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Kac's Program in Kinetic Theory

Stéphane Mischler, Clément Mouhot

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KAC'S PROGRAM IN KINETIC THEORY

(VERSION OF July 22, 2012)

S. MISCHLER AND C. MOUHOT

ABSTRACT. This paper is devoted to the study of propagation of chaos and mean-field limits for systems of indistinguishable particles, undergoing collision processes. The prime examples we will consider are the many-particle jump processes of Kac and McKean [42, 53] giving rise to the Boltzmann equation. We solve the conjecture raised by Kac [42], motivating his program, on the rigorous connection between the long-time behavior of a collisional many-particle system and the one of its mean-field limit, for bounded as well as unbounded collision rates.

Motivated by the inspirative paper by Grünbaum [35], we develop an abstract method that reduces the question of propagation of chaos to that of proving a purely functional estimate on generator operators (*consistency estimates*), along with differentiability estimates on the flow of the nonlinear limit equation (*stability estimates*). This allows us to exploit dissipativity at the level of the mean-field limit equation rather than the level of the particle system (as proposed by Kac).

Using this method we show: (1) Quantitative estimates, that are uniform in time, on the chaoticity of a family of states. (2) Propagation of *entropic chaoticity*, as defined in [10]. (3) Estimates on the time of relaxation to equilibrium, that are *independent of the number of particles in the system*. Our results cover the two main Boltzmann physical collision processes with unbounded collision rates: hard spheres and *true* Maxwell molecules interactions. The proof of the *stability estimates* for these models requires significant analytic efforts and new estimates.

Keywords: Kac's program; kinetic theory; master equation; mean-field limit; quantitative; uniform in time; jump process; collision process; Boltzmann equation; Maxwell molecules; non cutoff; hard spheres.

AMS Subject Classification: 82C40 Kinetic theory of gases, 76P05 Rarefied gas flows, Boltzmann equation, 54C70 Entropy, 60J75 Jump processes.

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Boltzmann is best known for the equation bearing his name in kinetic theory [7, 8]. Inspired by Maxwell's discovery [51] of (what is now called) the Boltzmann equation and its "Maxwellian" (i.e. Gaussian) equilibrium, Boltzmann [7] discovered the " H -theorem" (the entropy must increase under the time evolution of the equation), which explained why the solutions should be driven towards the equilibrium of Maxwell. In the same work he also proposed the deep idea of "stosszahlansatz" (molecular chaos) to explain how the irreversible Boltzmann equation can emerge from Newton's laws of the dynamics of the particle system. Giving a precise mathematical meaning to this notion and proving this

limit remains a tremendous open problem to this date; the best and astonishing result so far [45] is only valid for very short times.

In 1956, Kac [42] proposed the simpler, and seemingly more tractable, question of deriving the *spatially homogeneous* Boltzmann equation from a many-particle jump process. To do so, he introduced a rigorous notion of “molecular chaos”¹ in this context. The “chaoticity” of the many-particle equilibrium with respect to the Maxwellian distribution, i.e. the fact that the first marginals of the uniform measure on the sphere $\mathbb{S}^{N-1}(\sqrt{N})$ converges to a Gaussian function as N goes to infinity, has been known for a long time (at least since Maxwell).² However in [42] Kac proposed the first proof of the *propagation of chaos* for a simplified collision evolution process for which series expansion of the solution exists, and he showed how the many-particle limit rigorously follows from this property of propagation of chaos. This proof was later extended to a more realistic collision model, the so-called cutoff Maxwell molecules, by McKean [53].

Since in this setting both the many-particle system and the limit equation are dissipative, Kac raised the natural question of relating their asymptotic behaviors. In his mind this program was to be achieved by understanding dissipativity at the level of the linear many-particle jump process and he insisted on the importance of estimating how its relaxation rate depends on the number of particles. This has motivated beautiful works on the “Kac spectral gap problem” [40, 50, 11, 13, 9], i.e. the study of this relaxation rate in a L^2 setting. However, so far this linear strategy has proved to be unsuccessful in relating the asymptotic behavior of the many-particle process and that of the limit equation (cf. the discussion in [10]).

In the time of Kac the study of nonlinear partial differential equations was rather young and it was plausible that the study of a linear many-dimension Markov process would be easier. However the mathematical developpement somehow followed the reverse direction and the theory of existence, uniqueness and relaxation to equilibrium for the spatially homogeneous Boltzmann is now well-developed (see the many references along this paper). This paper (together with its companion paper [58]) is thus an attempt to develop a quantitative theory of mean-field limit which *strongly relies on detailed knowledge of the nonlinear limit equation, rather than on detailed properties of the many-particle Markov process*.

The main outcome of our theory will be to find quantitative estimates to the propagation of chaos that are uniform in time, as well as propagation of entropic chaos. We also prove estimates on the relaxation rates, measured in the Wasserstein distance and relative entropy, that are *independent of the number of particles*. All this is done for the two important, realistic and achetypal models of collision with unbounded collision rates, namely hard spheres and the true (i.e. without cutoff) Maxwell molecules. This provides a first answer to the question raised by Kac. However, our answer is an “inverse” answer in the sense that our methodology is “top-down” from the limit equation to the many-particle system rather than “bottom-up” as was expected by Kac.

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¹Kac in fact called this notion “Boltzmann’s property” in [42] as a clear tribute to the fundamental intuition of Boltzmann.

²We refer to [20] for a bibliographic discussion, see also [10] where [54] is quoted as the first paper proving this result.

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1. INTRODUCTION AND MAIN RESULTS

1.1. The Boltzmann equation. The Boltzmann equation (Cf. [17] and [18]) describes the behavior of a dilute gas when the only interactions taken into account are binary collisions. It is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \quad x \in \Omega, \quad v \in \mathbb{R}^d, \quad t \geq 0,$$

where $Q = Q(f, f)$ is the bilinear *collision operator* acting only on the velocity variable, Ω is the spatial domain and $d \geq 2$ is the dimension. Some appropriate boundary conditions need to be imposed.

In the case where the distribution function is assumed to be independent of the position x , we obtain the so-called *spatially homogeneous Boltzmann equation*:

$$(1.1) \quad \frac{\partial f}{\partial t}(t, v) = Q(f, f)(t, v), \quad v \in \mathbb{R}^d, \quad t \geq 0,$$

which will be studied in this paper.

Let us now focus on the collision operator Q . It is defined by the bilinear symmetrized form

$$(1.2) \quad Q(g, f)(v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos \theta) (g'_* f' + g' f'_* - g_* f - g f_*) dv_* d\sigma,$$

where we use the shorthands $f = f(v)$, $f' = f(v')$, $g_* = g(v_*)$ and $g'_* = g(v'_*)$. Moreover, v' and v'_* are parametrized by

$$(1.3) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^{d-1}.$$

Finally, $\theta \in [0, \pi]$ is the *deviation angle* between $v' - v'_*$ and $v - v_*$ defined by

$$\cos \theta = \sigma \cdot \hat{u}, \quad u = v - v_*, \quad \hat{u} = \frac{u}{|u|},$$

and B is the *collision kernel* determined by the physical context of the problem.

The Boltzmann equation has the following fundamental formal properties: first, it conserves mass, momentum and energy, i.e.

$$\frac{d}{dt} \int_{\mathbb{R}^d} f \phi(v) dv = \int_{\mathbb{R}^d} Q(f, f) \phi(v) dv = 0, \quad \phi(v) = 1, v, |v|^2.$$

Second, it satisfies Boltzmann's celebrated *H*-theorem:

$$-\frac{d}{dt} \int_{\mathbb{R}^d} f \ln f dv = - \int_{\mathbb{R}^d} Q(f, f) \ln(f) dv \geq 0.$$

We shall consider collision kernels of the form

$$B = \Gamma(|v - v_*|) b(\cos \theta)$$

where Γ, b are nonnegative functions. In dimension $d = 3$, we give a short overview of the main collision kernels appearing in physics, highlighting the key models we consider in this paper.

- (1) Short (finite) range interaction are usually modeled by the *hard spheres collision kernel*

$$(1.4) \quad (\mathbf{HS}) \quad B(|v - v_*|, \cos \theta) = C |v - v_*|, \quad C > 0.$$

- (2) Long-range interactions are usually modeled by collision kernels derived from interaction potentials

$$V(r) = C r^{-s}, \quad s > 2, \quad C > 0.$$

They satisfy the formula

$$\Gamma(z) = |z|^\gamma \text{ with } \gamma = (s - 4)/s$$

and

$$b(\cos \theta) \sim_{\theta \sim 0} C_b \theta^{-2-\nu} \text{ with } \nu = 2/s, \quad C_b > 0$$

(b is in L^1 away from $\theta = 0$). More informations about this type of interactions can be found in [17]. This general class of collision kernels includes the *true Maxwell molecules collision kernel* when $\gamma = 0$ and $\nu = 1/2$, i.e.

$$(1.5) \quad (\mathbf{tMM}) \quad B(|v - v_*|, \cos \theta) = b(\cos \theta) \sim_{\theta \sim 0} C_b \theta^{-5/2}.$$

It also includes the so-called *Grad's cutoff Maxwell molecules* when the singularity in the θ variable is removed. For simplicity we will consider the model where

$$(1.6) \quad (\mathbf{GMM}) \quad B(|v - v_*|, \cos \theta) = 1$$

which is an archetype of such collision kernels.

1.2. Deriving the Boltzmann equation from many-particle systems. The question of deriving the Boltzmann equation from particle systems (interacting via Newton's laws) is a famous problem. It is related to the so-called 6-th Hilbert problem proposed by Hilbert at the International Congress of Mathematics at Paris in 1900: axiomatize mechanics by “*developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua*”.

At least at the formal level, the correct limiting procedure has been identified by Grad [33] in the late forties and a clear mathematical formulation of the open problem was proposed in [16] in the early seventies. It is now called the *Boltzmann-Grad* or *low density* limit. However the original question of Hilbert remains largely open, in spite of a striking breakthrough due to Lanford [45], who proved the limit for short times (see also Illner and Pulvirenti [39] for a close-to-vacuum result). The tremendous difficulty underlying this limit is the *irreversibility* of the Boltzmann equation, whereas the particle system interacting via Newton's laws is a *reversible* Hamiltonian system.

In 1954-1955, Kac [42] proposed a simpler and more tractable problem: start from the Markov process corresponding to collisions only, and try to prove the limit towards the *spatially homogeneous* Boltzmann equation. *Kac's jump process* runs as follows: consider N particles with velocities $v_1, \dots, v_N \in \mathbb{R}^d$. Assign a random time for each pair of particles (v_i, v_j) following an exponential law with parameter $\Gamma(|v_i - v_j|)$, and take the smallest. Next, perform a collision $(v_i, v_j) \rightarrow (v_i^*, v_j^*)$ according to a random choice of direction parameter, whose law is related to $b(\cos \theta)$, and start again with the post collision velocities. This process can be considered on \mathbb{R}^{dN} ; however it has some invariant submanifolds of \mathbb{R}^{dN} (depending on the number of quantities conserved under the collision), and can be restricted to them. For instance, in the original simplified model of Kac (scalar velocities, i.e. $d = 1$) the process can be restricted to $\mathbb{S}^{N-1}(\sqrt{\mathcal{E}N})$, the sphere with radius $\sqrt{\mathcal{E}N}$, where \mathcal{E} is the energy. In the more realistic models of the hard spheres or Maxwell molecules

(when $d = 3$) the process can be restricted to the following submanifold of \mathbb{R}^{dN} associated to elastic collisions invariants for $\mathcal{M} \in \mathbb{R}^d$, $\mathcal{E} \geq 0$:

$$(1.7) \quad \mathcal{S}^N(\mathcal{M}, \mathcal{E}) := \left\{ \frac{|v_1 - \mathcal{M}|^2 + \cdots + |v_N - \mathcal{M}|^2}{N} = \mathcal{E} \right\} \cap \left\{ \frac{v_1 + \cdots + v_N}{N} = \mathcal{M} \right\}.$$

Without loss of generality, we will consider the case $\mathcal{M} = 0$, using Galilean invariance. We will denote by $\mathcal{S}^N(\mathcal{E}) := \mathcal{S}^N(0, \mathcal{E})$ and refer to these submanifolds as *Boltzmann spheres*.

Kac then formulated the following notions of *chaos* and *propagation of chaos*: Consider a sequence $(f^N)_{N \geq 1}$ of probabilities on E^N , where E is some given Polish space (e.g. $E = \mathbb{R}^d$): the sequence is said to be *f-chaotic* if

$$f^N \sim f^{\otimes N} \quad \text{when } N \rightarrow \infty$$

for some given one-particle probability f on E . The convergence is to be understood as the convergence of the ℓ -th marginal of f^N to $f^{\otimes \ell}$, for any $\ell \in \mathbb{N}^*$, in the weak measure topology. This is a *low correlation* assumption.

It is an elementary calculation to see that if the probability densities f_t^N of the N -particle system were perfectly tensorized *during some time interval* $t \in [0, T]$ (i.e. had the form of an N -fold tensor product of a one particle probability density f_t), then the latter f_t would satisfy the nonlinear limit Boltzmann equation during that time interval. But generally interactions between particles instantaneously destroy this “tensorization” property and leave no hope to show its propagation in time. Nevertheless, following Boltzmann’s idea of molecular chaos, Kac suggested that the weaker property of chaoticity can be expected to propagate in time, in the correct scaling limit.

This framework is our starting point. Let us emphasize that the limit performed in this setting is different from the Boltzmann-Grad limit. It is in fact a *mean-field limit*. This limiting process is most well-known for deriving Vlasov-like equations. In a companion paper [58] we develop systematically our new functional approach to the study of mean-field limits for Vlasov equations, McKean-Vlasov equations, and granular gases equations.

1.3. The notion of chaos and how to measure it. Our goal in this paper is to set up a general and robust method for proving the propagation of chaos with *quantitative rates* in terms of the number of particles N and of the final time of observation T .

Let us briefly discuss the notion of *chaoticity*. The original formulation in [42] is: A sequence $f^N \in P_{\text{sym}}(E^N)$ of symmetric³ probabilities on E^N is *f-chaotic*, for a given probability $f \in P(E)$, if for any $\ell \in \mathbb{N}^*$ and any $\varphi \in C_b(E)^{\otimes \ell}$ there holds

$$\lim_{N \rightarrow \infty, N \geq \ell} \left\langle f^N, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle = \left\langle f^{\otimes \ell}, \varphi \right\rangle.$$

Together with additional assumptions on the moments, this weak convergence can be expressed for instance in Wasserstein distance as:

$$\forall \ell \geq 1, \quad \lim_{N \rightarrow \infty, N \geq \ell} W_1 \left(\Pi_\ell (f^N), f^{\otimes \ell} \right) = 0$$

where Π_ℓ denotes the marginal on the ℓ first variables. We call this notion *finite-dimensional chaos*.

In contrast with most previous works on this topic, we are interested here in *quantitative chaos*: Namely, we say that f^N is *f-chaotic* with rate $\varepsilon(N)$, where $\varepsilon(N) \rightarrow 0$ as $N \rightarrow \infty$ (typically $\varepsilon(N) \sim N^{-r}$, $r > 0$ or $\varepsilon(N) \sim (\ln N)^{-r}$, $r > 0$), if for any $\ell \in \mathbb{N}^*$ there exists $K_\ell \in (0, \infty)$ such that

$$(1.8) \quad W_1 \left(\Pi_\ell (f^N), f^{\otimes \ell} \right) \leq K_\ell \varepsilon(N).$$

³i.e. invariant according to permutations of the particles.

Similar statements can be formulated for other metrics. For instance, a convenient way to measure chaoticity is through duality: for some normed space of *smooth functions* $\mathcal{F} \subset C_b(E)$ (to be specified) and for any $\ell \in \mathbb{N}^*$ there exists $K_\ell \in (0, \infty)$ such that for any $\varphi \in \mathcal{F}^{\otimes \ell}$, $\|\varphi\|_{\mathcal{F}} \leq 1$, there holds

$$|\langle \Pi_\ell [f^N] - f^{\otimes \ell}, \varphi \rangle| \leq K_\ell \varepsilon(N).$$

The Wasserstein distance W_1 is recovered when \mathcal{F} is the space of Lipschitz functions.

Observe that in the latter statements the number of variables ℓ considered in the marginal is kept fixed as N goes to infinity. It is a natural question to know how the rate depends on ℓ . As we will see, the answer to this question is essential to the estimation of a relaxation time that will be uniform in the number of particles. We therefore introduce a stronger notion of *infinite-dimensional chaos*, based on *extensive*⁴ functionals. We consider the following definition:

$$\lim_{N \rightarrow \infty} \frac{W_1(f^N, f^{\otimes N})}{N} = 0$$

(with corresponding quantitative formulations...). This amounts to say that one can prove a sublinear control on K_ℓ in terms of ℓ in (1.8). Variants for other extensive metrics could easily be considered as well.

Finally one can formulate an even stronger notion of (*infinite-dimensional*) *entropic chaos* (see [10] and definition (1.9) of the relative entropy below):

$$\frac{1}{N} H(f^N | \gamma^N) \xrightarrow{N \rightarrow \infty} H(f | \gamma),$$

where γ^N is an invariant measure of the N -particle system, which is γ -chaotic with γ an invariant measure of the limit equation. This notion of chaos obviously admits quantitative versions as well. Moreover, it is particularly interesting as it corresponds to the derivation of Boltzmann's entropy and H -theorem from the entropies of the many-particle system. We shall come back to this point.

Now, considering a sequence of symmetric N -particle densities

$$f^N \in C([0, \infty); P_{\text{sym}}(E^N))$$

and a 1-particle density of the expected mean field limit

$$f \in C([0, \infty); P(E)),$$

we say that there is *propagation of chaos* on some time interval $[0, T]$, $T \in (0, +\infty]$, if the f_0 -chaoticity of the initial family f_0^N implies the f_t -chaoticity of the family f_t^N for any time $t \in [0, T]$, *according to one of the definitions of chaoticity above*.

Note that the *support* of the N -particle distributions matters. Indeed the energy conservation implies that the evolution is entirely decoupled on the different subspaces associated with this invariant, e.g. each $\mathcal{S}^N(\mathcal{E})$ for the different values of \mathcal{E} (we consider here centered distributions). On each such subspace the N -particle process is ergodic and admits a unique invariant measure γ^N , which is constant. However when considered on \mathbb{R}^{dN} the N -particle process admits infinitely many invariant measures. Therefore, in the study of the long-time behavior we will often consider N -particle distributions that are supported on $\mathcal{S}^N(\mathcal{E})$ for appropriate energy \mathcal{E} . We shall discuss the construction of such chaotic initial data in Section 4.

⁴By extensivity we mean here that the functional measuring the distance between two distributions should behave additively with respect to the tensor product.

1.4. Kac's program. As was mentioned before, Kac proposed to derive the spatially homogeneous Boltzmann equation from a many-particle Markov jump process with binary collisions, via its master equation (the equation on the law of the process). Intuitively, this amounts to considering the spatial variable as a hidden variable inducing ergodicity and markovian properties on the velocity variable. Although the latter point has not been proved so far to our knowledge, it is worth noting that it is a very natural guess and an extremely interesting open problem (and possibly a very difficult one). Hence Kac introduced artificial stochasticity as compared to the initial Hamiltonian evolution, and raised a fascinating question: *if we have to introduce stochasticity, at least can we keep it under control through the process of deriving the Boltzmann equation and relate it to the dissipativity of the limit equation?*

Let us briefly summarize the main questions raised in or motivated by [42]:

- (1) Kac's combinatorical proof (later to be extended to collision processes that preserve momentum and energy [53]) had the unfortunate non-physical restriction that the collision kernel is bounded. These proofs were based on an infinite series "tree" representation of the solution according to the collision history of the particles, as well as some sort of Leibniz formula for the N -particle operator acting on tensor product. The first open problem raised was: *can one prove propagation of chaos for the hard spheres collision process?*
- (2) Following closely the spirit of the previous question, it is natural to ask whether *one can prove propagation of chaos for the true Maxwell molecules collision process?* (this is the other main physical model showing an unbounded collision kernel). The difficulty here lies in the fact that the particle system can undergo infinite number of collisions in a finite time interval, and no "tree" representation of solutions is available. This is related to the interesting physical situation of *long-range interaction*, as well as the interesting mathematical framework of *fractional derivative operators* and *Lévy walk*.
- (3) Kac conjectures [42, Eq. (6.39)] that (in our notations) the convergence

$$\frac{H(f_t^N | \gamma^N)}{N} \rightarrow H(f_t | \gamma)$$

is propagated in time, which would imply Boltzmann's H -theorem from the monotonicity of $H(f^N | \gamma^N)/N$ for the Markov process. He concludes with: *"If the above steps could be made rigorous we would have a thoroughly satisfactory justification of Boltzmann's H -theorem."* In our notation the question is: *can one prove propagation of entropic chaos in time?*

- (4) Eventually, Kac discusses the relaxation times, with the goal of deriving the relaxation times of the limit equation from those of the many-particle system. This requires the estimations to be *independent of the number of particles* on this relaxation times. As a first natural step he raises the question of obtaining such uniform estimates on the L^2 spectral gap of the Markov process. This question has triggered many beautiful works (see the next subsection), however it is easy to convince oneself (see the discussion in [10] for instance) that there is no hope of passing to the limit $N \rightarrow \infty$ in this spectral gap estimate, even if the spectral gap is independent of N . The L^2 norm is catastrophic in infinite dimension. Therefore we reframe this question in a setting which "tensorizes correctly in the limit $N \rightarrow \infty$ ", that is in our notation: *can one prove relaxation times independent of the number of particles on the normalized Wasserstein distance $W_1(f^N, \gamma^N)/N$ or on the normalized relative entropy $H(f^N | \gamma^N)/N$?*

This paper is concerned with solving these four questions.

1.5. Review of known results. Kac [42]-[43] has proved point (1) in the case of his one-dimensional toy model. The key point in his analysis is a clever combinatorial use of an exponential formula for the solution, expressing it in terms of an abstract derivation operator (reminiscent of Wild sums [78]). It was generalized by McKean [53] to the Boltzmann collision operator but only for “Maxwell molecules with cutoff”, i.e. roughly when the collision kernel B is constant. In this case the convergence of the exponential formula is easily proved and this combinatorial argument can be extended. Kac raised in [42] the question of proving propagation of chaos in the case of hard spheres and more generally unbounded collision kernels, although his method seemed impossible to extend (the problem is the convergence of this exponential formula, as discussed in [53] for instance).

During the seventies, in a very abstract and compact paper [35], Grünbaum proposed another method for dealing with the hard spheres model, based on the Trotter-Kato formula for semigroups and a clever functional framework (partially reminiscent of the tools used for mean-field limits for McKean-Vlasov equations). Unfortunately this paper was incomplete for two reasons: (1) It was based on two “unproved assumptions on the Boltzmann flow” (page 328): (a) existence and uniqueness for measure solutions and (b) a smoothness assumption. Assumption (a) was recently proved in [31] using Wasserstein metrics techniques and in [28], adapting the classical DiBlasio trick [19]. Assumption (b), while inspired by the cutoff Maxwell molecules (for which it is true), fails for the hard spheres model (cf. the counterexample constructed by Lu and Wennberg [49]) and is somehow too “rough” in this case. (2) A key part in the proof in this paper is the expansion of the “ H_f ” function, which is a clever idea by Grünbaum — however, it is again too coarse for the hard spheres case (though adaptable to the cutoff Maxwell molecules). Nevertheless it is the starting point for our idea of developing a differential calculus in spaces of probability measures in order to control fluctuations.

A completely different approach was undertaken by Sznitman in the eighties [68] (see also Tanaka [71] for partial results concerning non-cutoff Maxwell molecules). Starting from the observation that Grünbaum’s proof was incomplete, he took on to give a full proof of propagation of chaos for hard spheres by another approach. His work was based on: (1) a new uniqueness result for measures for the hard spheres Boltzmann equation (based on a probabilistic reasoning on an enlarged space of “trajectories”); (2) an idea already present in Grünbaum’s approach: reduce the question of the propagation of chaos to a law of large numbers on measures by using a combinatorial argument on symmetric probabilities; (3) a new compactness result at the level of the empirical measures; (4) the identification of the limit by an “abstract test function” construction showing that the (infinite particle) system has trajectories included in the chaotic ones. While Sznitman’s method proves propagation of chaos for the hard spheres, it doesn’t provide any rate for chaoticity as defined previously.

Let us also emphasize that in [52] McKean studied fluctuations around deterministic limit for 2-speed Maxwellian gas and for the usual hard spheres gas. In [34] Graham and Méléard obtained a rate of convergence (of order $1/N$ for the ℓ -th marginal) on any bounded finite interval of the N -particle system to the deterministic Boltzmann dynamics in the case of Maxwell molecules under Grad’s cut-off hypothesis. Lastly, in [29, 30] Fournier and Méléard obtained the convergence of the Monte-Carlo approximation (with numerical cutoff) of the Boltzmann equation for true Maxwell molecules with a rate of convergence (depending on the numerical cutoff and on the number N of particles).

Kac was raising the question of how to control the asymptotic behavior of the particle system in the many-particle limit. As a first step, he suggested to study the behavior of the spectral gap in $L^2(\mathbb{S}^{N-1}(\sqrt{N}))$ of the Markov process as N goes to infinity and conjectured it to be bounded away from zero uniformly in terms of N . This question has been answered only recently in [40, 11] (see also [50, 13]). However the L^2 norm

behaves geometrically in terms of N for tensorized data; this leave no hope to use it for estimating the long-time behavior as N goes to infinity, as the time-decay estimates degenerate beyond times of order $O(1/N)$. In the paper [10], the authors suggested to make use of the relative entropy for estimating the relaxation to equilibrium, and replace the L^2 spectral gap by a linear inequality between the entropy and the entropy production. They constructed entropically chaotic initial data, following the definition mentioned above, but did not succeed in proving the propagation in time of this chaoticity property. Moreover the linear inequality between the entropy and the entropy production is conjectured to be wrong for any physical collision kernels in [75], see for instance [23, 24] for partial confirmations to this conjecture.

After we had finished writing our paper, we were told about the recent book [44] by Kolokoltsov. This book focuses on fluctuation estimates of central limit theorem type. It does not prove quantitative propagation of chaos but weaker estimates (and on finite time intervals), and moreover we were not able to extract from it a full proof for the cases with unbounded collision kernels (e.g. hard spheres). However the comparison of generators for the many-particle and the limit semigroup is reminiscent of our work.

1.6. The abstract method. Our initial inspiration was Grünbaum's paper [35]. Our original goal was to construct a general and robust method able to deal with mixture of jump and diffusion processes, as it occurs in granular gases (see our companion paper [58] for results in this direction). This lead us to develop a new theory, inspired from more recent tools such as the course of Lions on "Mean-field games" at Collège de France, and the master courses of Méléard [55] and Villani [73] on mean-field limits. One of the byproduct of our paper is that we make Grünbaum's original intuition fully rigorous in order to prove propagation of chaos for the Boltzmann velocities jump process for hard spheres.

Like Grünbaum [35], we use a duality argument. We introduce the semigroup S_t^N associated to the flow of the N -particle system in $P(E^N)$, and the semigroup T_t^N in $C_b(E^N)$ in duality with it. We also introduce the (nonlinear) semigroup S_t^{NL} in $P(E)$ associated to the mean-field dynamics (the exponent "NL" signifies the nonlinearity of the limit semigroup, due to the interaction) as well as the associated (linear) "pullback" semigroup T_t^∞ in $C_b(P(E))$ (see Subsection 2.3 for the definition). Then we prove stability and convergence estimates between the linear semigroups T_t^N and T_t^∞ as N goes to infinity.

The preliminary step consists of defining a common functional framework in which the N -particle dynamics and the limit dynamics make sense so that we can compare them. Hence we work at the level of the "full" limit space $P(P(E))$ and, at the dual level, $C_b(P(E))$. Then we identify the regularity required in order to prove the "consistency estimate" between the generators G^N and G^∞ of the dual semigroups T_t^N and T_t^∞ , and finally prove a corresponding "stability estimate" on T_t^∞ , based on stability estimates at the level of the limit semigroup S_t^{NL} .

The latter crucial step leads us to introduce an abstract differential calculus for functions acting on measures endowed with various metrics associated with the weak or strong topologies. More precisely, we shall define functions of class $C^{1,\eta}$ on the space of probability measures by working on subspaces of the space of probability measures, for which the tangent space has a Banach space structure. This "stratification" of subspaces is related to the conservation properties of the flows S_t^N and S_t^{NL} . This notion is related but different from the notions of differentiability developed in the theory of gradient flow by Ambrosio, Otto, Villani and co-authors in [2, 41, 62], and from the one introduced by Lions in [46].

Another viewpoint on our method is to consider it as some kind of accurate version (in the sense that it establishes a rate of convergence) of the BBGKY hierarchy method for proving propagation of chaos and mean-field limit on statistical solutions. This viewpoint is extensively explored and made rigorous in Section 8 where we revisit the BBGKY

hierarchy method in the case of a collisional many-particle system, as was considered for instance in [3]. The proof of uniqueness for statistical solutions to the hierarchy becomes straightforward within our framework by using differentiability of the limit semigroup as a function acting on probabilities.

This general method is first exposed at an abstract level in Section 3 (see in particular Theorem 3.1). This method is, we hope, interesting per se for several reasons: (1) it is fully quantitative, (2) it is highly flexible in terms of the functional spaces used in the proof, (3) it requires a minimal amount of informations on the N -particle systems but more stability informations on the limit PDE (we intentionally presented the assumptions in a way resembling the proof of the convergence of a numerical scheme, which was our “methodological model”), (4) the “differential stability” conditions that are required on the limit PDE seem (to our knowledge) new, at least for the Boltzmann equation.

1.7. Main results. Let us give some simplified versions of the main results in this paper. All the abstract objects will be fully introduced in the next sections with more details.

Theorem 1.1 (Summary of the main results). *Consider some centered initial distribution $f_0 \in P(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with finite energy \mathcal{E} , and with compact support or moment bounds. Consider the corresponding solution f_t to the spatially homogeneous Boltzmann equation for hard spheres or Maxwell molecules, and the solution f_t^N of the corresponding Kac's jump process starting either from the N -fold tensorization of f_0 or the latter conditioned to $\mathcal{S}^N(\mathcal{E})$.*

One can classify the results established into three main statements:

- (1) **Quantitative uniform in time propagation of chaos, finite or infinite dimensional, in weak measure distance** (cf. Theorems 5.1-5.2-6.1-6.2):

$$\forall N \geq 1, \forall 1 \leq \ell \leq N, \sup_{t \geq 0} \frac{W_1 \left(\Pi_\ell f_t^N, \left(f_t^{\otimes \ell} \right) \right)}{\ell} \leq \alpha(N)$$

for some $\alpha(N) \rightarrow 0$ as $N \rightarrow \infty$. In the hard spheres case, the uniformity in time of this estimate is only proved when the distribution is conditioned to $\mathcal{S}^N(\mathcal{E})$. Moreover, the proof provides explicit estimates on the rate α , which are controlled by a power law for Maxwell molecules, and by a power of a logarithm for hard spheres.

- (2) **Propagation of entropic chaos** (cf. Theorem 7.1-(i)): *Consider the case where the initial datum of the many-particle system has support included in $\mathcal{S}^N(\mathcal{E})$. If the initial datum is entropically chaotic in the sense*

$$\frac{1}{N} H(f_0^N | \gamma^N) \xrightarrow{N \rightarrow +\infty} H(f_0 | \gamma)$$

with

$$(1.9) \quad H(f_0^N | \gamma^N) := \int_{\mathcal{S}^N(\mathcal{E})} \ln \frac{df_0^N}{d\gamma^N} df_0^N(v) \quad \text{and} \quad H(f_0 | \gamma) := \int_{\mathbb{R}^d} f_0 \ln \frac{f_0}{\gamma} dv$$

where γ is the Gaussian equilibrium with energy \mathcal{E} and γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$, then the solution is also entropically chaotic for any later time:

$$\forall t \geq 0, \quad \frac{1}{N} H(f_t^N | \gamma^N) \xrightarrow{N \rightarrow +\infty} H(f_t | \gamma).$$

This proves the derivation of the H -theorem in this context, i.e. the monotonic decay in time of $H(f_t | \gamma)$, since for any $N \geq 2$, the functional $H(f_t^N | \gamma^N)$ is monotone decreasing in time for the Markov process.

- (3) **Quantitative estimates on relaxation times, independent of the number of particles** (cf. Theorems 5.2-6.2 and Theorem 7.1-(ii)): Consider the case where the initial datum of the many-particle system has support included in $\mathcal{S}^N(\mathcal{E})$. Then we have

$$\forall N \geq 1, \forall 1 \leq \ell \leq N, \forall t \geq 0, \quad \frac{W_1(\Pi_\ell f_t^N, \Pi_\ell(\gamma^N))}{\ell} \leq \beta(t)$$

for some $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$. Moreover, the proof provides explicit estimates on the rate β , which are controlled by a power law for Maxwell molecules, and by a power of a logarithm for hard spheres.

Finally in the case of Maxwell molecules, if we assume furthermore that the Fisher information of the initial datum f_0 is finite:

$$(1.10) \quad I(f_0) := \int_{\mathbb{R}^d} \frac{|\nabla_v f_0|^2}{f_0} dv < +\infty,$$

then the following estimate also holds:

$$\forall N \geq 1, \quad \frac{1}{N} H(f_t^N | \gamma^N) \leq \beta(t)$$

for some function $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$, with the same kind of estimates.

1.8. Some open questions. Here are a few questions among those raised by this work:

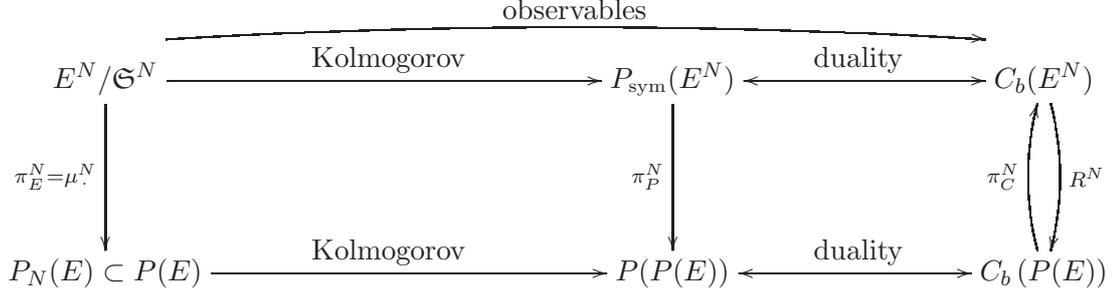
- (1) What about the optimal rate in the chaoticity estimates along time? Our method reduces this question to the chaoticity estimates at initial time, and therefore to the optimal rate in the quantitative law of large numbers for measures according to various weak measure distances.
- (2) What about the optimal rate in the relaxation times (uniformly in the number of particles)? Spectral gap studies predict exponential rates, both for the many-particle system and for the limit system, however our rates are far from it!
- (3) Can uniform in time propagation of chaos be proved for non-reversible jump processes (such as inelastic collision processes) for which the invariants measures γ^N and γ are not explicitly known (e.g. granular gases)?

1.9. Plan of the paper. In Section 2 we set the abstract functional framework together with the general assumption and in Section 3 we state and prove our main abstract theorem (Theorem 3.1). In Section 4 we present some tools and results on weak measure distances, on the construction of initial data with support on the Boltzmann sphere $\mathcal{S}^N(\mathcal{E})$ for the N -particle system, and on the sampling process of the limit distribution by empirical measures. In Section 5 we apply the method to (true) Maxwell molecules (Theorems 5.1 and 5.2). In Section 6 we apply the method to hard spheres (Theorems 6.1 and 6.2). Section 7 is devoted to the study of entropic chaos. Lastly, in Section 8 we revisit the BBGKY hierarchy method for the spatially homogeneous Boltzmann equation in the light of our framework.

2. THE ABSTRACT SETTING

In this section we shall state and prove the key abstract result. This will motivate the introduction of a general functional framework.

2.1. The general functional framework of the duality approach. Let us set up the framework. Here is a diagram which sums up the duality approach (norms and duality brackets shall be specified in Subsections 2.4):



In this diagram:

- E denotes a *Polish space*:

This is a separable completely metrizable topological space. We shall denote by d_E the distance on this space in the sequel.

- \mathfrak{S}^N denotes the N -*permutation group*.

- $P_{\text{sym}}(E^N)$ denotes the set of *symmetric probabilities* on E^N :

Given a permutation $\sigma \in \mathfrak{S}^N$, a vector

$$V = (v_1, \dots, v_N) \in E^N,$$

a function $\varphi \in C_b(E^N)$ and a probability $f^N \in P(E^N)$, we successively define

$$V_\sigma = (v_{\sigma(1)}, \dots, v_{\sigma(N)}) \in E^N,$$

and

$$\varphi_\sigma \in C_b(E^N) \text{ by setting } \varphi_\sigma(V) = \varphi(V_\sigma)$$

and finally

$$f_\sigma^N \in P(E^N) \text{ by setting } \langle f_\sigma^N, \varphi \rangle = \langle f^N, \varphi_\sigma \rangle.$$

We then say that a probability f^N on E^N is symmetric if it is invariant under permutations:

$$\forall \sigma \in \mathfrak{S}^N, \quad f_\sigma^N = f^N.$$

- The probability measure μ_V^N denotes the *empirical measure*:

$$\mu_V^N := \frac{1}{N} \sum_{i=1}^N \delta_{v_i}, \quad V = (v_1, \dots, v_N) \in E^N$$

where δ_{v_i} denotes the Dirac mass on E at point $v_i \in E$.

- $P_N(E)$ denotes the subset $\{\mu_V^N, V \in E^N\}$ of empirical measures of $P(E)$.

- $P(P(E))$ denotes the space of probabilities on the Polish space $P(E)$ (endowed for instance with the Prokhorov distance), and this is again a Polish space.

- $C_b(P(E))$ denotes the space of continuous and bounded functions on $P(E)$:

This space shall be endowed with either the weak or strong topologies (see Subsection 2.4), and later with some metric differential structure.

- The map π_E^N from E^N/\mathfrak{S}^N to $P_N(E)$ is defined by

$$\forall V \in E^N/\mathfrak{S}^N, \quad \pi_E^N(V) := \mu_V^N.$$

- The map π_C^N from $C_b(P(E))$ to $C_b(E^N)$ is defined by

$$\forall \Phi \in C_b(P(E)), \quad \forall V \in E^N, \quad (\pi_C^N \Phi)(V) := \Phi(\mu_V^N).$$

- The map R^N from $C_b(E^N)$ to $C_b(P(E))$ is defined by:

$$\forall \varphi \in C_b(E^N), \forall \rho \in P(E), \quad R_\varphi^N(\rho) := \langle \rho^{\otimes N}, \varphi \rangle.$$

- The map π_P^N from $P_{\text{sym}}(E^N)$ to $P(P(E))$ is defined by:

$$\langle \pi_P^N f^N, \Phi \rangle = \langle f^N, \pi_C^N \Phi \rangle = \int_{E^N} \Phi(\mu_V^N) \, df^N(v)$$

for any $f^N \in P_{\text{sym}}(E^N)$ and any $\Phi \in C_b(P(E))$, where the first bracket means $\langle \cdot, \cdot \rangle_{P(P(E)), C_b(P(E))}$ and the second bracket means $\langle \cdot, \cdot \rangle_{P(E^N), C_b(E^N)}$.

Let us now discuss the ‘‘horizontal’’ arrows:

- The arrows pointing from the first column to the second one consists in writing the *Kolmogorov* equation associated with the many-particle stochastic Markov process.

- The arrows pointing from the second column to the third column consists in writing the dual evolution semigroup (note that the N -particle dynamics is linear). As we shall discuss later the dual spaces of the spaces of probabilities on the phase space can be interpreted as the spaces of observables on the original systems.

Our functional framework shall be applied to weighted spaces of probability measures rather than directly in $P(E)$. More precisely, for a given weight function $m : E \rightarrow \mathbb{R}_+$ we shall use affine subsets of the weighted space of probability measures

$$\{f \in P(E); M_m(f) := \langle f, m \rangle < \infty\}$$

as our basis functional spaces. Typical examples are $m(v) := \tilde{m}(d_E(v, v_0))$ for some fixed $v_0 \in E$ with $\tilde{m}(z) = z^k$ or $\tilde{m}(z) = e^{a|z|^k}$, $a, k > 0$. More specifically when $E = \mathbb{R}^d$, we shall use $m(v) = \langle v \rangle^k := (1 + |v|^2)^{k/2}$ or $m(v) = e^{a|z|^k}$, $a, k > 0$.

We shall sometimes abuse notation by writing M_k for M_m when $\tilde{m}(z) = z^k$ or $m(v) = \langle v \rangle^k$ in the examples above. We shall denote by M^1 the space of finite signed measures endowed with the total variation norm, and M_m^1 the space of finite signed measures h whose variation $|h|$ satisfies $\langle |h|, m \rangle < +\infty$, and endowed with the total variation norm. Again we contract the notation as M_k^1 when $\tilde{m}(z) = z^k$ or $m(v) = \langle v \rangle^k$.

2.2. The N -particle semigroups. Let us introduce the mathematical semigroups describing the evolution of objects living in these spaces, for any $N \geq 1$.

Step 1. Consider the trajectories $\mathcal{V}_t^N \in E^N$, $t \geq 0$, of the particles (Markov process viewpoint). We make the further assumption that this flow commutes with permutations:

$$\text{For any } \sigma \in \mathfrak{S}^N, \text{ the solution at time } t \text{ starting from } (\mathcal{V}_0^N)_\sigma \text{ is } (\mathcal{V}_t^N)_\sigma.$$

This mathematically reflects the fact that particles are indistinguishable.

Step 2. This flow on E^N yields a corresponding semigroup S_t^N acting on $P_{\text{sym}}(E^N)$ for the probability density of particles in the phase space E^N (statistical viewpoint), defined through the formula

$$\forall f_0^N \in P_{\text{sym}}(E^N), \varphi \in C_b(E^N), \quad \langle S_t^N(f_0^N), \varphi \rangle = \mathbf{E}(\varphi(\mathcal{V}_t^N))$$

where the bracket obviously denotes the duality bracket between $P(E^N)$ and $C_b(E^N)$ and \mathbf{E} denotes the expectation associated to the space of probability measures in which the process \mathcal{V}_t^N is built. In other words, $f_t^N := S_t^N(f_0^N)$ is nothing but the law of \mathcal{V}_t^N . Since the flow (\mathcal{V}_t^N) commutes with permutation, the semigroup S_t^N acts on $P_{\text{sym}}(E^N)$: if the law f_0^N of \mathcal{V}_0^N belongs to $P_{\text{sym}}(E^N)$, then for later times the law f_t^N of \mathcal{V}_t^N also belongs

to $P_{\text{sym}}(E^N)$. To the C_0 -semigroup S_t^N on $P_{\text{sym}}(E^N)$ one can associate a linear evolution equation with a generator denoted by A_N :

$$\partial_t f^N = A^N f^N, \quad f^N \in P_{\text{sym}}(E^N),$$

which is the *forward Kolmogorov (or Master) equation* on the law of \mathcal{V}_t^N .

