}\partial_{lk}\rho](t,Y_t)\,d\overline{W}_t^r - [\sigma_{kr}\partial_{kj}\rho\partial_l\sigma_{jr}+\partial_k\rho\partial_l\bar{b}_k](t,Y_t)\,dt \\ &= [\sigma_{kr}\partial_{lk}\rho](t,Y_t)\,d\overline{W}_t^r - \big[\tfrac12\partial_{kj}\rho\partial_l a_{kj}+\partial_k\rho\partial_l\bar{b}_k\big](t,Y_t)\,dt. \end{aligned} \tag{B.6}$$

For the remainder of the proof, the argument (t,Y_t) will be omitted for notational simplicity. By Itô 's formula we get $d\sigma_{li}=[\sigma_{mr}\partial_m\sigma_{li}]\,d\overline{W}_t^r+[\bar{b}_m\partial_m\sigma_{li}+\frac12 a_{mk}\partial_{mk}\sigma_{li}]\,dt$. We then have

$$\begin{aligned} d[\sigma_{li}\partial_l\rho] &= \sigma_{li}\,d\partial_l\rho+\partial_l\rho d\sigma_{li}+d\langle\partial_l\rho,\sigma_{li}\rangle \\ &= \partial_k[\partial_l\rho\sigma_{li}]\sigma_{kr}\,d\overline{W}^r+\partial_l\rho\big[\bar{b}_m\partial_m\sigma_{li}+\tfrac12 a_{mk}\partial_{mk}\sigma_{li}\big] \\ &\quad -\sigma_{li}[\sigma_{kr}\partial_{kj}\rho\partial_l\sigma_{jr}+\partial_k\rho\partial_l\bar{b}_k]+a_{mk}\partial_{lk}\rho\partial_m\sigma_{li}, \end{aligned}$$

where we used in the stochastic integral the fact that $\partial_l\rho\sigma_{mr}\partial_m\sigma_{li}+\sigma_{li}\sigma_{kr}\partial_{lk}\rho=\partial_l\rho\sigma_{kr}\partial_k\sigma_{li}+\sigma_{li}\sigma_{kr}\partial_{lk}\rho=\partial_k[\partial_l\rho\sigma_{li}]\sigma_{kr}$. It follows that

$$\begin{aligned} &d\big[\nabla^*\rho a\nabla\rho\big] \\ &\quad = d[\sigma_{li}\partial_l\rho\sigma_{l'i}\partial_{l'}\rho] \\ &\quad = 2\big\{[\sigma_{l'i}\partial_{l'}\rho a_{mk}\partial_m\sigma_{li}\partial_{lk}\rho]+\sigma_{l'i}\partial_{l'}\rho\partial_l\rho\big[\bar{b}_m\partial_m\sigma_{li}+\tfrac12 a_{mk}\partial_{mk}\sigma_{li}\big] \\ &\qquad\qquad - a_{ll'}\partial_{l'}\rho[\sigma_{kr}\partial_{kj}\rho\partial_l\sigma_{jr}+\partial_k\rho\partial_l\bar{b}_k]\big\}\,dt \\ &\qquad + a_{kk'}\partial_k[\partial_l\rho\sigma_{li}]\partial_{k'}[\partial_{l'}\rho\sigma_{l'i}]\,dt+2\sigma_{l'i}\partial_{l'}\rho\partial_k[\sigma_{li}\partial_l\rho]\sigma_{kr}\,d\overline{W}^r. \end{aligned}$$

On the other hand, using (B.2) at $x=Y_0$ we have $dU''_\delta(\rho)=U^{(3)}_\delta(\rho)\sigma_{nr}\partial_n\rho\,d\overline{W}^r+\frac12 U^{(4)}_\delta(\rho)a_{nj}\partial_n\rho\partial_j\rho\,dt$, which combined with the previous expression yields

$$\begin{aligned} &d\big[U''_\delta(\rho)\nabla^*\rho a\nabla\rho\big] \\ &\quad = d\widehat{M}^{(\delta)}+\tfrac12 U^{(4)}_\delta(\rho)\big|\nabla^*\rho a\nabla\rho\big|^2\,dt \end{aligned} \tag{B.7}$$