Step 3. We also consider the Markov semigroup T_t^N acting on the functions space $C_b(E^N)$ of *observables* on the evolution system (\mathcal{V}_t^N) on E^N (see the discussion in the next remark), which is *in duality* with the semigroup S_t^N , in the sense that:

$$\forall f^N \in P(E^N), \varphi \in C_b(E^N), \quad \langle f^N, T_t^N(\varphi) \rangle = \langle S_t^N(f^N), \varphi \rangle.$$

To the C_0 -semigroup T_t^N on $C_b(E^N)$ we can associate the following *linear* evolution equation with a generator denoted by G^N :

$$\partial_t \varphi = G^N(\varphi), \quad \varphi \in C_b(E^N),$$

which is the *backward Kolmogorov equation*.

2.3. The mean-field limit semigroup. We now define the evolution of the limit mean-field equation.

Step 1. Consider a semigroup S_t^{NL} acting on $P(E)$ associated with an evolution equation and some operator Q :

For any $f_0 \in P(E)$ (assuming possibly some additional moment bounds), then $S_t^{NL}(f_0) := f_t$ where $f_t \in C(\mathbb{R}_+, P(E))$ is the solution to

$$(2.1) \quad \partial_t f_t = Q(f_t), \quad f|_{t=0} = f_0.$$

This semigroup and the operator Q are typically *nonlinear* for mean-field models, namely bilinear in case of Boltzmann's collisions interactions.

Step 2. Consider then the associated *pullback semigroup* T_t^∞ acting on $C_b(P(E))$:

$$\forall f \in P(E), \Phi \in C_b(P(E)), \quad T_t^\infty[\Phi](f) := \Phi(S_t^{NL}(f)).$$

(Again additional moment bounds can be required on f in order to make this definition rigorous.) Note carefully that T_t^∞ is always *linear* as a function of Φ , although of course $T_t^\infty[\Phi](f)$ is not linear in general as a function of f . We shall associate (when possible) the following *linear* evolution equation on $C_b(P(E))$ with some generator denoted by G^∞ :

$$\partial_t \Phi = G^\infty(\Phi).$$

Remark 2.1. The semigroup T_t^∞ can be interpreted physically as the semigroup of the evolution of *observables* of the nonlinear equation (2.1). Let us give a short heuristic explanation. Consider a nonlinear ordinary differential equation

$$\frac{dv}{dt} = F(V) \quad \text{on } \mathbb{R}^d \quad \text{with } \nabla_v \cdot F \equiv 0 \quad \text{and } V|_{t=0} = v$$

with divergence-free vector field for simplicity. One can then define formally the *linear* Liouville transport partial differential equation

$$\partial_t f + \nabla_v \cdot (F f) = 0,$$

where $f = f_t(v)$ is a time-dependent probability density over the phase space \mathbb{R}^d , whose solution is given (at least formally) by following the characteristics backward $f_t(v) = f_0(V_{-t}(v))$. Now, instead of the Liouville viewpoint, one can adopt the viewpoint of *observables*, that is functions depending on the position of the system in the phase space (e.g. energy, momentum, etc ...). For some observable function φ_0 defined on \mathbb{R}^d , the

evolution of the value of this observable along the trajectory is given by $\varphi_t(v) = \varphi_0(V_t(v))$ and φ_t is solution to the following *dual* linear PDE

$$\partial_t \varphi - F \cdot \nabla_v \varphi = 0.$$

Now let us consider a nonlinear evolution system

$$\frac{df}{dt} = Q(f) \quad \text{in an abstract space } f \in \mathcal{H}.$$

By analogy we define two linear evolution systems on the larger functional spaces $P(\mathcal{H})$ and $C_b(\mathcal{H})$: first the abstract Liouville equation

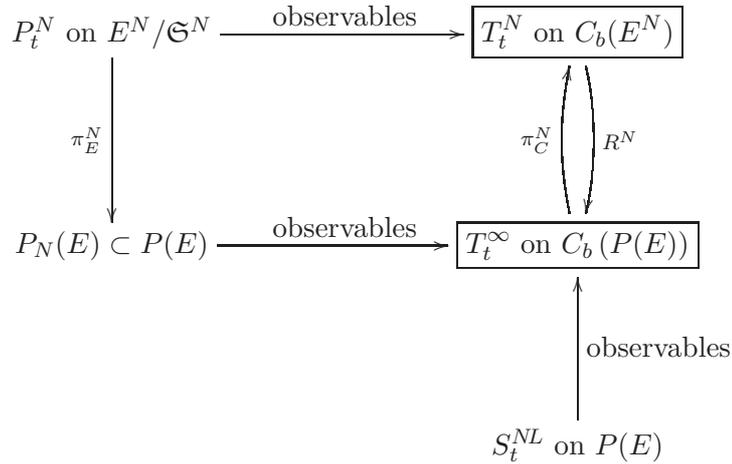
$$\partial_t \pi + \frac{\delta}{\delta f} \cdot (Q(f) \pi) = 0, \quad \pi \in P(\mathcal{H})$$

and second the abstract equation for the evolution of observables

$$\partial_t \Phi - Q(f) \cdot \frac{\delta \Phi}{\delta f}(f) = 0, \quad \Phi \in C_b(\mathcal{H}).$$

However in order to make sense of this heuristic, the scalar product have to be defined correctly as duality brackets, and, most importantly, a differential calculus on \mathcal{H} has to be defined rigorously. Taking $\mathcal{H} = P(E)$, this provides an intuition for our functional construction, as well as for the formula of the generator G^∞ below (compare the previous equation with formula (2.9)). Be careful that when $\mathcal{H} = P(E)$, the abstract Liouville and observable equations refers to trajectories *in the space of probabilities* $P(E)$ (i.e. solutions to the nonlinear equation (2.1)), and not trajectories of a particle in E . Note also that for a *dissipative equation* at the level of \mathcal{H} (such as the Boltzmann equation), it seems more convenient to use the observable equation rather than the Liouville equation since “forward characteristics” can be readily used in order to construct the solutions to this observable equation.

Summing up we obtain the following picture for the semigroups:



Hence a key point of our construction is that, through the evolution of *observables*, we shall “interface” the two evolution systems (the nonlinear limit equation and the N -particle system) via the applications π_C^N and R^N . From now on we shall write $\pi^N = \pi_C^N$.

2.4. The metric issue. $C_b(P(E))$ is our fundamental space in which we shall compare (through their observables) the semigroups of the N -particle system and the limit mean-field equation. Let us make the topological and metric structures used on $P(E)$ more precise. At the topological level there are two canonical choices (which determine two different sets $C_b(P(E))$):

- (1) The strong topology which is associated to the total variation norm, denoted by $\|\cdot\|_{M^1}$; the corresponding set shall be denoted by $C_b(P(E), TV)$.
- (2) The weak topology, i.e. the trace on $P(E)$ of the weak topology on $M^1(E)$ (the space of Radon measures on E with finite mass) induced by $C_b(E)$; the corresponding set shall be denoted by $C_b(P(E), w)$.

It is clear that

$$C_b(P(E), w) \subset C_b(P(E), TV).$$

The supremum norm $\|\Phi\|_{L^\infty(P(E))}$ does *not* depend on the choice of topology on $P(E)$, and induces a Banach topology on the space $C_b(P(E))$. The transformations π^N and R^N satisfy:

$$(2.2) \quad \|\pi^N \Phi\|_{L^\infty(E^N)} \leq \|\Phi\|_{L^\infty(P(E))} \quad \text{and} \quad \|R_\phi^N\|_{L^\infty(P(E))} \leq \|\phi\|_{L^\infty(E^N)}.$$

The transformation π^N is well defined from $C_b(P(E), w)$ to $C_b(E^N)$, but in general, it does not map $C_b(P(E), TV)$ into $C_b(E^N)$ since the map

$$V \in E^N \mapsto \mu_V^N \in (P(E), TV)$$

is not continuous.

In the other way round, the transformation R^N is well defined from $C_b(E^N)$ to $C_b(P(E), w)$, and therefore also from $C_b(E^N)$ to $C_b(P(E), TV)$: for any $\phi \in C_b(E^N)$ and for any sequence $f_k \rightharpoonup f$ weakly, we have $f_k^{\otimes N} \rightharpoonup f^{\otimes N}$ weakly, and then $R^N[\phi](f_k) \rightarrow R^N[\phi](f)$.

There are many different possible metric structures inducing the weak topology on $C_b(P(E), w)$. The mere notion of continuity does not require discussing these metrics, but any subspace of $C_b(P(E), w)$ with differential regularity shall strongly depend on this choice, which motivates the following definitions.

Definition 2.2. For a given *weight function* $m : E \rightarrow \mathbb{R}_+$, we define the subspaces of probabilities:

$$\mathcal{P}_m := \{f \in P(E); \langle f, m \rangle < \infty\}.$$

As usual we contract the notation as \mathcal{P}_k when $E = \mathbb{R}^d$ and $m(v) = \langle v \rangle^k := (1 + |v|^2)^{k/2}$, $k \in \mathbb{R}$, $v \in \mathbb{R}^d$.

We also define the corresponding bounded subsets for $a > 0$

$$\mathcal{BP}_{m,a} := \{f \in \mathcal{P}_m; \langle f, m \rangle \leq a\}.$$

For a given *constraint function* $\mathbf{m} : E \rightarrow \mathbb{R}^D$ such that $\langle f, \mathbf{m} \rangle$ is well defined for any $f \in \mathcal{P}_m$ and a given *space of constraints* $\mathcal{R}_{m,\mathbf{m}} \subset \mathbb{R}^D$, for any $\mathbf{r} \in \mathcal{R}_{m,\mathbf{m}}$, we define the corresponding (possibly empty) *constrained subsets*

$$\mathcal{P}_{m,\mathbf{m},\mathbf{r}} := \{f \in \mathcal{P}_m; \langle f, \mathbf{m} \rangle = \mathbf{r}\},$$

and the corresponding (possibly empty) *bounded constrained subsets*

$$\mathcal{BP}_{m,\mathbf{m},a,\mathbf{r}} := \{f \in \mathcal{BP}_{m,a}; \langle f, \mathbf{m} \rangle = \mathbf{r}\}.$$

We also define the corresponding *space of increments*

$$\mathcal{IP}_{m,\mathbf{m},\mathcal{R}_{m,\mathbf{m}}} := \{f_2 - f_1; \exists \mathbf{r} \in \mathcal{R}_{m,\mathbf{m}} \text{ s.t. } f_1, f_2 \in \mathcal{P}_{m,\mathbf{m},\mathbf{r}}\}.$$

Be careful that the space of increments is not a vector space in general. Let us now define the notion of distances over probabilities that we shall consider.

Definition 2.3. Consider a weight function m_G , a constraint function \mathbf{m}_G and a set of constraints \mathcal{R}_G . We shall use for the associated spaces of the previous definition the following simplified contracted notation: \mathcal{P}_G for \mathcal{P}_m , $\mathcal{BP}_{G,a}$ for $\mathcal{BP}_{m,a}$, \mathcal{R}_G for $\mathcal{R}_{m,\mathbf{m}}$, $\mathcal{P}_{G,\mathbf{r}}$ for $\mathcal{P}_{m,\mathbf{m},\mathbf{r}}$, $\mathcal{BP}_{G,a,\mathbf{r}}$ for $\mathcal{BP}_{m,\mathbf{m},a,\mathbf{r}}$ and \mathcal{IP}_G for $\mathcal{IP}_{m,\mathbf{m},\mathcal{R}_{m,\mathbf{m}}}$.

We shall consider a distance d_G which

- (1) either is defined on the whole space $\mathcal{P}_{\mathcal{G}}$ (i.e. whatever the values of the constraints),
(2) or such that there is a Banach space $\mathcal{G} \supset \mathcal{IP}_{\mathcal{G}}$ endowed with a norm $\|\cdot\|_{\mathcal{G}}$ such that $d_{\mathcal{G}}$ is defined for any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}}$ on $\mathcal{P}_{\mathcal{G},\mathbf{r}}$, by setting

$$\forall f, g \in \mathcal{P}_{\mathcal{G},\mathbf{r}}, \quad d_{\mathcal{G}}(f, g) := \|g - f\|_{\mathcal{G}}.$$

Let us finally define a quantitative Hölder notion of equivalence for the distances over probabilities.

Definition 2.4. Consider some weight and constraint functions $m_{\mathcal{G}}, \mathbf{m}_{\mathcal{G}}$. We say that two metrics d_0 and d_1 defined on $\mathcal{P}_{\mathcal{G}}$ are *Hölder equivalent on bounded sets* if there exists $\kappa \in (0, \infty)$ and, for any $a \in (0, \infty)$, there exists $C_a \in (0, \infty)$ such that

$$\forall f, g \in \mathcal{BP}_{\mathcal{G},a}, \quad d_0(f, g) \leq C_a [d_1(f, g)]^{\kappa}, \quad d_1(f, g) \leq C_a [d_0(f, g)]^{\kappa}$$

for some constant C_a depending on $a > 0$.

If d_0 and d_1 are, as in the previous definition, only defined on $\mathcal{P}_{\mathcal{G},\mathbf{r}}$ for given values of the constraints $\mathbf{r} \in \mathcal{R}_{\mathcal{G}}$, we modify this definition as follows:

$$\forall \mathbf{r} \in \mathcal{R}_{\mathcal{G}}, \forall f, g \in \mathcal{BP}_{\mathcal{G},\mathbf{r},a}, \quad d_0(f, g) \leq C_a [d_1(f, g)]^{\kappa}, \quad d_1(f, g) \leq C_a [d_0(f, g)]^{\kappa}$$

for some $\kappa \in (0, \infty)$ and some constant C_a depending on $a > 0$.

Example 2.5. The choice

$$m_{\mathcal{G}} := 1, \quad \mathbf{m}_{\mathcal{G}} := 0, \quad \mathcal{R}_{\mathcal{G}} := \{0\}, \quad \|\cdot\|_{\mathcal{G}} := \|\cdot\|_{M^1}$$

recovers $\mathcal{P}_{\mathcal{G}}(E) = P(E)$. More generally one can choose

$$m_{\mathcal{G}_k}(v) := d_E(v, v_0)^k, \quad \mathbf{m}_{\mathcal{G}_k} := 0, \quad \mathcal{R}_{\mathcal{G}_k} := \{0\}, \quad \|\cdot\|_{\mathcal{G}_k} := \left\| \cdot d_E(v, v_0)^k \right\|_{M^1}.$$

For $k_1 > k_2, k_3 \geq 0$, the distances $d_{\mathcal{G}_{k_2}}$ and $d_{\mathcal{G}_{k_3}}$ are Hölder equivalent on bounded sets of $\mathcal{P}_{\mathcal{G}_{k_1}}$.

Example 2.6. There are many distances on $P(E)$ which induce the weak topology, see for instance [65]. In the next section, we present some of them which have a practical interest for us, and which are all topologically uniformly equivalent on bounded sets of $P(E)$ in the sense of the previous definition, with the choice of a convenient (strong enough) weight function.

2.5. Distances on probabilities. Let us discuss some well-known distances on $P(\mathbb{R}^d)$ (or defined on subsets of $P(\mathbb{R}^d)$), which shall be useful in the sequel. These distances are all topologically equivalent to the weak topology $\sigma(P(E), C_b(E))$ on the sets $\mathcal{BP}_{k,a}(E)$ for k large enough and for any $a \in (0, \infty)$, and they are all uniformly topologically equivalent (see [72, 15] and section 2.5.6). We refer to [65, 76, 15] and the references therein for more details on these distances.

2.5.1. Dual-Hölder –or Zolotarev’s– Distances. Denote by d_E a distance on E and let us fix $v_0 \in E$ (e.g. $v_0 = 0$ when $E = \mathbb{R}^d$ in the sequel). Denote by $C_0^{0,s}(E)$, $s \in (0, 1)$ (resp. $\text{Lip}_0(E)$) the set of s -Hölder functions (resp. Lipschitz functions) on E vanishing at one arbitrary point $v_0 \in E$ endowed with the norm

$$[\varphi]_s := \sup_{x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{d_E(x, y)^s}, \quad s \in (0, 1], \quad [\varphi]_{\text{Lip}} := [\varphi]_1.$$

We then define the dual norm: take $m_{\mathcal{G}} := 1$, $\mathbf{m}_{\mathcal{G}} := 0$, $\mathcal{R}_{\mathcal{G}} := \{0\}$, and $\mathcal{P}_{\mathcal{G}}(E)$ endowed with

$$\forall f, g \in \mathcal{P}_{\mathcal{G}}, \quad [g - f]_s^* := \sup_{\varphi \in C_0^{0,s}(E)} \frac{\langle g - f, \varphi \rangle}{[\varphi]_s}.$$

2.5.2. *Wasserstein distances.* Given $q \in [1, \infty)$, define W_q on

$$\mathcal{P}_{\mathcal{G}}(E) = \mathcal{P}_q(E) := \{f \in P(E); \langle f, d_E(\cdot, v_0)^q \rangle < \infty\}$$

by

$$\forall f, g \in \mathcal{P}_q(E), \quad W_q(f, g) := \inf_{\pi \in \Pi(f, g)} \left(\int_{E \times E} d_E(x, y)^q d\pi(x, y) \right)^{1/q},$$

where $\Pi(f, g)$ denotes the set of probability measures $\pi \in P(E \times E)$ with marginals f and g :

$$\pi(A, E) = f(A) \quad \text{and} \quad \pi(E, A) = g(A) \quad \text{for any Borel set } A \subset E.$$

Note that for $V_1, V_2 \in E^N$ and any $q \in [1, \infty)$, one has

$$(2.3) \quad W_q(\mu_{V_1}^N, \mu_{V_2}^N) = d_{\ell^q(E^N/\mathfrak{S}_N)}(V_1, V_2) := \min_{\sigma \in \mathfrak{S}_N} \left(\frac{1}{N} \sum_{i=1}^N d_E((V_1)_i, (V_2)_{\sigma(i)})^q \right)^{1/q},$$

and that

$$\forall f, g \in \mathcal{P}_1(E), \quad W_1(f, g) = [f - g]_1^* = \sup_{\varphi \in \text{Lip}_0(E)} \langle f - g, \varphi \rangle.$$

2.5.3. *Fourier-based norms.* Given $E = \mathbb{R}^d$, $m_{\mathcal{G}_1} := |v|$, $\mathbf{m}_{\mathcal{G}_1} := 0$, $\mathcal{R}_{\mathcal{G}_1} := \{0\}$, let us define

$$\forall f \in \mathcal{IP}_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{|\xi|^s}, \quad s \in (0, 1],$$

where \hat{f} denotes the Fourier transform of f defined through the expression

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} df(x).$$

Similarly, given $E = \mathbb{R}^d$, $m_{\mathcal{G}_2} := |v|^2$, $\mathbf{m}_{\mathcal{G}_2} := v$, $\mathcal{R}_{\mathcal{G}_2} := \mathbb{R}^d$, we define

$$(2.4) \quad \forall f \in \mathcal{IP}_{\mathcal{G}_2}, \quad \|f\|_{\mathcal{G}_2} = |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{|\xi|^s}, \quad s \in (1, 2].$$

Obviously higher-order versions of this norm could be defined similarly by increasing the number of constraints. However we shall see in the next subsection how to extend this notion of distance without constraints.

For any given $a > 0$ and any constraint

$$\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1, a} := \left\{ \mathbf{r} \in \mathbb{R}^d; \exists f \in \mathcal{P}_2(\mathbb{R}^d), \langle f, |v|^2 \rangle \leq a, \langle f, v \rangle = \mathbf{r} \right\} = \left\{ \mathbf{r} \in \mathbb{R}^d, |\mathbf{r}|^2 \leq a \right\},$$

we observe that on the set

$$\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}} := \left\{ f \in \mathcal{P}_2(\mathbb{R}^d), \langle f, |v|^2 \rangle \leq a, \langle f, v \rangle = \mathbf{r} \right\},$$

the distance $d_{\mathcal{G}_2}$ is bounded, so that the diameter of $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ is bounded: for any $f_1, f_2 \in \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ there holds

$$\begin{aligned} \|f_1 - f_2\|_{\mathcal{G}_2} &= \sup_{\xi \in \mathbb{R}^d} \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^d} (e^{-i\xi \cdot v} - 1 + i\xi \cdot v) (df_1 - df_2)(v) \right| \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 |(df_1 - df_2)(v)| \leq a. \end{aligned}$$

2.5.4. *More Fourier-based norms.* More generally, given $E = \mathbb{R}^d$ and $k \in \mathbb{N}^*$, we set

$$m_{\mathcal{G}} := |v|^k, \quad \mathbf{m}_{\mathcal{G}} := (v^\alpha)_{\alpha \in \mathbb{N}^d, |\alpha| \leq k-1}, \quad \mathcal{R}_{\mathcal{G}} := \mathbb{R}^D, \quad D := d + \dots + (k-1)d,$$

with $|\alpha| = \alpha_1 + \dots + \alpha_d$ and

$$v^\alpha = \left(v_1^{\alpha_1}, \dots, v_d^{\alpha_d} \right), \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d,$$

and we define

$$\forall f \in \mathcal{IP}_{\mathcal{G}}, \quad \|f\|_{\mathcal{G}} = |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{|\xi|^s}, \quad s \in (0, k].$$

In fact, we may extend the above norm to $M_k^1(\mathbb{R}^d)$ in the following way. We first define for

$$f \in M_{k-1}^1(\mathbb{R}^d) \quad \text{and} \quad \alpha \in \mathbb{N}^d, \quad |\alpha| = \alpha_1 + \dots + \alpha_d \leq k-1$$

the following moment:

$$M_\alpha[f] := \int_{\mathbb{R}^d} v^\alpha \, df(v).$$

Consider a fixed (once for all) function $\chi = \chi(\xi) \in C_c^\infty(\mathbb{R}^d)$ (smooth with compact support), such that $\chi \equiv 1$ on the set $\{\xi \in \mathbb{R}^d, |\xi| \leq 1\}$. This implies in particular

$$\int_{\mathbb{R}^d} \mathcal{F}^{-1}(\chi)(v) \, dv = \chi(0) = 1.$$

Then we define the following function $\mathcal{M}_k[f]$ through its Fourier transform

$$\hat{\mathcal{M}}_k[f](\xi) := \chi(\xi) \left(\sum_{|\alpha| \leq k-1} M_\alpha[f] \frac{\xi^\alpha}{\alpha!} \right), \quad \alpha! := \alpha_1! \dots \alpha_d!$$

Note that this is a mollified version of the $(k-1)$ -Taylor expansion of \hat{f} at $\xi = 0$. Then we may define the norm

$$(2.5) \quad \| |f| \|_k := |f - \mathcal{M}_k[f]|_k + \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k-1} |M_\alpha[f]|$$

where

$$|g|_k := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{g}(\xi)|}{|\xi|^k}$$

is defined for a signed measure whose Fourier transform Taylor expansion at zero cancels up to the order $k-1$.

2.5.5. *Negative Sobolev norms.* Given $s \in (d/2, d/2 + 1/2)$ and

$$E = \mathbb{R}^d, \quad m_{\mathcal{G}_1} := |v|, \quad \mathbf{m}_{\mathcal{G}_1} := 0, \quad \mathcal{R}_{\mathcal{G}_1} := \{0\},$$

we define the following negative homogeneous Sobolev norm

$$\forall f \in \mathcal{IP}_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = \|f\|_{\dot{H}^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat{f}(\xi)}{|\xi|^s} \right\|_{L^2}.$$

Similarly, given $s \in [d/2 + 1/2, d/2 + 1)$ and

$$E = \mathbb{R}^d, \quad m_{\mathcal{G}_2} := |v|^2, \quad \mathbf{m}_{\mathcal{G}_2} := v, \quad \mathcal{R}_{\mathcal{G}_2} := \mathbb{R}^d,$$

we define

$$\forall f \in \mathcal{IP}_{\mathcal{G}_2}, \quad \|f\|_{\mathcal{G}_2} = \|f\|_{\dot{H}^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat{f}(\xi)}{|\xi|^s} \right\|_{L^2(\mathbb{R}^d)}.$$

It is also possible to use the non-homogeneous Sobolev space $H^{-s}(\mathbb{R}^d)$. Observe that probabilities are included in $H^{-s}(\mathbb{R}^d)$ as soon as $s > d/2$.

2.5.6. *Comparison of distances when $E = \mathbb{R}^d$.* All the previous distances are *Hölder equivalent on bounded sets* in the sense of Definition 2.4. Precise quantitative statements of these equivalences are given in Lemma 4.1 in Section 4.1.

2.6. Differential calculus in the space of probability measures. We start with a purely metric definition in the case of usual Hölder regularity.

Definition 2.7. Given some metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, some weight function

$$\Lambda : \tilde{\mathcal{G}}_1 \mapsto [1, +\infty),$$

we denote by

$$UC_\Lambda(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$$

the weighted space of uniformly continuous functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$, that is the set of functions $\mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ such that there exists a modulus of continuity ω so that

$$(2.6) \quad \forall f_1, f_2 \in \tilde{\mathcal{G}}_1, \quad d_{\mathcal{G}_2}(\mathcal{S}(f_1), \mathcal{S}(f_2)) \leq \Lambda(f_1, f_2) \omega(d_{\mathcal{G}_1}(f_1, f_2)),$$

with

$$\Lambda(f_1, f_2) := \max\{\Lambda(f_1), \Lambda(f_2)\}$$

and where $d_{\mathcal{G}_k}$ denotes the metric of $\tilde{\mathcal{G}}_k$. Note that the tilde sign in the notation of the distance has been removed in order to present unified notation with the next definition.

For any $\eta \in (0, 1]$, we denote by

$$C_\Lambda^{0,\eta}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$$

the weighted space of functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$ with η -Hölder regularity, that are the uniformly continuous functions for which the modulus of continuity satisfies $\omega(s) \leq C s^\eta$ for some constant $C > 0$. We then define the *semi-norm*

$$[S]_{C_\Lambda^{0,\eta}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)} \quad \text{for } S \in C_\Lambda^{0,\eta}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$$

as the infimum of the constants $C > 0$ such that (2.6) holds with $\omega(s) = C s^\eta$.

We now define a first order differential calculus, for which we require a norm structure on the functional spaces.

Definition 2.8. Given some *Banach* spaces $\mathcal{G}_1, \mathcal{G}_2$ and some *metric* sets $\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2$ such that

$$\mathcal{I}\mathcal{G}_i := \tilde{\mathcal{G}}_i - \tilde{\mathcal{G}}_i \subset \mathcal{G}_i, \quad i = 1, 2,$$

and where all vectorial lines of \mathcal{G}_i intersect $\mathcal{I}\mathcal{G}_i$, some weight function

$$\Lambda : \tilde{\mathcal{G}}_1 \mapsto [1, \infty),$$

we define

$$UC_\Lambda^1(\tilde{\mathcal{G}}_1, \mathcal{G}_1; \tilde{\mathcal{G}}_2, \mathcal{G}_2)$$

(later simply denoted by $UC_\Lambda^1(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$), the space of continuously differentiable functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$, whose derivative satisfies some weighted uniform continuity.

In a more explicit way, this is the set of uniformly continuous functions

$$\mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$$

such that there exists a map

$$D\mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \mathcal{L}(\mathcal{G}_1, \mathcal{G}_2)$$

(where $\mathcal{L}(\mathcal{G}_1, \mathcal{G}_2)$ denotes the space of linear applications from \mathcal{G}_1 to \mathcal{G}_2), some modulus of continuity

$$\Omega_c : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \Omega_c(s) \rightarrow 0 \quad \text{as } s \rightarrow 0$$

and some modulus of differentiability

$$\Omega_d : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \frac{\Omega_d(s)}{s} \rightarrow 0 \quad \text{as } s \rightarrow 0$$

so that for any $f_1, f_2 \in \tilde{\mathcal{G}}_1$:

$$(2.7) \quad \|\mathcal{S}(f_2) - \mathcal{S}(f_1)\|_{\mathcal{G}_2} \leq \Lambda(f_1, f_2) \Omega_c(\|f_2 - f_1\|_{\mathcal{G}_1})$$

$$(2.8) \quad \|\mathcal{S}(f_2) - \mathcal{S}(f_1) - \langle D\mathcal{S}[f_1], f_2 - f_1 \rangle\|_{\mathcal{G}_2} \leq \Lambda(f_1, f_2) \Omega_d(\|f_2 - f_1\|_{\mathcal{G}_1}).$$

For any $\eta \in (0, 1]$, we also denote by

$$C_\Lambda^{1,\eta}(\tilde{\mathcal{G}}_1, \mathcal{G}_1; \tilde{\mathcal{G}}_2, \mathcal{G}_2)$$

(later simply denoted by $C_\Lambda^{1,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$), the space of continuously differentiable functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$, so that

$$\Omega_c(s) = C_c s^{\eta'} \quad \text{and} \quad \Omega_d(s) = C_d s^{1+\eta}$$

for some constants $C_c, C_d > 0$ and $\eta' \in [\eta, 1]$.

We define respectively $C_c^{\mathcal{S}}, C_d^{\mathcal{S}}$, as the infimum of the constants $C_c, C_d > 0$ such that respectively (2.7), (2.8) holds with the above choice of modulus Ω_i . We then define the *semi-norms*

$$[\mathcal{S}]_{C_\Lambda^{0,\eta'}} := C_c^{\mathcal{S}}, \quad [\mathcal{S}]_{C_\Lambda^{1,\eta}} := C_d^{\mathcal{S}}$$

and the norm

$$\|\mathcal{S}\|_{C_\Lambda^{1,\eta}} := C_c^{\mathcal{S}} + C_d^{\mathcal{S}}.$$

In the sequel we omit the subscript Λ or we replace it by the subscript b in the case when $\Lambda \equiv 1$. We also omit the second space when it is \mathbb{R} : $C_\Lambda^{1,\eta}(\tilde{\mathcal{G}}) := C_\Lambda^{1,\eta}(\tilde{\mathcal{G}}; \mathbb{R})$.

Remarks 2.9. (1) Due to the different notions of distances used for \mathcal{G}_1 and \mathcal{G}_2 on the one hand, and the lack of a vector space structure on $\mathcal{I}\mathcal{G}_1 = \tilde{\mathcal{G}}_1 - \tilde{\mathcal{G}}_1$ on the other hand, our definition differs from the usual one, in the sense that it does not imply the Lipschitz property or the boundedness of $D\mathcal{S}[f_1]$ for instance.

(2) In the sequel, we shall apply this abstract differential calculus with some suitable subspaces $\tilde{\mathcal{G}}_i \subset P(E)$, i.e. in our applications sets of probabilities some *prescribed* moments and some *moment bounds*. This choice of subspaces is crucial in order to make rigorous the intuition of Grünbaum [35] (see the — unjustified — expansion of H_f in [35]).

(3) It is worth emphasizing that our differential calculus is based on the idea of considering $P(E)$ (or subsets of $P(E)$) as “plunged sub-manifolds” of some larger normed spaces \mathcal{G}_i . We hence develop a differential calculus in the space of probability measures into a simple and robust framework, well suited to deal with the different objects we have to manipulate (1-particle semigroup, polynomial, generators...). Our approach thus differs from the approach of P.-L. Lions recently developed in his course at Collège de France [46] or the one developed by L. Ambrosio *et al.* [2] in order to deal with gradient flows PDEs in spaces of probability measures associated with the Wasserstein metric, as introduced by Otto *et al.* [41, 61].

(4) One novelty of our work is the use of this differential calculus in order to state some “differential” stability conditions on the limit semigroup. Roughly speaking the latter estimates measure how this limit semigroup handles fluctuations departing from chaoticity, they are the corner stone of our analysis.

This differential calculus behaves well for composition in the sense that for any given

$$\mathcal{U} \in C_{\Lambda_u}^{1,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2) \quad \text{and} \quad \mathcal{V} \in C_{\Lambda_v}^{1,\eta}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3)$$

there holds

$$\mathcal{S} := \mathcal{V} \circ \mathcal{U} \in C_{\Lambda_S}^{1,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_3)$$

for some appropriate weight function Λ_S . We conclude the section by stating a precise result adapted to our applications.

Lemma 2.10. *For any given*

$$\mathcal{U} \in C_{\Lambda}^{1,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2) \cap C_{\Lambda}^{0,(1+\eta)/2}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2), \quad \eta \in (0, 1], \quad \text{and} \quad \mathcal{V} \in C^{1,1}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3),$$

there holds

$$\mathcal{S} := \mathcal{V} \circ \mathcal{U} \in C_{\Lambda^2}^{1,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_3) \quad \text{and} \quad D\mathcal{S}[f] = D\mathcal{V}[\mathcal{U}(f)] \circ D\mathcal{U}[f].$$

More precisely, there holds

$$[\mathcal{S}]_{C_{\Lambda}^{0,1}} \leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_{\Lambda}^{0,1}},$$

and

$$[\mathcal{S}]_{C_{\Lambda^2}^{1,\eta}} \leq \|\mathcal{V}\|_{C^{1,1}} [\mathcal{U}]_{C_{\Lambda}^{1,\eta}} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]_{C_{\Lambda}^{0,(1+\eta)/2}}^2.$$

Proof of Lemma 2.10. For $f_1, f_2 \in \tilde{\mathcal{G}}_1$, we have

$$\mathcal{U}(f_2) = \mathcal{U}(f_1) + \langle D\mathcal{U}[f_1], f_2 - f_1 \rangle + \mathcal{T}\mathcal{U}(f_1, f_2)$$

with

$$\|\mathcal{T}\mathcal{U}(f_1, f_2)\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_{\Lambda}^{1,\eta}} \Lambda(f_1, f_2) \|f_2 - f_1\|_{\mathcal{G}_1}^{1+\eta},$$

as well as

$$\|\mathcal{U}(f_2) - \mathcal{U}(f_1)\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_{\Lambda}^{0,(1+\eta)/2}} \Lambda(f_1, f_2) \|f_2 - f_1\|_{\mathcal{G}_1}^{(1+\eta)/2}.$$

A similar Taylor expansion holds for \mathcal{V} : for $g_1, g_2 \in \tilde{\mathcal{G}}_2$

$$\mathcal{V}(g_2) = \mathcal{V}(g_1) + \langle D\mathcal{V}[g_1], g_2 - g_1 \rangle + \mathcal{T}\mathcal{V}(g_1, g_2)$$

with

$$\|\mathcal{T}\mathcal{V}(g_1, g_2)\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{1,1}} \|g_2 - g_1\|_{\mathcal{G}_2}^2.$$

We then write

$$\|\mathcal{S}(f_2) - \mathcal{S}(f_1)\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{0,1}} \|\mathcal{U}(f_2) - \mathcal{U}(f_1)\|_{\mathcal{G}_2} \leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_{\Lambda}^{0,1}} \Lambda(f_1, f_2) \|f_2 - f_1\|_{\mathcal{G}_1}$$

which implies

$$[\mathcal{S}]_{C_{\Lambda}^{0,1}} \leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_{\Lambda}^{0,1}},$$

and

$$\begin{aligned} \mathcal{S}(f_2) &= (\mathcal{V} \circ \mathcal{U})(f_2) = \mathcal{V}\left(\mathcal{U}(f_1) + \langle D\mathcal{U}[f_1], f_2 - f_1 \rangle + \mathcal{T}\mathcal{U}(f_1, f_2)\right) \\ &= \mathcal{V}(\mathcal{U}(f_1)) + \mathcal{T}\mathcal{V}(\mathcal{U}(f_2), \mathcal{U}(f_1)) \\ &\quad + \left\langle D\mathcal{V}[\mathcal{U}(f_1)], \langle D\mathcal{U}[f_1], f_2 - f_1 \rangle + \mathcal{T}\mathcal{U}(f_1, f_2) \right\rangle \end{aligned}$$

which implies

$$\langle D\mathcal{S}[f_1], f_2 - f_1 \rangle = \left\langle D\mathcal{V}[\mathcal{U}(f_1)], (\langle D\mathcal{U}[f_1], f_2 - f_1 \rangle) \right\rangle.$$

Observe that since $\mathcal{V} \in C^{1,1}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3)$, one has

$$\forall h \in \mathcal{I}\mathcal{G}_2, \quad \left\| \left\langle D\mathcal{V}[\mathcal{U}(f_1)], h \right\rangle \right\|_{\mathcal{G}_3} \leq ([\mathcal{V}]_{C^{0,1}} + [\mathcal{V}]_{C^{1,1}}) \|h\|_{\mathcal{G}_2}$$

and therefore by scaling (and using that all vectorial lines of $\mathcal{I}\mathcal{G}_2$ intersects $\mathcal{I}\mathcal{G}_2$ at a non-zero point) it extends to

$$\forall h \in \mathcal{G}_2, \quad \left\| \left\langle D\mathcal{V}[\mathcal{U}(f_1)], h \right\rangle \right\|_{\mathcal{G}_3} \leq ([\mathcal{V}]_{C^{0,1}} + [\mathcal{V}]_{C^{1,1}}) \|h\|_{\mathcal{G}_2}.$$

Finally we estimate the remaining term:

$$\begin{aligned}
& \|\mathcal{S}(f_2) - \mathcal{S}(f_1) - \langle D\mathcal{S}[f_1], f_2 - f_1 \rangle\|_{\mathcal{G}_3} \\
&= \|\mathcal{T}_{\mathcal{V}}(\mathcal{U}(f_2), \mathcal{U}(f_1)) + \langle D\mathcal{V}[\mathcal{U}(f_1)], \mathcal{T}_{\mathcal{U}}(f_1, f_2) \rangle\|_{\mathcal{G}_3} \\
&\leq [\mathcal{V}]_{C^{1,1}} \|\mathcal{U}(f_1) - \mathcal{U}(f_2)\|_{\mathcal{G}_2}^2 + \|\mathcal{V}\|_{C^{1,1}} \|\mathcal{T}_{\mathcal{U}}(f_1, f_2)\|_{\mathcal{G}_2}, \\
&\leq [\mathcal{V}]_{C^{1,1}} \left([\mathcal{U}]_{C_{\Lambda}^{0,(1+\eta)/2}} \Lambda(f_1, f_2) \|f_2 - f_1\|_{\mathcal{G}_1}^{(1+\eta)/2} \right)^2 \\
&\quad + \|\mathcal{V}\|_{C^{1,1}} \Lambda(f_1, f_2) [\mathcal{U}]_{C_{\Lambda}^{1,\eta}} \|f_2 - f_1\|_{\mathcal{G}_1}^{1+\eta},
\end{aligned}$$

and we conclude by recalling that $\Lambda \geq 1$. \square

2.7. The pullback generator. As a first application of this differential calculus, let us compute the generator of the pullback limit semigroup.

Lemma 2.11. *Given some Banach space \mathcal{G} and some space of probability measures $\mathcal{P}_{\mathcal{G}}(E)$ (see Definitions 2.2-2.3) associated to a weight function m and constraint function \mathbf{m} , and endowed with the metric induced from \mathcal{G} , then for some $\zeta \in (0, 1]$ and some $\bar{a} \in (0, \infty)$ we assume that for any $a \in (\bar{a}, \infty)$ and any choice of constraints $\mathbf{r} \in \mathcal{R}_{\mathcal{G}}$, there holds:*

(i) *The equation (2.1) generates a semigroup*

$$S_t^{NL} : \mathcal{BP}_{\mathcal{G},a,\mathbf{r}} \rightarrow \mathcal{BP}_{\mathcal{G},a,\mathbf{r}}$$

which is ζ -Hölder continuous locally uniformly in time, in the sense that for any $\tau \in (0, \infty)$ there exists $C_{\tau} \in (0, \infty)$ such that

$$\forall f, g \in \mathcal{BP}_{\mathcal{G},a,\mathbf{r}}, \quad \sup_{t \in [0, \tau]} \|S_t^{NL} f - S_t^{NL} g\|_{\mathcal{G}_1} \leq C_{\tau} \|f - g\|_{\mathcal{G}_1}^{\zeta}.$$

(ii) *The application Q is bounded and ζ -Hölder continuous from $\mathcal{BP}_{\mathcal{G},a,\mathbf{r}}$ into \mathcal{G} .*

Then for any $a \in (\bar{a}, \infty)$, $\mathbf{r} \in \mathcal{R}_{\mathcal{G}}$ the pullback semigroup T_t^{∞} defined by

$$\forall f \in \mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E), \quad \Phi \in UC_b(\mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E)), \quad T_t^{\infty}[\Phi](f) := \Phi(S_t^{NL}(f))$$

is a C_0 -semigroup of contractions on the Banach space $UC_b(\mathcal{BP}_{\mathcal{G},\mathbf{r},a}(E))$.

Its generator G^{∞} is an unbounded linear operator on $UC_b(\mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E))$ with domain denoted by $\text{Dom}(G^{\infty})$ and containing $UC_b^1(\mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E))$. On the latter space, it is defined by the formula

$$(2.9) \quad \forall \Phi \in UC_b^1(\mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E)), \quad \forall f \in \mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E), \quad (G^{\infty}\Phi)(f) := \langle D\Phi[f], Q(f) \rangle.$$

Remark 2.12. Note that the restriction to uniformly continuous functions Φ on space of probability measures will be harmless in the sequel for two reasons: first in most cases our choice of weight, constraints and distance yields a compact space $\mathcal{BP}_{\mathcal{G},a}(E)$, and second and most importantly we shall only manipulate this pullback semigroup for functions Φ having at least uniform Hölder regularity.

Proof of Lemma 2.11. The proof is split in several steps.

Step 1. Hölder regularity in time for $S_t^{NL}(f_0)$. We claim that for any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}}$ and $f_0 \in \mathcal{BP}_{\mathcal{G},a,\mathbf{r}}(E)$ and $\tau > 0$ the application

$$\mathcal{S}(f_0) : [0, \tau) \rightarrow \mathcal{P}_{\mathcal{G}}, \quad t \mapsto S_t^{NL}(f_0)$$

is right differentiable in $t = 0$ with

$$\mathcal{S}(f_0)'(0^+) = Q(f_0).$$

Let us write $f_t := S_t^{NL} f_0$. First, since $f_t \in \mathcal{BP}_{\mathcal{G},a,r}$ for any $t \in [0, \tau]$ (assumption (i)) and Q is bounded on $\mathcal{BP}_{\mathcal{G},a,r}(E)$ (assumption (ii)), we deduce that

$$(2.10) \quad \|f_t - f_0\|_{\mathcal{G}} = \left\| \int_0^t Q(f_s) ds \right\|_{\mathcal{G}} \leq K t$$

for a constant K which is uniform according to $f_0 \in \mathcal{BP}_{\mathcal{G},a,r}(E)$.

We then use the previous inequality together with the fact that Q is ζ -Hölder continuous (assumption (ii) again), to get

$$\begin{aligned} \|f_t - f_0 - t Q(f_0)\|_{\mathcal{G}} &= \left\| \int_0^t (Q(f_s) - Q(f_0)) ds \right\|_{\mathcal{G}} \\ &= L \int_0^t \|f_s - f_0\|_{\mathcal{G}}^{\zeta} ds \\ &\leq L \int_0^t (K s)^{\zeta} ds = L K^{\zeta} \frac{t^{1+\zeta}}{1+\zeta}, \end{aligned}$$

which implies the claim.

Then the semigroup property of (S_t^{NL}) implies that $t \mapsto f_t$ is continuous from \mathbb{R}_+ into $\mathcal{P}_{\mathcal{G}}(E)$ and right differentiable at any point.

Step 2. Contraction property for T_t^{∞} . We claim that (T_t^{∞}) is a C_0 -semigroup of contractions on $UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E))$. First for any $\Phi \in UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E))$, we denote by ω_{Φ} the modulus of continuity of Φ . We have thanks to the assumption (i):

$$\begin{aligned} \forall t \in [0, \tau], \quad |(T_t^{\infty} \Phi)(g) - (T_t^{\infty} \Phi)(f)| &= |\Phi(S_t^{NL}(g)) - \Phi(S_t^{NL}(f))| \\ &\leq \omega_{\Phi} \left(\|S_t^{NL}(g) - S_t^{NL}(f)\|_{\mathcal{G}_1} \right) \\ &\leq \omega_{\Phi} \left(C_{\tau} \|g - f\|_{\mathcal{G}_1}^{\zeta} \right), \end{aligned}$$

so that $T_t^{\infty} \Phi \in UC_b(\mathcal{BP}_{\mathcal{G}_1,a,r}(E))$ for any $t \in [0, \tau]$, and then, by iteration, for any $t \geq 0$. Next, we have

$$\|T_t^{\infty}\| = \sup_{\|\Phi\| \leq 1} \|T_t^{\infty} \Phi\| = \sup_{\|\Phi\| \leq 1} \sup_{f \in \mathcal{BP}_{\mathcal{G},a,r}(E)} |\Phi(S_t^{NL}(f))| \leq 1$$

since

$$\|\Phi\| = \sup_{f \in \mathcal{BP}_{\mathcal{G},a,r}(E)} |\Phi(f)|$$

and S_t^{NL} maps $\mathcal{BP}_{\mathcal{G},a,r}(E)$ to itself.

Finally, from (2.10), for any $\Phi \in UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E))$, we have

$$\|T_t^{\infty} \Phi - \Phi\| = \sup_{f \in \mathcal{BP}_{\mathcal{G},a,r}(E)} |\Phi(S_t^{NL}(f)) - \Phi(f)| \leq \omega_{\Phi}(K t) \rightarrow 0.$$

As a consequence (T_t^{∞}) has a closed generator G^{∞} with dense domain

$$\text{Dom}(G^{\infty}) \subset UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E)), \quad \overline{\text{Dom}(G^{\infty})} = UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E))$$

(see for instance [63, Chapter 1, Corollary 2.5]).

Step 3. We shall now identify this generator on a subset of its domain. Let us construct a natural candidate provided by the heuristic of Remark 2.1. Let us define $\tilde{G}^{\infty} \Phi$ by

$$\forall \Phi \in UC_b^1(\mathcal{BP}_{\mathcal{G},a}(E)), \quad \forall f \in \mathcal{BP}_{\mathcal{G},a}(E), \quad (\tilde{G}^{\infty} \Phi)(f) := \langle D\Phi[f], Q(f) \rangle.$$

The right-hand side is well defined since

$$D\Phi[f] \in \mathcal{L}(\mathcal{G}, \mathbb{R}) = \mathcal{G}' \quad \text{and} \quad Q(f) \in \mathcal{G}.$$

Moreover, since both applications

$$f \mapsto D\Phi[f] \quad \text{and} \quad f \mapsto Q(f)$$

are uniformly continuous on $\mathcal{BP}_{\mathcal{G},a,r}(E)$, so is the application

$$f \mapsto (\tilde{G}^\infty \Phi)(f).$$

Hence $\tilde{G}^\infty \Phi \in UC_b(\mathcal{BP}_{\mathcal{G},a,r}(E))$.

Step 4. Finally, by composition, for any fixed $\Phi \in UC_b^1(\mathcal{BP}_{\mathcal{G},a,r}(E))$ and $f \in \mathcal{BP}_{\mathcal{G},a,r}(E)$, the map

$$t \mapsto T_t^\infty \Phi(f) = \Phi \circ S_t^{NL}(f)$$

is right differentiable in $t = 0$ and

$$\begin{aligned} \left(\frac{d}{dt} (T_t^\infty \Phi)(f) \right) \Big|_{t=0} &:= \left(\frac{d}{dt} (\Phi \circ S_t^{NL}(f)(t)) \right) \Big|_{t=0} \\ &= \left\langle D\Phi[S_0^{NL}(f)], \left(\frac{d}{dt} S_t^{NL}(f) \right) \Big|_{t=0} \right\rangle \\ &= \langle D\Phi[f], Q(f) \rangle = (\tilde{G}^\infty \Phi)(f), \end{aligned}$$

which precisely means that $\Phi \in \text{Dom}(G^\infty)$ and that (2.9) holds. \square

2.8. Duality inequalities. Our transformations π^N and R^N behave nicely for the supremum norm on $C_b(P(E), TV)$, see (2.2). More generally we shall consider “*duality pairs*” of metric spaces as follows:

Definition 2.13. We say that a pair $(\mathcal{F}, \mathcal{P}_{\mathcal{G}})$ of a normed vectorial space $\mathcal{F} \subset C_b(E)$ endowed with the norm $\|\cdot\|_{\mathcal{F}}$ and a space of probability measures $\mathcal{P}_{\mathcal{G}} \subset P(E)$ endowed with a metric $d_{\mathcal{G}}$ satisfy a *duality inequality* if

$$(2.11) \quad \forall f, g \in \mathcal{P}_{\mathcal{G}}, \quad \forall \varphi \in \mathcal{F}, \quad |\langle g - f, \varphi \rangle| \leq C d_{\mathcal{G}}(f, g) \|\varphi\|_{\mathcal{F}},$$

where here $\langle \cdot, \cdot \rangle$ stands for the usual duality brackets between probabilities and continuous functions. In the case where the distance $d_{\mathcal{G}}$ is associated with a normed vector space \mathcal{G} , this amounts to the usual duality inequality $|\langle h, \varphi \rangle| \leq \|h\|_{\mathcal{G}} \|\varphi\|_{\mathcal{F}}$.

The “compatibility” of the transformation R^N for any such pair follows from the multilinearity: if \mathcal{F} and \mathcal{G} are in duality, $\mathcal{F} \subset C_b(E)$ and $\mathcal{P}_{\mathcal{G}}$ is endowed with the metric associated to $\|\cdot\|_{\mathcal{G}}$, then for any

$$\varphi = \varphi_1 \times \cdots \times \varphi_N \in \mathcal{F}^{\otimes N},$$

the polynomial function R_φ^N in $C_b(P(E))$ is $C^{1,1}(\mathcal{P}_{\mathcal{G}}, \mathbb{R})$. Indeed, given $f_1, f_2 \in \mathcal{P}_{\mathcal{G}_1}$, we define

$$\mathcal{G} \rightarrow \mathbb{R}, \quad h \mapsto DR_\varphi^N[f_1](h) := \sum_{i=1}^N \left(\prod_{j \neq i} \langle f_1, \varphi_j \rangle \right) \langle h, \varphi_i \rangle,$$

and we have

$$R_\varphi^N(f_2) - R_\varphi^N(f_1) = \sum_{i=1}^N \left(\prod_{1 \leq k < i} \langle f_2, \varphi_k \rangle \right) \langle f_2 - f_1, \varphi_i \rangle \left(\prod_{i < k \leq \ell} \langle f_1, \varphi_k \rangle \right),$$

and

$$\begin{aligned} & R_\varphi^N(f_2) - R_\varphi^N(f_1) - DR_\varphi^N[f_1](f_2 - f_1) = \\ &= \sum_{1 \leq j < i \leq N} \left(\prod_{1 \leq k < j} \langle f_2, \varphi_k \rangle \right) \langle f_2 - f_1, \varphi_j \rangle \left(\prod_{j < k < i} \langle f_1, \varphi_k \rangle \right) \langle f_2 - f_1, \varphi_i \rangle \left(\prod_{i < k \leq \ell} \langle f_1, \varphi_k \rangle \right). \end{aligned}$$

Hence for instance $R_\varphi^N \in C^{1,1}(\mathcal{P}_\mathcal{G}; \mathbb{R})$ with

$$\begin{aligned} |R_\varphi^N(f_2) - R_\varphi^N(f_1)| &\leq N \|\varphi\|_{\mathcal{F} \otimes (L^\infty)^{N-1}} \|f_2 - f_1\|_{\mathcal{G}}, \\ |DR_\varphi^N[f_1](h)| &\leq N \|\varphi\|_{\mathcal{F} \otimes (L^\infty)^{N-1}} \|h\|_{\mathcal{G}}, \end{aligned}$$

and

$$(2.12) \quad |R_\varphi^N(f_2) - R_\varphi^N(f_1) - DR_\varphi^N[f_1](f_2 - f_1)| \leq \frac{N(N-1)}{2} \|\varphi\|_{\mathcal{F}^2 \otimes (L^\infty)^{N-2}} \|f_2 - f_1\|_{\mathcal{G}}^2,$$

where we have defined

$$\|\varphi\|_{\mathcal{F}^k \otimes (L^\infty)^{N-k}} := \max_{i_1, \dots, i_k \text{ distincts in } [1, N]} \left(\|\varphi_{i_1}\|_{\mathcal{F}} \dots \|\varphi_{i_k}\|_{\mathcal{F}} \prod_{j \neq (i_1, \dots, i_k)} \|\varphi_j\|_{L^\infty(E)} \right).$$

Remarks 2.14. (1) It is easily seen in this computation that the assumption that φ is tensor product is not necessary. In fact it is likely that this assumption could be relaxed all along our proof.

(2) The assumption $\mathcal{F} \subset C_b(E)$ could also be relaxed. For instance, when

$$\mathcal{F} := \text{Lip}_0(E)$$

is the space of Lipschitz function which vanishes in some fixed point $x_0 \in E$, \mathcal{G} is its dual space, and

$$\mathcal{P}_\mathcal{G} := \{f \in \mathcal{P}_1(E); \langle f, \text{dist}_E(\cdot, x_0) \rangle \leq a\}$$

for some fixed $a > 0$, we have $R_\varphi^N \in C^{1,1}(\mathcal{P}_1(E); \mathbb{R})$ with

$$[R_\varphi^N]_{C^{0,1}} \leq N a^{N-2} \|\varphi\|_{\mathcal{F} \otimes N}, \quad [R_\varphi^N]_{C^{1,1}} \leq \frac{N(N-1)}{2} a^{N-1} \|\varphi\|_{\mathcal{F} \otimes N},$$

or equivalently $R_\varphi^N \in C_\Lambda^{1,1}(\mathcal{P}_1(E); \mathbb{R})$ with $\Lambda(f) := \|f\|_{M_1^1}^{N-2}$.

In the other way round, for the projection π^N it is clear that if the empirical measure map

$$X \in E^N \mapsto \mu_X^N \in P(E)$$

belongs to $C^{k,\eta}(E^N, \mathcal{P}_\mathcal{G})$ for some norm structure \mathcal{G} , then by composition one has

$$(2.13) \quad \|\pi^N(\Phi)\|_{C^{k,\eta}(E^N; \mathbb{R})} \leq C_\pi \|\Phi\|_{C^{k,\eta}(\mathcal{P}_\mathcal{G})}.$$

However the regularity of the empirical measure of course heavily depends on the choice of the metric \mathcal{G} .

Example 2.15. In the case $\mathcal{F} = (C_b(E), L^\infty)$ and $\mathcal{G} = (M^1(E), TV)$, (2.13) trivially holds with $C^{k,\eta}$ replaced by C_b .

Example 2.16. When $\mathcal{F} = \text{Lip}_0(E)$ endowed with the norm $\|\phi\|_{\text{Lip}}$ and $\mathcal{P}_\mathcal{G}(E)$ (constructed in Subsubsection 2.5.2) is endowed with the Wasserstein distance W_1 with linear cost, one has (2.13) with $k = 0$, $\eta = 1$:

$$|\Phi(\mu_X^N) - \Phi(\mu_Y^N)| \leq \|\Phi\|_{C^{0,1}(\mathcal{P}_\mathcal{G})} W_1(\mu_X^N, \hat{\mu}_Y^N) \leq \|\Phi\|_{C^{0,1}(\mathcal{P}_\mathcal{G})} \|X - Y\|_{\ell^1},$$

where we use (2.3), which proves that

$$\|\pi^N(\Phi)\|_{C^{0,1}(E^N)} \leq \|\Phi\|_{C^{0,1}(\mathcal{P}_{\mathcal{G}})},$$

when E^N is endowed with the ℓ^1 distance defined in (2.3).

3. THE ABSTRACT THEOREM

3.1. Assumptions of the abstract theorem. Let us list the assumptions that we need for our main abstract theorem.

(A1) Assumptions on the N -particle system.

G^N and T_t^N are well defined on $C_b(E^N)$ and are invariant under permutation so that the evolution f_t^N is well defined. We moreover assume the following moment conditions:

- (i) *Conservation constraint:* There exists a constraint function $\mathbf{m}_{\mathcal{G}_1} : E \rightarrow \mathbb{R}^D$ and a subset $\mathcal{R}_{\mathcal{G}_1} \subset \mathbb{R}^D$ such that defining the sequence of constraint sets

$$\mathbb{E}_N := \{V \in E^N; \langle \mu_V^N, \mathbf{m}_{\mathcal{G}_1} \rangle \in \mathcal{R}_{\mathcal{G}_1}\}$$

there holds

$$\forall t \in [0, T), \quad \text{Supp } f_t^N \subset \mathbb{E}_N.$$

- (ii) *Propagation of an integral moment bound:* There exists a weight function $m_{\mathcal{G}_1}$, a time $T \in (0, \infty]$ and a constant $C_{1,T} \in (0, \infty)$, possibly depending on T and $m_{\mathcal{G}_1}$, but not on the number of particles N , such that

$$\forall N \geq 1, \quad \sup_{0 \leq t < T} \langle f_t^N, M_{m_{\mathcal{G}_1}}^N \rangle \leq C_{m_{\mathcal{G}_1}, T},$$

where for any weight function m on E we define the *mean weight function* M_m^N on E^N by

$$\forall V \in E^N, \quad M_m^N(V) := \frac{1}{N} \sum_{i=1}^N m(v_i) = M_m(\mu_V^N) = \langle \mu_V^N, m \rangle.$$

- (iii) *Support moment bound at initial time:* There exists a weight function $m_{\mathcal{G}_3}$ and a constant $C_3^N \in (0, +\infty)$, possibly depending on the number of particles N , such that

$$\text{Supp } f_0^N \subset \left\{ V \in E^N; M_{m_{\mathcal{G}_3}}^N(V) \leq C_{m_{\mathcal{G}_3}, 0}^N \right\}.$$

Note that the name of the weights functions $m_{\mathcal{G}_1}$ and $m_{\mathcal{G}_3}$ and of the constraint function $\mathbf{m}_{\mathcal{G}_1}$ in (A1) above are chosen bearing in mind the coherence with the functional framework introduced in Definition 2.2 (in particular $\langle f, \mathbf{m}_{\mathcal{G}_1} \rangle$ is well defined as a vector of \mathbb{R}^D for any $f \in \mathcal{P}_{\mathcal{G}_1}$) and with the functional spaces in the other assumptions. Observe moreover that the support condition \mathbb{E}^N will be useful for controlling the moments of the empirical measures sampled out of the distribution f^N : $V \in \mathbb{E}_N$ implies $\mu_V^N \in \mathcal{R}_{\mathcal{G}_1}$, where this set of constraints is defined in Definitions 2.2 and 2.3.

(A2) Assumptions for the existence of the pullback semigroup.

We consider a weight function $m_{\mathcal{G}_1}$, a constraint function $\mathbf{m}_{\mathcal{G}_1} : E \rightarrow \mathbb{R}^D$ and a set of constraints $\mathcal{R}_{\mathcal{G}_1} \subset \mathbb{R}^D$ as in **(A1)-(i)** and **(A1)-(ii)**. We then consider the corresponding space of probability measures $\mathcal{P}_{\mathcal{G}_1}(E)$ and the corresponding vectorial *space of increments* $\mathcal{IP}_{\mathcal{G}_1}$ according to Definition 2.2. We finally consider a Banach space $\mathcal{G}_1 \supset \mathcal{IP}_{\mathcal{G}_1}$ so that $\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}(E)$ is endowed with the distance $d_{\mathcal{G}_1}$ induced by the norm $\|\cdot\|_{\mathcal{G}_1}$ for any constraint vector $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$.

Then we assume the following: there are $\zeta \in (0, 1]$ and $\bar{a} \in (0, \infty)$ such that for any $a \in (\bar{a}, \infty)$ and $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$ one has

- (i) The equation (2.1) generates a semigroup

$$S_t^{NL} : \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}} \rightarrow \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$$

which is ζ -Hölder continuous locally uniformly in time, in the sense that for any $\tau \in (0, \infty)$ there exists $C_\tau \in (0, \infty)$ such that

$$\forall f, g \in \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}, \quad \sup_{t \in [0, \tau]} \|S_t^{NL} f - S_t^{NL} g\|_{\mathcal{G}_1} \leq C_\tau \|f - g\|_{\mathcal{G}_1}^\zeta.$$

- (ii) The application Q is bounded and ζ -Hölder continuous from $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ into \mathcal{G}_1 .

The important consequence of this assumption is that the semigroups S_t^{NL} and T_t^∞ are well defined as well as the generators G^N and G^∞ thanks to Lemma 2.11.

We then need the key following *consistency* assumption. It intuitively states that the N -particle *approximation* of the limit mean-field equation is consistent. More rigorously this means a convergence of the generators of the N -particle approximation towards the generator of the limit pullback semigroup within the abstract functional framework we have introduced.

(A3) Convergence of the generators.

We consider $\mathcal{P}_{\mathcal{G}_1}$, $m_{\mathcal{G}_1}$ and $\mathbf{m}_{\mathcal{G}_1}$ introduced in **(A2)**, and we also define a weight function

$$1 \leq m'_{\mathcal{G}_1} \leq C m_{\mathcal{G}_1}$$

possibly weaker than $m_{\mathcal{G}_1}$ and we define the associated weight on the distribution:

$$\Lambda_1(f) := \langle f, m'_{\mathcal{G}_1} \rangle.$$

Then we assume that for some function

$$\varepsilon(N) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{and} \quad \eta \in (0, 1]$$

the generators G^N and G^∞ satisfy

$$\begin{aligned} \forall \Phi \in \bigcap_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}), \quad & \left\| \left(M_{m_{\mathcal{G}_1}}^N \right)^{-1} (G^N \pi_N - \pi_N G^\infty) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \\ & \leq \varepsilon(N) \sup_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} [\Phi]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}})}. \end{aligned}$$

Note the following aspect, which shall be a source of difficulties in the applications of the abstract theorem: *the loss of weight in the consistency estimate (A3) has to be matched by the support constraints on the N -particle system in the assumption (A1)-(i) and the moment bounds propagated on the N -particle system in the assumption (A1)-(ii).* (In fact the loss of weight $m_{\mathcal{G}_1}$ in the consistency estimate can be even slightly higher

than the weight $m'_{\mathcal{G}_1}$ for which the stability estimates are performed, due to the energy conservation, see later.)

Moreover, the best we are able to prove uniformly in N on the N -particle system are *polynomial* moment bounds. This thus constraints the kind of loss of weight we can afford in the following stability estimate.

We now state the second key *stability* assumption. Intuitively this corresponds to the abstract regularity that needs to be transported along the flow of the limit mean-field equation so that the fluctuations around chaoticity can be controlled. More rigorously this means some differential regularity on the pullback limit semigroup, which corresponds to some differential regularity on the limit nonlinear semigroup *according to the initial data and in the space of probability measures*.

(A4) Differential stability of the limit semigroup.

We consider some Banach space $\mathcal{G}_2 \supset \mathcal{G}_1$ (where \mathcal{G}_1 was defined in **(A2)**) and the corresponding space of probability measures $\mathcal{P}_{\mathcal{G}_2}(E)$ (see Definitions 2.2-2.3) with the weight function $m_{\mathcal{G}_2}$ and the constraint function $\mathbf{m}_{\mathcal{G}_2}$, and endowed with the metric induced from \mathcal{G}_2 .

We assume that the flow S_t^{NL} is $C_{\Lambda_2}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}}, \mathcal{P}_{\mathcal{G}_2})$ for any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$, in the sense that there exists $C_T^\infty > 0$ such that

$$\sup_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} \int_0^T \left([S_t^{NL}]_{C_{\Lambda_2}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}}, \mathcal{P}_{\mathcal{G}_2})} + [S_t^{NL}]_{C_{\Lambda_2}^{0,(1+\eta)/2}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}}, \mathcal{P}_{\mathcal{G}_2})}^2 \right) dt \leq C_T^\infty,$$

with $\Lambda_2 := \Lambda_1^{1/2}$, where $\eta \in (0, 1)$ and Λ_1 are the same as in **(A3)**.

We finally state a weaker stability assumption on the limit semigroup. It shall be used intuitively for proving that the initial error made in the law of large numbers when approximating a probability by empirical measures is propagated by the limit semigroup. The reason for dissociating this assumption from the previous one is because we need flexibility with different choices of distances for them.

(A5) Weak stability of the limit semigroup.

We assume that, for some probabilistic space $\mathcal{P}_{\mathcal{G}_3}(E)$ associated to the weight function $m_{\mathcal{G}_3}$ (as in **(A1)**-**(iii)**), a constraint function $\mathbf{m}_{\mathcal{G}_3}$, a set of constraints $\mathcal{R}_{\mathcal{G}_3}$ and some metric structure $d_{\mathcal{G}_3}$, for any $a, T > 0$ there exists a concave modulus of continuity $\Theta_{a,T}$ (i.e. $\Theta_{a,T} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous concave with $\Theta_{a,T}(0) = 0$) such that we have

$$\forall \mathbf{r} \in \mathcal{R}_{\mathcal{G}_3}, \forall f_1, f_2 \in \mathcal{BP}_{\mathcal{G}_3,a,\mathbf{r}}(E), \quad \sup_{[0,T]} d_{\mathcal{G}_3}(S_t^{NL}(f_1), S_t^{NL}(f_2)) \leq \Theta_{a,T}(d_{\mathcal{G}_3}(f_1, f_2)).$$

Observe that in the latter assumption, we require that $f_1, f_2 \in \mathcal{BP}_{\mathcal{G}_3,a}(E)$ which requires in particular the bounds

$$M_{m_{\mathcal{G}_3}}(f_i) = \int_{\mathbb{R}^d} f_i m_{\mathcal{G}_3} dv \leq a, \quad i = 1, 2.$$

When applying the assumption to some empirical measure for one of the argument f_1 or f_2 , this requires the pointwise control of terms like $M_{m_{\mathcal{G}_3}}^N$. This is the reason for the assumption **(A1)**-**(iii)**.

3.2. Statement of the result. We are now in position to state the main abstract result. This result can be considered intuitively as a *convergence* in approximation theory, in the sense of proving that approximation errors between the N -particle system and the limit mean-field system are propagated along time without instability amplification mechanism. More specifically the approximation error means in the present context some kind of distance between the discrete N -particle system and the limit mean-field system, within our abstract functional framework. This result implies in particular the propagation of chaos. In terms of method, we aim at treating the N -particle system as a perturbation (in a very degenerated sense) of the limit problem, and minimizing assumptions on the many-particle systems in order to avoid complications of many dimensions dynamics.

Theorem 3.1. *Consider a family of N -particle initial conditions*

$$f_0^N \in P_{\text{sym}}(E^N), \quad N \geq 1,$$

and the associated solutions

$$f_t^N = S_t^N(f_0^N).$$

Consider a 1-particle initial condition $f_0 \in P(E)$ and the associated solution

$$f_t = S_t^{NL}(f_0)$$

of the limit mean-field equation.

Assume that the assumptions (A1)-(A2)-(A3)-(A4)-(A5) hold for some spaces $\mathcal{P}_{\mathcal{G}_k}$, \mathcal{G}_k and \mathcal{F}_k , $k = 1, 2, 3$ with $\mathcal{F}_k \subset C_b(E)$, and where \mathcal{F}_k and \mathcal{G}_k are in duality (that is (2.11) holds).

Assume also that the 1-particle distribution satisfies the moment bound

$$M_{m_{\mathcal{G}_3}}(f_0) = \langle f_0, m_{\mathcal{G}_3} \rangle < +\infty.$$

Then there is an explicit absolute constant $C \in (0, \infty)$ such that for any $N, \ell \in \mathbb{N}^$, with $N \geq 2\ell$, and for any*

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$$

we have

$$(3.1) \quad \sup_{[0, T]} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \\ \leq C \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + C_{m_{\mathcal{G}_1}, T} C_T^\infty \varepsilon(N) \ell^2 \|\varphi\|_{\mathcal{F}_2^{\otimes 2} \otimes (L^\infty)^{\ell-2}} \right. \\ \left. + \ell \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} \Theta_{a^N, T} \left(\mathcal{W}_{d_{\mathcal{G}_3}}(\pi_P^N f_0^N, \delta_{f_0}) \right) \right],$$

where $a^N > 0$ depends on $C_{\mathcal{G}_3, 0}^N$ and $M_{\mathcal{G}_3}(f_0)$, and where $\mathcal{W}_{d_{\mathcal{G}_3}}$ stands for an abstract Monge-Kantorovich distance in $P(\mathcal{P}_{\mathcal{G}_3}(E))$ (see the third point in the next remark)

$$(3.2) \quad \mathcal{W}_{d_{\mathcal{G}_3}}(\pi_P^N f_0^N, \delta_{f_0}) = \int_{E^N} d_{\mathcal{G}_3}(\mu_V^N, f_0) \, d f_0^N(v).$$

Remark 3.2. In the applications the worst decay rate in the right-hand side of (3.1) is always the last one. This last term controls two kind of errors: (1) the chaoticity of the initial data, that is how well $f_0^N \sim f_0^{\otimes N}$, (2) the rate of convergence in the law of large numbers for measures in the distance $d_{\mathcal{G}_3}$.

Let us discuss more the meaning of this last term and the related issue of sampling by empirical measures in statistics (see also Section 4). Following the abstract definition of the optimal transport Wasserstein distance we define

$$\forall \mu_1, \mu_2 \in P(\mathcal{P}_{\mathcal{G}_3}), \quad \mathcal{W}_{d_{\mathcal{G}_3}}(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}} d_{\mathcal{G}_3}(\rho_1, \rho_2) \, d\pi(\rho_1, \rho_2),$$

where $\Pi(\mu_1, \mu_2)$ denotes the set of probability measures on the product space $\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}$ with first marginal μ_1 and second marginal μ_2 . In the case when $\mu_2 = \delta_{f_0}$ then

$$\Pi(\mu_1, \delta_{f_0}) = \{\mu_1 \otimes \delta_{f_0}\}$$

has only one element, and therefore

$$\begin{aligned} \mathcal{W}_{d_{\mathcal{G}_3}}(\pi_P^N f_0^N, \delta_{f_0}) &= \inf_{\pi \in \Pi(\pi_P^N f_0^N, \delta_{f_0})} \int_{\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}} d_{\mathcal{G}_3}(\rho_1, \rho_2) \, d\pi(\rho_1, \rho_2) \\ &= \int_{E^N} d_{\mathcal{G}_3}(\mu_V^N, f_0) \, df_0^N(v), \end{aligned}$$

which explains the notation (3.2). We simply write in the tensorized case:

$$(3.3) \quad \mathcal{W}_{d_{\mathcal{G}_3}}^N(f_0) := \mathcal{W}_{d_{\mathcal{G}_3}}(\pi_P^N f_0^{\otimes N}, \delta_{f_0})$$

Comparisons of the \mathcal{W}_d^N functionals and estimates on the rate

$$\mathcal{W}_d^N(f) \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

depending on the choice of the distance d are discussed in Subsection 4.2.

3.3. Proof of the abstract theorem. For a given function

$$\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell},$$

we break up the term to be estimated into three parts:

$$\begin{aligned} & \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \otimes 1^{\otimes N-\ell} \right\rangle \right| \leq \\ & \leq \left| \left\langle S_t^N(f_0^N), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_0^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \quad (=:\mathcal{T}_1) \\ & + \left| \left\langle f_0^N, T_t^N(R_\varphi^\ell \circ \mu_V^N) \right\rangle - \left\langle f_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N \right\rangle \right| \quad (=:\mathcal{T}_2) \\ & + \left| \left\langle f_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N \right\rangle - \left\langle (S_t^{NL}(f_0))^{\otimes \ell}, \varphi \right\rangle \right| \quad (=:\mathcal{T}_3). \end{aligned}$$

We deal separately with each part step by step:

- \mathcal{T}_1 is controlled by a purely combinatorial arguments introduced in [35]. Roughly speaking it is the combinatorial price we have to pay when we use the injection π_E^N based on empirical measures.
- \mathcal{T}_2 is controlled thanks to the consistency estimate **(A3)** on the generators which are well defined thanks to assumption **(A2)** and Lemma 2.11, the differential stability assumption **(A4)** on the limit semigroup, the support constraint **(A1)-(i)** and the propagation of integral moment bounds **(A1)-(ii)**.
- \mathcal{T}_3 is controlled in terms of the chaoticity of the initial data thanks to the weak stability assumption **(A5)** on the limit semigroup and the support moment bounds at initial time **(A1)-(iii)**.

Step 1: Estimate of the first term \mathcal{T}_1 . Let us prove that for any $t \geq 0$ and any $N \geq 2\ell$ there holds

$$(3.4) \quad \mathcal{T}_1 := \left| \left\langle S_t^N(f_0^N), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_0^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Since $S_t^N(f_0^N)$ is a symmetric probability measure, estimate (3.4) is a direct consequence of the following lemma.

Lemma 3.3. *For any $\varphi \in C_b(E^\ell)$ we have*

$$(3.5) \quad \forall N \geq 2\ell, \quad \left| \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell} \right)_{\text{sym}} - \pi_N R_\varphi^\ell \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}$$

where for a function $\phi \in C_b(E^N)$, we define its symmetrized version ϕ_{sym} as:

$$\phi_{\text{sym}} = \frac{1}{|\mathfrak{S}^N|} \sum_{\sigma \in \mathfrak{S}^N} \phi_\sigma$$

where we recall that \mathfrak{S}^N is the set of N -permutations.

As a consequence for any symmetric measure we have

$$(3.6) \quad \forall f^N \in P_{\text{sym}}(E^N), \quad \left| \langle f^N, R_\varphi^\ell(\mu_V^N) \rangle - \langle f^N, \varphi \rangle \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Proof of Lemma 3.3. This lemma is a simple and classical combinatorial computation. We briefly sketch the proof for the sake of completeness.

For a given $\ell \leq N/2$ we introduce

$$A_{N,\ell} := \left\{ (i_1, \dots, i_\ell) \in [1, N]^\ell : \forall k \neq k' \in [1, \ell], i_k \neq i_{k'} \right\}$$

and

$$B_{N,\ell} := A_{N,\ell}^c = \left\{ (i_1, \dots, i_\ell) \in [1, N]^\ell \right\} \setminus A_{N,\ell}.$$

Since there are $N!/(N-\ell)!$ ways of choosing ℓ distinct indices among $\{1, \dots, N\}$, we get

$$\begin{aligned} \frac{|B_{N,\ell}|}{N^\ell} &= 1 - \frac{N!}{(N-\ell)! N^\ell} \\ &= 1 - \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{\ell-1}{N}\right) = 1 - \exp\left(\sum_{i=0}^{\ell-1} \ln\left(1 - \frac{i}{N}\right)\right) \\ &\leq 1 - \exp\left(-2 \sum_{i=0}^{\ell-1} \frac{i}{N}\right) \leq 2 \sum_{i=0}^{\ell-1} \frac{i}{N} \leq \frac{\ell^2}{N}, \end{aligned}$$

where we have used

$$\forall x \in [0, 1/2], \quad \ln(1-x) \geq -2x \quad \text{and} \quad \forall x \in \mathbb{R}, \quad e^{-x} \geq 1-x.$$

Then we compute

$$\begin{aligned} R_\varphi^\ell(\mu_V^N) &= \frac{1}{N^\ell} \sum_{i_1, \dots, i_\ell=1}^N \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell} \sum_{(i_1, \dots, i_\ell) \in A_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) + \frac{1}{N^\ell} \sum_{(i_1, \dots, i_\ell) \in B_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell} \frac{1}{(N-\ell)!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{\ell^2}{N} \|\varphi\|_{L^\infty}\right). \end{aligned}$$

We now use the same estimate

$$1 - \frac{N!}{(N-\ell)! N^\ell} \leq \frac{\ell^2}{N}$$

as above to get

$$\begin{aligned} R_\varphi^\ell(\mu_V^N) &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{2\ell^2}{N} \|\varphi\|_{L^\infty}\right) \\ &= \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell}\right)_{\text{sym}} + \mathcal{O}\left(\frac{2\ell^2}{N} \|\varphi\|_{L^\infty}\right) \end{aligned}$$

and the proof of (3.5) is complete.

Next for any $f^N \in P_{\text{sym}}(E^N)$ we have

$$\langle f^N, \varphi \rangle = \left\langle f^N, \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell}\right)_{\text{sym}} \right\rangle,$$

and (3.6) trivially follows from (3.5). \square

Step 2: Estimate of the second term \mathcal{T}_2 . Let us prove that for any $t \in [0, T)$ and any $N \geq 2\ell$ there holds

$$(3.7) \quad \begin{aligned} \mathcal{T}_2 &:= \left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N\right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell\right) \circ \mu_V^N \right\rangle \right| \\ &\leq C_{m_{\mathcal{G}_1}, T} C_T^\infty \varepsilon(N) \ell^2 \|\varphi\|_{\mathcal{F}_2^2 \otimes (L^\infty)^{\ell-2}}. \end{aligned}$$

Observing that the semigroup T_t^∞ and its generator G^∞ are well defined thanks to assumption **(A2)** and Lemma 2.11, we start with the following identity

$$T_t^N \pi_N - \pi_N T_t^\infty = - \int_0^t \frac{d}{ds} (T_{t-s}^N \pi_N T_s^\infty) ds = \int_0^t T_{t-s}^N [G^N \pi_N - \pi_N G^\infty] T_s^\infty ds.$$

We then use assumptions **(A1)-(i)**, **(A1)-(ii)** and **(A3)** and we get for any $t \in [0, T)$

$$(3.8) \quad \begin{aligned} &\left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N\right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell\right) \circ \mu_V^N \right\rangle \right| \\ &\leq \int_0^T \left| \left\langle M_{m_{\mathcal{G}_1}}^N S_{t-s}^N (f_0^N), \left(M_{m_{\mathcal{G}_1}}^N\right)^{-1} [G^N \pi_N - \pi_N G^\infty] \left(T_s^\infty R_\varphi^\ell\right) \right\rangle \right| ds \\ &\leq \left(\sup_{0 \leq t < T} \left\langle f_t^N, M_{m_{\mathcal{G}_1}}^N \right\rangle \right) \times \\ &\quad \left(\int_0^T \left\| \left(M_{m_{\mathcal{G}_1}}^N\right)^{-1} [G^N \pi_N - \pi_N G^\infty] \left(T_s^\infty R_\varphi^\ell\right) \right\|_{L^\infty(\mathbb{E}_N)} ds \right) \\ &\leq C_{m_{\mathcal{G}_1}, T} \varepsilon(N) \sup_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} \int_0^T \left[T_s^\infty R_\varphi^\ell \right]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}})} ds. \end{aligned}$$

Now, let us fix $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$. Since

$$T_t^\infty(R_\varphi^\ell) = R_\varphi^\ell \circ S_t^{NL} \quad \text{with } S_t^{NL} \in C_{\Lambda_2}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathcal{P}_{\mathcal{G}_2})$$

and $R_\varphi^\ell \in C^{1,1}(\mathcal{P}_{\mathcal{G}_2})$ because $\varphi \in \mathcal{F}_2^{\otimes \ell}$ (see subsection 2.8), we can apply Lemma 2.10 and use assumption **(A4)** to obtain

$$T_t^\infty(R_\varphi^\ell) \in C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}})$$

with

$$\left[T_s^\infty \left(R_\varphi^\ell\right) \right]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}})} \leq \left(\left[S_t^{NL} \right]_{C_{\Lambda_2}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}, \mathcal{P}_{\mathcal{G}_2})} + \left[S_t^{NL} \right]_{C_{\Lambda_2}^{0, (1+\eta)/2}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}, \mathcal{P}_{\mathcal{G}_2})}^2 \right) \left\| R_\varphi^\ell \right\|_{C^{1,1}(\mathcal{P}_{\mathcal{G}_2})}$$

and $\Lambda_2 = \Lambda_1^{1/2}$. We then deduce thanks to (2.12) and assumption **(A4)**:

$$(3.9) \quad \int_0^T [T_s^\infty(R_\varphi^\ell)]_{C_{\Lambda_1}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1,r})} ds \leq C_T^\infty \ell^2 \left(\|\varphi\|_{\mathcal{F}_2 \otimes (L^\infty)^{\ell-2}} + \|\varphi\|_{\mathcal{F}_2 \otimes (L^\infty)^{\ell-1}} \right).$$

Then we go back to the computation (3.8), and plugging (3.9) we deduce (3.7).

Step 3: Estimate of the third term \mathcal{T}_3 . Let us prove that for any $t \geq 0$ and $N \geq \ell$ we have

$$\begin{aligned} \mathcal{T}_3 &:= \left| \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle - \left\langle \left(S_t^{NL}(f_0) \right)^{\otimes \ell}, \varphi \right\rangle \right| \\ &\leq [R_\varphi]_{C^{0,1}} \Theta_{a^N, T} \left(\mathcal{W}_{1, \mathcal{P}_{\mathcal{G}_3}} \left(\pi_P^N f_0^N, \delta_{f_0} \right) \right) \end{aligned}$$

where $\Theta_{a,T}$ was introduced in assumption **(A5)**, and $a = a^N$ is defined by

$$a^N := \max \left\{ M_{\mathcal{G}_3}(f_0), C_{m_{\mathcal{G}_3,0}}^N \right\},$$

where $C_{m_{\mathcal{G}_3,0}}^N$ was introduced in assumption **(A1)-(iii)**.

Assumption **(A1)-(iii)** indeed implies that

$$\text{Supp } f_0^N \subset \mathcal{K} := \left\{ V \in \mathbb{R}^{dN} \text{ s. t. } M_{m_{\mathcal{G}_3}}^N(\mu_V^N) = \frac{1}{N} \sum_{i=1}^N m_{\mathcal{G}_3}(v_i) \leq C_{m_{\mathcal{G}_3,0}}^N \right\}.$$

Hence we are in position to apply **(A5)** for the functions f_0 and μ_V^N on the support of f_0^N since $M_{m_{\mathcal{G}_3}}(f_0)$ is bounded by assumption, and $M_{\mathcal{G}_3}(\mu_V^N)$ is bounded by $C_{m_{\mathcal{G}_3,0}}^N$ when restricting to $V \in \mathcal{K}$ thanks to the previous equation.

Let us also recall that $R_\varphi^\ell \in C^{0,1}(\mathcal{P}_{\mathcal{G}_3}, \mathbb{R})$ because $\varphi \in \mathcal{F}_3^{\otimes \ell}$.

We then write

$$\begin{aligned} \mathcal{T}_3 &= \left| \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(\mu_V^N) \right) \right\rangle - \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(f_0) \right) \right\rangle \right| \\ &= \left| \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(\mu_V^N) \right) - R_\varphi^\ell \left(S_t^{NL}(f_0) \right) \right\rangle \right| \\ &\leq [R_\varphi]_{C^{0,1}(\mathcal{P}_{\mathcal{G}_3})} \left\langle f_0^N, d_{\mathcal{G}_3} \left(S_t^{NL}(f_0), S_t^{NL}(\mu_V^N) \right) \right\rangle. \end{aligned}$$

We now apply **(A5)** to get

$$\forall t \in [0, T], \quad d_{\mathcal{G}_3} \left(S_t^{NL}(f_0), S_t^{NL}(\mu_V^N) \right) \leq \Theta_{a^N, T} \left(d_{\mathcal{G}_3} \left(f_0, \mu_V^N \right) \right)$$

on the support of f_0^N , and therefore

$$\mathcal{T}_3 \leq [R_\varphi]_{C^{0,1}(\mathcal{P}_{\mathcal{G}_3})} \left\langle f_0^N, \Theta_{a^N, T} \left(d_{\mathcal{G}_3} \left(f_0, \mu_V^N \right) \right) \right\rangle.$$

We then obtain from the concavity of the $\Theta_{a^N, T}$ function:

$$\mathcal{T}_3 \leq [R_\varphi]_{C^{0,1}(\mathcal{P}_{\mathcal{G}_3})} \Theta_{a^N, T} \left(\left\langle f_0^N, d_{\mathcal{G}_3} \left(f_0, \mu_V^N \right) \right\rangle \right)$$

which concludes the proof of this step.

The proof of Theorem 3.1 is complete by combining the previous steps.

4. THE N -PARTICLE APPROXIMATION AT INITIAL TIME

4.1. Comparison of distances on probabilities. In the following lemma we compare the different metrics and norms defined Subsection 2.5. Let us write

$$M_k(f, g) := \max \left\{ \langle f, \langle v \rangle^k \rangle; \langle g, \langle v \rangle^k \rangle \right\}.$$

Lemma 4.1. *Let $f, g \in P(\mathbb{R}^d)$ and $k \in (0, \infty)$, then the following estimates hold:*

(i) *For any $q \in (1, +\infty)$ and any $k \in [q - 1, \infty)$:*

$$(4.1) \quad W_1(f, g) \leq W_q(f, g) \leq 2^{\frac{k+1}{q}} M_{k+1}(f, g)^{\frac{q-1}{qk}} W_1(f, g)^{\frac{1}{q}(1-\frac{q-1}{k})}.$$

(ii) *For any $s \in (0, 1]$,*

$$(4.2) \quad |f - g|_s \leq 2^{(1-s)} W_s(f, g) \leq 2^{(1-s)} W_1(f, g)^s.$$

(iii) *For any $s \in (d/2, d/2 + 1)$,*

$$(4.3) \quad \|f - g\|_{\dot{H}^{-s}}^2 \leq \frac{8 |\mathbb{S}^{d-1}|}{(2s - d)} \left(\frac{(2s - d)}{4(d + 2 - 2s)} \right)^{s - \frac{d}{2}} |f - g|_1^{2s-d}.$$

(iv) *For any $s > 0$ and $k > 0$ we have*

$$(4.4) \quad [f - g]_1^* \leq C(d, s, k) M_{k+1}(f, g)^{\frac{d}{d+k(d+s)}} |f - g|_s^{\frac{k}{d+k(d+s)}}$$

for some constant $C(d, s, k) > 0$ depending on d, s and k .