$$
\begin{aligned}
&+ 2U_\delta^{(3)}(\rho)\sigma_{l'i}\partial_{l'}\rho\partial_k[\sigma_{li}\partial_l\rho]a_{jk}\partial_j\rho\,dt \\
&+ 2U_\delta''(\rho)\big\{[\sigma_{l'i}\partial_{l'}\rho a_{mk}\partial_m\sigma_{li}\partial_{lk}\rho] \\
&\qquad + \sigma_{l'i}\partial_{l'}\rho\partial_l\rho\big[\bar{b}_m\partial_m\sigma_{li} + \tfrac{1}{2}a_{mk}\partial_{mk}\sigma_{li}\big] \\
&\qquad - a_{ll'}\partial_{l'}\rho[\sigma_{kr}\partial_{kj}\rho\partial_l\sigma_{jr} + \partial_k\rho\partial_l\bar{b}_k]\big\}\,dt \\
&+ U_\delta''(\rho)a_{kk'}\partial_k[\partial_l\rho\sigma_{li}]\partial_{k'}[\partial_{l'}\rho\sigma_{l'i}]\,dt.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
&d\big[U_\delta''(\rho)\nabla^*\rho a\nabla\rho\big] \\
&\quad = d\widehat{M}^{(\delta)} + \mathrm{tr}[\Lambda_\delta\Gamma]\,dt \\
&\qquad + 2U_\delta''(\rho)\big\{\partial_{l'}\rho\partial_l\rho\big[\tfrac{1}{4}(\partial_k\sigma_{lj}a_{km}\partial_m\sigma_{l'j} - \sigma_{ki}\partial_k\sigma_{lj}\sigma_{mj}\partial_m\sigma_{l'i}) \\
&\qquad\qquad + \tfrac{1}{2}\bar{b}_m\partial_m a_{ll'} + \tfrac{1}{2}\sigma_{l'i}a_{mk}\partial_{mk}\sigma_{li} - a_{kl'}\partial_k\bar{b}_l\big] \\
&\qquad\qquad + [\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho\big\}\,dt.
\end{aligned}
$$

B.2. Proof of Theorem 2.4. Let us check (2.2). Since U'' is continuous and nonincreasing in $(0,\infty)$ by Remark 2.1, one has $U_\delta''(r) \nearrow U''(r)$ for each $r>0$ as $\delta\to 0$. It is therefore enough to obtain (the integrated version of) inequality (2.2) with U_δ'' instead of U'', monotone convergence allowing us to pass to the limit as $\delta\to 0$ on both sides. For $0\le r\le t<T$, we have by Proposition 2.2 that

$$
\big[U_\delta''(\rho)\nabla^*\rho a\nabla\rho\big](t,Y_t) - \big[U_\delta''(\rho)\nabla^*\rho a\nabla\rho\big](r,Y_r)
$$

$$
\begin{aligned}
&\ge \widehat{M}_t^{(\delta)} - \widehat{M}_r^{(\delta)} + 2\int_r^t U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho\,ds \\
&\quad + 2\int_r^t U_\delta''(\rho)\partial_{l'}\rho\partial_l\rho\bigg(\frac{1}{4}(\partial_k\sigma_{lj}a_{km}\partial_m\sigma_{l'j} - \sigma_{ki}\partial_k\sigma_{lj}\sigma_{mj}\partial_m\sigma_{l'i}) \\
&\qquad + \frac{1}{2}[\bar{b}_m\partial_m a_{ll'} + \sigma_{l'i}a_{mk}\partial_{mk}\sigma_{li}] - a_{ml'}\partial_m\bar{b}_l\bigg)\,ds.
\end{aligned}
\tag{B.8}
$$

Since $\partial_{kl'}\rho U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}] = 0$ and

$$
\partial_k\big(U_\delta''(\rho)\big)\partial_{l'}\rho[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}] = U_\delta^{(3)}(\rho)\partial_k\rho\partial_{l'}\rho[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}] = 0,
$$

one has

$$
\begin{aligned}
&U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho \\
&\quad = \frac{1}{p_\infty}\partial_k\big(\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li}p_\infty\big) \\
&\qquad - \frac{\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)}{p_\infty}\partial_k\big([a_{mk}\sigma_{l'i} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li}p_\infty\big).
\end{aligned}
\tag{B.9}
$$

Setting

$$\Sigma_{ll'} \stackrel{\text{def}}{=} \frac{1}{4}(\partial_k \sigma_{lj} a_{km} \partial_m \sigma_{l'j} - \sigma_{ki} \partial_k \sigma_{lj} \sigma_{mj} \partial_m \sigma_{l'i}) + \frac{1}{2}[\bar{b}_m \partial_m a_{ll'} + \sigma_{l'i} a_{mk} \partial_{mk} \sigma_{li}] - a_{ml'} \partial_m \bar{b}_l - \frac{1}{p_\infty} \partial_k \biggl[\biggl(\frac{1}{2} a_{mk} \partial_m a_{ll'} - \sigma_{ki} a_{ml'} \partial_m \sigma_{li} \biggr) p_\infty \biggr],$$

we deduce that

$$\begin{aligned} &[U''_\delta(\rho) \nabla^* \rho a \nabla \rho](t, Y_t) - [U''_\delta(\rho) \nabla^* \rho a \nabla \rho](r, Y_r) \\ &\quad \geq \widehat{M}^{(\delta)}_t - \widehat{M}^{(\delta)}_r + 2 \int_r^t U''_\delta(\rho) \Sigma_{ll'} \partial_{l'} \rho \partial_l \rho \, ds \\ &\qquad + 2 \int_r^t \frac{1}{p_\infty} \partial_k \bigl(\partial_l \rho \partial_{l'} \rho U''_\delta(\rho) [\sigma_{l'i} a_{mk} - \sigma_{ki} a_{ml'}] \partial_m \sigma_{li} p_\infty \bigr) \, ds. \end{aligned} \tag{B.10}$$

Using (2.1) and the identity $\sigma_{ki} \partial_{k'} \sigma_{li} = \partial_{k'} a_{kl} - \partial_{k'} \sigma_{ki} \sigma_{li}$, one can check that

$$\begin{aligned} \Theta_{ll'} &= \frac{1}{2} \bar{b}_{k'} \partial_{k'} a_{ll'} + \frac{1}{2}(a_{kl'} \partial_k \bar{b}_l + a_{kl} \partial_k \bar{b}_{l'}) + \frac{1}{4} a_{k'k} \partial_{k'k} a_{ll'} \\ &\quad - \frac{1}{4}(a_{k'k} \partial_{k'} \sigma_{li} \partial_k \sigma_{l'i} + \sigma_{ki} \partial_k \sigma_{lj} \sigma_{k'j} \partial_{k'} \sigma_{l'i}) \\ &\quad + \frac{1}{2} \sigma_{ki} (\partial_{k'} \sigma_{li} a_{k'l'} + \partial_{k'} \sigma_{l'i} a_{k'l}) \partial_k \ln(p_\infty) - \frac{1}{2} a_{k'k} \partial_{k'} a_{ll'} \partial_k \ln(p_\infty) \\ &\quad + \frac{1}{2} \partial_k \bigl[\sigma_{ki} (\partial_{k'} \sigma_{li} a_{k'l'} + \partial_{k'} \sigma_{l'i} a_{k'l}) - a_{k'k} \partial_{k'} a_{ll'} \bigr] \\ &= \frac{\Sigma_{ll'} + \Sigma_{l'l}}{2}, \end{aligned} \tag{B.11}$$

and therefore, the second integral on the right-hand side of (B.10) rewrites as $2 \int_r^t U''_\delta(\rho) \Theta_{ll'} \partial_{l'} \rho \partial_l \rho \, ds$.

Now, the quadratic variation of $\widehat{M}^{(\delta)}$ is bounded above in $[0, T)$ by a constant times

$$\int_0^t [|U^{(3)}_\delta(\rho)|^2 |\nabla^* \rho a \nabla \rho|^3 (Y_s) + (U''_\delta(\rho))^2 \nabla^* (\nabla^* \rho a \nabla \rho) a \nabla (\nabla^* \rho a \nabla \rho)](Y_s) \, ds.$$

This fact and our assumptions imply that $\widehat{M}^\delta$ is a martingale in $[0, T)$ for all $\delta > 0$ sufficiently small. Indeed, we have from Remark 2.1 that $U''_\delta(r) \leq U''(\delta) \wedge U''(r)$ and $|U^{(3)}_\delta(r)| \leq |U^{(3)}(\delta)| \wedge |U^{(3)}(r)|$ for all $r \geq 0$. Therefore (since $U'' > 0$) we have $U''_\delta(r) \leq (U''(r) \wedge 1) \mathbf{1}_{U''(\delta) \leq 1} + U''(\delta)(U''(r)/U''(\delta)) \wedge 1) \mathbf{1}_{U''(\delta) > 1}$ whence $U''_\delta(r) \leq (U''(\delta) + 1)(U''(r) \wedge 1)$. As $U^{(3)}$ is nondecreasing and nonpositive, either $|U^{(3)}(\delta)| \neq 0$ for all δ sufficiently small, in which case we similarly get