(v) *For any*

$$s \in \left(\max \left\{ \frac{d}{2}; 1 \right\}, \frac{d}{2} + 1 \right)$$

and $k > 0$ we have

$$(4.5) \quad [f - g]_1^* \leq C(d, s, k) M_{k+1}(f, g)^{\frac{d}{d+2ks}} \|f - g\|_{\dot{H}^{-s}}^{\frac{2k}{d+2ks}}$$

for some constant $C(d, s, k) > 0$ depending on d, s and k .

Proof of Lemma 4.1. Let us consider each inequality one by one.

Point (i). The inequality (4.1) is well-known in optimal transport theory, we refer for instance to [72, 15].

Point (ii). Let us prove inequality (4.2). Let $\pi \in \Pi(f, g)$. We write

$$\begin{aligned} |\hat{f}(\xi) - \hat{g}(\xi)| &= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(e^{-i v \cdot \xi} - e^{-i w \cdot \xi} \right) d\pi(v, w) \right| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| e^{-i v \cdot \xi} - e^{-i w \cdot \xi} \right| d\pi(v, w) \\ &\leq 2^{(1-s)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - w|^s |\xi|^s d\pi(v, w), \end{aligned}$$

which yields the first inequality in (4.2) by taking the supremum in $\xi \in \mathbb{R}^d$ and the infimum in $\pi \in \Pi(f, g)$. The second inequality is then immediate from the concavity estimate

$$W_s(f, g) \leq (W_1(f, g))^s.$$

Point (iii). Let us prove the inequality (4.3). Consider $R > 0$ and the ball

$$B_R := \left\{ x \in \mathbb{R}^d; |x| \leq R \right\},$$

and write

$$\begin{aligned} \|f - g\|_{\dot{H}^{-s}}^2 &= \int_{B_R} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|^2}{|\xi|^{2s}} d\xi + \int_{B_R^c} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|^2}{|\xi|^{2s}} d\xi \\ &\leq |f - g|_1^2 \int_{B_R} \frac{d\xi}{|\xi|^{2(s-1)}} + 4 \int_{B_R^c} \frac{d\xi}{|\xi|^{2s}} \\ &\leq |\mathbb{S}^{d-1}| R^{d-2s} \left(\frac{R^2}{(d+2-2s)} |f - g|_1^2 + \frac{4}{(2s-d)} \right). \end{aligned}$$

We optimize this estimate by choosing

$$R = \left(\frac{4(d+2-2s)}{(2s-d)} \right)^{\frac{1}{2}} |f - g|_1^{-1}$$

which yields

$$\|f - g\|_{\dot{H}^{-s}}^2 \leq \frac{8 |\mathbb{S}^{d-1}|}{(2s-d)} \left(\frac{(2s-d)}{4(d+2-2s)} \right)^{s-\frac{d}{2}} |f - g|_1^{2s-d}$$

which concludes the proof of (4.3).

Point (iv). Let us now prove inequality (4.4). We introduce a truncation function

$$\chi_R(x) = \chi\left(\frac{x}{R}\right), \quad R > 0$$

where

$$\chi \in C_c^\infty(\mathbb{R}^d), \quad [\chi]_1 \leq 1, \quad \text{and} \quad 0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } B(0,1)$$

and a mollifier

$$\gamma_\varepsilon(x) = \varepsilon^{-d} \gamma\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0 \quad \text{with} \quad \gamma(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}.$$

In particular we have an explicit formula for the Fourier transform of this mollifier:

$$\hat{\gamma}_\varepsilon(\xi) = \hat{\gamma}(\varepsilon\xi) = e^{-\varepsilon^2 \frac{|\xi|^2}{2}}.$$

Fix $\varphi \in W^{1,\infty}(\mathbb{R}^d)$ such that $[\varphi]_1 \leq 1$, $\varphi(0) = 0$ and define

$$\varphi_R := \varphi \chi_R, \quad \varphi_{R,\varepsilon} = \varphi_R * \gamma_\varepsilon$$

and write

$$\begin{aligned} &\int_{\mathbb{R}^d} \varphi(df - dg) \\ &= \int_{\mathbb{R}^d} \varphi_{R,\varepsilon}(df - dg) + \int_{\mathbb{R}^d} (\varphi_R - \varphi_{R,\varepsilon})(df - dg) + \int_{\mathbb{R}^d} (\varphi - \varphi_R)(df - dg). \end{aligned}$$

For the last term, we have

$$(4.6) \quad \begin{aligned} \left| \int_{\mathbb{R}^d} (\varphi_R - \varphi)(df - dg) \right| &\leq \int_{\mathbb{R}^d} (1 - \chi_R) |\varphi| (df + dg) \\ &\leq \int_{B_R^c} [\varphi]_1 \frac{|x|^{k+1}}{R^k} (df + dg) \leq \frac{M_{k+1}(f, g)}{R^k}. \end{aligned}$$

Concerning the second term, we observe that

$$|\nabla \varphi_R(x)| \leq \chi\left(\frac{x}{R}\right) + |\varphi(x)| |\nabla(\chi_R)(x)| \leq \chi\left(\frac{x}{R}\right) + \frac{|x|}{R} \left| \nabla \chi\left(\frac{x}{R}\right) \right|,$$

so that

$$\forall q \in [1, \infty], \quad \|\nabla \varphi_R\|_{L^q} \leq C(q, d, \chi) R^{\frac{d}{q}},$$

for some constant depending on q , d and χ . Next, using that

$$\begin{aligned} \|\varphi_R - \varphi_{R,\varepsilon}\|_\infty &\leq \|\nabla\varphi_R\|_\infty \left(\int_{\mathbb{R}^d} \gamma_\varepsilon(x) |x| dx \right) \\ &= \varepsilon \|\nabla\varphi_R\|_\infty \left(\int_{\mathbb{R}^d} \gamma(x) |x| dx \right) \leq C(q, d, \chi) \varepsilon, \end{aligned}$$

we find

$$(4.7) \quad \left| \int_{\mathbb{R}^d} (\varphi_R - \varphi_{R,\varepsilon}) (df - dg) \right| \leq C(q, d, \chi) \varepsilon.$$

Concerning the first term, using Parseval's identity, we have (the "hat" denotes as usual the Fourier transform)

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \varphi_{R,\varepsilon} (df - dg) \right| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \hat{\varphi}_R \hat{\gamma}_\varepsilon \overline{(\hat{f} - \hat{g})} d\xi \right| \\ &\leq \frac{1}{(2\pi)^d} \|\nabla\varphi_R\|_{L^1} \left\| \frac{\hat{f} - \hat{g}}{|\xi|^s} \right\|_{L^\infty} \left(\int_{\mathbb{R}^d} |\xi|^{s-1} e^{-\varepsilon^2 \frac{|\xi|^2}{2}} d\xi \right) \\ &\leq C(d, \chi) R^d |f - g|_s \varepsilon^{-(d+s-1)} \\ (4.8) \quad &\leq C R^d \varepsilon^{-(d+s-1)} |f - g|_s. \end{aligned}$$

Gathering (4.6), (4.7) and (4.8), we get

$$|f - g|_1^* \leq C(q, d, \chi) \left(\frac{M_{k+1}(f, g)}{R^k} + \varepsilon + R^d \varepsilon^{-(d+s-1)} |f - g|_s \right).$$

This yields (4.4) by optimizing the parameters ε and R with

$$R = M_{k+1}(f, g)^{\frac{1}{d+k}} |f - g|_s^{-\frac{1}{d+k}} \varepsilon^{\frac{d+s-1}{d+k}}$$

and then

$$\varepsilon = M_{k+1}(f, g)^{\frac{d}{d+k(d+s)}} |f - g|_s^{\frac{k}{d+k(d+s)}}.$$

Point (v). Let us now prove the inequality (4.5).

Let us consider some smooth φ such that $[\varphi]_1 \leq 1$, $\varphi(0) = 0$ and let us perform the same decomposition as before:

$$\int_{\mathbb{R}^d} \varphi (df - dg) = \int_{\mathbb{R}^d} \varphi_{R,\varepsilon} (df - dg) + \int_{\mathbb{R}^d} (\varphi_R - \varphi_{R,\varepsilon}) (df - dg) + \int_{\mathbb{R}^d} (\varphi - \varphi_R) (df - dg).$$

The first term is controlled by

$$\left| \int_{\mathbb{R}^d} \varphi_{R,\varepsilon} (df - dg) \right| = \left| \int_{\mathbb{R}^d} \hat{\varphi}_{R,\varepsilon} |\xi|^s \frac{(\hat{f} - \hat{g})}{|\xi|^s} d\xi \right| \leq \|\varphi_{R,\varepsilon}\|_{\dot{H}^s} \|f - g\|_{\dot{H}^{-s}}$$

with

$$\begin{aligned} \|\varphi_{R,\varepsilon}\|_{\dot{H}^s} &= \left(\int_{\mathbb{R}^d} |\xi|^2 |\widehat{\varphi_R}|^2 |\xi|^{2(s-1)} |\hat{\gamma}_\varepsilon|^2 d\xi \right)^{1/2} \\ &\leq \|\nabla(\varphi_R)\|_{L^2} \|\xi|^{s-1} \hat{\gamma}_\varepsilon(\xi)\|_{L^\infty} \\ &\leq C(s) \|\nabla(\varphi_R)\|_{L^2} \varepsilon^{1-s} \leq C(d, s) R^{\frac{d}{2}} \varepsilon^{-(s-1)} \end{aligned}$$

with the same arguments as above.

The second term and the last term are controlled exactly as in (4.6) and (4.7), which yields

$$|f - g|_1^* \leq C \left(\varepsilon + \frac{M_{k+1}(f, g)}{R^k} + R^{\frac{d}{2}} \varepsilon^{-(s-1)} \|f - g\|_{\dot{H}^{-s}} \right).$$

We deduce (4.5) by optimizing the parameters ε and R with

$$R = M_{k+1}(f, g)^{\frac{2}{d+2k}} \|f - g\|_{\dot{H}^{-s}}^{-\frac{2}{d+2k}} \varepsilon^{\frac{2(s-1)}{d+2k}}$$

and then

$$\varepsilon = M_{k+1}(f, g)^{\frac{d}{d+2ks}} \|f - g\|_{\dot{H}^{-s}}^{\frac{2k}{d+2ks}}.$$

□

4.2. Quantitative law of large numbers for measures. Let us recall and extend the definition of the functional $\mathcal{W}_d^N(f)$ which was introduced in (3.3). For any function

$$D : \mathcal{P}_k(E) \times \mathcal{P}_k(E) \rightarrow \mathbb{R}_+, \quad (f, g) \mapsto D(f, g)$$

(where $k \geq 0$ is the index of a polynomial weight possibly required for the correct definition of D) such that

$$D(f, g) = 0 \quad \text{if and only if} \quad f = g$$

it is legitimate to define

$$\forall f \in \mathcal{P}_k(\mathbb{R}^d), \quad \mathcal{W}_D^N(f) := \langle f^{\otimes N}, D(\pi_E^N, f) \rangle = \int_{E^N} D(\mu_V^N, f) \, df^{\otimes N}(v).$$

For well chosen function D , the goal of the next lemma is to quantify the rate of convergence

$$\mathcal{W}_D^N(f) \xrightarrow{N \rightarrow +\infty} 0 \quad \text{in the case } E = \mathbb{R}^d.$$

Lemma 4.2. *We have the following rates for the \mathcal{W} function:*

(i) *Let us consider*

$$\forall f, g \in \mathcal{P}_2(\mathbb{R}^d), \quad D_1(f, g) := \|f - g\|_{\dot{H}^{-s}}^2.$$

Then for any $s \in (d/2, d/2 + 1)$ and $N \geq 1$ there holds

$$(4.9) \quad \forall f \in \mathcal{P}_2(\mathbb{R}^d), \quad \mathcal{W}_{D_1}^N(f) = \int_{\mathbb{R}^{dN}} \|\mu_V^N - f\|_{\dot{H}^{-s}}^2 \, df^{\otimes N}(V) \leq \frac{C}{N}$$

for some constant C depending on the second moment of f .

(ii) *Let us consider*

$$\forall f, g \in \mathcal{P}_2(\mathbb{R}^d), \quad D_2(f, g) := \|f - g\|_{H^{-s}}^2.$$

Then for any $s > d/2$ and $N \geq 1$ there holds

$$(4.10) \quad \forall f \in \mathcal{P}_2(\mathbb{R}^d), \quad \mathcal{W}_{D_2}^N(f) = \int_{\mathbb{R}^{dN}} \|\mu_V^N - f\|_{H^{-s}}^2 \, df^{\otimes N}(V) \leq \frac{C}{N}$$

for some constant C depending on the second moment of f .

(iii) *Let us consider*

$$\forall f, g \in \mathcal{P}_1(\mathbb{R}^d), \quad D_3(f, g) := W_1(f, g).$$

Then for any $\eta > 0$ there exists $k \geq 1$ such that for any $N \geq 1$

$$(4.11) \quad \forall f \in \mathcal{P}_k(\mathbb{R}^d), \quad \mathcal{W}_{D_3}^N(f) = \int_{\mathbb{R}^{dN}} W_1(\mu_V^N, f) \, df^{\otimes N}(V) \leq \frac{C}{N^{\frac{1}{\max\{d, 2\} + \eta}}}$$

for some constant C depending on η and the k -th moment of f .

(iv) *Let us consider*

$$\forall f, g \in \mathcal{P}_2(\mathbb{R}^d), \quad D_4(f, g) := (W_2(f, g))^2.$$

Then for any $\eta > 0$ there exists $k \geq 2$ such that for any $N \geq 1$

$$(4.12) \quad \forall f \in \mathcal{P}_k(\mathbb{R}^d), \quad \mathcal{W}_{D_4}^N(f) = \int_{\mathbb{R}^{dN}} (W_2(\mu_V^N, f))^2 \, df^{\otimes N}(V) \leq \frac{C}{N^{\frac{1}{\max\{d, 2\} + \eta}}}$$

for some constant C depending on η and the k -th moment of f .

Remarks 4.3. (1) Estimate (4.12) has to be compared with the following classical estimate (see e.g. [65]): for any $N \geq 1$ there holds

$$(4.13) \quad \forall f \in \mathcal{P}_{d+5}(\mathbb{R}^d), \quad \mathcal{W}_{W_2^N}^N(f) \leq \frac{C}{N^{\frac{1}{d+4}}}$$

where the constant $C > 0$ depends on the $(d+5)$ -th moment of f . It is worth mentioning that our estimate (4.12) improves on (4.13) when $d \leq 3$ and k is large enough.

Similarly, if one tries to translate (4.13) into an estimate for the W_1 distance through a Hölder inequality, it yields for any $N \geq 1$

$$\forall f \in \mathcal{P}_{d+5}(\mathbb{R}^d), \quad \mathcal{W}_{W_1^N}^N(f) \leq \frac{C}{N^{\frac{1}{d+4}}}$$

where the constant $C > 0$ depends on the $(d+5)$ -th moment of f . Again observe that for any d this last estimate is weaker than (4.11) as soon as the probability f belongs to $\mathcal{P}_k(\mathbb{R}^d)$ with k large enough.

- (2) When $f, g \in P(\mathbb{R}^d)$ are compactly supported, observe that the estimate (4.5) improves into

$$\forall s \geq 1, \quad [f - g]_1^* \leq C \|f - g\|_{H^{-s}}^{1/s},$$

for a constant C depending on s and on a common bound R of the support of f and g .

If furthermore $d = 1$, we can take $s = 1$ in order to apply (4.9) in the proof of (4.11) below and we obtain the “*optimal rate*” of convergence in the functional law of large numbers in Wasserstein distance W_1 :

$$\forall N \geq 1, \quad \mathcal{W}_{W_1^N}^N(f) \leq \frac{C}{\sqrt{N}}.$$

In higher dimension $d \geq 2$, the restriction $s > d/2$ means that one does not produce a better estimate than (4.11) by this line of argument.

- (3) As was kindly pointed out by M. Hauray, estimate (4.12) should also be compared with some estimates in [22] where the related quantity

$$\mathcal{Z}^N(f) := \int_{\mathbb{R}^{2dN}} W_1(\mu_{V_1}^N, \mu_{V_2}^N) \, df^{\otimes N}(V_1) \, df^{\otimes N}(V_2)$$

is considered. When $f \in P(\mathbb{R}^d)$ has compact support and $d \geq 3$, they prove that

$$\mathcal{Z}^N(f) \leq \frac{C}{N^{1/d}}$$

where the constant depends on the support of f .

Since for any $f, g \in \mathcal{P}_1(\mathbb{R}^d)$ and for any $\varphi \in \text{Lip}_1(\mathbb{R}^d)$ we have

$$\begin{aligned} \int_{\mathbb{R}^{dN}} W_1(g, \mu_V^N) \, df^{\otimes N}(V) &\geq \int_{\mathbb{R}^{d(N+1)}} \varphi(v) \, (dg - d\mu_V^N)(v) \, df^{\otimes N}(V) \\ &= \int_{\mathbb{R}^d} \varphi(v) \, dg(v) - \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \varphi(v_i) \, df^{\otimes N}(V) \\ &= \int_{\mathbb{R}^d} \varphi(v) \, (dg - df)(v), \end{aligned}$$

we deduce by minimizing in φ that

$$W_1(f, g) \leq \int_{\mathbb{R}^{dN}} W_1(g, \mu_V^N) \, df^{\otimes N}(V),$$

and therefore

$$\mathcal{W}_{W_1^N}^N(f) \leq \mathcal{Z}^N(f).$$

As a consequence, when $f \in P(\mathbb{R}^d)$ has compact support and $d \geq 3$ we obtain from this line of argument the stronger estimate

$$\mathcal{W}_{W_1}^N(f) \leq \frac{C}{N^{1/d}}.$$

It is likely that one could obtain similar estimates to (4.11) by tracking the formula for the constants in the results of [22] and combining them with moment bounds and some interpolation.

On the other hand, observe that our estimate (4.11) is *almost optimal* in the sense that we can not expect a better convergence rate than (4.11) with $\eta = 0$, as it is stressed in [64, Appendix].

Proof of Lemma 4.2. We split the proof into two steps.

Step 1. Let us prove (4.9) (note that (4.10) is then readily implied by (4.9)).

Let us fix $f \in \mathcal{P}_2(\mathbb{R}^d)$. We write in Fourier transform

$$\left(\hat{\mu}_V^N - \hat{f}\right)(\xi) = \frac{1}{N} \sum_{j=1}^N \left(e^{-i v_j \cdot \xi} - \hat{f}(\xi)\right),$$

which implies

$$\begin{aligned} \mathcal{W}_{\|\cdot\|_{\dot{H}^{-s}}}^N(f) &= \int_{\mathbb{R}^{Nd}} \left(\int_{\mathbb{R}^d} \frac{|\hat{\mu}_V^N - \hat{f}|^2}{|\xi|^{2s}} d\xi \right) df^{\otimes N}(V) \\ &= \frac{1}{N^2} \sum_{j_1, j_2=1}^N \int_{\mathbb{R}^{(N+1)d}} \frac{\left(e^{-i v_{j_1} \cdot \xi} - \hat{f}(\xi)\right) \overline{\left(e^{-i v_{j_2} \cdot \xi} - \hat{f}(\xi)\right)}}{|\xi|^{2s}} d\xi df^{\otimes N}(V). \end{aligned}$$

Observe then that

$$\int_{\mathbb{R}^d} \left(e^{-i v_j \cdot \xi} - \hat{f}(\xi)\right) df(v_j) = 0, \quad j = 1, \dots, d,$$

which implies that

$$\int_{\mathbb{R}^{(N+1)d}} \frac{\left(e^{-i v_{j_1} \cdot \xi} - \hat{f}(\xi)\right) \overline{\left(e^{-i v_{j_2} \cdot \xi} - \hat{f}(\xi)\right)}}{|\xi|^{2s}} d\xi df^{\otimes N}(V) = 0$$

as soon as $j_1 \neq j_2$, and

$$\begin{aligned} \int_{\mathbb{R}^d} \left|e^{-i v \cdot \xi} - \hat{f}(\xi)\right|^2 df(v) &= \int_{\mathbb{R}^d} \left[1 - e^{-i v \cdot \xi} \overline{\hat{f}(\xi)} - e^{i v \cdot \xi} \hat{f}(\xi) + |\hat{f}(\xi)|^2\right] df(v) \\ &= 1 - |\hat{f}(\xi)|^2. \end{aligned}$$

We deduce that

$$\begin{aligned} \mathcal{W}_{\|\cdot\|_{\dot{H}^{-s}}}^N(f) &= \frac{1}{N^2} \sum_{j=1}^N \int_{\mathbb{R}^{(N+1)d}} \frac{\left|e^{-i v_j \cdot \xi} - \hat{f}(\xi)\right|^2}{|\xi|^{2s}} d\xi df^{\otimes N}(V) \\ &= \frac{1}{N} \int_{\mathbb{R}^{2d}} \frac{\left|e^{-i v \cdot \xi} - \hat{f}(\xi)\right|^2}{|\xi|^{2s}} d\xi df(v) \\ &= \frac{1}{N} \int_{\mathbb{R}^d} \frac{(1 - |\hat{f}(\xi)|^2)}{|\xi|^{2s}} d\xi. \end{aligned}$$

Finally, denoting

$$M_2 := \int_{\mathbb{R}^d} \langle v \rangle^2 df(v)$$

we have

$$\hat{f}(\xi) = 1 + i \langle f, v \rangle \cdot \xi + \mathcal{O}(M_2 |\xi|^2),$$

and therefore

$$\begin{aligned} |\hat{f}(\xi)|^2 &= (1 + i \langle f, v \rangle \cdot \xi + \mathcal{O}(M_2 |\xi|^2)) \left(1 - i \langle f, v \rangle \cdot \xi + \overline{\mathcal{O}(M_2 |\xi|^2)}\right) \\ &= 1 + \mathcal{O}(M_2 |\xi|^2), \end{aligned}$$

which implies

$$\begin{aligned} \mathcal{W}_{\|\cdot\|_{\dot{H}^{-s}}^2}^N(f) &= \frac{1}{N} \left(\int_{|\xi| \leq 1} \frac{(1 - |\hat{f}(\xi)|^2)}{|\xi|^{2s}} d\xi + \int_{|\xi| \geq 1} \frac{(1 - |\hat{f}(\xi)|^2)}{|\xi|^{2s}} d\xi \right) \\ &= \frac{1}{N} \left(\int_{|\xi| \leq 1} \frac{M_2}{|\xi|^{2(s-1)}} d\xi + \int_{|\xi| \geq 1} \frac{1}{|\xi|^{2s}} d\xi \right) \leq \frac{C}{N} \end{aligned}$$

from which (4.9) follows.

Step 2. Let us now prove (4.11).

We use first (4.5) in order to get

$$\begin{aligned} \mathcal{W}_{W_1}^N(f) &= \int_{\mathbb{R}^{dN}} [\mu_V^N - f]_1^* df^{\otimes N}(V) \\ &\leq C \int_{\mathbb{R}^{dN}} (M_{k+1}(f) + M_{k+1}(\mu_V^N))^{\frac{d}{d+2ks}} \left(\|\mu_V^N - f\|_{\dot{H}^{-s}}^2 \right)^{\frac{k}{d+2ks}} df^{\otimes N}(V). \end{aligned}$$

We then perform a Hölder inequality with exponents

$$p = \frac{d+2ks}{k}, \quad p' = \frac{d+2ks}{d+k(2s-1)}$$

and get

$$\begin{aligned} \mathcal{W}_{W_1}^N(f) &\leq C \left(\int_{\mathbb{R}^{dN}} (M_{k+1}(f) + M_{k+1}(\mu_V^N))^{\frac{d}{d+k(2s-1)}} df^{\otimes N}(V) \right)^{\frac{d+k(2s-1)}{d+2ks}} \times \\ &\quad \left(\int_{\mathbb{R}^{dN}} \|\mu_V^N - f\|_{\dot{H}^{-s}}^2 df^{\otimes N}(V) \right)^{\frac{k}{d+2ks}}. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\mathbb{R}^{dN}} (M_{k+1}(f) + M_{k+1}(\mu_V^N))^{\frac{d}{d+k(2s-1)}} df^{\otimes N}(V) \\ &\leq \int_{\mathbb{R}^{dN}} (M_{k+1}(f) + M_{k+1}(\mu_V^N)) df^{\otimes N}(V) \\ &\leq M_{k+1}(f) + \int_{\mathbb{R}^{dN}} M_{k+1}(\mu_V^N) df^{\otimes N}(V) \\ &\leq M_{k+1}(f) + \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle v_i \rangle^{k+1} df^{\otimes N}(V) \leq 2 M_{k+1}(f) \end{aligned}$$

we deduce by using (4.9) that

$$\mathcal{W}_{W_1}^N(f) \leq \frac{C(f, k)}{N^{\frac{k}{d+2ks}}}$$

where the constant $C(f, k)$ depends on the $(k+1)$ -th moment of f .

This allows to conclude the proof of (4.11) since

- if $d = 1$ we can take $s = 1$ in (4.5) and then k large enough so that $k/(d+2ks) = 2 + \eta$ with some $\eta > 0$ as small as wanted,

- if $d \geq 2$ we take s close to $d/2$ and then k large enough so that $k/(d+2ks) = 1/d+\eta$ with some $\eta > 0$ as small as wanted.

Then the estimate (4.12) follows from (4.11) with the help of (4.1) and a Hölder inequality. \square

4.3. Chaotic initial data with prescribed energy and momentum. In many aspects, the simplest N -particle initial data is the sequence of tensorized initial data $f^{\otimes N}$, $N \geq 1$, where f is a 1-particle distribution. This means perfect chaoticity. On the other hand it has a drawback: since in all applications we shall use pointwise bounds on the energy of the N -particle system (and also sometimes pointwise higher moment bound as in **(A1)-(iii)**), this implies for this kind of initial data that f has to be compactly supported. There is another “natural” choice of initial data, by restricting it to one of the subspaces left invariant by the dynamics as defined in (1.7). Without loss of generality we shall always set $\mathcal{M} = 0$ in this formula in the sequel, and therefore we shall consider

$$(4.14) \quad \mathcal{S}^N(\mathcal{E}) := \left\{ V \in \mathbb{R}^{dN} \text{ s. t. } \frac{1}{N} \sum_{i=1}^N |v_i|^2 = \mathcal{E}, \quad \frac{1}{N} \sum_{i=1}^N v_i = 0 \right\}.$$

The drawback is now that an initial data on $\mathcal{S}^N(\mathcal{E})$ *cannot* be perfectly tensorized, and some additional chaoticity error is paid at initial time. However an advantage of this viewpoint is that it is simpler to study the asymptotic behavior of both the N -particle and the limit mean-field equation in this setting. Moreover it has historical value since this approach was introduced by Kac (see the discussion in [42, Section 5 “Distributions having Boltzmann’s property”]), although in his case there was only one conservation law, namely the energy, and therefore $\mathcal{S}^N(\mathcal{E})$ was replaced by $\mathbb{S}^{N-1}(\sqrt{N})$. Finally in the case of hard spheres it is easy to check that the relaxation rate degenerates as $\mathcal{E} \rightarrow 0$ both for the N -particle system and the limit equation, but uniform in time chaoticity can be achieved by avoiding the zero energy distributions thanks to the restriction to $\mathcal{S}^N(\mathcal{E})$ with $\mathcal{E} > 0$.

We shall present some results on the construction of chaotic initial data on $\mathcal{S}^N(\mathcal{E})$, whose proofs are mostly extensions of the precise statements and estimates recently established in [10] on this issue in the setting of Kac on $\mathbb{S}^{N-1}(\sqrt{N})$. We refer to the work [14] where an extensive study and precise computations of rates are performed. Without loss of generality we only consider the case $\mathcal{M} = 0$ for simplicity.

Lemma 4.4. *Consider $\mathcal{E} > 0$ and an initial data*

$$f_0 \in \mathcal{P}_4(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

which fulfills some moment conditions

$$M_{m_{\mathcal{G}_1}}(f) = \langle f, m_{\mathcal{G}_1} \rangle < +\infty, \quad M_{m_{\mathcal{G}_3}}(f) = \langle f, m_{\mathcal{G}_3} \rangle < +\infty$$

for some positive radially symmetric increasing weight functions $m_{\mathcal{G}_1}$ and $m_{\mathcal{G}_3}$, and

$$\int_{\mathbb{R}^d} v f_0(v) dv = 0, \quad \int_{\mathbb{R}^d} |v|^2 f_0(v) dv = \mathcal{E}.$$

Let us define a non-decreasing sequence $(\alpha_N)_{N \geq 1}$ as follows:

- *If f_0 has compact support*

$$(4.15) \quad \text{Supp} f_0 \subset \left\{ v \in \mathbb{R}^d, |v| \leq A \right\}$$

for some $A > 0$, then

$$\forall N \geq 1, \quad \alpha_N := m_{\mathcal{G}_3}(A).$$

- If f_0 has non-compact support, then $(\alpha_N)_{N \geq 1}$ can be any non-decreasing sequence such that

$$\lim_{N \rightarrow \infty} \alpha_N = +\infty$$

(note in particular that this sequence can grow as slowly as wanted).

Then there exists

$$f_0^N \in P(\mathbb{R}^{dN}), \quad N \geq 1,$$

such that

- (i) The sequence $(f_0^N)_{N \geq 1}$ is f_0 -chaotic.
- (ii) Its support satisfies $\text{Supp } f_0^N \subset \mathcal{S}^N(\mathcal{E})$.
- (iii) It satisfies the following integral moment bound based on m_1 :

$$\forall N \geq 0, \quad \langle f_0^N, M_{m_{\mathcal{G}_1}}^N \rangle \leq C \langle f_0, m_{\mathcal{G}_1} \rangle$$

where the constant $C > 0$ depends on M_{0, m_1}^{NL} .

- (iv) It satisfies the following “support moment bound”:

$$\text{Supp } f_0^N \subset \left\{ V \in \mathbb{R}^{dN}; M_{m_{\mathcal{G}_3}}^N(V) \leq \alpha_N \right\}.$$

- (v) It satisfies a uniform relative entropy bound

$$\frac{H(f_0^N | \gamma^N)}{N} \leq C,$$

for some constant $C > 0$ (see (1.9) for notations).

- (vi) If furthermore the Fisher information associated to f_0 is bounded, that is $I(f_0) < \infty$ (see (1.10) for notations), then f_0^N can be built in such a way that it satisfies a uniform relative Fisher information bound

$$I(f_0^N | \gamma^N) := \frac{1}{N} \int_{\mathcal{S}^N(\mathcal{E})} \left| \nabla \ln \frac{df_0^N}{d\gamma^N} \right|^2 df_0^N \leq C,$$

for some constant $C > 0$, where the gradient in this formula stands for the Riemannian gradient on the manifold $\mathcal{S}^N(\mathcal{E})$.

Proof of Lemma 4.4. We aim at defining our initial data f_0^N by conditioning the tensorized initial data $f_0^{\otimes N}$ to $\mathcal{S}^N(\mathcal{E})$:

$$f_0^N(V) = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} := \left(\frac{\prod_{j=1}^N f_0(v_j)}{F_N(\sqrt{N})} \right) \Big|_{\mathcal{S}^N(\mathcal{E})}$$

with

$$F^N(r) := \int_{\mathbb{S}^{dN-1}(r) \cap \{\sum_{i=1}^N v_i = 0\}} \left(\prod_{j=1}^N f_0(v_j) \right) d\omega.$$

Such a construction obviously satisfies (ii).

It is proved similarly as in [10] (see for instance Theorem 9 in this reference) that this conditioned measure is well-defined, and that it is f_0 -chaotic, which proves (i).

Remark 4.5. Among many interesting intermediate steps and other results, it is also proved in [10] the following estimate: assume for simplicity that $d = 1$ and that f_0 has energy 1, then the function

$$\bar{F}^N(r) := \frac{F^N(r)}{\gamma^N(r)}$$

is asymptotically divergent except for $r = \sqrt{N}$, for which

$$\bar{F}^N(\sqrt{N}) \sim_{N \rightarrow +\infty} \frac{\sqrt{2}}{\Sigma}$$

with

$$\Sigma = \sqrt{\int_{\mathbb{R}} (v^2 - 1)^2 df(v)}.$$

(In fact this result was sketched by Kac [42] but the proof is made more precise in [10]).

This shows in particular that the sequence of chaotic initial data $f_0^{\otimes N}$, $N \geq 1$, as considered many times in the sequel, asymptotically concentrates on the Boltzmann spheres $\mathcal{S}^N(\mathcal{E})$. This manifestation of the central limit theorem explains why the construction of Kac (to condition to a given energy sphere) is very natural. It also enlightens why it is possible to expect the kind of uniform in time propagation of chaos results that we shall prove in the next sections for such chaotic initial data.

Point (iii) is just a consequence of the chaoticity with the test function $M_{m_{\mathcal{G}_1}}^N$ (actually an easy truncation and passage to the limit procedure is needed in full rigor).

Concerning point (iv), first if f_0 is compactly supported (4.15) we deduce that

$$\text{Supp } f_0^N \subset \left\{ V \in \mathbb{R}^{dN}, M_{m_{\mathcal{G}_3}}^N(V) \leq m_{\mathcal{G}_3}(A) \right\}$$

and (iv) holds.

In the non compactly supported case, for any increasing sequence $(A_k)_{k \geq 1}$ of positive reals (with A_0 big enough for the following to be well-defined) we define

$$f_{0,k} := \frac{f_0 \mathbf{1}_{|v| \leq A_k}}{f_0(\{|v| \leq A_k\})}.$$

Using the previous we know that

$$f_{0,k}^N := \left[f_{0,k}^{\otimes N} \right]_{\mathcal{S}^N(\mathcal{E})}$$

(conditioning to $\mathcal{S}^N(\mathcal{E})$) is $f_{0,k}$ -chaotic. Conditions (ii) and (iii) will therefore be immediately satisfied.

We now want to choose a sequence $k_N \rightarrow \infty$ such that (iv) is satisfied and at the same proving chaoticity *towards* f_0 . It is clear that

$$\text{Supp } f_{0,k}^N \subset \left\{ V \in \mathbb{R}^{dN}; M_{m_3}^N(V) \leq m_3(A_k) \right\}.$$

For any given sequence (α_N) which tends to infinity, we define k_N in such a way that $m_3(A_{k_N}) \leq \alpha_N$ so that $k_N \rightarrow \infty$ when $N \rightarrow \infty$. The chaos property is equivalent to the weak convergence of the 2-marginal, which can be expressed in Wasserstein distance for instance:

$$W_1 \left((f_{0,k}^N)_2, f_0^{\otimes 2} \right) \leq W_1 \left((f_{0,k}^N)_2, f_{0,k}^{\otimes 2} \right) + W_1 \left(f_{0,k}^{\otimes 2}, f_0^{\otimes 2} \right).$$

The last term of the RHS converges to zero only depending on $k \rightarrow 0$, while the first term in the RHS converges to zero for fixed k as $N \rightarrow 0$ from the previous part of the proof. Therefore, maybe at the price of a slower increasing sequence k_N we can have both the support moment condition (iv) and

$$W_1 \left((f_{0,k_N}^N)_2, f_0^{\otimes 2} \right) \xrightarrow{N \rightarrow 0} 0$$

which shows the chaoticity and concludes the proof.

For the proof of (v) and (vi) we refer to [14]. □

Remarks 4.6. (1) We note that if one only wants to get rid of the compact support requirement in f_0 (used for deriving the support bounds on f_0^N on the energy and $m_{\mathcal{G}_3}$), and not necessarily to prescribe a given energy, another strategy could have been to simply perform the cutoff in the end of the previous proof. In principle it could allow to get better information on the rate of convergence. However a drawback of this approach is that, in the absence of conditioning to an energy sphere, the bound on the support of the energy of f_0^N shall grow with N . In our applications it induces a growth in N of the moment bounds that we prove *along time* on the N -particle system. This growth should be matched by the decay of the scheme and a precise optimized balance could be searched for. We do not pursue this line of research.

(2) Observe that the process of conditioning on the energy sphere is obviously compatible with the equilibrium states in the following sense: if one denotes by γ a centered Gaussian equilibrium of the limit equation with energy \mathcal{E} , then

$$\gamma^N(V) := [\gamma^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})}$$

is the uniform measure on $\mathcal{S}^N(\mathcal{E})$, i.e. an equilibrium of the N -particle system.

Let us also state a refinement of the previous lemma which is needed for the applications.

Lemma 4.7. *We use the same setting and assumptions as in Lemma 4.4.*

Then the sequence (f_0^N) , $N \geq 1$ of Lemma 4.4 satisfies the more precise chaoticity estimate:

$$(4.16) \quad \mathcal{W}_{W_1}(\pi_P^N(f_0^N), f_0) = \int_{\mathbb{R}^{dN}} W_1(\mu_V^N, f_0) \, df_0^N(V) = \int_{\mathcal{S}^N(\mathcal{E})} W_1(\mu_V^N, f_0) \, df_0^N(V) \xrightarrow{N \rightarrow +\infty} 0$$

with explicit polynomial rate.

Remarks 4.8. (1) As an easy consequence, for any function $\Theta_a(x)$ such that

$$\forall a > 0, \quad \Theta_a(x) \xrightarrow{x \rightarrow +\infty} 0.$$

(where the parameter a should be thought as keeping track of dependency of this functional on moment estimates on the distributions), we have by a diagonal extraction process

$$(4.17) \quad \Theta_{a^N}(\mathcal{W}_{W_1}(\pi_P^N(f_0^N), f_0)) \xrightarrow{N \rightarrow +\infty} 0$$

for some explicit rate in terms of (4.16) and Θ , with $a^N = \max\{\alpha_N; \langle f_0, m_{\mathcal{G}_3} \rangle\}$.

(2) Using Lemma 4.1 it would be immediate to extend the previous statement to the other weak measure distances we have discussed so far.

Proof of Lemma 4.7. Let us give two proofs. The first one is non-constructive and inspired from [69]. First, thanks to the well-known result [69, Proposition 2.2] and the fact that the sequence f_0^N constructed in Lemma 4.4 is f_0 -chaotic, we deduce that

$$\pi_P^N f_0^N \rightharpoonup \delta_{f_0} \text{ in } P(P(\mathbb{R}^d))$$

(which means convergence when testing against functions in $C(P(\mathbb{R}^d))$).

Next, thanks to [76, Theorem 7.12], (4.16) boils down to prove the tightness estimate

$$(4.18) \quad \lim_{R \rightarrow \infty} \sup_{N \in \mathbb{N}^*} \int_{W_1(\rho, f_0) \geq R} W_1(\rho, f_0) \, d(\pi_P^N f_0^N)(\rho) = 0.$$

Let us prove that it easily follows from the following bound

$$(4.19) \quad \sup_{N \in \mathbb{N}^*} \int_{E^N} (W_1(\mu_V^N, f_0))^2 \, df_0^N(V) < \infty.$$

Indeed (4.19) implies that uniformly in $N \geq 1$

$$\begin{aligned} \int_{W_1(\rho, f_0) \geq R} W_1(\rho, f_0) d(\pi_P^N f_0^N)(\rho) &= \int_{\{V \in E^N \text{ s.t. } W_1(\mu_V^N, f_0) \geq R\}} W_1(\mu_V^N, f_0) df_0^N(V) \\ &\leq \frac{1}{R} \int_{E^N} (W_1(\mu_V^N, f_0))^2 df_0^N(V) \leq \frac{C}{R} \xrightarrow{R \rightarrow \infty} 0 \end{aligned}$$

which concludes the proof of (4.18).

In order to show (4.19), we infer that from [76, Theorem 7.10]

$$\begin{aligned} (W_1(\mu_V^N, f_0))^2 &\leq \|\mu_V^N - f_0\|_{M_1^1}^2 \leq 2\|\mu_V^N\|_{M_1^1}^2 + 2\|f_0\|_{M_1^1}^2 \\ &\leq 2(M_1^N(V))^2 + 2\|f_0\|_{M_1^1}^2 \leq 2M_2^N(V) + 2\|f_0\|_{M_1^1}^2, \end{aligned}$$

which implies

$$\int_{E^N} (W_1(\mu_V^N, f_0))^2 df_0^N(V) \leq 2\|f_0\|_{M_1^1}^2 + 2\langle f_0^N, M_2^N \rangle,$$

which, together with (ii) in Lemma 4.4, ends the proof of (4.19) and then of (4.16).

Let us now give an alternative explicit argument, even if less self-contained. From [14, Theorem 3, (i)] we deduce that

$$\forall \ell \geq 1, \quad W_1(\Pi_\ell f_0^N, f_0^{\otimes \ell}) \leq \frac{C}{\sqrt{N}}$$

for some explicit constant $C > 0$ uniform in ℓ and N . Then we use the [36, Theorem 2.4] to deduce that

$$\int_{E^N} (W_1(\mu_V^N, f_0))^2 df_0^N(V) \leq C \left(W_1(\Pi_2 f_0^N, f_0^{\otimes 2}) + \frac{1}{N} \right)^{C'}$$

for some explicit constants $C, C' > 0$ depending on the energy bound on f_0 , which concludes the proof. \square

5. TRUE MAXWELL MOLECULES

5.1. The model. Let us consider $E = \mathbb{R}^d$, $d \geq 2$, and a N -particle system undergoing space homogeneous random Boltzmann type collisions according to a collision kernel

$$B(z, \cos \theta) = \Gamma(z) b(\cos \theta)$$

(see Subsection 1.1). More precisely, given a pre-collisional system of velocity variables

$$V = (v_1, \dots, v_N) \in E^N = (\mathbb{R}^d)^N,$$

the stochastic process is:

- (i) for any $i' \neq j'$, pick a random time $T_{\Gamma(|v_{i'} - v_{j'}|)}$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$, and then choose the collision time T_1 and the colliding pair (v_i, v_j) (which is a.s. well-defined) in such a way that

$$T_1 = T_{\Gamma(|v_i - v_j|)} := \min_{1 \leq i' \neq j' \leq N} T_{\Gamma(|v_{i'} - v_{j'}|)};$$

- (ii) then pick a random unit vector $\sigma \in S^{d-1}$ according to the law $b(\cos \theta_{ij})$, where

$$\cos \theta_{ij} = \sigma \cdot (v_j - v_i) / |v_j - v_i|;$$

- (iii) the new state after collision at time T_1 becomes

$$(5.1) \quad V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N),$$

where only velocities labelled i and j have changed, according to the rotation

$$(5.2) \quad v_i^* = \frac{v_i + v_j}{2} + \frac{|v_i - v_j| \sigma}{2}, \quad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_i - v_j| \sigma}{2}.$$

The associated Markov process

$$(\mathcal{V}_t^N)_{t \geq 0} \text{ on } (\mathbb{R}^d)^N$$

is then built by iterating the above construction.