$|U_\delta^{(3)}(r)| \le (|U^{(3)}(\delta)|+1)(|U^{(3)}(r)| \wedge 1)$, or otherwise $U_\delta^{(3)}$ identically vanishes for all δ. Assumption $(H6')_{p_\infty}$ and the previous then ensure that $\langle M^{(\delta)}\rangle_t$ has finite expectation for $t \in [0,T)$.

In order to conclude that inequality (2.2) holds for the function U_δ, noting that $\nabla\rho_t$ vanishes on $\{\rho_t = 0\}$, it is enough to show that the last integral in (B.10) has (well-defined) null expectation. Using (B.9) and assumption $(H6')_{p_\infty}$ we obtain [with the same estimation for $U_\delta''(r)$ as before] that

$$
\begin{aligned}
&\mathbb{E}_\infty^T \int_r^t \left| \frac{1}{p_\infty} \partial_k\big(\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li}p_\infty\big)\right|(Y_s)\,ds \\
&\quad = \int_r^t \int_{\mathbb{R}^d} \big|\partial_k\big(\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li}p_\infty\big)\big|\,dx\,ds \qquad \text{(B.12)}\\
&\quad < \infty,
\end{aligned}
$$

which shows that the expectation of the last term in (B.10) is well defined. Moreover, the (everywhere-defined) spatial divergence of $g(s,x) := \partial_l\rho_s\partial_{l'}\rho_s U_\delta''(\rho_s) \times [\sigma_{l'i}a_{m\bullet} - \sigma_{\bullet i}a_{ml'}]\partial_m\sigma_{li}p_\infty$ is $L^1(dx,\mathbb{R}^d)$ for a.e. s. For such s and $\phi_n \in C_0^\infty(\mathbb{R}^d)$ satisfying $0 \le \phi_n \le 1$, $0 \le |\nabla\phi_n| \le 1$, $\phi_n(x) = 1$ for $x \in B(0,n)$ and $\phi_n(x) = 0$ for $x \in B(0,2n)^c$, we have

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^d} \nabla\cdot\big(\phi_n(x)g(s,x)\big)\,dx \\
&= \int_{\mathbb{R}^d} \phi_n(x)\nabla\cdot g(s,x)\,dx + \int_{\mathbb{R}^d} \nabla\phi_n(x).g(s,x)\,dx.
\end{aligned}
$$

Since by Lebesgue's theorem, the second term on the right-hand side tends to 0 as $n\to\infty$, the limit $\int_{\mathbb{R}^d}\nabla\cdot g(s,x)\,dx$ of the first term is equal to 0.

APPENDIX C

In this section we compare our results on the dissipation of the Fisher information with the computations and results in [1].

The form of the term $\mathrm{tr}(\Lambda_\delta\Gamma)$ in Proposition 2.2 is inspired from the term $\mathrm{tr}(\mathbf{XY})$ in [1], pages 163–164, where $\mathbf{X} = 2\Lambda_\delta$. One has

$$
\begin{aligned}
\Gamma_{12} &= \big(\nabla^*\rho a\big)_j \partial_j(\sigma_{ki}\partial_k\rho)\sigma_{li}\partial_l\rho \\
&= \tfrac{1}{2}\big(\nabla^*\rho a\big)_j\big[\partial_j(\sigma_{ki}\partial_k\rho)\sigma_{li}\partial_l\rho + \partial_j(\sigma_{li}\partial_l\rho)\sigma_{ki}\partial_k\rho\big] \\
&= \tfrac{1}{2}\big(\nabla^*\rho a\big)_j\partial_j[\partial_l\rho a_{kl}\partial_k\rho] = \tfrac{1}{2}\big(\nabla^*\rho a\big)\nabla\big(\nabla^*\rho a\nabla\rho\big)
\end{aligned}
$$

which, with $\frac{\partial v}{\partial x} := (\partial_j v_i)_{i,j}$ denoting the Jacobian matrix of vector field v, equals