After rescaling time $t \rightarrow t/N$ in order that the number of interactions is of order $\mathcal{O}(1)$ on finite time interval (see [67]) we denote by f_t^N the law of \mathcal{V}_t^N and S_t^N the associated semigroup. We recall that G^N and T_t^N respectively denotes the dual generator and dual semigroup, as in the previous abstract construction.

The so-called *Master equation* on the law f_t^N is given in dual form by

$$(5.3) \quad \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

with

$$(5.4) \quad (G^N \varphi)(V) = \frac{1}{N} \sum_{1 \leq i < j \leq N} \Gamma(|v_i - v_j|) \int_{S^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] d\sigma$$

$$\text{where } \varphi_{ij}^* = \varphi(V_{ij}^*) \text{ and } \varphi = \varphi(V) \in C_b(\mathbb{R}^{Nd}).$$

This collision process is invariant under velocities permutations and satisfies the microscopic conservations of momentum and energy at any collision time

$$\sum_{j=1}^N v_j^* = \sum_{j=1}^N v_j \quad \text{and} \quad |V^*|^2 = \sum_{j=1}^N |v_j^*|^2 = \sum_{j=1}^N |v_j|^2 = |V|^2.$$

As a consequence, for any symmetric initial law $f_0^N \in P_{\text{sym}}((\mathbb{R}^d)^N)$ the law f_t^N at later times is also a symmetric probability. Moreover the evolution conserves momentum and energy: for any measurable function $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$

$$\forall \alpha = 1, \dots, d, \quad \int_{\mathbb{R}^{dN}} \phi \left(\sum_{j=1}^N v_{j,\alpha} \right) df_t^N(v) = \int_{\mathbb{R}^{dN}} \phi \left(\sum_{j=1}^N v_{j,\alpha} \right) df_0^N(v),$$

where $(v_{j,\alpha})_{1 \leq \alpha \leq d}$ denote the components of $v_j \in \mathbb{R}^d$, and

$$(5.5) \quad \int_{\mathbb{R}^{dN}} \phi(|V|^2) df_t^N(v) = \int_{\mathbb{R}^{dN}} \phi(|V|^2) df_0^N(v),$$

(equalities between possibly infinite non-negative quantities).

The (expected) limit nonlinear homogeneous Boltzmann equation is defined by (1.1), (1.2), (1.3). The equation generates a nonlinear semigroup

$$S_t^{NL}(f_0) := f_t \text{ for any } f_0 \in \mathcal{P}_2(\mathbb{R}^d)$$

where $\mathcal{P}_2(\mathbb{R}^d)$ denotes the space of probabilities with bounded second moment.

Concerning the Cauchy theory for the limit Boltzmann equation:

- In the case (**GMM**) (Maxwell molecules with angular cutoff), see equation (1.6) in Subsection 1.1, we refer to [70];
- In the case (**tMM**) (true Maxwell molecules without angular cutoff), see equation (1.5) in Subsection 1.1, we refer to [72];
- In the case (**HS**) of hard spheres, see equation (1.4) in Subsection 1.1, we refer to [59] ($L^1(\mathbb{R}^d)$ theory) and [28, 31, 48] ($P(\mathbb{R}^d)$ theory).

For these solutions, one has the conservation of momentum and energy

$$\forall t \geq 0, \quad \int_{\mathbb{R}^d} v \, df_t(v) = \int_{\mathbb{R}^d} v \, df_0(v), \quad \int_{\mathbb{R}^d} |v|^2 \, df_t(v) = \int_{\mathbb{R}^d} |v|^2 \, df_0(v).$$

Observe that the change of variable

$$\sigma \in \mathbb{S}^{d-1} \mapsto -\sigma \in \mathbb{S}^{d-1}$$

maps the domain

$$\theta \in [-\pi, \pi/2] \cap [\pi/2, \pi] \quad \text{in} \quad \theta \in [-\pi/2, \pi/2].$$

Therefore without restriction we can consider, for the limit equation as well as the N -particle system, kernel function b such that $\text{Supp } b \subset [0, 1]$. We still denote by b the symmetrized version of b by a slight abuse of notation.

In this section we aim at considering the case of the *Maxwell molecules kernel*. We shall indeed make the following general assumption:

$$(5.6) \quad \begin{cases} \Gamma \equiv 1, & b \in L_{\text{loc}}^\infty([0, 1]) \\ \forall \alpha > 0, & C_\alpha(b) := \int_{\mathbb{S}^{d-1}} b(\cos \theta) (1 - \cos \theta)^{\frac{1}{4} + \alpha} \, d\sigma < \infty. \end{cases}$$

Let us show that Maxwell molecules model (1.5) enters this general framework. Indeed for any positive real function ψ and any given vector $u \in \mathbb{R}^d$ we have

$$\int_{\mathbb{S}^{d-1}} \psi(\hat{u} \cdot \sigma) \, d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi \psi(\cos \theta) \sin^{d-2} \theta \, d\theta.$$

Therefore the model (1.5) satisfies (in dimension $d = 3$)

$$b(z) \sim K (1 - z)^{-5/4} \quad \text{as} \quad z \rightarrow 1,$$

which hence fulfills (5.6). This assumption also trivially includes the Grad's cutoff Maxwell molecules model (1.6).

5.2. Statement of the results. Our main propagation of chaos estimate result for this model then states as follows:

Theorem 5.1 (Maxwell molecules detailed chaos estimates). *Assume that the collision kernel b satisfies (5.6). Let us consider a family of N -particle initial conditions $f_0^N \in P_{\text{sym}}((\mathbb{R}^d)^N)$, $N \geq 2$, and the associated N -particle system dynamics*

$$f_t^N = S_t^N (f_0^N).$$

Let us also consider a 1-particle initial distribution $f_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with zero momentum and energy $\mathcal{E} \in (0, +\infty)$

$$\int_{\mathbb{R}^d} v f_0 \, dv = 0, \quad \mathcal{E} := \int_{\mathbb{R}^d} |v|^2 f_0 \, dv \in (0, \infty),$$

and the associated solution

$$f_t = S_t^{\text{NL}} (f_0)$$

of the limit mean-field equation.

Then for any $\delta \in (0, 1)$, for any $\ell \in \mathbb{N}^$ and for any*

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \varphi_i \in \mathcal{F}, \quad i = 1, \dots, \ell,$$

where \mathcal{F} shall be specified below, we have, for some constant $C_\delta > 0$ (possibly blowing up as $\delta \rightarrow 0$) depending only on δ , on the collision kernel, on the size of the support and on some moments of f_0 :

- (i) Cases (GMM) and (tMM): Assume that $f_0^N = f_0^{\otimes N}$ is a tensorized N -particle initial datum and that f_0 has compact support, and take

$$\mathcal{F} := \left\{ \varphi : \mathbb{R}^d \rightarrow \mathbb{R}; \|\varphi\|_{\mathcal{F}} := \int_{\mathbb{R}^d} (1 + |\xi|^4) |\hat{\varphi}(\xi)| d\xi < \infty \right\}.$$

Then we have

$$(5.7) \quad \forall N \geq 2\ell, \quad \sup_{t \geq 0} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \\ \leq C_\delta \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + \frac{\ell^2}{N^{1-\delta}} \|\varphi\|_{\mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} + \ell \|\varphi\|_{W^{1,\infty} \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{W_2}^N(f_0) \right].$$

We deduce the following rate of convergence as N goes infinity by using (4.12)-(4.13):

$$\sup_{t \geq 0} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \leq \frac{C_\delta \ell^2}{N^{\kappa(d,\delta)}} \|\varphi\|_{\mathcal{F}^{\otimes \ell}}$$

with

$$\kappa(d, \delta) := \begin{cases} \frac{1}{4} - \delta & \text{if } d \leq 2, \\ \frac{1}{6} - \delta & \text{if } d = 3, \\ \frac{1}{d+4} & \text{if } d \geq 4. \end{cases}$$

This proves the propagation of chaos, uniformly in time and with explicit polynomial rates.

- (ii) Case (GMM) with optimal rate for finite time: On a finite time interval $[0, T]$, the following variant is available: consider tensorized initial data $f_0^N = f_0^{\otimes N}$ for the N -particle system and assume that f_0 has compact support, and take $\mathcal{F} = H^s$ with $s > d/2$ high enough. Then we have

$$\sup_{0 \leq t \leq T} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \\ \leq C_\delta \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + C_{T,4}^N \frac{C_{\delta,\infty}^\infty}{N^{1-\delta}} \ell^2 \|\varphi\|_{\mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} + \ell \|\varphi\|_{H^s \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{H^{-s}}^N(f_0) \right].$$

By using (4.9), this proves

$$\sup_{t \in [0, T]} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \leq \ell^2 \frac{C_{\delta, T}}{\sqrt{N}} \|\varphi\|_{\mathcal{F}^{\otimes \ell}}$$

with $\mathcal{F} = H^s$, and where the constant $C_{\delta, T}$ can also depends on the final time of observation T . This proves the propagation of chaos, on finite time intervals, but with the optimal rate of the law of large numbers.

- (iii) Cases (GMM) and (tMM) conditioned to the sphere: Finally consider \mathcal{F} as in (i), assume that the 1-particle initial datum f_0 belongs to $\mathcal{P}_6(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and consider the associated N -particle initial data $(f_0^N)_{N \geq 1}$ constructed in Lemma 4.4 and 4.7 by conditioning to $\mathcal{S}^N(\mathcal{E})$. Then the solution $f_t^N = S_t^N(f_0^N)$ has its support included in $\mathcal{S}^N(\mathcal{E})$ for all times

$$(5.8) \quad \forall t \geq 0, \quad \text{Supp } f_t^N \subset \mathcal{S}^N(\mathcal{E})$$

and we have the estimate

$$\sup_{t \geq 0} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \leq C_\delta \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + \frac{\ell^2}{N^{1-\delta}} \|\varphi\|_{\mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} + \ell \|\varphi\|_{W^{1,\infty} \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{W_2}^N(\pi_P^N f_0^N, \delta_{f_0}) \right]$$

which goes to zero as N goes infinity with polynomial rate thanks to Lemma 4.7. This proves the propagation of chaos, uniformly in time and with explicit polynomial rates.

We now state another version of the propagation of chaos estimate, in Wasserstein distance, but most importantly which is valid for *any number of marginals*, at the price of a possibly worse (but still constructive) rate. Combined with previous results on the relaxation of the N -particle system we also deduce some estimate of relaxation to equilibrium independent of N and, again, for any number of marginals.

Theorem 5.2 (Maxwell molecules Wasserstein chaos). *We consider the same setting as in Theorem 5.1, where the initial data are chosen as follows:*

- (a) either f_0 is compactly supported and $f_0^N = f_0^{\otimes N}$,
- (b) or $f_0 \in \mathcal{P}_6(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with zero momentum and energy \mathcal{E} , and f_0^N is constructed by Lemma 4.4 by conditioning to the sphere $\mathcal{S}^N(\mathcal{E})$.

Then we have

$$(5.9) \quad \forall N \geq 1, \forall 1 \leq \ell \leq N, \quad \sup_{t \geq 0} \frac{W_1(\Pi_\ell f_t^N, f_t^{\otimes \ell})}{\ell} \leq \alpha(N)$$

for some polynomial rate $\alpha(N) \rightarrow 0$ as $N \rightarrow \infty$.

Moreover in the case (b) the solution $f_t^N = S_t^N(f_0^N)$ has its support included in $\mathcal{S}^N(\mathcal{E})$ for all times and we have

$$(5.10) \quad \forall N \geq 1, \forall 1 \leq \ell \leq N, \forall t \geq 0, \quad \frac{W_1(\Pi_\ell f_t^N, \Pi_\ell(\gamma^N))}{\ell} \leq \beta(t)$$

for some polynomial rate $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$, where γ is the centered Gaussian equilibrium with energy \mathcal{E} and γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

In order to prove Theorem 5.1, we shall establish the assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** of Theorem 3.1 with $T < \infty$ or $T = \infty$ for a suitable choice of functional spaces. The application of the latter theorem then exactly yields Theorem 5.1 by following carefully each constant computed below. Then the proof of Theorem 5.2 will be done in Subsection 5.9: it is deduced from Theorem 5.1 by using Lemma 4.1 together with a result from [36].

5.3. Proof of condition (A1). When the collision kernel B is bounded the operator G^N is a linear bounded operator on $C(B_R)$ with $B_R := \{V \in \mathbb{R}^{dN}; |V| \leq R\}$ for any $R \in (0, \infty)$ with an operator norm independent of R . As a consequence, G^N is also well defined and bounded on

$$C_{-k,0}^0(\mathbb{R}^{dN}) := \left\{ \varphi \in C(\mathbb{R}^{dN}) \quad \text{s. t.} \quad \frac{\varphi(V)}{|V|^k} \rightarrow 0 \quad \text{as} \quad |V| \rightarrow \infty \right\}$$

endowed with the norm

$$\|\varphi\|_{L_{-k}^\infty} := \sup_{V \in \mathbb{R}^{dN}} |\varphi(V)| |V|^{-k}$$

for any $k \in \mathbb{R}$. It is also easy (and classical) to verify that G^N is dissipative in the sense that

$$\forall \varphi \in C_{-k,0}^0(\mathbb{R}^{dN}), \quad \forall \lambda > 0, \quad \|(\lambda - G^N) \varphi\|_{L_{-k}^\infty} \leq \lambda \|\varphi\|_{L_{-k}^\infty}.$$

From the Hille-Yosida theory we deduce that G^N generates a Markov type semigroup T_t^N on $C_{-k,0}^0(\mathbb{R}^{dN})$ and we may also define S_t^N by duality as a semigroup on $\mathcal{P}_k(\mathbb{R}^{dN})$. The nonlinear semigroup S_t^{NL} is also well defined on $\mathcal{P}_k(\mathbb{R}^d)$, see for instance [72, 31, 28, 48].

For the true Maxwell molecules model, the operator G^N is not bounded anymore and some additional explanations are needed. The simplest argument is to say that B can be approximated by a sequence of bounded collision kernels

$$B_\varepsilon := b_\varepsilon(\cos \theta) \quad \text{with} \quad b_\varepsilon \in L^\infty \quad \text{and} \quad b_\varepsilon \nearrow b.$$

We may then define the associated generator $G^{N,\varepsilon}$, the associated semigroups $T_t^{N,\varepsilon}$ on continuous functions and $S_t^{N,\varepsilon}$ on probabilities, and the nonlinear semigroup $S_t^{NL,\varepsilon}$ on probabilities. We first write estimate (5.7) for any fixed $\varepsilon > 0$. Then since (a) the right-hand side term in (5.7) does not depend on $\varepsilon > 0$ (as a consequence of the estimates established in the proof below), (b) $S^{NL,\varepsilon}(f_0) \rightharpoonup S^{NL}(f_0)$ weakly in $P(\mathbb{R}^d)$, and (c) $S^{N,\varepsilon}(f_0^N) \rightharpoonup f_t^N$ weakly in $P(\mathbb{R}^{Nd})$, we can conclude that (5.7) holds for the true Maxwell molecules model by letting ε go to 0.

Possible other direct arguments (without using approximations) could be (a) to establish and use stability estimates in Wasserstein distance of the many-particle equation, or (b) use the core $\mathcal{C} := W_{k+2}^{1,\infty}$ and prove that $\varphi \in W_{k+2}^{1,\infty}$ implies $G^N \varphi \in C_{k,0}$ (this follows from an easy decomposition between singular and non-singular angles in the formula for G^N).

Hence the semigroups T_t^N and

$$S_t^N = (T_t^N)^* = T_t^N$$

are well defined on $C_{-k,0}^0(\mathbb{R}^{dN})$. Moreover since for $\varphi \in L^2(\mathcal{S}^N(\mathcal{E}))$ we have

$$\langle G^N \varphi, \varphi \rangle_{L^2(\mathcal{S}^N(\mathcal{E}))} = -\frac{1}{N} \sum_{i,j=1}^N \int_{\mathcal{S}^N(\mathcal{E})} \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi]^2 d\sigma d\gamma^N(v) \leq 0,$$

it is easily seen by arguing similarly as above that they are C_0 -semigroups of contractions on this space $L^2(\mathcal{S}^N(\mathcal{E}))$.

Then it remains to prove bounds on the polynomial moments of the N -particle system. We shall prove the following more general lemma:

Lemma 5.3. *Consider the collision kernel*

$$B = |v - v_*|^\gamma b(\cos \theta) \quad \text{with} \quad \gamma = 0 \text{ or } 1$$

and $b \geq 0$ such that

$$C_b := \int_0^1 b(z) (1-z)^2 dz < +\infty.$$

This covers the three cases **(HS)**, **(tMM)** and **(GMM)**.

Assume that the initial datum f_0^N of the N -particle system satisfies:

$$\text{Supp } f_0^N \subset \left\{ V \in \mathbb{R}^{Nd}; M_2^N(V) \leq \mathcal{E}_0 \right\} \quad \text{and} \quad \langle f_0^N, M_k^N \rangle \leq C_{0,k} < \infty, \quad k > 2,$$

with

$$\forall V \in \mathbb{R}^{dN}, \quad M_k^N(V) = \frac{1}{N} \sum_{j=1}^N |v_j|^k.$$

Then we have

$$(5.11) \quad \forall t \geq 0, \quad \text{Supp } f_t^N \subset \left\{ V \in \mathbb{R}^{Nd}; M_2^N(V) \leq \mathcal{E}_0 \right\}.$$

and

$$\sup_{t \geq 0} \langle f_t^N, M_k^N \rangle \leq \max \{C_{0,k}; \bar{a}_k\}$$

where $\bar{a}_k \in (0, \infty)$ depends on k and \mathcal{E}_0 . Moreover, when

$$\text{Supp } f_0^N \subset \mathcal{S}^N(\mathcal{E}) = \left\{ V \in \mathbb{R}^{Nd}; M_2(\mu_V^N) = \mathcal{E}, \langle \mu_V^N, z \rangle = 0 \right\},$$

then

$$(5.12) \quad \forall t \geq 0, \quad \text{Supp } f_t^N \subset \mathcal{S}^N(\mathcal{E}) = \left\{ V \in \mathbb{R}^{Nd}; M_2(\mu_V^N) = \mathcal{E}, \langle \mu_V^N, z \rangle = 0 \right\}.$$

Proof of Lemma 5.3. By using (5.5) with the function of the energy

$$\phi : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \phi(z) := \mathbf{1}_{z > N \mathcal{E}_0}$$

and the assumptions on f_0^N we deduce (5.11). On the other hand, by using (5.5) with the functions of the momentum and energy

$$\phi_\varepsilon : \mathbb{R}^{d+1} \rightarrow \mathbb{R}, \quad \phi_\varepsilon(z) := \mathbf{1}_{|z_0/N - \mathcal{E}| + |z_1| + \dots + |z_d| > \varepsilon}, \quad \forall \varepsilon > 0,$$

and the assumptions on f_0^N we deduce (5.12).

Next, we write the differential equality on the k -th moment:

$$\frac{d}{dt} \left\langle f_t^N, \frac{1}{N} \sum_{j=1}^N |v_j|^k \right\rangle = \frac{1}{N^2} \sum_{j_1 \neq j_2}^N \langle f_t^N, |v_{j_1} - v_{j_2}|^\gamma \mathcal{K}(v_{j_1}, v_{j_2}) \rangle,$$

with

$$\mathcal{K}(v_{j_1}, v_{j_2}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\theta_{j_1 j_2}) \left[|v_{j_1}^*|^k + |v_{j_2}^*|^k - |v_{j_1}|^k - |v_{j_2}|^k \right] d\sigma.$$

We then apply the so-called *Povzner Lemma* proved in [59, Lemma 2.2] (valid for singular collision kernel as in our case) which implies

$$\mathcal{K}(v_{j_1}, v_{j_2}) \leq C_1 \left(|v_{j_1}|^{k-1} |v_{j_2}| + |v_{j_1}| |v_{j_2}|^{k-1} \right) - C_2 \left(|v_{j_1}|^k + |v_{j_2}|^k \right)$$

for some constants $C_1, C_2 \in (0, \infty)$ depending only on k and b .

By using the inequalities $|v_{j_1} - v_{j_2}| \geq |v_{j_1}| - |v_{j_2}|$ and $|v_{j_1} - v_{j_2}| \geq |v_{j_2}| - |v_{j_1}|$ in order to estimate the last term when $\gamma = 1$, we then deduce

$$\begin{aligned} |v_{j_1} - v_{j_2}|^\gamma \mathcal{K}(v_{j_1}, v_{j_2}) &\leq C_3 \left[(1 + |v_{j_1}|^{k+\gamma-1}) (1 + |v_{j_2}|^2) \right. \\ &\quad \left. + (1 + |v_{j_1}|^2) (1 + |v_{j_2}|^{k+\gamma-1}) \right] - C_2 (|v_{j_1}|^{k+\gamma} + |v_{j_2}|^{k+\gamma}), \end{aligned}$$

for a constant C_3 depending on C_1 and C_2 .

Using (symmetry hypothesis) that

$$\forall k \geq 0, \quad \left\langle f_t^N, |v_1|^k \right\rangle = \left\langle f_t^N, M_k^N \right\rangle,$$

and (5.11) we get

$$\begin{aligned} \frac{d}{dt} \left\langle f_t^N, |v_1|^k \right\rangle &\leq 2C_3 \left\langle f_t^N, (1 + M_{k+\gamma-1}^N) (1 + M_2^N) \right\rangle - 2C_2 \left\langle f_t^N, M_{k+\gamma}^N \right\rangle \\ &\leq 2C_3 (1 + \mathcal{E}) \left(1 + \left\langle f_t^N, |v_1|^{k+\gamma-1} \right\rangle \right) - 2C_2 \left\langle f_t^N, |v_1|^{k+\gamma} \right\rangle. \end{aligned}$$

Using finally Hölder's inequality

$$\left\langle f_1^N, |v|^{k-\gamma+1} \right\rangle \leq \left\langle f_1^N, |v|^{k+\gamma} \right\rangle^{(k-\gamma+1)/(k+\gamma)}$$

we conclude that $y(t) = \langle f_t^N, |v_1|^k \rangle$ satisfies a differential inequality of the following kind

$$y' \leq -K_1 y^{\theta_1} + K_2 y^{\theta_2} + K_3$$

with $\theta_1 \geq 1$ and $\theta_2 < \theta_1$, and for some constants $K_1, K_2, K_3 > 0$, which concludes the proof of the lemma by a maximum principle argument. \square

We have

$$\text{Supp} f_0^N \subset \mathbb{E}_N := \{V \in E^N, M_2^N(V) \leq \mathcal{E}_0\},$$

with $\mathcal{E}_0 = A^2$, where A is such that $\text{Supp} f_0 \subset B_{\mathbb{R}^d}(0, A)$ under assumptions (i) or (ii) in Theorem 5.1, and $\mathcal{E}_0 = \mathcal{E}$ under assumption (iii) in Theorem 5.1. Lemma 5.3 proves **(A1)-(i)** with the constraint function $\mathbf{m}_{\mathcal{G}_1} : \mathbb{R}^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$, $\mathbf{m}_{\mathcal{G}_1}(v) := (|v|^2, v)$ for all $v \in \mathbb{R}^d$ and with the set of constraints

$$\mathcal{R}_{\mathcal{G}_1} := \{(r_0, r') \in \mathbb{R}_+ \times \mathbb{R}^d; |r'|^2 \leq r_0 \leq \mathcal{E}_0\}.$$

Lemma 5.3 also proves **(A1)-(ii)** with

$$m_{\mathcal{G}_1}(v) := \langle v \rangle^6 = (1 + |v|^2)^3.$$

Moreover we do not need **(A1)-(iii)** in the present case and we may take $m_{\mathcal{G}_3} \equiv 0$. Finally, under assumption (iii) in Theorem 5.1, the support condition (5.8) is nothing but (5.12) and is also proved by Lemma 5.3.

5.4. Proof of condition (A2). Let us define the space of probability measures

$$\mathcal{P}_{\mathcal{G}_1} := \left\{ f \in P(\mathbb{R}^d); M_6(f) < +\infty \right\},$$

for $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$, $\mathbf{r} = (r_0, r')$, $r' = (r_1, \dots, r_d)$, the constrained space

$$\mathcal{P}_{\mathcal{G}_1, \mathbf{r}} := \left\{ f \in \mathcal{P}_6(\mathbb{R}^d); \langle f, |v|^2 \rangle = r_0, \forall i = 1, \dots, d, \langle f, v_i \rangle = r_i \right\}$$

and the vector space

$$\mathcal{G}_1 := \left\{ h \in M_6^1(\mathbb{R}^d); \forall i = 1, \dots, d, \langle h, v_i \rangle = \langle h, 1 \rangle = \langle h, |v|^2 \rangle = 0 \right\}$$

endowed with the modified Fourier-based norm $\|\cdot\|_{\mathcal{G}_1} := \|\cdot\|_2$ defined in (2.5). We also define

$$\mathcal{BP}_{\mathcal{G}_1, a} := \{f \in \mathcal{P}_{\mathcal{G}_1}; M_6(f) \leq a\}$$

as well as for any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$

$$\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}} := \left\{ f \in \mathcal{BP}_{\mathcal{G}_1, a}; \langle f, |v|^2 \rangle = r_0, \forall i = 1, \dots, d, \langle f, v_i \rangle = r_i \right\}.$$

These spaces of probability measures are endowed with the distance $d_{\mathcal{G}_1}$ associated to the norm $\|\cdot\|_{\mathcal{G}_1}$. Observe that for any $f, g \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$, $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$, the fact that the two distributions have equal momentum implies that $d_{\mathcal{G}_1}(f, g) = \|f - g\|_{\mathcal{G}_1} = \|f - g\|_2$, where $\|\cdot\|_2$ is the usual Fourier-based norm defined in (2.4).

Let us recall the following result proved in [70, 32, 12, 72]. We briefly outline its proof for the sake of completeness but also, most importantly, since we shall need to modify it in order to adapt it to our purpose in the next sections.

Lemma 5.4. *For any $f_0, g_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with same momentum (first-order moments), the associated solutions f_t and g_t to the Boltzmann equation for Maxwell molecules satisfy*

$$(5.13) \quad \sup_{t \geq 0} |f_t - g_t|_2 \leq |f_0 - g_0|_2.$$

Moreover, there exists $\bar{a} \in (0, \infty)$ such that for any $a \in [\bar{a}, \infty)$ and any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$, the nonlinear semigroup S_t^{NL} maps $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ into $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$.

Proof of Lemma 5.4. We only prove (5.13). The fact that S_t^{NL} maps $\mathcal{BP}_{G_1, a, r}$ into $\mathcal{BP}_{G_1, a, r}$ follows from the conservations of momentum and energy and higher moment estimates similar to Lemma 5.3 but on the limit equation. We refer to [70, 32, 12, 72] for this classical results.

We recall Bobylev's identity for Maxwellian collision kernel (cf. [6])

$$\mathcal{F}(Q^+(f, g))(\xi) = \hat{Q}^+(F, G)(\xi) =: \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) [F^+ G^- + F^- G^+] d\sigma,$$

with

$$F = \hat{f}, \quad G = \hat{g}, \quad F^\pm = F(\xi^\pm), \quad G^\pm = G(\xi^\pm), \quad \hat{\xi} = \frac{\xi}{|\xi|}$$

and

$$\xi^+ = \frac{1}{2}(\xi + |\xi| \sigma), \quad \xi^- = \frac{1}{2}(\xi - |\xi| \sigma).$$

With the shorthand notation $D = \hat{g} - \hat{f}$, $S = \hat{g} + \hat{f}$, the following equation holds

$$(5.14) \quad \partial_t D = \hat{Q}(S, D) = \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left[\frac{D^+ S^-}{2} + \frac{D^- S^+}{2} - D \right] d\sigma.$$

We perform the following cutoff decomposition on the angular collision kernel:

$$b = b_K + b_K^c \quad \text{with} \quad \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) d\sigma = K, \quad b_K = b \mathbf{1}_{|\theta| \geq \delta(K)}$$

for some well-chosen $\delta(K)$. As in [72] observe that

$$R_K(\xi) = \int_{\mathbb{S}^{d-1}} b_K^c(\sigma \cdot \hat{\xi}) \left[\frac{D^+ S^-}{2} + \frac{D^- S^+}{2} - D \right] d\sigma$$

satisfies

$$\forall \xi \in \mathbb{R}^d, \quad |R_K(\xi)| \leq r_K |\xi|^2 \quad \text{where} \quad r_K \xrightarrow{K \rightarrow \infty} 0$$

and r_K depends on moments of order 2 on d and s (hence bounded by the energy).

Using that $\|S\|_\infty \leq 2$, we deduce in distributional sense

$$\frac{d}{dt} \frac{|D|}{|\xi|^2} + K \frac{|D|}{|\xi|^2} \leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D|}{|\xi|^2} \right) \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \left(|\hat{\xi}^+|^2 + |\hat{\xi}^-|^2 \right) d\sigma \right) + r_K$$

with

$$|\hat{\xi}^+| = \frac{1}{\sqrt{2}} \left(1 + \sigma \cdot \hat{\xi} \right)^{1/2}, \quad |\hat{\xi}^-| = \frac{1}{\sqrt{2}} \left(1 - \sigma \cdot \hat{\xi} \right)^{1/2}.$$

By using

$$|\hat{\xi}^+|^2 + |\hat{\xi}^-|^2 = 1,$$

we deduce

$$\frac{d}{dt} \frac{|D|}{|\xi|^2} + K \frac{|D|}{|\xi|^2} \leq K \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D|}{|\xi|^2} \right) + r_K$$

which implies

$$\sup_{\xi \in \mathbb{R}^d} \frac{|D_t(\xi)|}{|\xi|^2} \leq \sup_{\xi \in \mathbb{R}^d} \frac{|D_0(\xi)|}{|\xi|^2} + C \frac{r_K}{K}$$

for any value of the cutoff parameter K . Therefore by relaxing $K \rightarrow \infty$, we deduce (5.13). \square

Hence we deduce from the previous lemma that S_t^{NL} is Lipschitz on $\mathcal{BP}_{G_1, a, r}$ (uniformly in time) and **(A2)-(i)** is proved.

Lemma 5.5. *Consider $f, g \in \mathcal{P}_2(\mathbb{R}^d)$ two probabilities with same momentum (first order moments). Then we have*

$$(5.15) \quad |Q(f, f)|_2 \leq C \left(\int_{\mathbb{R}^d} (1 + |v|^2) \, df(v) \right)^2,$$

$$(5.16) \quad |Q(f + g, f - g)|_2 \leq C \left(\int_{\mathbb{R}^d} (1 + |v|) \, (df(v) + dg(v)) \right) \left(|f - g|_2 + |(f - g)v|_1 \right).$$

*As a consequence, the assumption **(A2)-(ii)** is satisfied in the sense that Q is Hölder continuous on $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ for any $a > 0$, $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$: there exists $C > 0$ and $\zeta \in (0, 1)$ so that*

$$\forall f, g \in \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}, \quad |Q(f, f) - Q(g, g)|_2 \leq C |f - g|_2^\zeta.$$

Proof of Lemma 5.5. We split the proof into two steps.

Step 1: proof of (5.15) and (5.16). We prove the second inequality (5.16). The first inequality (5.15) then follows immediately by writing

$$Q(f, f) = Q(f, f) - Q(\gamma, \gamma) = Q(f - \gamma, f + \gamma)$$

where γ is the Maxwellian distribution with same momentum and energy as f , and then applying (5.16) with $f - \gamma$ and $f + \gamma$.

We write in Fourier:

$$\begin{aligned} \mathcal{F}(Q(f + g, f - g)) &= \hat{Q}(D, S) \\ &= \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(S(\xi^+) D(\xi^-) + S(\xi^-) D(\xi^+) - 2D(\xi) \right) \end{aligned}$$

where \hat{Q} is the Fourier form the symmetrized collision operator Q , which we can rewrite

$$\frac{|\hat{Q}(D, S)|}{|\xi|^2} \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.$$

We have

$$\mathcal{T}_1 \leq \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) |S(\xi^+)| \frac{|D(\xi^-)|}{|\xi^-|^2} \frac{|\xi^-|^2}{|\xi|^2} \, d\sigma \leq C |D|_2$$

for some constant $C > 0$, where we have used

$$\frac{|\xi^-|^2}{|\xi|^2} = (1 - \cos \theta)^2$$

which permits to control the angular singularity of b .

Similarly we compute

$$\mathcal{T}_2 \leq \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{|D(\xi^+)|}{|\xi^+|} \frac{|S(\xi^-) - 2|}{|\xi^-|} \frac{|\xi^-|}{|\xi|} \, d\sigma \leq C |D|_1 \left(\int_{\mathbb{R}^d} (1 + |v|) \, (df(v) + dg(v)) \right)$$

for some constant $C > 0$, and

$$\begin{aligned} \mathcal{T}_3 &\leq 2 \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{|D(\xi^+) - D(\xi)|}{|\xi|} \, d\sigma \\ &\leq \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{|\xi^-|}{|\xi|} \int_0^1 \frac{|\nabla D(\theta \xi + (1 - \theta)\xi^+)|}{|\theta \xi + (1 - \theta)\xi^+|} \, d\theta \, d\sigma \leq C |(f - g)v|_1 \end{aligned}$$

for some constant $C > 0$. This concludes the proof of (5.16) by combining these estimates.

Step 2: proof of (A2)-(ii). First, for $f, g \in \mathcal{BP}_{\mathcal{G}_1, a, r}$ and $1 \leq \alpha \leq d$, we have for any $R > 0$

$$\begin{aligned} |(f - g) v_\alpha|_1 &= \sup_{\xi \in \mathbb{R}^d} \frac{1}{|\xi|} \left| \int_{\mathbb{R}^d} v_\alpha \left(e^{-i v \cdot \xi} - 1 \right) (df - dg)(v) \right| \\ &\leq \sup_{\xi \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \varphi_\xi(v) (df - dg)(v) \right| + \sup_{\xi \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \psi_\xi(v) (df - dg)(v) \right| \end{aligned}$$

with

$$\varphi_\xi(v) := \frac{1}{|\xi|} \chi_R v_\alpha \left(e^{-i v \cdot \xi} - 1 \right), \quad \psi_\xi(v) := \frac{1}{|\xi|} (1 - \chi_R) v_\alpha \left(e^{-i v \cdot \xi} - 1 \right),$$

and where χ_R is a truncation function just as in the proof of point (iv) in Lemma 4.1.

Next, we observe that for $R \geq 1$ we have

$$\forall v, \xi \in \mathbb{R}^d, \quad |\nabla_v \varphi_\xi(v)| \leq C_1 R^2, \quad |\psi_\xi(v)| \leq |v|^2 (1 - \chi_R).$$

Using the bound on the sixth moment of f and g , we deduce

$$\begin{aligned} |(f - g) v_\alpha|_1 &\leq \inf_{R \geq 1} \left\{ C_1 [f - g]_1^* R^2 + C_2 \frac{a}{R^4} \right\} \\ &\leq C_3 \min \left\{ a^{1/3} ([f - g]_1^*)^{2/3}, [f - g]_1^* \right\} \\ &\leq C'_3 a^{1/3} ([f - g]_1^*)^{2/3}. \end{aligned}$$

From (4.4), we then obtain

$$f, g \in \mathcal{BP}_{\mathcal{G}_1, a, r}, \quad |(f - g) v_\alpha|_1 \leq C |f - g|_2^\zeta,$$

for some constants $C > 0$ and $\zeta \in (0, 1)$ depending on d and a .

Gathering this last estimate with (5.16) we conclude that for any $f, g \in \mathcal{BP}_{\mathcal{G}_1, a, r}$

$$\begin{aligned} |Q(f, f) - Q(g, g)|_2 &= |Q(f + g, f - g)|_2 \\ &\leq C M_1(f + g) \left(|f - g|_2 + |f - g|_2^\zeta \right) \leq C' |f - g|_2^\zeta, \end{aligned}$$

for some constant $C' > 0$ depending on d and a . □

5.5. Proof of condition (A3). Let us define $m'_{\mathcal{G}_1} = \langle v \rangle^4$ and

$$\Lambda_1(f) := \langle f, m'_{\mathcal{G}_1} \rangle = \langle f, \langle v \rangle^4 \rangle$$

for any $f \in \mathcal{P}_{\mathcal{G}_1}$.

Let us prove that for any

$$(5.17) \quad \Phi \in \bigcap_{r \in \mathcal{R}_{\mathcal{G}_1}} C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, r})$$

we have

$$\left\| \left(M_{m_{\mathcal{G}_1}}^N(V) \right)^{-1} (G^N \pi^N - \pi^N G^\infty) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \leq \frac{C_1 \mathcal{E}_0}{N^\eta} \sup_{r \in \mathcal{R}_{\mathcal{G}_1}} [\Phi]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, r})},$$

for some constant $C_1 > 0$.

First, consider velocities $v, v_*, w, w_* \in \mathbb{R}^d$ such that

$$w = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad w_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^{d-1}.$$

Then $\delta_v + \delta_{v_*} - \delta_w - \delta_{w_*} \in \mathcal{IP}_{\mathcal{G}_1}$. Performing Taylor expansions, we get

$$\begin{aligned}
& e^{iv \cdot \xi} + e^{iv_* \cdot \xi} - e^{iw \cdot \xi} - e^{iw_* \cdot \xi} \\
&= i(w - v) \xi e^{iv \cdot \xi} + \mathcal{O}(|w - v|^2 |\xi|^2) + i(w_* - v_*) \xi e^{iv_* \cdot \xi} + \mathcal{O}(|w_* - v_*|^2 |\xi|^2) \\
&= i(w - v) \xi e^{iv \cdot \xi} + \mathcal{O}(|w - v|^2 |\xi|^2) \\
&\quad + i(w_* - v_*) \xi (e^{iv \cdot \xi} + \mathcal{O}(|v - v_*| |\xi|)) + \mathcal{O}(|w_* - v_*|^2 |\xi|^2) \\
&= \mathcal{O}(|v - v_*|^2 |\xi|^2 \sin \theta / 2)
\end{aligned}$$

thanks to the impulsion conservation and the fact that

$$|w - v| = |w_* - v_*| = |v - v_*| \sin \frac{\theta}{2}.$$

We hence deduce

$$|\delta_v + \delta_{v_*} - \delta_w - \delta_{w_*}|_2 = \sup_{\xi \in \mathbb{R}^d} \frac{|e^{iv \cdot \xi} + e^{iv_* \cdot \xi} - e^{iw \cdot \xi} - e^{iw_* \cdot \xi}|}{|\xi|^2} \leq C |v - v_*|^2 (1 - \cos \theta).$$

As an immediate consequence, for any $V \in \mathbb{E}_N$ and V_{ij}^* defined by (5.2), we have

$$(5.18) \quad \left| \mu_{V_{ij}^*}^N - \mu_V^N \right|_2 \leq \frac{C}{N} |v_i - v_j|^2 (1 - \cos \theta_{ij})$$

and

$$\mathbf{r}_V := (\langle \mu_V^N, |z|^2 \rangle, \langle \mu_V^N, z_1 \rangle, \dots, \langle \mu_V^N, z_d \rangle) \in \mathcal{R}_{\mathcal{G}_1},$$

where $z = (z_1, \dots, z_d) \in \mathbb{R}^d$ has to be understood as the blind integration variable in the duality bracket.

Then for given a Φ satisfying (5.17), for any $V \in \mathbb{E}_N$, and any $1 \leq i, j \leq N$, we set $\phi := D\Phi[\mu_V^N]$, $u_{ij} = (v_i - v_j)$ and we compute:

$$\begin{aligned}
G^N(\Phi \circ \mu_V^N) &= \frac{1}{2N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \left[\Phi(\mu_{V_{ij}^*}^N) - \Phi(\mu_V^N) \right] b(\cos \theta_{ij}) \, d\sigma \\
&= \frac{1}{2N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \langle \mu_{V_{ij}^*}^N - \mu_V^N, \phi \rangle b(\cos \theta_{ij}) \, d\sigma \\
&\quad + \frac{[\Phi]_{C_\Lambda^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}_V})}}{2N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \left[M_{m'_{\mathcal{G}_1}}(\mu_{V_{ij}^*}^N) + M_{m'_{\mathcal{G}_1}}(\mu_V^N) \right] \mathcal{O}\left(\left| \mu_{V_{ij}^*}^N - \mu_V^N \right|_2^{1+\eta} \right) b(\cos \theta_{ij}) \, d\sigma \\
&=: I_1(V) + I_2(V).
\end{aligned}$$

Concerning the first term $I_1(V)$, thanks to Lemma 2.11, we have

$$\begin{aligned}
I_1(V) &= \frac{1}{2N^2} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\phi(v_i^*) + \phi(v_j^*) - \phi(v_i) - \phi(v_j)] \, d\sigma \\
&= \frac{1}{2} \int_v \int_w \int_{\mathbb{S}^{d-1}} b(\cos \theta) [\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)] \, d\mu_V^N(v) \, d\mu_V^N(w) \, d\sigma \\
&= \langle Q(\mu_V^N, \mu_V^N), \phi \rangle = (G^\infty \Phi)(\mu_V^N).
\end{aligned}$$

For the second term $I_2(V)$, using estimate (5.18) and the following inequality which holds for any $k \geq 2$ and any $V \in \mathbb{E}^N$

$$\begin{aligned}
(5.19) \quad M_k \left(\mu_{V_{ij}^*}^N \right) &:= \frac{1}{N} \sum_{\ell=1}^N \langle (V_{ij^*})_{\ell} \rangle^k \\
&\leq 2^{k/2} \left(1 + \frac{1}{N} \left(\left(\sum_{\ell \neq i,j} |v_{\ell}|^k \right) + |v_i^*|^k + |v_j^*|^k \right) \right) \\
&\leq 2^{k/2} \left(1 + \frac{1}{N} \left(\left(\sum_{\ell \neq i,j} |v_{\ell}|^k \right) + (|v_i|^2 + |v_j|^2)^{k/2} \right) \right) \\
&\leq 2^k \left(1 + \frac{1}{N} \sum_{\ell=1}^N |v_{\ell}|^k \right) \leq 2^k M_k(\mu_V^N) = 2^k M_k^N(V),
\end{aligned}$$

we deduce, for some constant $C > 0$ depending on k and $\eta \in (0, 1]$,

$$\begin{aligned}
|I_2(V)| &\leq \frac{C}{N^{2+\eta}} M_{m'_{\mathcal{G}_1}}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})} \\
&\quad \times \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) (1 + |v_i|^2 + |v_j|^2)^{1+\eta} (1 - \sigma \cdot \hat{u}_{ij}) \, d\sigma \\
&\leq \frac{C}{N^{\eta}} M_{m'_{\mathcal{G}_1}}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})} C_b M_4^N(V).
\end{aligned}$$

We finally use

$$\forall V \in \mathbb{E}_N, \quad M_{m'_{\mathcal{G}_1}}^N(V) = M_4^N(V) \leq M_2^N(V)^{1/2} M_6^N(V)^{1/2} = \mathcal{E}_0^{1/2} M_{m_{\mathcal{G}_1}}^N(V)^{1/2}$$

by Cauchy-Schwartz inequality and the energy constraint, which implies

$$|I_2(V)| \leq \frac{C_1 \mathcal{E}_0}{N^{\eta}} M_{m_{\mathcal{G}_1}}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})}$$

and concludes the proof.

5.6. Proof of condition (A4) uniformly in time. Let us consider some 1-particle initial data $f_0, g_0 \in \mathcal{P}_4(\mathbb{R}^d)$ (space of probability measures with bounded fourth moment) and the associated solutions f_t and g_t to the nonlinear Boltzmann equation (1.1) under the assumption (5.6) as well as

$$h_t := \mathcal{D}_t^{NL} [f_0] (g_0 - f_0)$$

the solution to the linearized Boltzmann equation around f_t . Those solutions are given by

$$\begin{cases} \partial_t f_t = Q(f_t, f_t), & f_{|t=0} = f_0 \\ \partial_t g_t = Q(g_t, g_t), & g_{|t=0} = g_0 \\ \partial_t h_t = 2Q(h_t, f_t), & h_{|t=0} = h_0 := g_0 - f_0. \end{cases}$$

We shall now *expand the limit nonlinear semigroup* in terms of the initial data, around f_0 .

Lemma 5.6. *There exists $\lambda \in (0, \infty)$ and, for any $\eta \in (1/2, 1)$, there exists $C_{\eta} > 0$ such that for any*

$$\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1} \quad \text{and} \quad f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$$

we have

$$(5.20) \quad |f_t - g_t|_2 \leq C_{\eta} e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{\frac{1}{2}} |f_0 - g_0|_2^{\eta},$$

and

$$(5.21) \quad |h_t|_2 \leq C_\eta e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{\frac{1}{2}} |f_0 - g_0|_2^\eta,$$

where we recall that

$$\forall f \in P(\mathbb{R}^d), \quad M_4(f) := \langle f, \langle v \rangle^4 \rangle.$$

Remark 5.7. Observe that the decay rate λ in this statement is uniform in terms of the energy $\mathcal{E} \geq 0$.

Proof of Lemma 5.6. We shall proceed in several steps.

Step 1. Estimate in $|\cdot|_4$. We closely follow ideas in [38, 70, 32, 12]. We shall use the notation

$$\mathcal{M} = \mathcal{M}_4, \quad \hat{\mathcal{M}} = \hat{\mathcal{M}}_4,$$

introduced in Example 2.5.4, as well as

$$d := f - g, \quad s := f + g$$

and

$$\tilde{d} := d - \mathcal{M}[d], \quad D := \mathcal{F}(d), \quad S := \mathcal{F}(s) \quad \text{and} \quad \tilde{D} := \mathcal{F}(\tilde{d}) = D - \hat{\mathcal{M}}[d].$$

The equation satisfied by \tilde{D} is

$$(5.22) \quad \begin{aligned} \partial_t \tilde{D} &= \hat{Q}(D, S) - \partial_t \hat{\mathcal{M}}[d] \\ &= \hat{Q}(\tilde{D}, S) + \left(\hat{Q}(\hat{\mathcal{M}}[d], S) - \hat{\mathcal{M}}[Q(d, s)] \right). \end{aligned}$$

We infer from [38, 70] that for any $\alpha \in \mathbb{N}^d$, there exists some absolute coefficients $(a_{\alpha, \beta})$, $\beta \leq \alpha$ (which means $\beta_i \leq \alpha_i$ for any $1 \leq i \leq d$), depending on the collision kernel b through

$$(5.23) \quad \int_{\mathbb{S}^{d-1}} b(\cos \theta) [(v^\alpha)' + (v^\alpha)'_* - (v^\alpha) - (v^\alpha)_*] d\sigma = \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} (v^\beta) (v^{\alpha-\beta})_*$$

where $\alpha, \beta \in \mathbb{N}^d$ are *coordinates* indices and

$$v^\alpha := v_1^{\alpha_1} v_2^{\alpha_2} \dots v_d^{\alpha_d}.$$

These multi-indices are compared through the usual lexicographical order, and we use the standart notation

$$|\alpha| := \sum_{k=1}^d \alpha_k.$$

We deduce that

$$\forall |\alpha| \leq 3, \quad \nabla_\xi^\alpha \hat{\mathcal{M}}[Q(d, s)]|_{\xi=0} = M_\alpha[Q(d, s)] = \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[d] M_{\alpha-\beta}[s]$$

together with

$$\begin{aligned} \forall |\alpha| \leq 3, \quad \nabla_\xi^\alpha \hat{Q}(\hat{\mathcal{M}}[d], S)|_{\xi=0} &= M_\alpha[Q(\mathcal{M}[d], s)] \\ &= \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[\mathcal{M}[d]] M_{\alpha-\beta}[s] = \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[d] M_{\alpha-\beta}[s] \end{aligned}$$

since

$$M_\alpha[\mathcal{M}[d]] = M_\alpha[d]$$

for any $|\alpha| \leq 3$ by construction. As a consequence, we get

$$(5.24) \quad \forall \xi \in \mathbb{R}^d, \quad \left| \hat{\mathcal{M}}[Q(d, s)] - \hat{Q}(\hat{\mathcal{M}}[d], S) \right| \leq C |\xi|^4 \left(\sum_{|\alpha| \leq 3} |M_\alpha[f - g]| \right).$$

On the other hand, from [70, Theorem 8.1] and its corollary, we know that there exists some constants $C > 0$ and $\lambda_1 > 0$ (given by $\lambda_1 := \min_{\alpha, |\alpha| \leq 3} \{-a_{\alpha, \alpha}\} \in (0, \infty)$) so that

$$(5.25) \quad \forall t \geq 0, \quad \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_t - g_t]| \right) \leq C e^{-\lambda_1 t} \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right).$$

We perform the same decomposition on the angular collision kernel

$$b = b_K + b_K^c \quad \text{with} \quad \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \, d\sigma = K, \quad b_K = b \mathbf{1}_{|\theta| \geq \delta(K)}$$

as in the proof of Lemma 5.4 and use the straightforward estimate

$$R_K(\xi) := \hat{Q}_{b_K^c}(\tilde{D}, S)(\xi)$$

satisfies

$$\forall \xi \in \mathbb{R}^d, \quad |R_K(\xi)| \leq r_K |\xi|^4 \quad \text{where} \quad r_K \xrightarrow{K \rightarrow \infty} 0$$

where $Q_{b_K^c}$ denotes the collision operator associated with the part b_K^c of the decomposition of the angular collision kernel, and where r_K depends on moments of order 4 on d and s .

Then we gather (5.22), (5.24) and (5.25) and we have

$$\begin{aligned} \frac{d}{dt} \frac{|\tilde{D}(\xi)|}{|\xi|^4} + K \frac{|\tilde{D}(\xi)|}{|\xi|^4} &\leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{D}(\xi)|}{|\xi|^4} \right) \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4 \right) d\sigma \right) \\ &\quad + C e^{-\lambda_1 t} \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right) + r_K. \end{aligned}$$

Let us compute (the supremum has been dropped thanks to the spherical invariance)

$$\lambda_K := \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4 \right) d\sigma = \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \frac{1 + (\sigma \cdot \hat{\xi})^2}{2} d\sigma,$$

so that

$$\begin{aligned} \lambda_K - K &= - \int_{\mathbb{S}^{d-1}} b_K(\sigma \cdot \hat{\xi}) \frac{1 - (\sigma \cdot \hat{\xi})^2}{2} d\sigma \\ &\xrightarrow{K \rightarrow \infty} - \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{1 - (\sigma \cdot \hat{\xi})^2}{2} d\sigma := -\bar{\lambda} \in (-\infty, 0), \end{aligned}$$

where in the last step we have used the (5.6).

Then, thanks to Gronwall lemma, we get

$$\begin{aligned} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{D}_t(\xi)|}{|\xi|^4} \right) &\leq e^{(\lambda_K - K)t} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{D}_0(\xi)|}{|\xi|^4} \right) \\ &\quad + C_3 \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right) \left(\frac{e^{-\lambda_1 t}}{K - \lambda_K - \lambda} - \frac{e^{(\lambda_K - K)t}}{K - \lambda_K - \lambda} \right) + C \frac{r_K}{K(K - \lambda_K)}. \end{aligned}$$

Therefore, passing to the limit $K \rightarrow \infty$ and choosing (without restriction) $\lambda_2 \in (0, \min\{\lambda_1; \bar{\lambda}\})$, we obtain

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{D}_t(\xi)|}{|\xi|^4} \leq C e^{-\lambda_2 t} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{D}_0(\xi)|}{|\xi|^4} + \sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right)$$

from which we conclude thanks to (5.25) and with the notations of Example 2.5.4

$$(5.26) \quad |||d_t|||_4 \leq C e^{-\lambda_2 t} |||d_0|||_4.$$

Step 2. From $|\cdot|_4$ to $|\cdot|_2$ on the difference. From the preceding step and a straightforward interpolation argument, we have

$$(5.27) \quad \begin{aligned} |f - g|_2 &\leq |f - g - \mathcal{M}[f - g]|_2 + C \sum_{|\alpha| \leq 3} |M_\alpha[f - g]| \\ &\leq \left\| \hat{f} - \hat{g} - \hat{\mathcal{M}}[f - g] \right\|_{L^\infty(\mathbb{R}^d)}^{1/2} |f - g - \mathcal{M}[f - g]|_4^{1/2} \\ &\quad + C \sum_{|\alpha| \leq 3} |M_\alpha[f - g]| \\ &\leq C M_4(f_0 + g_0) e^{-\frac{\lambda_2}{2} t} \end{aligned}$$

where we have used $|||d|||_4 \leq C M_4(d)$.

Then by writing

$$|f - g|_2 \leq |f - g|_2^\eta |f - g|_2^{1-\eta}$$

with $\eta \in (1/2, 1)$, using Lemma 5.4 for the first term of the right-hand side and the previous decay estimate (5.26) for the second term, and $(1 - \eta) \leq 1/2$, we obtain (5.20).

Step 3. From the difference to the linearized semigroup. A similar line of argument imply the same estimate on h_t as on the difference $(f_t - g_t)$, that is inequality (5.21).

Let us briefly sketch the argument. We define

$$\tilde{h} := h - \mathcal{M}[h], \quad H := \mathcal{F}(h), \quad F := \mathcal{F}(f) \quad \text{and} \quad \tilde{H} := \mathcal{F}(\tilde{h}) = H - \hat{\mathcal{M}}[h].$$

The equation satisfied by H is

$$\partial_t H = Q(H, F)$$

and arguing exactly as in Lemma 5.4 one deduces

$$(5.28) \quad \forall t \geq 0, \quad |h_t|_2 \leq |h_0|_2 = |f_0 - g_0|_2.$$

Then the equation satisfied by \tilde{H} is

$$\begin{aligned} \partial_t \tilde{H} &= \hat{Q}(H, F) - \partial_t \hat{\mathcal{M}}[h] \\ &= \hat{Q}(\tilde{H}, F) + \left(\hat{Q}(\hat{\mathcal{M}}[h], F) - \hat{\mathcal{M}}[Q(h, f)] \right). \end{aligned}$$

We infer from (5.23) again that

$$\forall |\alpha| \leq 3, \quad \nabla_\xi^\alpha \hat{\mathcal{M}}[Q(h, f)]|_{\xi=0} = M_\alpha[Q(h, f)] = \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[h] M_{\alpha-\beta}[f]$$

together with

$$\begin{aligned} \forall |\alpha| \leq 3, \quad \nabla_\xi^\alpha \hat{Q}(\hat{\mathcal{M}}[h], F)|_{\xi=0} &= M_\alpha[Q(\mathcal{M}[h], f)] \\ &= \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[\mathcal{M}[h]] M_{\alpha-\beta}[f] = \sum_{\beta, \beta \leq \alpha} a_{\alpha, \beta} M_\beta[h] M_{\alpha-\beta}[f]. \end{aligned}$$

As a consequence, we get

$$\forall \xi \in \mathbb{R}^d, \quad \left| \hat{\mathcal{M}}[Q(h, f)] - \hat{Q}(\hat{\mathcal{M}}[h], F) \right| \leq C |\xi|^4 \left(\sum_{|\alpha| \leq 3} |M_\alpha[h]| \right).$$

On the other hand, arguing as in [70, Theorem 8.1] and its corollary on the linearized equation around f_t , we deduce that there exists some constants $C > 0$ and $\lambda_1 > 0$ (given by $\lambda_1 := \min_{\alpha, |\alpha| \leq 3} \{-a_{\alpha, \alpha}\} \in (0, \infty)$) so that

$$(5.29) \quad \forall t \geq 0, \quad \left(\sum_{|\alpha| \leq 3} |M_\alpha[h_t]| \right) \leq C e^{-\lambda_1 t} \left(\sum_{|\alpha| \leq 3} |M_\alpha[h_0]| \right).$$

Then by exactly the same proof as before we deduce

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{H}_t(\xi)|}{|\xi|^4} \leq C e^{-\lambda_2 t} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\tilde{H}_0(\xi)|}{|\xi|^4} + \sum_{|\alpha| \leq 3} |M_\alpha[h_0]| \right)$$

for some $\lambda_2 \in (0, \lambda_1)$, from which we conclude

$$(5.30) \quad \|h_t\|_4 \leq C e^{-\lambda_2 t} \|d_0\|_4$$

thanks to (5.29), and using that $h_0 = d_0$.

Next we write

$$\begin{aligned} |h|_2 &\leq |h - \mathcal{M}[h]|_2 + C \sum_{|\alpha| \leq 3} |M_\alpha[h]| \\ &\leq \left\| \hat{h} - \hat{\mathcal{M}}[h] \right\|_{L^\infty(\mathbb{R}^d)}^{1/2} |h - \mathcal{M}[h]|_4^{1/2} + C \sum_{|\alpha| \leq 3} |M_\alpha[h]| \\ &\leq C M_4(f_0 + g_0) e^{-\frac{\lambda_2}{2} t} \end{aligned}$$

and

$$|h|_2 \leq |h|_2^\eta |h|_2^{1-\eta}$$

with $\eta \in (1/2, 1)$. Using Lemma 5.4 for the first term of the right-hand side and the previous decay estimate (5.30) for the second term, and $(1-\eta) \leq 1/2$, we obtain (5.21). \square

We can now consider the *second-order* term in the expansion of the semigroup. Let us recall that the crucial point here is to prove that this second-order term is controlled in terms of some power *strictly greater than 1* of the initial difference.

Lemma 5.8. *There exists $\lambda \in (0, \infty)$ and, for any $\eta \in (1/2, 1)$, there exists C_η such that for any*

$$\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1} \quad \text{and} \quad f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}},$$

we have

$$|\omega_t|_4 \leq C e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{\frac{1}{2}} |g_0 - f_0|_2^{1+\eta}$$

where

$$\omega_t := g_t - f_t - h_t = S_t^{NL}(g_0) - S_t^{NL}(f_0) - \mathcal{D}_t^{NL}[f_0](g_0 - f_0).$$

Remark 5.9. As proved below ω_t always has vanishing moments up to order 3, which implies that the norm $|\omega_t|_4$ is well-defined.

Proof of Lemma 5.8. We consider the angular cutoff decomposition as in Lemma 5.4. Consider the error term

$$\omega := g - f - h, \quad \Omega := \hat{\omega}.$$

which satisfies the evolution equation

$$\partial_t \omega_t = Q(\omega_t, f + g) - Q^+(h, f - g), \quad \omega_0 = 0$$

and (in Fourier variable)

$$\partial_t \Omega = \hat{Q}(\Omega, S) - \hat{Q}^+(H, D).$$

Let us prove that

$$\forall |\alpha| \leq 3, \forall t \geq 0, \quad M_\alpha[\omega_t] := \int_{\mathbb{R}^d} v^\alpha d\omega_t(v) = 0.$$

We shall use again the fact that, for Maxwell molecules, the α -th moment of $Q(f_1, f_2)$ is a sum of terms given by product of moments of f_1 and f_2 whose orders sum to $|\alpha|$, see equation (5.23).

We obtain

$$\forall |\alpha| \leq 3, \quad \frac{d}{dt} M_\alpha[\omega_t] = \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta[\omega_t] M_{\alpha-\beta}[f_t + g_t] + \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta[h_t] M_{\alpha-\beta}[f_t - g_t]$$

and since

$$\forall |\alpha| \leq 1, \quad M_\alpha[h_t] = M_\alpha[f_t - g_t] = 0,$$

we deduce

$$\forall |\alpha| \leq 3, \quad \frac{d}{dt} M_\alpha[\omega_t] = \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta[\omega_t] M_{\alpha-\beta}[f_t + g_t].$$

This concludes the proof of the claim about the moments of ω_t since $\omega_0 = 0$.

We now consider the equation in Fourier form

$$\partial_t \Omega = \hat{Q}(\Omega, S) - \hat{Q}^+(H, D)$$

and we deduce in distributional sense

$$\left(\frac{d}{dt} \frac{|\Omega(\xi)|}{|\xi|^4} + K \frac{|\Omega(\xi)|}{|\xi|^4} \right) \leq \mathcal{T}_1 + \mathcal{T}_2 + r_K, \quad r_K \xrightarrow{K \rightarrow \infty} 0$$

(depending on some moments of order 1 of ω , h , d), and

$$\begin{aligned} \mathcal{T}_1 &:= \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b(\sigma \cdot \hat{\xi})}{|\xi|^4} \left(\left| \frac{\Omega(\xi^+) S(\xi^-)}{2} \right| + \left| \frac{\Omega(\xi^-) S(\xi^+)}{2} \right| \right) d\sigma \\ &\leq \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(\frac{|\Omega(\xi^+)|}{|\xi^+|^4} \frac{|\xi^+|^4}{|\xi|^4} + \frac{|\Omega(\xi^-)|}{|\xi^-|^4} \frac{|\xi^-|^4}{|\xi|^4} \right) d\sigma \\ &\leq \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|^2}{|\xi|^2} \right) \left(\sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4 \right) d\sigma \right) \\ &\leq \lambda_K \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|}{|\xi|^4} \right), \end{aligned}$$

where λ_K was defined in Lemma 5.4, and

$$\begin{aligned} \mathcal{T}_2 &:= \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b(\sigma \cdot \hat{\xi})}{|\xi|^4} |H(\xi^+) D(\xi^-) + H(\xi^-) D(\xi^+)| d\sigma \\ &\leq \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(\frac{|H(\xi^+)|}{|\xi^+|^2} \frac{|D(\xi^-)|}{|\xi^-|^2} \frac{|\xi^-|^2}{|\xi|^2} + \frac{|D(\xi^+)|^2}{|\xi^+|^2} \frac{|H(\xi^-)|^2}{|\xi^-|^2} \frac{|\xi^-|^2}{|\xi|^2} \right) d\sigma \\ &\leq \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}_0) \left(1 - \sigma \cdot \hat{\xi}_0 \right) d\sigma |h_t|_2 |d_t|_2 \leq C_b |d_0|_2^{1+\eta} |d_t|^{1-\eta} \\ &\leq C e^{-(1-\eta) \frac{\lambda_2}{2} t} M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{1+\eta} \end{aligned}$$

by using the estimates (5.28) and (5.27).

Hence we obtain

$$\left(\frac{d}{dt} \frac{|\Omega(\xi)|}{|\xi|^4} + K \frac{|\Omega(\xi)|}{|\xi|^4} \right) \leq \lambda_K \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|}{|\xi|^4} \right) + C e^{-(1-\eta) \frac{\lambda_2}{2} t} M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{1+\eta} + r_K.$$

We then deduce from the Gronwall inequality, relaxing the cutoff parameter K as in Lemma 5.6 and choosing without restriction $\lambda > 0$ so that $(1 - \eta)\lambda \leq \min\{(1 - \eta)\lambda_2/2; \bar{\lambda}\}$, that

$$\left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega_t(\xi)|}{|\xi|^4} \right) \leq C e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |g_0 - f_0|_2^{1+\eta}.$$

This concludes the proof (using $(1 - \eta) \leq 1/2$ on the moment exponent). \square

5.7. Proof of condition (A5) uniformly in time in Wasserstein distance. We know from [70] that for f_0 and g_0 with same momentum and energy one has

$$\sup_{t \geq 0} W_2(S_t^{NL} f_0, S_t^{NL} g_0) \leq W_2(f_0, g_0).$$

As a consequence, by using

$$[\cdot]_1^* = W_1 \leq W_2,$$

we deduce that (A5) holds with

$$\Theta(x) = x, \quad \mathcal{F}_3 = \text{Lip}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{P}_{\mathcal{G}_3} = \mathcal{P}_2(\mathbb{R}^d)$$

endowed with the distance $d_{\mathcal{G}_3} = W_2$ and the constraints corresponding to the momentum and energy.

By using Theorem 3.1 whose assumptions have been proved above, this proves point (i) in Theorem 5.1 and the rate follows from the estimate on $\mathcal{W}_{W_2^N}^N(f)$ from Lemma 4.2.

By using Lemma 4.1 in order to relate $\mathcal{W}_{W_2^N}(\pi_P^N(f_0^N), \delta_{f_0})$ with $\mathcal{W}_{W_1}(\pi_P^N(f_0^N), \delta_{f_0})$ and then Lemma 4.7 in order to estimate

$$\mathcal{W}_{W_1}(\pi_P^N(f_0^N), f_0) \xrightarrow{N \rightarrow \infty} 0$$

for the sequence of initial data conditioned on the energy sphere constructed in Lemma 4.4, we then deduce point (iii) in Theorem 5.1.

5.8. Proof of condition (A5) with time growing bounds in Sobolev norms. It is also possible (and in fact easier) to prove, in the cutoff case, that the weak stability holds in negative Sobolev spaces with *non-uniform-in-time estimates*.

Lemma 5.10. *For any $T \geq 0$ and $s > d/2$ there exists $C_{T,s}$ such that for any f_t, g_t solutions of the Boltzmann equation for Maxwell molecules (5.6) and initial data f_0 and g_0 , there holds*

$$\sup_{[0,T]} \|f_t - g_t\|_{H^{-s}} \leq C_{T,s} \|f_0 - g_0\|_{H^{-s}}.$$

Sketch the proof of Lemma 5.10. We integrate (5.14) against $D/(1 + |\xi|^2)^s$:

$$\frac{d}{dt} \|D\|_{H^{-k}}^2 = \frac{1}{2} \int_{\xi} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{[D^- S^+ D + D^+ S^- D - 2|D|^2]}{(1 + |\xi|^2)^s} d\sigma d\xi$$

and we use Young's inequality together with the bounds

$$\|S^+\|_{\infty}, \quad \|S^-\|_{\infty} \leq \|f + g\|_{M^1} \leq 2$$

to conclude. \square

This proves (A5) with the alternate choice

$$\Theta(x) = x, \quad \mathcal{F}_3 = H^s(\mathbb{R}^d) \quad \text{and} \quad \mathcal{P}_{\mathcal{G}_3} = \mathcal{P}_2(\mathbb{R}^d)$$

endowed with the distance of the normed space $\mathcal{G}_3 = H^s(\mathbb{R}^d)$. Then point (ii) in Theorem 5.1 follows from the abstract theorem 3.1 where the (optimal) rate is provided by the estimate on $\mathcal{W}_{\|\cdot\|_{H^{-s}}}^N(f)$ from Lemma 4.2.

5.9. Proof of infinite-dimensional Wasserstein chaos. We shall prove Theorem 5.2 in this subsection. We only present the proof in the case of assumption (b), since the case of assumption (a) is similar. Let us proceed in several steps. Let us emphasize that we do not search for optimality on the rate functions in this subsection.

Step 1: Finite-dimensional Wasserstein chaos. It is immediate that Theorem 5.1 implies that, under one of the two possible assumptions on the initial data, for any given $\ell \geq 1$, one has

$$\sup_{t \geq 0} \left\| \Pi_\ell [f_t^N] - f_t^{\otimes \ell} \right\|_{H^{-s}} \leq \alpha_0(\ell, N)$$

for some power law rate function $\alpha_0(\ell, N) \rightarrow 0$ as $N \rightarrow \infty$.

Then by using Lemma 4.1 we deduce that

$$\sup_{t \geq 0} W_1 \left(\Pi_\ell [f_t^N], f_t^{\otimes \ell} \right) \leq \alpha(\ell, N)$$

for some power law rate function $\alpha(\ell, N) \rightarrow 0$ as $N \rightarrow \infty$.

Note carefully that at this point our rate function still depends on ℓ and in fact a quick look at Theorem 5.1 shows that they scale like ℓ^2 , therefore making impossible to choose $\ell \sim N$.

Step 2: Infinite-dimensional Wasserstein chaos. We shall use here the following result obtained in [36], see also [56, Théorème 2.1]: for any $f \in P(\mathbb{R}^d)$ and sequence $f^N \in P_{\text{sym}}(\mathbb{R}^d)$ we have

$$\forall 1 \leq \ell \leq N, \quad \frac{W_1(\Pi_\ell [f^N], f^{\otimes \ell})}{\ell} \leq C \left(W_1(\Pi_2 [f^N], f^{\otimes 2})^{\alpha_1} + \frac{1}{N^{\alpha_2}} \right)$$

for some constructive constant $C, \alpha_1, \alpha_2 > 0$.

By combining this estimate with the previous step we immediately obtain

$$\sup_{1 \leq \ell \leq N} \sup_{t \geq 0} \frac{W_1(\Pi_\ell [f_t^N], f_t^{\otimes \ell})}{\ell} \leq \alpha(N)$$

for some power law rate function $\alpha(N) \rightarrow 0$ as $N \rightarrow \infty$. This concludes the proof of (5.9).

Step 3: Relaxation in Wasserstein distance. We shall prove (5.10) and then we shall consider here initial data f_0^N constructed by conditioning $f_0^{\otimes N}$ to the Boltzmann sphere $\mathcal{S}^N(\mathcal{E})$. We first write

$$\frac{W_1(f_t^N, \gamma^N)}{N} \leq \frac{W_1(f_t^N, f_t^{\otimes N})}{N} + \frac{W_1(f_t^{\otimes N}, \gamma^{\otimes N})}{N} + \frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N}.$$

Since $f_t^N \rightarrow \gamma^N$ in L^2 and $f_t \rightarrow \gamma$ in L^1 as $t \rightarrow +\infty$, one can pass to the limit in the Wasserstein distance and get from the previous step

$$\frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N} \leq \alpha(N).$$

Moreover it is immediate that

$$\frac{W_1(f_t^{\otimes N}, \gamma^{\otimes N})}{N} = W_1(f_t, \gamma).$$

Finally it was proved in [32, 12] that under our assumptions on f_0 one has

$$\|(f_t - \gamma) \langle v \rangle\|_{L^1} \leq C e^{-\lambda_1 t}$$

for some constants $C > 0$ and $\lambda_1 > 0$ which implies

$$W_1(f_t, \gamma) \leq \|(f_t - \gamma) \langle v \rangle\|_{L^1} \leq C e^{-\lambda_1 t}.$$

Hence, gathering these three estimates, we deduce that

$$(5.31) \quad \frac{W_1(f_t^N, \gamma^N)}{N} \leq 2\alpha(N) + C e^{-\lambda_1 t}$$

for some polynomial rate $\alpha(N) \rightarrow 0$ as $N \rightarrow +\infty$.

On the other hand, it was proved in [13] that there exists $\lambda_2 > 0$ such that

$$\forall N \geq 1, \forall t \geq 0, \quad \|h^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)} \leq e^{-\lambda_2 t} \|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)},$$

where $h^N = df^N/d\gamma^N$ is the Radon-Nikodym derivative of f^N with respect to the measure γ^N so that $f^N = h^N \gamma^N$. When $f_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})}$ with $f_0 \in \mathcal{P}_4(\mathbb{R}^d)$ we easily bound from above the right-hand side term by

$$\|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)} \leq A^N,$$

where $A = A(f_0) > 1$. Then by the Cauchy-Schwartz inequality we also have

$$\|h^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)} \geq \|h^N - 1\|_{L^1(\mathcal{S}^N(\mathcal{E}), \gamma^N)}$$

and the Wasserstein distance can be controlled as

$$\begin{aligned} W_1(f^N, \gamma^N) &= \sup_{\|\varphi\|_{C^{0,1}(\mathbb{R}^{dN})} \leq 1} \int_{\mathbb{R}^{dN}} \varphi (df^N - d\gamma^N) \\ &= \sup_{\|\varphi\|_{C^{0,1}(\mathbb{R}^{dN})} \leq 1} \int_{\mathbb{R}^{dN}} (\varphi - \varphi(0)) (df^N - d\gamma^N) \\ &\leq \int_{\mathbb{R}^{dN}} \left(\sum_{i=1}^N |v_i| \right) |df^N - d\gamma^N| \\ &\leq N \mathcal{E}^{1/2} \|h^N - 1\|_{L^1(\mathcal{S}^N(\mathcal{E}), \gamma^N)}. \end{aligned}$$

We hence deduce

$$(5.32) \quad \forall N \geq 1, \forall t \geq 0, \quad \frac{W_1(f_t^N, \gamma^N)}{N} \leq A^N e^{-\lambda_2 t}.$$

Finally by combining (5.31) when

$$N \geq N(t) := \frac{\lambda_2 t}{2 \ln A},$$

and (5.32) when $N \leq N(t)$, we easily obtain

$$\forall N \geq 1, \forall t \geq 0, \quad \frac{W_1(f_t^N, \gamma^N)}{N} \leq \min \left\{ e^{-\frac{\lambda_2}{2} t}; \alpha(N(t)) + C e^{-\lambda_1 t} \right\} =: \beta(t)$$

for some polynomial rate $\beta(t) \rightarrow 0$ as $t \rightarrow +\infty$, which concludes the proof of (5.10).

6. HARD SPHERES

6.1. The model. The limit equation was introduced in Subsection 1.1 and the stochastic model has been already discussed Subsection 5.1.

We consider here the case of the Master equation (5.3), (5.4) and the limit nonlinear homogeneous Boltzmann equation (1.1), (1.2), (1.3) with

$$(6.1) \quad B(z, \cos \theta) = \Gamma(z) b(\cos \theta) = \Gamma(z) = |z|.$$

6.2. Statement of the result. Our fluctuations estimate result for this model then states as follows:

Theorem 6.1 (Hard spheres detailed chaos estimates). *Assume that the collision kernel B satisfies (6.1). Let us consider a family of N -particle initial conditions $f_0^N \in P_{\text{sym}}((\mathbb{R}^d)^N)$ and the associated N -particle system dynamics*

$$f_t^N = S_t^N (f_0^N).$$

Let us also consider a centered 1-particle initial distribution $f_0 \in P(\mathbb{R}^d)$ with energy $\mathcal{E} \in (0, +\infty)$

$$\int_{\mathbb{R}^d} v \, df_0 = 0, \quad \mathcal{E} := \int_{\mathbb{R}^d} |v|^2 \, df_0 \in (0, \infty),$$

and the associated solution

$$f_t = S_t^{NL} (f_0)$$

of the limit mean-field equation.

Let us finally fix some $\delta \in (0, 1)$. Then we have the following results:

(i) *Suppose that f_0 has compact support*

$$\text{Supp } f_0 \subset \left\{ v \in \mathbb{R}^d, |v| \leq A \right\}$$

and that the N -particle initial data are tensorized

$$\forall N \geq 1, \quad f_0^N = f_0^{\otimes N}.$$

Then for any $T \in (0, \infty)$ there are

- some constants $k_1 \geq 2$ depending on δ and A ;*
- some constant $C_{\delta, T} > 0$ depending on δ , T and A , and blowing up as $\delta \rightarrow 1$;*
- some constant $C_{b, T} > 0$ depending on the collision kernel and T ,*

such that for any $\ell \in \mathbb{N}^$, and for any*

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in W^{1, \infty}(\mathbb{R}^d)^{\otimes \ell},$$

we have

$$\begin{aligned} \forall N \geq 2\ell, \quad \sup_{t \in [0, T]} \left| \left\langle \left(S_t^N (f_0^N) - (S_t^{NL} (f_0))^{\otimes N} \right), \varphi \right\rangle \right| \\ \leq \|\varphi\|_{W^{1, \infty}(\mathbb{R}^d)^{\otimes \ell}} \left[\frac{2\ell^2}{N} + \frac{C_{\delta, T} \ell^2 \|f_0\|_{M_{k_1}^1}}{N^{1-\delta}} + \ell e^{C_{b, T} A} \theta(N) \right]. \end{aligned}$$

The last term of the right-hand side (which is also the dominant error term as N goes to infinity in our estimate) is given by

$$\theta(N) = \frac{C}{(1 + |\ln N|)^\alpha}$$

for some constants $C, \alpha > 0$.

(ii) *Under the same setting but assuming instead for the initial datum of the mean-field limit*

$$(6.2) \quad f_0 \in L^\infty(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} e^{z|v|} \, df_0(v) < +\infty$$

for some $z > 0$, and taking for the N -particle initial data the sequence $(f_0^N)_{N \geq 1}$ constructed in Lemma 4.4 and 4.7 by conditioning to the Boltzmann sphere $\mathcal{S}^N(\mathcal{E})$, then the solution $f_t^N = S_t^N (f_0^N)$ has its support included in $\mathcal{S}^N(\mathcal{E})$ for all times

$$(6.3) \quad \forall t \geq 0, \quad \text{Supp } f_t^N \subset \mathcal{S}^N(\mathcal{E})$$

and there are

- some constants $k_1 \geq 2$ depending on δ and \mathcal{E} ;*

- some constant $C_\delta > 0$ depending on δ and \mathcal{E} , and blowing up as $\delta \rightarrow 1$;
 - some constant $C_b > 0$ depending on the collision kernel and the above exponential moment bound on f ,
- such that for any $\ell \in \mathbb{N}^*$, and for any

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in W^{1,\infty}(\mathbb{R}^d)^{\otimes \ell},$$

we have

$$\begin{aligned} \forall N \geq 2\ell, \quad \sup_{t \geq 0} \left| \left\langle \left(S_t^N(f_0^N) - (S_t^{NL}(f_0))^{\otimes N} \right), \varphi \right\rangle \right| \\ \leq \|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)^{\otimes \ell}} \left[\frac{2\ell^2}{N} + \frac{C_\delta \ell^2 \|f_0\|_{M_{k_1}^1}}{N^{1-\delta}} + \ell e^{C_b A} \theta(N) \right], \end{aligned}$$

with the same estimate on the rate $\theta(N)$ as in (i). This proves the propagation of chaos, uniformly in time.

We now state again another version of the propagation of chaos estimate, in Wasserstein distance, but most importantly which is valid for any number of marginals, at the price of a possibly worse (but still constructive) rate. Combined with previous results on the relaxation of the N -particle system we also deduce some estimate of relaxation to equilibrium independent of N and, again, for any number of marginals.

Theorem 6.2 (Hard spheres Wasserstein chaos). *We consider the same setting as in Theorem 6.1, where the initial data are chosen as follows:*

- (a) either f_0 is compactly supported and $f_0^N = f_0^{\otimes N}$,
- (b) or f_0 satisfying (6.2) and f_0^N is constructed by Lemma 4.4 by conditioning to the Boltzmann sphere $\mathcal{S}^N(\mathcal{E})$.

Then in the case (a) we have for any $T > 0$

$$(6.4) \quad \forall N \geq 1, \forall 1 \leq \ell \leq N, \quad \sup_{t \in [0, T]} \frac{W_1(\Pi_\ell f_t^N, f_t^{\otimes \ell})}{\ell} \leq \alpha_T(N)$$

for some $\alpha_T(N) \rightarrow 0$ as $N \rightarrow \infty$ like a power of a logarithm, and possibly depending on T .

In the case (b) the solution $f_t^N = S_t^N(f_0^N)$ has its support included in $\mathcal{S}^N(\mathcal{E})$ for all times and this estimate can be made uniform in time:

$$(6.5) \quad \forall N \geq 1, \forall 1 \leq \ell \leq N, \quad \sup_{t \geq 0} \frac{W_1(\Pi_\ell f_t^N, f_t^{\otimes \ell})}{\ell} \leq \alpha(N)$$

for some $\alpha(N) \rightarrow 0$ as $N \rightarrow \infty$ like a power of a logarithm.

Moreover, still in the case (b), we have

$$(6.6) \quad \forall N \geq 1, \forall 1 \leq \ell \leq N, \forall t \geq 0, \quad \frac{W_1(\Pi_\ell f_t^N, \Pi_\ell(\gamma^N))}{\ell} \leq \beta(t)$$

for some rate $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$ like a power of logarithm, where γ is the centered Gaussian equilibrium with energy \mathcal{E} and γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

In order to prove Theorem 6.1, we shall prove assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** of Theorem 3.1 with $T < \infty$ or $T = \infty$, and with suitable functional spaces. The application of the latter theorem then exactly yields Theorem 6.1 by following carefully each constant computed below. We fix

$$\mathcal{F}_1 = \mathcal{F}_2 = C_b(\mathbb{R}^d) \quad \text{and} \quad \mathcal{F}_3 = \text{Lip}(\mathbb{R}^d).$$

Then the proof of Theorem 6.2 is deduced from Theorem 6.1 in a similar way as Theorem 5.2 was deduced from Theorem 5.1, see Subsection 6.9.

6.3. Proof of condition (A1). From the discussion made in section 5.3 we easily see that for the hard spheres model the operator G^N is bounded from $C_{-k+1,0}(\mathbb{R}^{dN})$ onto $C_{-k,0}(\mathbb{R}^{dN})$ for any $k \in \mathbb{R}$. Since G^N is close, dissipative and $C_{-k+1,0}(\mathbb{R}^{dN})$ is dense in $C_{-k,0}(\mathbb{R}^{dN})$, the Hille-Yosida theory implies that G^N generates a Markov type semigroup T_t^N on $C_{-k,0}(\mathbb{R}^{dN})$ and we may also define S_t^N by duality as a semigroup on $\mathcal{P}_k(\mathbb{R}^{dN})$. The nonlinear semigroup S_t^{NL} is also well defined on $\mathcal{P}_k(\mathbb{R}^d)$, $k \geq 2$, see for instance [31, 28, 48].

Lemma 5.3 was proved both for Maxwell molecules and hard spheres. It first shows that

$$\forall t \geq 0, \quad \text{Supp } f_t^N \subset \mathbb{E}_N := \{V \in (\mathbb{R}^d)^N; \langle \mu_V^N, \mathbf{m}_{\mathcal{G}_1} \rangle \in \mathcal{R}_{\mathcal{G}_1}\},$$

where $\mathbf{m}_{\mathcal{G}_1} : \mathbb{R}^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$, $\mathbf{m}_{\mathcal{G}_1}(v) = (|v|^2, v)$ and

$$(6.7) \quad \mathcal{R}_{\mathcal{G}_1} := \left\{ \mathbf{r} = (r_0, r') \in \mathbb{R}_+ \times \mathbb{R}^d, |r'|^2 \leq r_0 \leq \mathcal{E}_0 \right\} \text{ with } \mathcal{E}_0 = A^2 \text{ in case (i),}$$

and

$$(6.8) \quad \mathcal{R}_{\mathcal{G}_1} := \{(\mathcal{E}_0, 0)\} \subset \mathbb{R}_+ \times \mathbb{R}^d, \quad \mathcal{E}_0 := \mathcal{E}, \text{ in case (ii).}$$

It also proves that for any $k \geq 2$,

$$\sup_{t \geq 0} \langle f_t^N, M_k^N \rangle \leq C_k^N$$

where C_k^N depends on k , \mathcal{E}_0 , on the collision kernel and on the initial value

$$\langle f_0^N, M_k^N \rangle$$

which is uniformly bounded in N in terms of k and \mathcal{E}_0 . This shows that **(A1)-(ii)** holds with $m_1(v) := |v|^{k_1}$ for any $k_1 \geq 2$. The precise value of k_1 shall be chosen in Section 6.7.

As for **(A1)-(iii)**, we remark that for a given N -particle velocity

$$V = (v_1, \dots, v_N) \in \mathbb{R}^{dN},$$

we have

$$V \in \text{Supp } f_0^{\otimes N} \iff \forall i = 1, \dots, N, v_i \in \text{Supp } f_0$$

which implies

$$\forall i = 1, \dots, N, \quad m_{\mathcal{G}_3}(v_i) \leq m_{\mathcal{G}_3}(A) \text{ with } m_{\mathcal{G}_3}(v) := e^{a|v|}$$

for any constant $a > 0$, which shall be chosen later on.

We conclude that

$$\text{Supp } f_0^{\otimes N} \subset \left\{ V \in \mathbb{R}^{dN}; M_{m_{\mathcal{G}_3}}^N(V) \leq m_{\mathcal{G}_3}(A) \right\},$$

and **(A1)-(iii)** holds for the exponential growing weight $m_{\mathcal{G}_3}$.

6.4. Proof of condition (A2). For a given $k_1 \geq 2$, let us define the space of probability measures

$$\mathcal{P}_{\mathcal{G}_1} := \left\{ f \in P(\mathbb{R}^d); M_{k_1}(f) < +\infty \right\},$$

the sets of constraints $\mathcal{R}_{\mathcal{G}_1}$ given by (6.7) or (6.8), the constrained space (for $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$)

$$\mathcal{P}_{\mathcal{G}_1, \mathbf{r}} := \left\{ f \in \mathcal{P}_{k_1}(\mathbb{R}^d); \langle f, |v|^2 \rangle = r_0, \quad \forall i = 1, \dots, d, \langle f, v_i \rangle = r_i \right\},$$

and the vector space

$$\mathcal{G}_1 := \left\{ h \in M_{k_1}^1(\mathbb{R}^d); \quad \forall i = 1, \dots, d, \langle h, v_i \rangle = \langle h, 1 \rangle = \langle h, |v|^2 \rangle = 0 \right\}$$

endowed with the total variation norm $\|\cdot\|_{\mathcal{G}_1} := \|\cdot\|_{M^1}$. We also define

$$\mathcal{BP}_{\mathcal{G}_1, a} := \{f \in \mathcal{P}_{\mathcal{G}_1}; M_{k_1}(f) \leq a\}$$

as well as for any $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$ the (possibly empty) bounded constrained space

$$\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}} := \left\{ f \in \mathcal{BP}_{\mathcal{G}_1, a}; \quad \langle f, |v|^2 \rangle = r_0, \quad \forall i = 1, \dots, d, \quad \langle f, v_i \rangle = r_i \right\}$$

endowed with the distance $d_{\mathcal{G}_1}$ associated to the norm $\|\cdot\|_{\mathcal{G}_1}$.

The proof of the assertion **(A2)-(i)** is postponed to section 6.6, where we prove in (6.11) a Hölder continuity of the flow in $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$, $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$.