$$
\begin{aligned}
\frac{1}{2}\big(\nabla^*\rho a\big)_j\partial_j[\partial_k\rho a_{kl}\partial_l\rho] &= \frac{1}{2}\big(\nabla^*\rho a\big)_j\big(\partial_{kj}\rho a_{kl}\partial_l\rho + \partial_j[a_{kl}\partial_l\rho]\partial_k\rho\big) \\
&= \frac{1}{2}\nabla^*\rho a\frac{\partial(\nabla\rho)}{\partial x}a\nabla\rho + \frac{1}{2}\nabla^*\rho a\frac{\partial(a\nabla\rho)^*}{\partial x}\nabla\rho
\end{aligned}
$$

and corresponds to $4\mathbf{Y}_{12}$ in [1], page 164 [noting that in their notation, $\mathbf{D}(x) = a(x)/2$]. Similarly, $\Gamma_{22} = 4\mathbf{Y}_{22}$. However Γ_{11} cannot in general be identified with $4\mathbf{Y}_{11}$. For instance, in the case of scalar diffusion $\mathbf{D}(x) = a(x)/2 = D(x)I_d$ for some real valued function D, the term $\Gamma_{11}(x)$ above, when written in terms of D, reads

$$\frac{1}{2}|\nabla D|^2|\nabla\rho|^2 + \frac{1}{2}(\nabla D\cdot\nabla\rho)^2 + 4D\partial_j D\partial_i\rho\partial_{ij}\rho + 4D^2\sum_{ij}(\partial_{ij}\rho)^2$$

for the choice $\sigma(x) = \sqrt{2D(x)}I_d$, whereas

$$4\mathbf{Y}_{11} = 4\biggl(D^2\sum_{ij}(\partial_{ij}\rho)^2 + \biggl(\frac{d}{4}-\frac{1}{2}\biggr)(\nabla\rho\cdot\nabla D)^2 + 2D\partial_j D\partial_i\rho\partial_{ij}\rho \\ - D(\nabla\rho\cdot\nabla D)\triangle\rho + \frac{1}{2}|\nabla D|^2|\nabla\rho|^2\biggr).$$

Moreover, our term Γ_{11} is nonintrinsic, in the sense that it cannot in general be written in terms of the diffusion matrix a only (without making explicit use of σ), contrary to the term $\mathbf{Y}_{11}$ in the matrix of [1].

We will next check that the criterion in [1] can also be derived from the computations in Proposition 2.2 in case a is nonsingular, which amounts to making an alternative choice in the expression for $d[U_\delta''(\rho)\nabla^*\rho a\nabla\rho]$ of the quantities in the roles of the coefficient Γ_{11} and of the term $\bar{\theta}$. This will also allow us to compare and combine both criteria.

Recall first that the matrix $\mathbf{D}(x)$ in [1] equals half of our matrix $a(x)$, and notice that our forward drift term writes in their notation $b = -\mathbf{D}\nabla\phi - \mathbf{D}F + \nabla\cdot\mathbf{D}$, where $(\nabla\cdot\mathbf{D})_i = \partial_j\mathbf{D}_{ij}$, $e^{-\phi} = p_\infty$ is the invariant density, and F a is vector field satisfying $\nabla\cdot(\mathbf{D}Fe^{-\phi}) = 0$. Thus $\bar{b} = a\nabla\ln p_\infty + \nabla\cdot a - b = -\mathbf{D}\nabla\phi + \mathbf{D}F + \nabla\cdot\mathbf{D}$.