Let us prove the assertion **(A2)-(ii)**, that is the fact that the operator Q is bounded and Hölder continuous from $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ to \mathcal{G}_1 . For any $f, g \in \mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ we have

$$\begin{aligned} \|Q(g, g) - Q(f, f)\|_{M^1} &= \|Q(g - f, g + f)\|_{M^1} \\ &\leq 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b(\cos \theta) |v - v_*| |f - g| |f_* + g_*| d\sigma dv_* dv \\ &\leq 2(1+a) \|b\|_{L^1} \|(f - g) \langle v \rangle\|_{M^1}. \end{aligned}$$

We deduce that

$$\|Q(g, g) - Q(f, f)\|_{M^1} \leq 2(1+a)^{3/2} \|b\|_{L^1} \|f - g\|_{M^1}^{1/2}$$

which yields

$$Q \in C^{0, 1/2}(\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}; \mathcal{G}_1)$$

and also implies that Q is bounded on $\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}}$ since we can choose g to be a Maxwellian distribution, for which $Q(g, g) = 0$.

6.5. Proof of condition (A3). Let us define the weight

$$\Lambda_1(f) := M_{k_1}(f) = \langle f, \langle v \rangle^{k_1} \rangle$$

(this means that we choose $m'_{\mathcal{G}_1}(v) = \langle v \rangle^{k_1}$ in assumption **(A3)**).

We claim that there exists a constant $C_{k_1} > 0$ (depending on k_1) such that for any $\eta \in (0, 1)$ and any function

$$(6.9) \quad \Phi \in \bigcap_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathbb{R}),$$

we have

$$(6.10) \quad \forall V \in \mathbb{E}_N, \quad |G^N(\Phi \circ \mu_V^N) - (G^\infty \Phi)(\mu_V^N)| \leq C_{k_1} \mathcal{E}_0 \left(\sup_{\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}} [\Phi]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}})} \right) \frac{M_{k_1}^N(V)}{N^\eta}$$

where we recall that

$$M_{k_1}^N(V) := \frac{1}{N} \sum_{i=1}^N \langle v_i \rangle^{k_1}.$$

This would prove assumption **(A3)** with the rate

$$\varepsilon(N) = \frac{C_{k_1}}{N^\eta}, \quad \eta := 1 - \delta.$$

For a given Φ satisfying (6.9), for any $V \in \mathbb{E}_N$ let us set

$$\phi := D\Phi[\mu_V^N]$$

and remark that

$$\mathbf{r}_V := \left(\langle \mu_V^N, |z|^2 \rangle, \langle \mu_V^N, z_1 \rangle, \dots, \langle \mu_V^N, z_d \rangle \right) \in \mathcal{R}_{\mathcal{G}_1}$$

where $z = (z_1, \dots, z_d) \in \mathbb{R}^d$ has to be understood as the blind integration variable in the duality bracket.

We then compute

$$\begin{aligned}
G^N(\Phi \circ \mu_V^N) &= \frac{1}{2N} \sum_{i,j=1}^N |v_i - v_j| \int_{\mathbb{S}^{d-1}} \left[\Phi(\mu_{V_{ij}^*}^N) - \Phi(\mu_V^N) \right] b(\theta_{ij}) \, d\sigma \\
&= \frac{1}{2N} \sum_{i,j=1}^N |v_i - v_j| \int_{\mathbb{S}^{d-1}} \langle \mu_{V_{ij}^*}^N - \mu_V^N, \phi \rangle b(\theta_{ij}) \, d\sigma \\
&\quad + \frac{[\Phi]_{C_{\Lambda_1}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})}}{2N} \sum_{i,j=1}^N |v_i - v_j| \times \\
&\quad \int_{\mathbb{S}^{d-1}} \max \left\{ M_{m'_{\mathcal{G}_1}}(\mu_{V_{ij}^*}^N); M_{m'_{\mathcal{G}_1}}(\mu_V^N) \right\} \mathcal{O} \left(\left\| \mu_{V_{ij}^*}^N - \mu_V^N \right\|_{M^1}^{1+\eta} \right) \, d\sigma \\
&=: I_1(V) + I_2(V).
\end{aligned}$$

For the first term $I_1(V)$ we argue similarly than in the proof of **(A3)** for the Maxwell molecules case in section 5.5, and we get

$$I_1(V) = \langle Q(\mu_V^N, \mu_V^N), \phi \rangle = (G^\infty \Phi)(\mu_V^N).$$

As for the second term $I_2(V)$, using (5.19) and $\|\mu_V^N - \mu_{V_{ij}^*}^N\|_{M^1} \leq 4/N$, we deduce

$$\begin{aligned}
|I_2(V)| &\leq C_{k_1} M_{k_1}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})} \left(\frac{1}{2N} \sum_{i,j=1}^N |v_i - v_j| \left(\frac{4}{N} \right)^{1+\eta} \right) \\
&\leq C_{k_1} M_{k_1}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})} \left(\frac{1}{N^\eta} \frac{1}{N^2} \sum_{i,j=1}^N (\langle v_i \rangle + \langle v_j \rangle) \right) \\
&\leq \frac{C_{k_1}}{N^\eta} M_{k_1}^N(V) M_2^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})}.
\end{aligned}$$

We then use the elementary inequality the energy bound to deduce

$$|I_2(V)| \leq \frac{C_{k_1} \mathcal{E}_0}{N^\eta} M_{k_1}^N(V) [\Phi]_{C_{\Lambda}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1, r_V})}.$$

We conclude that (6.10) holds by combining the two last estimates on I_1 and I_2 .

6.6. Proof of condition (A4) with time growing bounds. Let us consider some 1-particle initial data

$$f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1}.$$

In a similar way as in the previous section, we then define (under the assumption (6.1) on the collision kernel) the associated solutions f_t and g_t to the nonlinear Boltzmann equation (1.1), as well as

$$h_t := \mathcal{D}_t^{NL} [f_0] (g_0 - f_0)$$

the solution to the linearized Boltzmann equation around f_t . These solutions are given by

$$\begin{cases} \partial_t f_t = Q(f_t, f_t), & f_{|t=0} = f_0 \\ \partial_t g_t = Q(g_t, g_t), & g_{|t=0} = g_0 \\ \partial_t h_t = 2Q(f_t, h_t), & h_{|t=0} = h_0 := g_0 - f_0. \end{cases}$$

We also define as before

$$\omega_t := g_t - f_t - h_t.$$

We shall now again *expand the limit nonlinear semigroup* in terms of the initial data, around f_0 . The goal is to prove assumption **(A4)**. This imposes the choice of weight

$$\Lambda_2(f) = \Lambda_1(f)^{\frac{1}{2}} = \sqrt{M_{k_1}(f)}.$$

Lemma 6.3. *For any given energy $\mathcal{E}_0 > 0$ and any $\eta \in (0, 1)$ there exists*

- *some constant $\bar{k}_1 \geq 2$ (depending on \mathcal{E}_0 and η),*
- *some constant $C > 0$ (depending on \mathcal{E}_0),*

such that for $k_1 \geq \bar{k}_1$, for any

$$\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1} \quad \text{and} \quad f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$$

and for any $t \geq 0$ we have

$$(6.11) \quad \|g_t - f_t\|_{M_2^1} \leq e^{C(1+t)} \sqrt{M_{k_1}(f_0 + g_0)} \|f_0 - g_0\|_{M_1^1}^\eta,$$

$$(6.12) \quad \|h_t\|_{M_2^1} \leq e^{C(1+t)} \sqrt{M_{k_1}(f_0 + g_0)} \|f_0 - g_0\|_{M_1^1}^\eta,$$

$$(6.13) \quad \|\omega_t\|_{M_2^1} \leq e^{C(1+t)} \sqrt{M_{k_1}(f_0 + g_0)} \|f_0 - g_0\|_{M_1^1}^{1+\eta}.$$

Proof of Lemma 6.3. We proceed in several steps and number the constants for clarity.

Let us define

$$\forall h \in M^1(\mathbb{R}^d), \quad \|h\|_{M_k^1} := \int_{\mathbb{R}^d} \langle v \rangle^k d|h|(v), \quad \|h\|_{M_{k,\ell}^1} := \int_{\mathbb{R}^d} \langle v \rangle^k (1 + \ln \langle v \rangle)^\ell d|h|(v).$$

Step 1. The strategy. Existence and uniqueness for f_t , g_t and h_t is a consequence of the following important stability argument that we use several times. This estimate is due to DiBlasio [19] in a L^1 framework, and it has been recently extended to a measure framework in [28, Lemma 3.2] (see also [31] and [48] for other argument of uniqueness for measure solutions of the spatially homogeneous Boltzmann equation).

Let us sketch the argument for h . We first write

$$(6.14) \quad \begin{aligned} \frac{d}{dt} \int \langle v \rangle^2 d|h_t|(v) &\leq \iiint d|h_t|(v) df_t(v_*) |u| b(\theta) \left[\langle v' \rangle^2 + \langle v_*' \rangle^2 - \langle v \rangle^2 - \langle v_* \rangle^2 \right] d\sigma \\ &\quad + 2 \iiint d|h_t|(v) df_t(v_*) |u| b(\theta) \langle v_* \rangle^2 d\sigma \end{aligned}$$

(this formal computation can be justified by a regularization procedure, we refer to [28] for instance). Since the first term vanishes, we deduce that

$$(6.15) \quad \frac{d}{dt} \|h_t\|_{M_2^1} \leq C_1 \|f\|_{M_3^1} \|h_t\|_{M_2^1}$$

for some constant $C_1 > 0$ only depending on b .

Then in the case when

$$(6.16) \quad \|f_s\|_{M_3^1} \in L^1(0, t) \quad \text{on some time interval } s \in [0, t]$$

we may integrate this differential inequality and we deduce that the solution h to the linear equation $\partial_t h = 2Q(f_t, h)$ is unique in M_2^1 .

More precisely, we have established

$$(6.17) \quad \sup_{s \in [0, t]} \|h_s\|_{M_2^1} \leq \|g_0 - f_0\|_{M_2^1} \exp \left(C_1 \int_0^t \|f_s\|_{M_3^1} ds \right),$$

and similar arguments imply

$$(6.18) \quad \sup_{s \in [0, t]} \|f_s - g_s\|_{M_2^1} \leq \|g_0 - f_0\|_{M_2^1} \exp \left(C_1 \int_0^t \|f_s + g_s\|_{M_3^1} ds \right).$$

It is worth mentioning that one cannot prove (6.16) under the sole assumption

$$\|f_0\|_{M_2^1} < \infty$$

on the initial data since it would contradict the non-uniqueness result of [49]. However, as we prove in (6.20) below, one may show (thanks to the Povzner inequality, as developed in [59, 47]) that (6.16) holds as soon as

$$\|f_0\|_{M_{2,1}^1} < \infty.$$

This will be a key step for establishing (6.11) and (6.12).

Now, our goal is to estimate the M_2^1 norm of

$$\omega_t := g_t - f_t - h_t$$

in terms of $\|g_0 - f_0\|_{M_2^1}$. The measure ω_t satisfies the evolution equation:

$$\partial_t \omega_t = Q(g_t, g_t) - Q(f_t, f_t) - Q(h_t, f_t) - Q(f_t, h_t), \quad \omega_0 = 0,$$

which can be rewritten as

$$\partial_t \omega_t = Q(\omega_t, f_t + g_t) + Q(h_t, g_t - f_t).$$

The same arguments as in (6.14)-(6.15) yield the following differential inequality

$$\frac{d}{dt} \|\omega_t\|_{M_2^1} \leq C_2 \|\omega_t\|_{M_2^1} \|f_t + g_t\|_{M_3^1} + \|Q(h_t, g_t - f_t)\|_{M_2^1}, \quad \|\omega_0\|_{M_2^1} = 0,$$

for some constant $C_2 > 0$ depending on b .

We deduce

$$\sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} \leq \left(\int_0^t \|Q(h_s, f_s - g_s)\|_{M_2^1} ds \right) \exp \left(C_2 \int_0^t \|f_s + g_s\|_{M_3^1} ds \right).$$

Since

$$\begin{aligned} \int_0^t \|Q(h_s, f_s - g_s)\|_{M_2^1} ds &\leq C_2 \left(\sup_{s \in [0, t]} \|h_s\|_{M_2^1} \right) \left(\int_0^t \|g_s - f_s\|_{M_3^1} ds \right) \\ &\quad + C_2 \left(\sup_{s \in [0, t]} \|g_s - f_s\|_{M_2^1} \right) \left(\int_0^t \|h_s\|_{M_3^1} ds \right), \end{aligned}$$

we deduce from (6.17) and (6.18)

$$(6.19) \quad \begin{aligned} \sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} &\leq C_2 \|g_0 - f_0\|_{M_2^1} \exp \left(C_2 \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds \right) \\ &\quad \times \left[\left(\int_0^t \|g_s - f_s\|_{M_3^1} ds \right) \exp \left(C_1 \int_0^t \|f_s\|_{M_3^1} ds \right) \right. \\ &\quad \left. + \left(\int_0^t \|h_s\|_{M_3^1} ds \right) \exp \left(C_1 \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds \right) \right]. \end{aligned}$$

Hence the problem now reduces to the obtaining of sharp enough time integral controls over

$$\|f_s\|_{M_3^1}, \quad \|g_s\|_{M_3^1}, \quad \|f_s - g_s\|_{M_3^1} \quad \text{and} \quad \|h_s\|_{M_3^1}.$$

Step 2. Time integral control of f and g in M_3^1 . In this step we prove

$$(6.20) \quad \int_0^t \|f_s\|_{M_{3,\ell-1}^1} dt \leq C_3(\mathcal{E}_0) t + C_4 \|f_0\|_{M_{2,\ell}^1} \quad \ell = 1, 2,$$

for the solution f_t , where $C_3(\mathcal{E}_0) > 0$ is a constant depending on the energy, and $C_4 > 0$ is a numerical constant. The same estimate obviously holds for the solution g_t .

The estimates (6.20) are a consequence of the accurate version of the Povzner inequality which has been proved in [59, 47]. Indeed it was shown in [59, Lemma 2.2] that for any function

$$\Psi : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \Psi(v) = \psi(|v|^2) \quad \text{with } \psi \text{ convex,}$$

the solution f_t to the hard spheres Boltzmann equation satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^d} \Psi(v) df_t(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} df_t(v) df_t(v_*) |v - v_*| K_\Psi(v, v_*)$$

with $K_\Psi = G_\Psi - H_\Psi$, where the term G_Ψ “behaves mildly” (see below) and the term H_Ψ is given by (see [59, formula (2.7)])

$$H_\Psi(v, v_*) = 2\pi \int_0^{\pi/2} \left[\psi(|v|^2 \cos^2 \theta + |v_*|^2 \sin^2 \theta) - \cos^2 \theta \psi(|v|^2) - \sin^2 \theta \psi(|v_*|^2) \right] d\theta.$$

Note that $H_\Psi \geq 0$ since its integrand is nonnegative because of the convexity of ψ .

More precisely, in the cases that we are interested with, namely

$$\Psi(v) = \psi_{2,\ell}(|v|^2) \quad \text{with } \psi_{k,\ell}(r) = r^{k/2} (\ln r)^\ell \quad \text{and } \ell = 1, 2,$$

it is established in [59] that (with obvious notation)

$$\forall v, v_* \in \mathbb{R}^d, \quad |G_{\psi_{2,\ell}}(v, v_*)| \leq C_5(\ell) \langle v \rangle (\ln \langle v \rangle)^\ell \langle v_* \rangle (\ln \langle v_* \rangle)^\ell$$

for some constant $C_5(\ell) > 0$ depending on ℓ .

On the other hand, in the case $\ell = 1$, we compute, with the help of the the notation $x := \cos^2 \theta$ and $u = |v_*|/|v|$,

$$\forall x \in [1/4, 3/4], \quad \forall u \in [0, 1/2],$$

$$\begin{aligned} & \psi_{2,1}(|v|^2 \cos^2 \theta + |v_*|^2 \sin^2 \theta) - \cos^2 \theta \psi_{2,1}(|v|^2) - \sin^2 \theta \psi_{2,1}(|v_*|^2) = \\ & = |v|^2 \left[(1-x) \psi_{2,1}(u^2) + x \psi_{2,1}(1) - \psi_{2,1}((1-x)u^2 + x) \right] \geq C_6 |v|^2, \end{aligned}$$

for some numerical constant $C_6 > 0$, which only depends on the strict convexity of the real function $\psi_{2,1}$. We deduce that there exists a constant $C_7 > 0$ such that

$$H_{\psi_{2,1}}(v, v_*) \geq C_7 |v|^2 \mathbf{1}_{|v| \geq 2|v_*|}.$$

Similarly, in the case $\ell = 2$, we have

$$\forall x \in [1/4, 3/4], \quad \forall u \in [0, 1/2],$$

$$\begin{aligned} & \psi_{2,2}(|v|^2 \cos^2 \theta + |v_*|^2 \sin^2 \theta) - \cos^2 \theta \psi_{2,2}(|v|^2) - \sin^2 \theta \psi_{2,2}(|v_*|^2) = \\ & = 2|v|^2 \ln |v|^2 \left\{ (1-x) \psi_{2,1}(u^2) + x \psi_{2,1}(1) - \psi_{2,1}((1-x)u^2 + x) \right\} \\ & + |v|^2 \left[(1-x) \psi_{2,2}(u^2) + x \psi_{2,2}(1) - \psi_{2,2}((1-x)u^2 + x) \right] \geq C_8 |v|^2 \ln |v|^2, \end{aligned}$$

for some constant $C_8 > 0$ depending on the strict convexity of $\psi_{2,1}$ and $\psi_{2,2}$. Hence we obtain for some constant $C_9 > 0$

$$H_{\psi_{2,2}}(v, v_*) \geq C_9 |v|^2 \ln |v|^2 \mathbf{1}_{|v| \geq 2|v_*|}.$$

Putting together the estimates obtained on $G_{2,\ell}$ and $H_{2,\ell}$ we deduce

$$|v - v_*| K_{2,\ell} \leq C_{10} |v - v_*| \langle v \rangle \langle v_* \rangle (\ln \langle v \rangle)^\ell (\ln \langle v_* \rangle)^\ell - C_{11} |v - v_*| |v|^2 (\ln |v|)^{\ell-1} \mathbf{1}_{|v| \geq 2|v_*|}$$

for some constants $C_{10}, C_{11} > 0$. Since

$$\begin{aligned} |v - v_*| |v|^2 (\ln |v|)^{\ell-1} \mathbf{1}_{|v| \geq 2|v_*|} &\geq \text{Cst.} \langle v \rangle^3 (\ln \langle v \rangle)^{\ell-1} \mathbf{1}_{|v| \geq 2|v_*|} - \text{Cst.} \\ &\geq \text{Cst.} \langle v \rangle^3 (\ln \langle v \rangle)^{\ell-1} - \text{Cst.} \langle v \rangle^2 \langle v_* \rangle^2 \end{aligned}$$

and

$$\begin{aligned} |v - v_*| \langle v \rangle \langle v_* \rangle (\ln \langle v \rangle)^\ell (\ln \langle v_* \rangle)^\ell &\leq \langle v \rangle^2 \langle v_* \rangle (\ln \langle v \rangle)^\ell (\ln \langle v_* \rangle)^\ell + \langle v \rangle \langle v_* \rangle^2 (\ln \langle v \rangle)^\ell (\ln \langle v_* \rangle)^\ell \\ &\leq \text{Cst.} \langle v \rangle^2 \langle v_* \rangle^2 (\ln \langle v \rangle)^\ell + \langle v \rangle^2 \langle v_* \rangle^2 (\ln \langle v_* \rangle)^\ell \end{aligned}$$

we easily deduce

$$(6.21) \quad |v - v_*| K_{2,\ell} \leq C_{12}(R) \langle v \rangle^2 \langle v_* \rangle^2 + C_{13}(R) \langle v \rangle^3 (\ln \langle v \rangle)^{\ell-1} \langle v_* \rangle^2 - C_{14} \langle v \rangle^3 (\ln \langle v \rangle)^{\ell-1}$$

for some free cutoff parameter $R > 0$, some constant $C_{12}(R) \rightarrow +\infty$ as $R \rightarrow +\infty$, $C_{13}(R) \rightarrow 0$ as $R \rightarrow +\infty$, and $C_{14} > 0$, and we finally obtain the differential inequality

$$\frac{d}{dt} \|f_t\|_{M_{2,\ell}^1} \leq C_{12}(R) (1 + \mathcal{E}_0)^2 + C_{13}(R) (1 + \mathcal{E}_0) M_{3,\ell-1} - C_{14} M_{3,\ell-1},$$

from which (6.20) follows by choosing $R > 0$ large enough.

Step 3. Exponential time integral control of f and g in M_3^1 . This step yields a proof of (6.11) and (6.12).

Let us first prove that

$$(6.22) \quad \forall t \geq 0, \quad e^{(1+2C_1+C_2) \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds} \leq C_{15}(\mathcal{E}_0) e^{C_{16}(\mathcal{E}_0) t} (\max \{M_k(f_0), M_k(g_0)\})^{\frac{1}{6}},$$

for some constants $C_{15}(\mathcal{E}_0), C_{16}(\mathcal{E}_0) > 0$ depending on the energy \mathcal{E}_0 , for any $k \geq k_{\mathcal{E}_0}$, with $k_{\mathcal{E}_0}$ big enough depending on the energy \mathcal{E}_0 .

We shall use the previous step and an interpolation argument. For any given probability measure

$$f \in \mathcal{P}_k(\mathbb{R}^d) \quad \text{with} \quad \int_{\mathbb{R}^d} |v|^2 df(v) \leq \mathcal{E}_0,$$

we have for any $a > 2$

$$\begin{aligned} \|f\|_{M_{2,1}^1} &= \int_{\mathbb{R}^d} \langle v \rangle^2 \left(1 + \frac{\ln(\langle v \rangle^2)}{2}\right) (\mathbf{1}_{\langle v \rangle^2 \leq a} + \mathbf{1}_{\langle v \rangle^2 \geq a}) df(v) \\ &\leq (1 + \mathcal{E}_0) \left(1 + \frac{\ln a}{2}\right) + \frac{1}{a} \int_{\mathbb{R}^d} \langle v \rangle^4 (1 + \ln \langle v \rangle) df(v) \\ &\leq (1 + \mathcal{E}_0) \left(1 + \frac{\ln a}{2}\right) + \frac{1}{a} \|f\|_{M_5^1} \end{aligned}$$

where we have used inequality $\ln x \leq x - 1$ for $x \geq 1$ in the last step.

By choosing

$$a := \|f\|_{M_5^1}^2,$$

we get

$$(6.23) \quad \|f\|_{M_{2,1}^1} \leq 2(1 + \mathcal{E}_0) \left(1 + \ln \|f\|_{M_5^1}\right).$$

Remark 6.4. Observe here that it was absolutely crucial to be able to control the right-hand side of (6.20) in terms of the $M_{2,1}^1$ moment, that is only a *logarithmic loss* of moment as compared to M_2^1 . This is what allows us to control this right-hand side in terms of the *logarithm* of a higher moment of f , so that the exponential in (6.22) can be controlled in terms of some *polynomial* moment of f , hence fulfilling the requirement on the loss of weight in the stability estimates on the semigroup. Recall indeed that the moment associated with the weight Λ_1 has to be *controlled* along the flow of the N -particle system.

And we have not been able to show the propagation of exponential moment bounds for such a high-dimension evolution.

On the other hand, the following elementary Hölder inequality holds

$$(6.24) \quad \forall k, k' \in \mathbb{N}, k' \leq k, \forall f \in M_k^1, \quad \|f\|_{M_{k'}^1} \leq \|f\|_{M^1}^{1-k'/k} \|f\|_{M_k^1}^{k'/k} \leq \|f\|_{M_k^1}^{k'/k}.$$

Then estimate (6.22) follows from (6.20), (6.23) and (6.24) with $k' = 5$ and $k = k_1 \geq 5$ large enough in such a way that

$$(1 + 2C_1 + C_2) C_4 2(1 + \mathcal{E}_0) \frac{5}{k} \leq \frac{1}{6}.$$

We then deduce (6.11) from (6.18), and (similarly) (6.12) from (6.17).

Step 4. Time integral control on d and h . Let us write as before

$$d_t := f_t - g_t.$$

Let us prove

$$(6.25) \quad \left(\int_0^t \|d_s\|_{M_3^1} ds \right) \quad \text{and} \quad \left(\int_0^t \|h_s\|_{M_3^1} ds \right) \\ \leq C_{20} \|d_0\|_{M_2^1} e^{C_1 \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds} \left(C_3(\mathcal{E}_0) t + C_4 \|f_0\|_{M_{2,2}^1} \right) + C_{21} \|d_0\|_{M_{2,1}^1}.$$

for some constants $C_{20}, C_{21} > 0$ defined later. Performing similar computations to those leading to (6.14), we obtain

$$\frac{d}{dt} \|h_t\|_{M_{2,1}^1} \leq \iint d|h_t|(v) df_t(v_*) |v - v_*| K_{2,1}(v, v_*) \\ + C_{17} \iiint d|h_t|(v) df_t(v_*) |v - v_*| \langle v_* \rangle^2 (1 + \ln \langle v_* \rangle)$$

for some constant $C_{17} > 0$ depending on b . Thanks to the Povzner inequality (6.21) (with $\ell = 1$), we deduce for some constants $C_{18}, C_{19} > 0$

$$\frac{d}{dt} \|h_t\|_{M_{2,1}^1} \leq C_{18} \|h_t\|_{M_2^1} \|f_t\|_{M_{3,1}^1} - C_{19} \|h_t\|_{M_3^1}.$$

Integrating this differential inequality yields

$$\|h_t\|_{M_{2,1}^1} + C_{19} \int_0^t \|h_s\|_{M_3^1} ds \leq C_{18} \left(\sup_{s \in [0, t]} \|h_s\|_{M_2^1} \right) \left(\int_0^t \|f_s\|_{M_{3,1}^1} ds \right) + \|h_0\|_{M_{2,1}^1}.$$

Using the previous pointwise control on $\|h_t\|_{M_2^1}$ and (6.20) (with $\ell = 2$) we get

$$\int_0^t \|h_s\|_{M_3^1} ds \leq \frac{C_{18}}{C_{19}} \|d_0\|_{M_2^1} e^{C_1 \int_0^t \|f_s\|_{M_3^1} ds} \left(C_3(\mathcal{E}_0) t + C_4 \|f_0\|_{M_{2,2}^1} \right) + \frac{1}{C_{19}} \|d_0\|_{M_{2,1}^1}.$$

Arguing similarly for d_t , we deduce (6.25).

Step 5. Conclusion. We first rewrite (6.19) as

$$\sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} \leq C_2 \|d_0\|_{M_2^1} e^{(C_1 + C_2) \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds} \left(\int_0^t (\|d_s\|_{M_3^1} + \|h_s\|_{M_3^1}) ds \right).$$

Then we use the estimate (6.25) for the last term and thus obtain

$$\sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} \leq C_{22} \|d_0\|_{M_2^1} \|d_0\|_{M_{2,1}^1} e^{(2C_1 + C_2) \int_0^t (\|f_s\|_{M_3^1} + \|g_s\|_{M_3^1}) ds} \left(1 + C_3(\mathcal{E}_0) t + C_4 \|f_0\|_{M_{2,2}^1} \right)$$

for some constant $C_{22} > 0$. Finally we use estimate (6.22) for the exponential term with $k = k_1$ and we obtain

$$\begin{aligned} \sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} &\leq C_{22} C_{15}(\mathcal{E}_0) \|d_0\|_{M_2^1} \|d_0\|_{M_{2,1}^1} \times \\ &\quad \times e^{C_{16}(\mathcal{E}_0)t} (\max\{M_{k_1}(f_0), M_{k_1}(g_0)\})^{\frac{1}{6}} \left(1 + C_3(\mathcal{E}_0)t + C_4 \|f_0\|_{M_{2,2}^1}\right). \end{aligned}$$

Then arguing as in the end of Step 3, for any $\eta \in (0, 1)$, using (6.24) with k_1 large enough, we have

$$\|d_0\|_{M_2^1} \|d_0\|_{M_{2,1}^1} \leq (\max\{M_{k_1}(f_0), M_{k_1}(g_0)\})^{\frac{1}{6}} \|d_0\|_{M^1}^{1+\eta}$$

and

$$\|f_0\|_{M_{2,2}^1} \leq (\max\{M_{k_1}(f_0), M_{k_1}(g_0)\})^{\frac{1}{6}}.$$

We therefore obtain the desired estimate (6.13)

$$\sup_{s \in [0, t]} \|\omega_s\|_{M_2^1} \leq e^{C(1+t)} \sqrt{\max\{M_{k_1}(f_0), M_{k_1}(g_0)\}} \|f_0 - g_0\|_{M^1}^{1+\eta}$$

which concludes the proof. \square

6.7. Proof of condition (A4) uniformly in time. Let us start from an auxiliary result from [60]. Let us define the linearized Boltzmann collision operator at γ

$$\mathcal{L}_\gamma(f) := 2Q(\gamma, f)$$

where

$$(6.26) \quad \gamma = \frac{e^{-\frac{|v|^2}{2(\mathcal{E}/d)}}}{(2\pi(\mathcal{E}/d))^d}$$

is the Maxwellian distribution with zero momentum and energy $\mathcal{E} > 0$.

Theorem 6.5 (Theorem 1.2 in [60]). *First the linearized Boltzmann semigroup $e^{\mathcal{L}_\gamma t}$ for hard spheres satisfies*

$$(6.27) \quad \|e^{\mathcal{L}_\gamma t}\|_{L^1(m_z)} \leq C_z e^{-\lambda t}$$

where

$$m_z(v) := e^{z|v|}, \quad z > 0,$$

and $\lambda = \lambda(\mathcal{E})$ is the optimal rate, given by the first non-zero eigenvalue of the linearized operator \mathcal{L}_γ in the smaller space $L^2(\gamma^{-1})$, and $C_z > 0$ is an explicit constant depending on z .

Second the nonlinear Boltzmann semigroup S_t^{NL} satisfies

$$(6.28) \quad \|S_t^{NL}(f_0) - \gamma\|_{L^1(m_z)} \leq C_{f_0} e^{-\lambda t}$$

for any $f_0 \in L^1(m_z)$ with zero momentum and energy $\mathcal{E} > 0$, where C_{f_0} is some constant possibly depending on z and $\|f_0\|_{L^1(m_z)}$, and $\lambda = \lambda(\mathcal{E})$ is the same rate function as for the linearized operator above.

Let us now prove *uniform in time* estimate for the expansion of the limit semigroup in terms of the initial data.

Lemma 6.6. *For any given energy $\mathcal{E} > 0$ and $\eta \in (0, 1)$, there exists*

- some constant $\bar{k}_1 \geq 2$ (depending on \mathcal{E} and η),
- some constant C (depending on \mathcal{E}),

such that for any $k_1 \geq \bar{k}_1$, for any

$$f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1}$$

satisfying

$$\langle f_0, |v|^2 \rangle = \langle g_0, |v|^2 \rangle = \mathcal{E} \quad \text{and} \quad \forall i = 1, \dots, d, \quad \langle f_0, v_i \rangle = \langle g_0, v_i \rangle = 0,$$

and for any $t \geq 0$, we have

$$(6.29) \quad \|g_t - f_t\|_{M_2^1} \leq C e^{-\frac{\lambda}{2}t} \sqrt{M_{k_1}(f_0 + g_0)} \|g_0 - f_0\|_{M^1}^\eta,$$

$$(6.30) \quad \|h_t\|_{M_2^1} \leq C e^{-\frac{\lambda}{2}t} \sqrt{M_{k_1}(f_0 + g_0)} \|g_0 - f_0\|_{M^1}^\eta,$$

$$(6.31) \quad \|\omega_t\|_{M_2^1} \leq C e^{-\frac{\lambda}{2}t} \sqrt{M_{k_1}(f_0 + g_0)} \|g_0 - f_0\|_{M^1}^{1+\eta}.$$

Note that under the assumption (ii) in Theorem 6.1, these estimates imply **(A4)** with $T = +\infty$, $\mathcal{P}_{\mathcal{G}_2} = \mathcal{P}_{\mathcal{G}_1}$, since the momentum and energy conditions are implied by $\mathbf{r} \in \mathcal{R}_{\mathcal{G}_1}$, with $\mathcal{R}_{\mathcal{G}_1}$ defined by (6.8).

Remark 6.7. In the following proof we shall use *moment production bounds* on the limit equation. Indeed once stability estimates for small times have been secured (as in Lemma 6.3), one can use, for $t \geq T_0 > 0$, moments production estimates whose bounds only depend on the energy of the solution. This, together with the linearized theory in L^1 setting with exponential moment bounds of Theorem 6.5, will be the key to the following proof.

Proof of Lemma 6.6. In the proof below, we restrict ourself to an initial datum $f_0 \in \mathcal{P}_{\mathcal{G}_1} \cap L^1(\mathbb{R}^d)$ for the sake of simplicity of the presentation, but the proof straightforwardly applies to measures. From the result of appearance of exponential moments for measure solutions [48, Theorem 1.2-(b)] (see also [1] for another simpler argument in L^1 , and [57] for earlier results of appearance of exponential moments), there exists some constants z, Z (only depending on the collision kernel and the energy of the solutions) such that

$$(6.32) \quad \sup_{t \geq 1} \left(\|f_t\|_{M_{m_{2z}}^1} + \|g_t\|_{M_{m_{2z}}^1} + \|h_t\|_{M_{m_{2z}}^1} \right) \leq Z, \quad m_{2z}(v) := e^{2z|v|}$$

(note that the proof in [48] applies to the solutions f_t and g_t , however it is straightforward to apply exactly the same proof to the linearized solution h_t around f_t , once exponential moment is known on f_t).

We also know from (6.28) that (maybe by choosing a larger Z)

$$(6.33) \quad \forall t \geq 1, \quad \|f_t - \gamma\|_{M_{m_{2z}}^1} + \|g_t - \gamma\|_{M_{m_{2z}}^1} \leq 2Z e^{-\lambda t}.$$

We write

$$\partial_t(f_t - g_t) = Q(f_t - g_t, f_t + g_t) = \mathcal{L}_\gamma(f_t - g_t) + Q(f_t - g_t, f_t - \gamma) + Q(f_t - g_t, g_t - \gamma)$$

and, using also (6.27) on the linearized semigroup, we deduce for

$$u(t) := \|f_t - g_t\|_{M_{m_z}^1}$$

the following differential inequality for $t \geq T_0 \geq 1$ and some constant $C \geq 1$:

$$\begin{aligned} u(t) &\leq C e^{-\lambda(t-T_0)} u(T_0) \\ &\quad + C \int_{T_0}^t e^{-\lambda(t-s)} \left(\|Q(f_s - g_s, f_s - \gamma)\|_{M^1(m_z)} + \|Q(f_s - g_s, g_s - \gamma)\|_{M^1(m_z)} \right) ds \end{aligned}$$

(this formal inequality and next ones can easily be justified rigorously by a regularizing procedure and using a uniqueness result for measure solutions such as [31, 28, 48]). Therefore we obtain

$$u(t) \leq C e^{-\lambda(t-T_0)} u(T_0) + C \int_{T_0}^t e^{-\lambda(t-s)} \left(\|f_s - \gamma\|_{M^1(\langle v \rangle m_z)} + \|g_s - \gamma\|_{M^1(\langle v \rangle m_z)} \right) \|f_s - g_s\|_{M^1(\langle v \rangle m_z)} ds.$$

We then use the control of $M^1(\langle v \rangle m_z)$ by $M^1(m_{2z})$ together with the controls (6.32)-(6.33), the decay control (6.27) and the estimate

$$e^{-\lambda s - \lambda(t-s)} \leq e^{-\frac{\lambda}{2}t - \frac{\lambda}{2}s}.$$

We get

$$u(t) \leq C e^{-\lambda(t-T_0)} u(T_0) + C e^{-\frac{\lambda}{2}t} \int_{T_0}^t e^{-\frac{\lambda}{2}s} \|f_s - g_s\|_{M^1(\langle v \rangle m_z)} ds.$$

We then use the following control for any $a > 0$:

$$\begin{aligned} \forall s \geq T_0, \quad \|f_s - g_s\|_{M^1_{\langle v \rangle m_z}} &= \int_{\mathbb{R}^d} |f_s - g_s| \langle v \rangle e^{z|v|} dv \\ &\leq a \int_{|v| \leq a} |f_s - g_s| e^{z|v|} dv + e^{-za} \int_{|v| \geq a} (f_s + g_s) e^{2z|v|} dv \\ &\leq a u(s) + e^{-za} Z. \end{aligned}$$

Hence we get for any $s \geq T_0$:

$$\|f_s - g_s\|_{M^1_{\langle v \rangle m_z}} \leq \begin{cases} u(s) + e^{-za} Z \leq (1+Z)u(s) & \text{when } u(s) \geq 1, \quad (\text{choosing } a := 1) \\ \frac{1}{z} |\ln u(s)| u(s) + u(s) Z & \text{when } u(s) \leq 1 \quad (\text{choosing } -za := \ln u(s)) \end{cases}$$

and we deduce

$$\forall s \geq T_0, \quad \|f_s - g_s\|_{M^1_{\langle v \rangle m_z}} \leq K u(s) (1 + (\ln u(s))_-), \quad K := 1 + \frac{1}{z} + Z.$$

Then for any $\delta \in (0, 1)$, we have, by choosing T_0 large enough,

$$\forall t \geq T_0, \quad e^{-\frac{\lambda}{2}t} \leq \delta e^{-\frac{\lambda}{4}t}$$

and we conclude with the following integral inequality

$$(6.34) \quad u(t) \leq C e^{-\lambda(t-T_0)} u(T_0) + \delta e^{-\frac{\lambda}{4}t} \int_{T_0}^t e^{-\frac{\lambda}{2}s} u_s (1 + (\ln u_s)_-) ds.$$

Let us prove that this integral inequality implies

$$(6.35) \quad \forall t \geq T_0, \quad u(t) \leq C e^{-\frac{\lambda}{4}t} u(T_0)^{1-\delta}.$$

Consider the case of equality in (6.34). Then we have

$$u(t) \geq C e^{-\lambda(t-T_0)} u(T_0) \geq e^{-\lambda(t-T_0)} u(T_0)$$

and therefore

$$(1 + (\ln u_t)_-) \leq (1 + (\ln u(T_0))_-) + \lambda(t - T_0).$$

We then have

$$\begin{aligned} U(t) &:= \int_{T_0}^t e^{-\frac{\lambda}{2}s} u_s (1 + (\ln u_s)_-) ds \\ &\leq \int_{T_0}^t e^{-\frac{\lambda}{2}s} u_s (1 + (\ln u(T_0))_- + \lambda(s - T_0)) ds \\ &\leq (3 + (\ln u(T_0))_-) \int_{T_0}^t e^{-\frac{\lambda}{4}s} u_s ds. \end{aligned}$$

By a Gronwall-like argument we can therefore obtain

$$u(t) \leq C e^{-\lambda(t-T_0)} u(T_0) + C \delta e^{-\frac{\lambda}{4}t} (3 + (\ln u(T_0))_-) u(T_0).$$

Then thanks to the inequality

$$\forall x \in (0, 1], \quad -(\ln x)x \leq \frac{x^{1-\delta}}{\delta}$$

we can prove (6.35) when $u(T_0) \leq 1$, and in the case when $u(T_0) \geq 1$, we can use (6.32) again to get

$$u(T_0) \leq (2Z)^\delta u(T_0)^{1-\delta}.$$

This concludes the proof of the claimed inequality (6.35).

Then estimate (6.29) follows by choosing δ small enough (in relation to η) and then connecting the last estimate (6.35) from time T_0 on together with the previous finite time estimate (6.11) from time 0 until time T_0 .

Then the estimate (6.30) is proved exactly in the same way by using the equation

$$\partial_t h_t = \mathcal{L}_\gamma(h_t) + Q(h_t, f_t - \gamma)$$

(which is even simpler than the equation for $f_t - g_t$).

Concerning the estimate (6.31) we start from the equation

$$\partial_t \omega_t = 2 \mathcal{L}_\gamma(\omega_t) + Q(\omega_t, f_t - \gamma) + Q(\omega_t, g_t - \gamma) + Q(h_t, d_t).$$

Then we establish on

$$y(t) := \|\omega_t\|_{M_{m_z}^1}$$

the following differential inequality

$$y(t) \leq C e^{-\lambda(t-T_0)} y(T_0) + C \delta e^{-\frac{\lambda}{4}t} (1 + (\ln y(T_0))_-) y(T_0) + C e^{-\frac{\lambda}{2}t} \|d_{T_0}\|_{M_{m_z}^1}^{1-\delta} \|h_{T_0}\|_{M_{m_z}^1}^{1-\delta}$$

which implies

$$y(t) \leq C e^{-\frac{\lambda}{4}t} \left(y(T_0)^{1-\delta} + \|d_{T_0}\|_{M_{m_z}^1}^{1-\delta} \|h_{T_0}\|_{M_{m_z}^1}^{1-\delta} \right).$$

Then estimate (6.31) follows by choosing δ small enough (in relation to η) and then connecting the last estimate from time T_0 on together with the previous finite time estimate (6.13) from time 0 until time T_0 . \square

6.8. Proof of condition (A5) uniformly in time. Let us prove that for any $\bar{z}, \mathcal{M}_{\bar{z}} \in (0, \infty)$ there exists some continuous function

$$\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \Theta(0) = 0,$$

such that for any $f_0, g_0 \in \mathcal{P}_{m_{\bar{z}}}(\mathbb{R}^d)$, $m_{\bar{z}}(v) := e^{\bar{z}|v|}$, with same momentum and energy, and such that

$$\|f_0\|_{M_{m_{\bar{z}}}^1} \leq \mathcal{M}_{\bar{z}}, \quad \|g_0\|_{M_{m_{\bar{z}}}^1} \leq \mathcal{M}_{\bar{z}},$$

there holds

$$\sup_{t \geq 0} W_1(S_t^{NL}(f_0), S_t^{NL}(g_0)) \leq \Theta(W_1(f_0, g_0)).$$

where W_1 stands for the Kantorovich-Rubinstein distance. Let us write

$$W_t := W_1(S_t^{NL}(f_0), S_t^{NL}(g_0)).$$

As we shall see, we may choose

$$(6.36) \quad \Theta(w) := \min \left\{ \bar{\Theta}, \bar{\Theta} e^{1-(1+|\ln w|)^{1/2}}, \frac{C_1}{(1+|\ln w|)^{\frac{\lambda}{2K}}} \right\}, \quad \Theta(0) = 0,$$

for some constants $\bar{\Theta}, C > 0$ (only depending on \bar{z} and $\mathcal{M}_{\bar{z}}$).

We start off with the inequality

$$\forall t \geq 0 \quad W_t \leq \|(f_t - g_t)v\|_{M^1} \leq \frac{1}{2} \|(f_t + g_t)v\|_{M^1}^2 = 1 + \mathcal{E} =: \bar{\Theta}.$$

Let us now improve this inequality for small value of W_0 . Therefore we assume without restriction that

$$W_0 \leq \frac{1}{2}$$

in the sequel.