The factor of $U_\delta''(\rho)$ in (B.7) takes the intrinsic form

$$\begin{aligned}
&a_{kk'}[\partial_{kl}\rho\sigma_{li}\partial_{k'l'}\rho\sigma_{l'i} + \partial_{kl}\rho\sigma_{li}\partial_{l'}\rho\partial_{k'}\sigma_{l'i} + \partial_l\rho\partial_k\sigma_{li}\partial_{k'l'}\rho\sigma_{l'i} \\
&\qquad\qquad + \partial_l\rho\partial_k\sigma_{li}\partial_{l'}\rho\partial_{k'}\sigma_{l'i}] \\
&\quad + 2\sigma_{l'i}\partial_{l'}\rho a_{k'k}\partial_{k'}\sigma_{li}\partial_{lk}\rho + \partial_l\rho\partial_{l'}\rho a_{kk'}\partial_{k'k}\sigma_{li}\sigma_{l'i} - 2a_{ll'}\partial_{l'}\rho\sigma_{kr}\partial_{kk'}\rho\partial_l\sigma_{k'r} \\
&\quad + \bar{b}_m\partial_m a_{ll'}\partial_l\rho\partial_{l'}\rho - 2a_{ll'}\partial_{l'}\rho\partial_k\rho\partial_l\bar{b}_k \\
&\qquad = a_{kk'}[\partial_{kl}\rho\partial_{k'l'}\rho a_{ll'} + 2\partial_{kl}\rho\partial_{l'}\rho\partial_{k'}a_{ll'}] + \tfrac{1}{2}a_{kk'}\partial_l\rho\partial_{l'}\rho\partial_{kk'}a_{ll'} \\
&\qquad\quad - a_{ll'}\partial_{l'}\rho\partial_{kk'}\rho\partial_l a_{kk'} + \bar{b}_m\partial_m a_{ll'}\partial_l\rho\partial_{l'}\rho - 2a_{ll'}\partial_{l'}\rho\partial_k\rho\partial_l\bar{b}_k,
\end{aligned}$$

where to the second and third terms in the bracket on the left-hand side, brought together, we have added the first term after the bracket, and moreover the fourth term in the bracket on the left-hand side was added to the second term outside the bracket. Hence, writing

$$Q_1 := -a_{ll'}\partial_{l'}\rho\partial_{kk'}\rho\partial_l a_{kk'} + \bar{b}_m\partial_m a_{ll'}\partial_l\rho\partial_{l'}\rho - 2a_{ll'}\partial_{l'}\rho\partial_k\rho\partial_l\bar{b}_k,$$
$$Q_2 := a_{kk'}[\partial_{kl}\rho\partial_{k'l'}\rho a_{ll'} + 2\partial_{kl}\rho\partial_{l'}\rho\partial_{k'}a_{ll'}] + \tfrac{1}{2}a_{kk'}\partial_l\rho\partial_{l'}\rho\partial_{kk'}a_{ll'},$$

and using the last expression for Γ_{12} above, we can write

$$\begin{aligned}
&\frac{1}{2}d\big[U_\delta''(\rho)\nabla^*\rho a\nabla\rho\big]\\
&\quad= \frac{1}{2}d\widehat{M}^{(\delta)} + \frac{U_\delta''(\rho)}{2}(Q_1+Q_2)\,dt + \frac{U_\delta^{(4)}(\rho)}{4}\big|\nabla^*\rho a\nabla\rho\big|^2\,dt\\
&\qquad + \frac{U_\delta^{(3)}(\rho)}{2}\bigg(\nabla^*\rho a\frac{\partial(\nabla\rho)}{\partial x}a\nabla\rho + \nabla^*\rho a\frac{\partial(a\nabla\rho)^*}{\partial x}\nabla\rho\bigg)\,dt\\
&\quad= \frac{1}{2}d\widehat{M}^{(\delta)} + \bigg[\frac{U_\delta^{(4)}(\rho)}{4}\big|\nabla^*\rho a\nabla\rho\big|^2\\
&\qquad\qquad + \frac{U_\delta^{(3)}(\rho)}{4}\bigg(\nabla^*\rho a\frac{\partial(\nabla\rho)}{\partial x}a\nabla\rho + \nabla^*\rho a\frac{\partial(a\nabla\rho)^*}{\partial x}\nabla\rho\bigg)\bigg]\,dt\\
&\qquad + \bigg[\frac{U_\delta''(\rho)}{2}Q_1\bigg]\,dt\\
&\qquad + \bigg[\frac{U_\delta''(\rho)}{2}Q_2\\
&\qquad\quad + \frac{U_\delta^{(3)}(\rho)}{4}\bigg(\nabla^*\rho a\frac{\partial(\nabla\rho)}{\partial x}a\nabla\rho + \nabla^*\rho a\frac{\partial(a\nabla\rho)^*}{\partial x}\nabla\rho\bigg)\bigg]\,dt.
\end{aligned}\tag{C.1}$$

The latter identity yields the expression for the dissipation of entropy dissipation computed in [1]. Indeed, denoting, respectively, by J_1, J_2 and J_3 the expectations of the first, second and third terms in square brackets on the right-hand side, we observe that J_1 is, up to time reversal $t \mapsto T-t$, exactly equal to the term $\widetilde{R}_1$, at the top of page 162 in [1]. Starting from the last expression of T_3, page 160, and the definition (2.23) of $\widetilde{R}_2$ and T_4, and replacing $\mathbf{D}F$ by its expression $\bar{b} - \frac{1}{2}(a\nabla\ln(p_\infty) + \nabla\cdot a)$ in our notation, we get that $\widetilde{R}_2 + T_3 + T_4$ is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^d}\frac{U_\delta''(\rho)Q_1}{2}p_\infty - \int_{\mathbb{R}^d}\bigg[\frac{U_\delta''(\rho)}{4}(\partial_i\rho\partial_j a_{ik}\partial_k\rho + 2\partial_{ij}\rho a_{ik}\partial_k\rho)\bigg]\times\partial_l(a_{lj}p_\infty)\\
&\qquad - \int_{\mathbb{R}^d}\frac{U_\delta''(\rho)}{2}[a_{lj}\partial_{ijl}\rho\partial_k\rho a_{ki}]p_\infty
\end{aligned}$$

up to time reversal. The first term corresponds to J_2. Integrating by parts the second term to get rid of the derivative, with respect to the lth coordinate in the second factor, one checks that its sum with the last one is equal to J_3. Hence, up to time reversal, we have $J_1 + J_2 + J_3 = (\widetilde{R}_1 + T_3) + (\widetilde{R}_2 + T_4)$, which is the expression for the dissipation of entropy dissipation computed in [1], page 160.

In order to recover the Bakry–Emery criterion in [1], we rewrite $Q_1 + Q_2 = K_1(\rho) + K_2(\rho)$ where

$$K_1(\rho) := \bar{b}_m \partial_m a_{ll'} \partial_l \rho \partial_{l'} \rho - 2a_{ll'} \partial_{l'} \rho \partial_k \rho \partial_l \bar{b}_k + \tfrac{1}{2} a_{kk'} \partial_l \rho \partial_{l'} \rho \partial_{kk'} a_{ll'}$$

and

$$K_2(\rho) := a_{kk'} \partial_{kl} \rho a_{ll'} \partial_{k'l'} \rho + 2a_{kk'} \partial_{kl} \rho \partial_{l'} \rho \partial_{k'} a_{ll'} - a_{k'l'} \partial_{l'} \rho \partial_{kl} \rho \partial_{k'} a_{kl}.$$

When a is nonsingular, introducing $G_{jk}(\rho) = \partial_{l'} \rho a_{k'l'} \partial_{k'} a_{jk}$ and $H_{lj}(\rho) = \partial_j a_{ll'} \partial_{l'} \rho$, we can write

$$\begin{aligned}
K_2(\rho) &= \operatorname{tr}\big[(a\nabla^2\rho)^2 + 2H(\rho)a\nabla^2\rho - G(\rho)\nabla^2\rho\big] \\
&= \operatorname{tr}\big[(a\nabla^2\rho)^2 + H(\rho)a\nabla^2\rho + aH(\rho)^*\nabla^2\rho - G(\rho)a^{-1}a\nabla^2\rho\big] \\
&= \operatorname{tr}\big[(a\nabla^2\rho)^2 + \tfrac{1}{2}\big(H(\rho)a\nabla^2\rho + aH(\rho)^*\nabla^2\rho - G(\rho)a^{-1}a\nabla^2\rho\big) \\
&\qquad + \tfrac{1}{2}\big(a\nabla^2\rho H(\rho) + a\nabla^2\rho aH(\rho)^* a^{-1} - a\nabla^2\rho G(\rho)a^{-1}\big)\big],
\end{aligned}$$

where we have used the cyclicity of the trace and its invariance by transposition. Following [1], we complete the trace of a squared sum of matrices to get

$$\begin{aligned}
K_2(\rho) = \operatorname{tr}&\big[a\nabla^2\rho + \tfrac{1}{2}\big(H(\rho) + aH(\rho)^*a^{-1} - G(\rho)a^{-1}\big)\big]^2 \\
&- \tfrac{1}{4}\operatorname{tr}\big[H(\rho) + aH(\rho)^*a^{-1} - G(\rho)a^{-1}\big]^2.
\end{aligned}$$

The finite variation part on the right-hand side of the first line in (C.1) therefore rewrites

$$\begin{aligned}
&\frac{U_\delta''(\rho)}{2}\left(K_1(\rho) - \frac{1}{4}\operatorname{tr}\big[H(\rho) + aH(\rho)^*a^{-1} - G(\rho)a^{-1}\big]^2\right) \\
&\quad + \frac{U_\delta''(\rho)}{2}\operatorname{tr}\left[a\nabla^2\rho + \frac{1}{2}\big(H(\rho) + aH(\rho)^*a^{-1} - G(\rho)a^{-1}\big)\right]^2 dt \\
&\quad + \frac{U_\delta^{(3)}(\rho)}{2}\left(\nabla^*\rho a \frac{\partial(\nabla\rho)}{\partial x} a\nabla\rho + \nabla^*\rho a \frac{\partial(a\nabla\rho)^*}{\partial x}\nabla\rho\right) dt \\
&\quad + \frac{U_\delta^{(4)}(\rho)}{4}\left|\nabla^*\rho a\nabla\rho\right|^2 dt.
\end{aligned} \tag{C.2}$$

The sum of the second, third and fourth lines correspond to the matrix product **XY** in [1] and is shown to be nonnegative on page 164 therein. We can then check that for a smooth function $v : \mathbb{R}^d \to \mathbb{R}$, the term $\frac{1}{2}(K_1(v) - \frac{1}{4}\operatorname{tr}[H(v) + aH(v)^*a^{-1} - G(v)a^{-1}]^2)$ is twice the expression on the left-hand side of inequality (2.13) on page 158 of [1] (with ∇v corresponding to their vector field "U"). Consequently, their Bakry–Emery criterion (2.13) corresponds, in our notation, to imposing the condition

$\exists\lambda > 0$ such that for all smooth functions $v : \mathbb{R}^d \to \mathbb{R}$ and all $x \in \mathbb{R}^d$,

$$\tfrac{1}{2}\big(K_1(v) - \tfrac{1}{4}\operatorname{tr}\big[H(v) + aH(v)^* a^{-1} - G(v)a^{-1}\big]^2\big)(x) \geq \lambda \nabla v^* a \nabla v(x),$$

which implies exponential convergence at rate 2λ of the U-Fisher information and the U-relative entropy.

We may combine this criterion with ours by introducing some C^1 function $\alpha : \mathbb{R}^d \to [0, 1]$ and writing the finite variation part on the right-hand side of the first line in (C.1) as $(1-\alpha)$ multiplied by the expression (C.2), plus $\frac{1}{2}\alpha$ multiplied by the finite variation part in the right-hand side of (B.8). Because of the integration by parts performed in the proof of Theorem 2.4, the mixed criterion involves the derivatives of α. Let

$$\Theta^{\alpha}_{ll'} := \alpha\Theta_{ll'} - \tfrac{1}{2}\partial_k\alpha\big([\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}]\partial_m\sigma_{li} + [\sigma_{li}a_{mk} - \sigma_{ki}a_{ml}]\partial_m\sigma_{l'i}\big).$$

This ultimate mixed criterion writes

$\exists\lambda > 0$ such that for all smooth function $v : \mathbb{R}^d \to \mathbb{R}$ and all $x \in \mathbb{R}^d$:

$$\begin{aligned}&\big(1-\alpha(x)\big)\big(\tfrac{1}{2}\big(K_1(v) - \tfrac{1}{4}\operatorname{tr}\big[H(v) + aH(v)^* a^{-1} - G(v)a^{-1}\big]^2\big)(x)\big)\\ &\qquad + \nabla v^* \Theta^{\alpha} \nabla v(x) \geq \lambda \nabla v^* a \nabla v(x)\end{aligned}$$

and also implies exponential convergence at rate 2λ of the U-Fisher information and the U-relative entropy.

Acknowledgments. We thank Tony Lelièvre (CERMICS) for pointing out to us the paper of Arnold, Carlen and Ju [1] at an early stage of this research. We also thank Anton Arnold (TU Wien) for suggesting an improvement of our nonintrinsic Bakry–Emery criterion; see Remark 2.3. The first author thanks CERMICS for their hospitality and partial support.

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DIM-CMM, UMI(2807) UCHILE-CNRS
UNIVERSIDAD DE CHILE
CASILLA 170-3, CORREO 3
SANTIAGO
CHILE
E-MAIL: fontbona@dim.uchile.cl

UNIVERSITÉ PARIS-EST
CERMICS (ENPC), INRIA
6-8 AV BLAISE PASCAL, CITÉ DESCARTES
CHAMPS SUR MARNE
77455 MARNE-LA-VALLÉE CEDEX 2
FRANCE
E-MAIL: jourdain@cermics.enpc.fr