On the one hand, it has been proved in [31, Theorem 2.2 and Corollary 2.3] that

$$(6.37) \quad W_t \leq W_0 + K \int_0^t W_s (1 + (\ln W_s)_-) ds,$$

for some constant K (depending on \bar{z} in the exponential. (To be more precise, (6.37) is proved in the more complicated case of hard potentials with angular cutoff in [31, Theorem 2.2], but the proof applies to the simpler case of hard spheres).

One can then check that the function

$$\bar{W}_t := e^{1-e^{-Kt}} (W_0)^{e^{-Kt}}$$

satisfies

$$\frac{d}{dt} \bar{W}_t = K (1 - \ln \bar{W}_t) \bar{W}_t, \quad \bar{W}_0 = W_0.$$

Therefore it is a super-solution of the differential inequality (6.37) as long as $W_t \leq 1$. It is an easy computation that this super-solution satisfies

$$\bar{W}_t \leq 1 \quad \text{as long as} \quad t \leq t_0 := \frac{\ln(1 + |\ln W_0|)}{K}.$$

Observe also that \bar{W}_t is increasing on $t \in [0, t_0]$.

We then define

$$(6.38) \quad t_1 := \frac{t_0}{2} = \frac{\ln(1 + |\ln W_0|)}{2K}$$

and we deduce the following bound on the solution of (6.37):

$$(6.39) \quad \forall t \in [0, t_1], \quad W_t \leq \bar{W}_t \leq \bar{W}_{t_1} = e^{1-(1+|\ln W_0|)^{1/2}}.$$

On the other hand, from (6.28), there are constants $\lambda, Z > 0, z \in (0, \bar{z})$ such that

$$(6.40) \quad \forall t \geq 0, \quad \|f_t - \gamma\|_{L_{m_z}^1} + \|g_t - \gamma\|_{L_{m_z}^1} \leq Z e^{-\lambda t},$$

where γ stands again for the normalized Maxwellian associated to f_0 and g_0 .

We deduce from (6.40)

$$(6.41) \quad \forall t \geq 0, \quad W_t \leq C e^{-\lambda t}$$

for a constant $C > 0$.

We then consider times $t \geq t_1$ and we deduce from (6.38) and (6.41) the following bound from above

$$(6.42) \quad \forall t \geq t_1, \quad W_t \leq C e^{-\lambda t_1} = \frac{C}{(1 + |\ln W_0|)^{\frac{\lambda}{2K}}}.$$

It is then straightforward to conclude the proof of **(A5)** uniformly in time for the function (6.36) by combining (6.39) and (6.42).

We have proved all the assumptions of Theorem 3.1.

- Together with the estimate on $\mathcal{W}_{W_1}^N(f)$ from Lemma 4.2, this concludes the proof of point (i) in Theorem 6.1 by using the non-uniform estimates for **(A4)**.
- Then we can conclude the proof of point (ii) in Theorem 6.1 by using
 - Lemma 4.4 for the construction of the sequence initial data f_0^N which satisfies the required integral and support moment bounds,
 - The previous steps in order to apply Theorem 3.1,
 - Lemma 4.7 in order to estimate

$$\mathcal{W}_{W_1}(\pi_P^N(f_0^N), f_0) \xrightarrow{N \rightarrow \infty} 0.$$

6.9. Proof of infinite-dimensional Wasserstein chaos. Let us now prove Theorem 6.2. Its proof is similar to Theorem 5.2.

First the proof of (6.4) follows from the point (i) in Theorem 6.1 and [36, Theorem 1.1] exactly in a similar way as we proved that (5.9) follows from the point (i) in Theorem 5.1 and [36, Theorem 1.1]. The proof of (6.5) follows similarly from the point (ii) in Theorem 6.1 and [36, Theorem 1.1].

Then the proof of (6.6) is also similar to the one of (5.10), the only difference being that one needs the following result of lower bound (independent of N) on the spectral gap of the N -particle system.

Theorem 6.8 ([9]). *Consider the operator L_{HS} for the hard spheres N -particle model with collision kernel $B(v-w) = |v-w|$. Then for any $\mathcal{E} > 0$ there is a constant $\lambda > 0$ (independent of N but depending on \mathcal{E}) such that for any probability f^N on $\mathcal{S}^N(\mathcal{E})$ one has*

$$\langle L_{HS}f^N, f^N \rangle_{L^2(\mathcal{S}^N(\mathcal{E}))} \leq -\lambda \|f^N\|_{L^2(\mathcal{S}^N)}.$$

where $\mathcal{S}^N(\mathcal{E})$ was defined in (4.14).

Then using Theorem 6.8 we deduce that

$$\forall N \geq 1, \forall t \geq 0, \quad \|h^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)} \leq e^{-\lambda t} \|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), \gamma^N)},$$

where $h^N = df^N/d\gamma^N$ is the Radon-Nikodym derivative of f^N with respect to the measure γ^N and the end of proof of (6.6) is then exactly similar to the one of (5.10) in the previous section.

7. H -THEOREM AND ENTROPIC CHAOS

This section is concerned with the H -theorem. We answer a question raised by Kac [42] about the derivation of the H -theorem.

7.1. Statement of the results. Our main results of this section state as follows:

Theorem 7.1. *Consider the Boltzmann collision process for Maxwell molecules (with or without cutoff) or hard spheres, and some initial data with zero momentum and energy \mathcal{E} satisfying*

$$f_0 \in L^\infty(\mathbb{R}^d) \quad \text{s. t.} \quad \int_{\mathbb{R}^d} e^{z|v|} df_0(v) < +\infty$$

for some $z > 0$, and the sequence of N -particle initial data $(f_0^N)_{N \geq 1}$ on $\mathcal{S}^N(\mathcal{E})$ constructed in Lemma 4.4 and 4.7.

Then we have:

- (i) In the case of Maxwell molecules with cut-off and hard spheres, if the initial data is entropically chaotic in the sense

$$\frac{1}{N} H(f_0^N | \gamma^N) \xrightarrow{N \rightarrow +\infty} H(f_0 | \gamma),$$

with

$$H(f_0^N | \gamma^N) := \int_{S^N(\varepsilon)} h_0^N \ln h_0^N d\gamma^N(v), \quad h_0^N := \frac{df_0^N}{d\gamma^N},$$

then the solution is also entropically chaotic for any later time:

$$\forall t \geq 0, \quad \frac{1}{N} H(f_t^N | \gamma^N) \xrightarrow{N \rightarrow +\infty} H(f_t | \gamma).$$

This proves the derivation of the H -theorem in this context, i.e. the monotonic decay in time of $H(f_t | \gamma)$, since for any $N \geq 2$, the functional $H(f_t^N | \gamma^N)$ is monotone decreasing in time for the Markov process.

- (ii) In the case of Maxwell molecules, and assuming moreover that the Fisher information of the initial data f_0 is finite:

$$\int_{\mathbb{R}^d} \frac{|\nabla_v f_0|^2}{f_0} dv < +\infty,$$

the following estimate on the relaxation induced by the H -theorem uniformly in the number of particles also holds:

$$\forall N \geq 1, \quad \frac{H(f_t^N | \gamma^N)}{N} \leq \beta(t)$$

for some polynomial function $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$.

- Remarks 7.2.* (1) The assumptions on the initial data could be relaxed to just $\mathcal{P}_6 \cap L^\infty$ as in point (iii) of Theorem 5.1 in the case of Maxwell molecules. However our assumptions allow for a unified statement for hard spheres and Maxwell molecules. We do not search for optimal statement here, but rather emphasize the strategy.
- (2) A stronger notion of entropic chaoticity could be

$$\frac{1}{N} H(f^N | [f^{\otimes N}]_{S^N(\varepsilon)}) \xrightarrow{N \rightarrow +\infty} 0.$$

The propagation of such property is an interesting open question. A partial answer is given in [14, Theorem 25].

- (3) The point (ii) holds for the hard spheres conditionally to a bound on the Fisher information uniformly in time and in the number of particle. However at present, it is an open problem to know whether such a bound holds for the many-particle hard spheres jump process.
- (4) In point (ii) we conjecture the better decay rate

$$\forall N \in \mathbb{N}^*, \quad \forall t \geq 0, \quad \frac{H(f_t^N | \gamma^N)}{N} \leq C e^{-\lambda t}$$

for some constant $\lambda > 0$.

7.2. Propagation of entropic chaos and derivation of the H -theorem. In this subsection we shall prove the point (i) of Theorem 7.1. Its proof relies on a convexity argument.

Let us define $h^N := df^N/d\gamma^N$ and then compute

$$\begin{aligned} \frac{d}{dt} \frac{H(f_t^N | \gamma^N)}{N} &= -D^N(f_t^N) \\ &:= -\frac{1}{2N^2} \int_{\mathcal{S}^N(\mathcal{E})} \sum_{i \neq j} \int_{\mathbb{S}^{d-1}} (h_t^N(V_{ij}^*) - h_t^N(V)) \ln \frac{h_t^N(V_{ij}^*)}{h_t^N(V)} B(v_i - v_j, \sigma) d\sigma d\gamma^N(v) \end{aligned}$$

where we recall that V_{ij}^* was defined in (5.1), which implies

$$(7.1) \quad \forall t \geq 0, \quad \frac{H(f_t^N | \gamma^N)}{N} + \int_0^t D^N(f_s^N) ds = \frac{H(f_0^N | \gamma^N)}{N}.$$

We also note that the same kind of equality is true at the limit (see e.g. [47])

$$\forall t \geq 0, \quad H(f_t | \gamma) + \int_0^t D^\infty(f_s) ds = H(f_0 | \gamma)$$

with

$$D^\infty(f) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} (f' f'_* - f f_*) \ln \frac{f' f'_*}{f f_*} B(v - v_*, \sigma) dv dv_* d\sigma$$

(be careful to the factor 1/2 in our definition of the collision operator (1.2) when computing the entropy production functional).

We then have the following lower semi-continuity property on these functionals, as a consequence of their convexity property.

Lemma 7.3. *The many-particle relative entropy and entropy production functionals defined above are lower semi-continuous: if the sequence $(f^N)_{N \geq 1}$ is f -chaotic then*

$$\liminf_{N \rightarrow \infty} \frac{H(f^N | \gamma^N)}{N} \geq H(f | \gamma)$$

and

$$\liminf_{N \rightarrow \infty} \frac{D^N(f^N)}{N} \geq D^\infty(f).$$

Let us first explain how to conclude the proof of point (i) of Theorem 7.1 with this lemma at hand. We first deduce from (7.1) and the entropic chaoticity of the initial data that

$$\forall t \geq 0, \quad \frac{H(f_t^N | \gamma^N)}{N} + \int_0^t D^N(f_s^N) ds \xrightarrow{N \rightarrow \infty} H(f_0 | \gamma) = H(f_t | \gamma) + \int_0^t D^\infty(f_s) ds.$$

Second we use Lemma 7.3 on the LHS to deduce that

$$\forall t \geq 0, \quad \liminf_{N \rightarrow \infty} \left(\frac{H(f_t^N | \gamma^N)}{N} + \int_0^t D^N(f_s^N) ds \right) \geq H(f_t | \gamma) + \int_0^t D^\infty(f_s) ds$$

where each of the limit of the two non-negative terms on the LHS is greater than the corresponding non-negative term in the RHS. We deduce from the two last equations that necessarily

$$\forall t \geq 0, \quad \frac{H(f_t^N | \gamma^N)}{N} \xrightarrow{N \rightarrow \infty} H(f_t | \gamma)$$

and

$$\forall t \geq 0, \quad \int_0^t D^N(f_s^N) ds \xrightarrow{N \rightarrow \infty} \int_0^t D^\infty(f_s) ds$$

which concludes the proof of point (i) of Theorem 7.1.

Proof of Lemma 7.3. These inequalities are consequences of convexity properties. The lower continuity property on the relative entropy on the spheres was proved in [10, Theorem 12] (actually the proof in this reference is performed on the sphere \mathbb{S}^{N-1} , but extending it to the invariant subspaces of our jump processes $\mathcal{S}^N(\mathcal{E})$ is straightforward). We refer to [14] for a detailed proof of the latter.

Let us now prove the inequality for the entropy production functional D^N . Denoting $Z = h^N(V_{12}^*)/h^N$, we first rewrite thanks to the symmetry of f^N as

$$D^N(f^N) = \frac{N(N-1)}{2N^2} \int_{\mathcal{S}^N(\mathcal{E})} \int_{\mathbb{S}^{d-1}} J(Z) B(v_1 - v_2, \sigma) f_2^N(v_1, v_2) d\sigma \frac{df^N(v)}{f_2^N(v_1, v_2)},$$

where $J(z) := (z-1) \ln z$ and f_2^N denotes the 2-marginal. Since the function $z \mapsto J(z)$ is convex, we can apply a Jensen inequality according to the variables v_3, \dots, v_N with reference probability measure f^N/f_2^N , which yields

$$D^N(f^N) \geq \frac{N(N-1)}{2N^2} \int_{v_1, v_2 \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} J(\bar{Z}) B(v_1 - v_2, \sigma) f_2^N(v_1, v_2) d\sigma dv_1 dv_2$$

with

$$\bar{Z}(v_1, v_2) := \int_{v_3, \dots, v_N \in \mathcal{S}^N(v_1, v_2)} Z \frac{df^N(V)}{f_2^N(v_1, v_2)} = \frac{f_2^N((V_{12}^*)_1, (V_{12}^*)_2)}{f_2^N(v_1, v_2)},$$

where $\mathcal{S}^N(v_1, v_2) := \{v_3, \dots, v_N \in E^{N-2}, (v_1, \dots, v_N) \in \mathcal{S}^N\}$. We therefore deduce a control from below of the N -particle entropy production functional in terms of the 2-particle entropy production functional, denoting $(f_2^N)^* = f_2^N((V_{12}^*)_1, (V_{12}^*)_2)$:

$$D^N(f^N) \geq \frac{N(N-1)}{2N^2} \int_{v_1, v_2 \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} ((f_2^N)^* - f_2^N) \ln \frac{(f_2^N)^*}{f_2^N} B(v_1 - v_2, \sigma) d\sigma dv_1 dv_2$$

Finally we take advantage of the convexity of the functional

$$h(x, y) = (x - y) \ln \frac{x}{y}$$

which implies that the function

$$(f_2, g_2) \rightarrow \int_{v_1, v_2 \in \mathbb{R}^d} \int_{\sigma \in \mathbb{S}^{d-1}} (f_2 - g_2) \ln \frac{f_2}{g_2} B(v_1 - v_2, \sigma)$$

is lower semi-continuous for the weak convergence of the 2-particle distributions f_2 and g_2 as proved in [21, Step 2 of the proof].

Hence we obtain thanks to the chaoticity of the second marginal

$$\begin{aligned} & \liminf_{N \rightarrow \infty} D^N(f^N) \\ & \geq \frac{1}{2} \int_{v, v_* \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(v')f(v'_*) - f(v)f(v_*)) \ln \frac{f(v')f(v'_*)}{f(v)f(v_*)} B(v - v_*, \sigma) d\sigma dv dv_* \\ & = D^\infty(f) \end{aligned}$$

which concludes the proof. \square

7.3. Many-particle relaxation rate in the H -theorem. In this subsection we shall prove point (ii) in Theorem 7.1. Its proof goes in two steps. First we shall prove that it follows from an estimate on the Fisher information thanks to the so-called ‘‘HWI’’ interpolation inequality [76]. Second we shall prove such a uniform bound on the Fisher information in the case of Maxwell molecules. Let us take the opportunity to thank Maxime Hauray who kindly communicated to us a proof for the latter step.

Let us define the Fisher informations for the N -particle distribution:

$$I(f^N) := \int_{\mathbb{R}^{dN}} \frac{|\nabla f^N|^2}{f^N} dv$$

and

$$I(f^N|\gamma^N) := \int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N(\mathcal{E})} h^N|^2}{h^N} d\gamma^N(v), \quad h^N := \frac{df^N}{d\gamma^N}$$

for a probability f^N having a density with respect to the Lebesgue measure in \mathbb{R}^{dN} and with respect to the measure γ^N respectively. The gradient in that last formula has to be understood as the usual Riemannian geometry gradient in the manifold $\mathcal{S}^N(\mathcal{E})$. The tangent space $T\mathcal{S}^N(\mathcal{E})_V$ (of dimension $Nd - 2$) at some given point $V \in \mathcal{S}^N$ is given by

$$T\mathcal{S}^N(\mathcal{E})_V = \left\{ W \in \mathbb{R}^{dN} \text{ s. t. } \sum_{i=1}^N w_i = 0 \text{ and } W \perp V \right\}.$$

For more informations and other results on the Fisher informations on $\mathcal{S}^N(\mathcal{E})$ we refer to [5]. We shall prove the following lemma whose proof is inspired from [74].

Lemma 7.4. *Consider the N -particle jump process (\mathcal{V}_t^N) for Maxwell molecules as defined in Subsection 5.1 for $N \geq 1$, and some initial law f_0^N with support included in $\mathcal{S}^N(\mathcal{E})$ and whose Fisher information is finite $I(f_0^N|\gamma^N) < +\infty$ on $\mathcal{S}^N(\mathcal{E})$. Then f_t^N has support included in $\mathcal{S}^N(\mathcal{E})$ for later times, and one has the following uniform in time bound on the Fisher information of the associated law*

$$\forall t \geq 0, \quad I(f_t^N|\gamma^N) \leq I(f_0^N|\gamma^N).$$

Proof of Lemma 7.4. We shall first consider the case of cutoff Maxwell molecules whose collision kernel b is integrable, and for a positive and smooth solution f^N on $\mathcal{S}^N(\mathcal{E})$. These assumptions can be relaxed by a mollification argument.

It is possible to study directly the estimate to be proved on the Boltzmann sphere $\mathcal{S}^N(\mathcal{E})$, however it means that one has to consider some local coordinates and a local basis for the tangent space. Another simpler method is to take advantage of the fact that the dynamics leaves the energy unchanged.

Starting from an initial data f_0^N on $\mathcal{S}^N(\mathcal{E}_0)$ we consider the flattened initial data

$$\tilde{f}_0^N := \alpha(\mathcal{E}(V)) f_0^N \left(\frac{V\sqrt{N\mathcal{E}_0}}{\sqrt{N\mathcal{E}(V)}} \right) \quad \text{with} \quad \mathcal{E}(V) = \frac{\sum_{i=1}^N |v_i|^2}{N}.$$

Observe that from the conservation of energy and momentum and the uniqueness of the solutions to the linear master N -particle equation

$$\forall t \geq 0, \quad \tilde{f}_t^N := \alpha(\mathcal{E}(V)) f_t^N \left(\frac{V\sqrt{N\mathcal{E}_0}}{\sqrt{N\mathcal{E}(V)}} \right)$$

where \tilde{f}_t^N denotes the solution in \mathbb{R}^{dN} starting from \tilde{f}_0^N . If the function α is regular and compactly supported, as well as f_0^N , this produces a smooth solution on \mathbb{R}^{dN} .

Assume that the result on the Fisher information is true in \mathbb{R}^{dN} :

$$I(\tilde{f}_t^N) := \int_{\mathbb{R}^{dN}} \frac{|\nabla \tilde{f}_t^N|^2}{\tilde{f}_t^N} dv \leq \int_{\mathbb{R}^{dN}} \frac{|\nabla \tilde{f}_0^N|^2}{\tilde{f}_0^N} dv = I(\tilde{f}_0^N).$$

Then we have the orthogonal decomposition of the gradient locally in terms of radial and ortho-radial directions

$$\nabla_{\mathbb{R}^{dN}} \tilde{f}_t^N = \nabla_{\mathcal{E}} \tilde{f}_t^N + \nabla_{\mathcal{S}^N(\mathcal{E})} \tilde{f}_t^N = (\nabla_{\mathcal{E}} \ln \alpha) \tilde{f}_t^N + \nabla_{\mathcal{S}^N(\mathcal{E})} \tilde{f}_t^N$$

that we can plug into the Fisher information inequality:

$$\begin{aligned} I(\tilde{f}_t^N) &:= |\nabla_{\mathcal{E}} \ln \alpha|^2 + \left(\int_{\mathbb{R}_+} \alpha(\mathcal{E}) (N\mathcal{E})^{(dN-1)/2} d\sqrt{\mathcal{E}} \right) \int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N(\mathcal{E})} h_t^N|^2}{h_t^N} d\gamma^N(v) \\ &\leq |\nabla_{\mathcal{E}} \ln \alpha|^2 + \left(\int_{\mathbb{R}_+} \alpha(\mathcal{E}) (N\mathcal{E})^{(dN-1)/2} d\sqrt{\mathcal{E}} \right) \int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N(\mathcal{E})} h_0^N|^2}{h_0^N} d\gamma^N(v) = I(\tilde{f}_0^N). \end{aligned}$$

Dropping the terms which do not depend on time we obtain the desired inequality on $\mathcal{S}^N(\mathcal{E})$.

Let us now prove the inequality on \mathbb{R}^{dN} . Let us first fix some notation: the N -particle solution f_t^N satisfies

$$\partial_t f^N = \frac{1}{N} \sum_{i,j=1, i \neq j}^N \int_{\mathbb{S}^{d-1}} (f^N(r_{ij,\sigma}(V)) b(\cos \theta_{ij}) d\sigma - f^N(V)) d\sigma =: NB(Q^{+,N}(f^N) - f^N)$$

where we use the following notations. We define

$$\begin{aligned} Q^{+,N}(f^N) &:= \frac{1}{N^2} \sum_{i,j=1, i \neq j}^N Q_{ij}^{+,N}(f^N), \quad Q_{ij}^{+,N}(f^N) := \int_{\mathbb{S}^{d-1}} f_{ij}^N b(\cos \theta_{ij}) d\sigma, \\ \cos \theta_{ij} &:= \sigma \cdot k_{ij} \quad \text{with} \quad k_{ij} = (v_i - v_j)/|v_i - v_j| \end{aligned}$$

and where we assume that b is even and that

$$\int_{\mathbb{S}^{d-1}} b(\sigma \cdot k) d\sigma = C_B \quad \text{for any } k, |k| = 1.$$

For any function g^N on \mathbb{R}^{dN} shall use the shorthand notation g_{ij}^N to denote the function $V \mapsto g(r_{ij,\sigma}(V))$, which depends also implicitly on σ . We shall make use of the measure preserving involution

$$\Theta_{ij} : \begin{cases} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{d-1} & \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{d-1} \\ (v_i, v_j, \sigma) & \mapsto (v'_i, v'_j, \sigma') \end{cases}$$

where $\sigma' = (v_i - v_j)/|v_i - v_j| = k_{ij}$.

Finally as in [74], we shall use the following endomorphism of \mathbb{R}^d

$$\begin{aligned} M_{\sigma k}(x) &= (k \cdot \sigma)x - (k \cdot x)\sigma \\ P_{\sigma k}(x) &= (\sigma \cdot x)k + M_{\sigma k} \end{aligned}$$

and we recall that $\|P_{\sigma k}(x)\| \leq \|x\|$ with equality only if x, σ, k are coplanar.

We claim that it is enough to prove that

$$(7.2) \quad I(Q^{+,N}(f^N)) = I\left(\frac{1}{N^2} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} f(r_{ij,\sigma}(V)) b(\cos \theta_{ij}) d\sigma\right) \leq C_B I(f).$$

Indeed with this result at hand, we can write for $\varepsilon > 0$:

$$f_{t+\varepsilon}^N = e^{-NC_B\varepsilon} f_t^N + NC_B \int_0^\varepsilon e^{NC_B(s-\varepsilon)} Q^{+,N}(f_{t+s}^N) ds$$

and therefore from the convexity of I

$$I(f_{t+\varepsilon}^N) \leq e^{-NC_B\varepsilon} I(f_t^N) + (1 - e^{-NC_B\varepsilon}) I\left(\int_0^\varepsilon Q^{+,N}(f_{t+s}^N) \frac{NC_B e^{NC_B(s-\varepsilon)}}{(1 - e^{-NC_B\varepsilon})} ds\right).$$

Observe that

$$\int_0^\varepsilon \frac{N C_B e^{N C_B (s-\varepsilon)}}{(1 - e^{-N C_B \varepsilon})} ds = 1$$

and then we can use the convexity of I again to get

$$I(f_{t+\varepsilon}^N) \leq e^{-N C_B \varepsilon} I(f_t^N) + \int_0^\varepsilon I(Q^{+,N}(f_{t+s}^N)) N C_B e^{N C_B (s-\varepsilon)} ds.$$

Finally using the claimed result (7.2) we obtain

$$\frac{I(f_{t+\varepsilon}^N) - I(f_t^N)}{\varepsilon} \leq -\frac{(1 - e^{-N C_B \varepsilon})}{\varepsilon} I(f_t^N) + \frac{1}{\varepsilon} \int_0^\varepsilon I(f_{t+s}^N) N C_B^2 e^{N C_B (s-\varepsilon)} ds.$$

Then taking $\varepsilon \rightarrow 0$ and using Lebesgue's theorem we deduce

$$\frac{d}{dt} I(f_t^N) \leq -N C_B I(f_t^N) + N C_B I(f_t^N) \leq 0$$

which concludes the proof.

Let us now focus on the proof of the claim (7.2). Taking advantage of the convexity of I , it is enough to prove

$$\forall i \neq j \in \llbracket 1, N \rrbracket, \quad I(Q_{ij}^{+,N}(f^N)) \leq C_B I(f^N).$$

Let us compute each partial derivative of $Q_{ij}^{+,N}(f^N)$. If $\ell \notin \{i, j\}$ then the derivative does not act on the kernel b and we obtain:

$$\begin{aligned} \nabla_{v_\ell} (Q_{ij}^{+,N}(f^N)) &= \int_{\mathbb{S}^{d-1}} \nabla_{v_\ell} (f_{ij}^N) b(\cos \theta_{ij}) d\sigma = \int_{\mathbb{S}^{d-1}} (\nabla_{v_\ell} f^N)_{ij} b(\cos \theta_{ij}) d\sigma \\ &= 2 \int_{\mathbb{S}^{d-1}} (\sqrt{f^N})_{ij} (\nabla_{v_\ell} \sqrt{f^N})_{ij} b(\cos \theta_{ij}) d\sigma. \end{aligned}$$

If $\ell \in \{i, j\}$, then it is slightly more complicated. Without restriction we perform calculations in the case $\ell = i$. Let us first prove the formula

$$(7.3) \quad \begin{aligned} \nabla_{v_i} (Q_{ij}^{+,N}(f^N)) &= \int_{\mathbb{S}^{d-1}} \left[(\nabla_{v_i} f^N)_{ij} + (\nabla_{v_j} f^N)_{ij} + P_{\sigma k} \left((\nabla_{v_i} f^N)_{ij} - (\nabla_{v_j} f^N)_{ij} \right) \right] b(\cos \theta_{ij}) d\sigma \end{aligned}$$

(the same equality obviously holds where i is replaced by j).

Simple computations (see for instance [74]) yield

$$\begin{aligned} \nabla_{v_i} (f_{ij}^N) &= \frac{1}{2} \left((\nabla_{v_i} f^N)_{ij} + (\nabla_{v_j} f^N)_{ij} \right) + \frac{1}{2} \left[\left((\nabla_{v_i} f^N)_{ij} - (\nabla_{v_j} f^N)_{ij} \right) \cdot \sigma \right] k_{ij}, \\ \nabla_{\sigma} (f_{ij}^N) &= \frac{|v_i - v_j|}{2} \left((\nabla_{v_i} f^N)_{ij} - (\nabla_{v_j} f^N)_{ij} \right), \\ \nabla_{v_i} [b(\cos \theta_{ij})] &= \frac{1}{|v_i - v_j|} b'(\sigma \cdot k_{ij}) \Pi_{k^\perp} \sigma, \end{aligned}$$

where Π_{k^\perp} is the projection on the hyperplane k^\perp . Using the first and third equality above, we get

$$(7.4) \quad \begin{aligned} \nabla_{v_i} (Q_{ij}^{+,N}(f^N)) &= \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) \left((\nabla_{v_i} f)_{ij} + (\nabla_{v_j} f)_{ij} + \left[\left((\nabla_{v_i} f)_{ij} - (\nabla_{v_j} f)_{ij} \right) \cdot \sigma \right] k \right) d\sigma \\ &\quad + \left(\int_{\mathbb{S}^{d-1}} b'(\cos \theta_{ij}) \frac{f_{ij}}{|v_i - v_j|} \Pi_{k^\perp} \sigma d\sigma \right) \end{aligned}$$

and we use the following formula of integration by part on the sphere \mathbb{S}^{d-1} (see [74, Lemma 2])

$$\int_{\mathbb{S}^{d-1}} b'(\cos \theta_{ij}) F(\sigma) \Pi_{k^\perp} \sigma \, d\sigma = \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) M_{\sigma k} (\nabla_\sigma F(\sigma)) \, d\sigma$$

and the second equality above to rewrite the term involving b' in (7.4) into

$$\frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) M_{\sigma k} \left((\nabla_{v_i} f)_{ij} - (\nabla_{v_j} f)_{ij} \right) \, d\sigma$$

Putting all together, we get formula (7.3).

We deduce that for $\ell \neq i, j$ we have by Cauchy-Schwarz

$$\left| \nabla_{v_\ell} \left(Q_{ij}^{+,N} (f^N) \right) \right|^2 \leq 4 \left(\int_{\mathbb{S}^{d-1}} f_{ij}^N b(\cos \theta_{ji}) \, d\sigma \right) \left(\int_{\mathbb{S}^{d-1}} \left| (\nabla_{v_\ell} \sqrt{f^N})_{ij} \right|^2 b(\cos \theta_{ij}) \, d\sigma \right)$$

and therefore

$$\frac{\left| \nabla_{v_\ell} \left(Q_{ij}^{+,N} (f^N) \right) \right|^2}{Q_{ij}^{+,N} (f^N)} \leq 4 \int_{\mathbb{S}^{d-1}} \left| (\nabla_{v_\ell} \sqrt{f^N})_{ij} \right|^2 b(\cos \theta_{ij}) \, d\sigma.$$

Now integrating in V we obtain

$$\begin{aligned} I_\ell \left(Q_{ij}^{+,N} (f^N) \right) &\leq 4 \int_{\mathbb{R}^{dN}} \int_{\mathbb{S}^{d-1}} \left| (\nabla_{v_\ell} \sqrt{f^N})_{ij} \right|^2 b(\cos \theta_{ij}) \, d\sigma \, dv \\ &\leq 4 \int_{\mathbb{R}^{dN}} \int_{\mathbb{S}^{d-1}} \left| (\nabla_{v_\ell} \sqrt{f^N}) \right|^2 b(\cos \theta_{ij}) \, d\sigma \, dv \\ &\leq 4 C_B \int_{\mathbb{R}^{dN}} \left| (\nabla_{v_\ell} \sqrt{f^N}) \right|^2 \, dv =: I_\ell (f^N) \end{aligned}$$

where we have used Cauchy-Schwartz's inequality and the change of variable Θ_{ij} , and where I_ℓ is defined from the last line (Fisher information restricted to the ℓ -th derivative).

When $\ell = i, j$, we use (7.3) to get

$$\begin{aligned} \left| \nabla_{v_\ell} \left(Q_{ij}^{+,N} (f^N) \right) \right|^2 &\leq \left(\int_{\mathbb{S}^{d-1}} f^N b(\cos \theta_{ji}) \, d\sigma \right) \\ &\quad \times \left(\int_{\mathbb{S}^{d-1}} \left| (\nabla_{v_i} \sqrt{f^N}) + (\nabla_{v_j} \sqrt{f^N}) \right. \right. \\ &\quad \left. \left. + P_{\sigma k} \left((\nabla_{v_j} \sqrt{f^N}) - (\nabla_{v_i} \sqrt{f^N}) \right) \right|^2 b(\cos \theta_{ij}) \, d\sigma \right) \end{aligned}$$

where we have used Cauchy-Schwartz's inequality and the change of variable Θ_{ij} .

Since for fixed V , $P_{\sigma k}$ is odd in σ and $b(\cos \theta_{ij})$ is even in σ , we have

$$\int_{\mathbb{S}^{d-1}} A \cdot P_{\sigma k}(B) \, d\sigma = 0$$

for any functions A, B independent of σ . Using finally that $P_{\sigma k}$ has norm less than 1 (for the subordinated norm to the euclidean norm) we get

$$\begin{aligned} \left| \nabla_{v_\ell} \left(Q_{ij}^{+,N} (f^N) \right) \right|^2 &\leq 2 \left(\int_{\mathbb{S}^{d-1}} f^N b(\cos \theta_{ji}) \, d\sigma \right) \\ &\quad \times \left(\int_{\mathbb{S}^{d-1}} \left| \nabla_{v_i} \sqrt{f^N} \right|^2 + \left| \nabla_{v_j} \sqrt{f^N} \right|^2 b(\cos \theta_{ij}) \, d\sigma \right) \end{aligned}$$

and therefore

$$I_\ell \left(Q_{ij}^{+,N} (f^N) \right) \leq C_B \frac{I_i (f^N) + I_j (f^N)}{2}.$$

Finally we end up with

$$I \left(Q_{ij}^{+,N} (f^N) \right) = C_B \sum_{\ell=1}^N I_\ell \left(Q_{ij}^{+,N} (f^N) \right) \leq C_B \sum_{\ell=1}^N I_\ell (f^N) = C_B I (f^N)$$

which concludes the proof. \square

Let us now conclude the proof of point (ii) in Theorem 7.1. We make use of the so-called ‘‘HWI’’ interpolation inequality on the manifold $\mathcal{S}^N(\mathcal{E})$. Observe that $\mathcal{S}^N(\mathcal{E})$ has positive Ricci curvature since it has positive curvature. Then [77, Theorem 30.21] implies that

$$\frac{1}{N} H (f^N | \gamma^N) \leq \frac{W_2 (f^N, \gamma^N)}{\sqrt{N}} \sqrt{\frac{I (f^N | \gamma^N)}{N}}.$$

We can then use the uniform bound on the Fisher information provided by Lemma 7.4 (vi) and the bound on the initial data to get:

$$\frac{I (f^N | \gamma^N)}{N} \leq \frac{I (f_0^N | \gamma^N)}{N} \leq C$$

for some constant $C > 0$ independent of N . Moreover Lemma 4.1 and the propagation of moments on the N -particle system in Lemma 5.3 imply that

$$\frac{W_2 (f^N, \gamma^N)}{\sqrt{N}} \leq C \left(\frac{W_1 (f^N, \gamma^N)}{N} \right)^\alpha$$

for some constant $C > 0$ and exponent $\alpha > 0$ independent of N . Then using Theorem 5.2 (case (b)) we deduce that

$$\frac{W_2 (f^N, \gamma^N)}{\sqrt{N}} \leq C \left(\frac{W_1 (f^N, \gamma^N)}{N} \right)^\alpha \leq \beta(t)$$

with a polynomial rate $\beta(t) \rightarrow 0$ as $t \rightarrow +\infty$, which implies that

$$\frac{1}{N} H (f^N | \gamma^N) \leq C \beta(t)$$

and concludes the proof.

8. THE BBGKY HIERARCHY METHOD REVISITED

The so-called BBGKY hierarchy method (Bogoliubov, Born, Green, Kirkwood and Yvon) is very popular in physics and mathematics for studying many-particle systems: see for instance [3] where this approach is used for Kac’s master equation for hard spheres, or see, among many other works, the recent series of papers [4, 25, 26, 27] where this approach is used for the derivation of nonlinear mean-field Schrödinger equations in quantum physics. The basic ideas underlying this approach to mean-field limit could be summarized as:

- (i) Write a BBGKY hierarchy on marginals of the N -particle system and prove that the N -particle system solutions converge to the solutions of an ‘‘infinite hierarchy’’ when N goes to infinity. The proof of this convergence often relies on a compactness argument.
- (ii) Prove that solutions to this infinite hierarchy are unique, which is the hardest part of this program.
- (iii) Then deduce the propagation of chaos by exhibiting, for any chaotic initial data to the infinite hierarchy, a solution to the infinite hierarchy obtained by the infinite tensorization of the 1-particle solution to the limit nonlinear mean-field equation.

This section revisits this BBGKY hierarchy method, under some appropriate regularity assumptions on the limit semigroup. We build a rigorous connection with *statistical solutions* and our pullback semigroup T_t^∞ , by showing (1) how these notions are included in our functional framework (cf. the abstract semigroup T_t^∞ defined in Section 2), and (2) how to give a proof of uniqueness and propagation of chaos based on them by using the functional tools we have introduced. We would like to take the opportunity to mention here the interesting paper [66] (pointed out to us by Golse) where some key ideas about the connection between the BBGKY hierarchy and the pullback semigroup were already presented.

8.1. The BBGKY hierarchy. Let us recall the master equation of the N -particle system undergoing a Boltzmann collision process (the notion of BBGKY hierarchy has wider application range, but we shall stick to this concrete case for clarity), see (5.3)-(5.4):

$$(8.1) \quad \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

with

$$(G^N \varphi)(V) = \frac{1}{N} \sum_{1 \leq i < j \leq N} \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] \, d\sigma$$

$$\text{where } \varphi_{ij}^* = \varphi(V_{ij}^*) \text{ and } \varphi = \varphi(V) \in C_b(\mathbb{R}^{Nd}).$$

Then the BBGKY hierarchy writes as follows. Let us recall the notation

$$f_\ell^N = \Pi_\ell[f^N] = \int_{v_{\ell+1}, \dots, v_N} df^N(v_{\ell+1}, \dots, v_N)$$

for the marginals. Then integrating the master equation (8.1) against some test function $\varphi = \varphi(v_1, \dots, v_\ell)$ depending only on the first ℓ variables leads to

$$\frac{d}{dt} \langle f_\ell^N, \varphi \rangle = \langle f_{\ell+1}^N, G_{\ell+1}^N(\varphi) \rangle$$

where

$$G_{\ell+1}^N(\varphi) := \frac{1}{N} \sum_{1 \leq i \leq \ell, 1 \leq j \leq N, i \neq j} \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] \, d\sigma.$$

Then with the notation

$$\mathcal{Z}_{ij}^N := \left\langle f^N, \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] \, d\sigma \right\rangle$$

we can further decompose this sum as

$$\frac{d}{dt} \langle f_\ell^N, \varphi \rangle = \frac{1}{N} \sum_{i,j \leq \ell} \mathcal{Z}_{ij}^N + \frac{1}{N} \sum_{i \leq \ell < j} \mathcal{Z}_{ij}^N = \frac{1}{N} \sum_{i,j \leq \ell} \mathcal{Z}_{ij}^N + \mathcal{O}\left(\frac{\ell^2}{N}\right)$$

by observing that $\mathcal{Z}_{ij}^N = 0$ for $i, j > \ell$. Using the symmetry of f^N we finally deduce

$$(8.2) \quad \frac{d}{dt} \langle f_\ell^N, \varphi \rangle = \frac{(N-\ell)}{N} \left(\sum_{i=1}^{\ell} \mathcal{Z}_{i(\ell+1)}^N \right) + \mathcal{O}\left(\frac{\ell^2}{N}\right).$$

We thus end with a series of N coupled equations on the marginals f_ℓ^N , where the ℓ -equation ($\ell \leq N-1$) depends on the $f_{\ell+1}^N$ marginal.

8.2. The infinite hierarchy and statistical solutions. Assume now that

$$\forall t \geq 0, \forall \ell \geq 1, \quad f_{t\ell}^N \rightharpoonup \pi_{t\ell} \text{ in } P(\mathbb{R}^{d\ell}).$$

Starting from (8.2) we obtain, for $\varphi = \varphi(v_1, \dots, v_\ell) \in C_b(\mathbb{R}^{d\ell})$ depending only on the first ℓ variables,

$$\frac{d}{dt} \langle \pi_\ell, \varphi \rangle = \langle \pi_{\ell+1}, G_{\ell+1}^\infty(\varphi) \rangle$$

where $G_{\ell+1}^\infty(\varphi) \in C_b(\mathbb{R}^{d(\ell+1)})$ is defined by

$$G_{\ell+1}^\infty(\varphi) := \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] d\sigma.$$

In a more compact form, we have the following set of linear coupled evolution equations

$$(8.3) \quad \forall \ell \geq 1, \quad \partial_t \pi_\ell = A_{\ell+1}^\infty(\pi_{\ell+1}) \text{ with } A_{\ell+1}^\infty := (G_{\ell+1}^\infty)^*.$$

Since the family of ℓ -particle probabilities π_ℓ is symmetric and compatible in the sense that

$$\forall \ell \geq 1, \quad \Pi_\ell[\pi_{\ell+1}] = \pi_\ell$$

(this follows from the construction), we can associate by Hewitt-Savage's Theorem [37] a unique $\pi \in P(P(\mathbb{R}^d))$ such that, for any $\ell \geq 1$ and $\varphi = \varphi(v_1, \dots, v_\ell) \in C_b(\mathbb{R}^{d\ell})$ depending only on the first ℓ variables,

$$\langle \pi, R_\varphi \rangle = \langle \pi_\ell, \varphi \rangle$$

and these evolution equations for the π_ℓ translate into an evolution equation

$$\partial_t \pi = A^\infty(\pi) \text{ on } P(P(\mathbb{R}^d))$$

of *statistical solutions* and the corresponding dual evolution

$$(8.4) \quad \partial_t \Phi = \bar{G}^\infty \Phi \text{ on } C_b(P(\mathbb{R}^d)).$$

In order to make this heuristic rigorous at an abstract level, one needs at least some tightness on the sequence $(f_\ell^N)_{N \geq \ell}$ for any ℓ , and some convergence

$$G_{\ell+1}^N(\varphi) \rightarrow G_{\ell+1}^\infty(\varphi)$$

on compact subset of $\mathbb{R}^{d(\ell+1)}$. Both are satisfied for Boltzmann collision processes considered in this paper (note that the tightness follows from the moment estimates in Lemma 5.3 for instance).

8.3. Uniqueness of statistical solutions and chaos. We now want, under appropriate abstract assumptions, to identify the limit evolution (8.4) in $C_b(P(\mathbb{R}^d))$ obtained from the hierarchy, and show that it coincides with the pullback evolution semigroup T_t^∞ introduced in Subsection 2.3. Meanwhile we shall prove that the statistical solutions to the infinite hierarchy are unique, and hence prove the propagation of chaos, without any rate, but also under weaker assumptions than previously. For the sake of clarity we do not include weights nor constraints in the following theorem, but it can easily be extended in this direction in a similar way as in Theorem 3.1. Our aim here is rather the conceptual presentation of the method. As a consequence, our result only applies (straightforwardly) to the **(GMM)** model. The **(HS)** model and the **(tMM)** model could be handled in a similar way by using an extended version of the theorem including weights and constraints.

We make the following assumptions:

(A1') Assumptions on the N -particle system.

G^N and T_t^N are well defined on $C_b(E^N)$ and invariant under permutation, and the associated solutions f_t^N satisfy:

$$\forall t \geq 0, \forall \ell \geq 1, \quad \text{the sequence } (\Pi_\ell f_t^N)_{N \geq \ell} \text{ is tight in } \mathcal{P}_{\mathcal{G}_1}(E)^{\otimes \ell}$$

where \mathcal{G}_1 is a Banach space and $\mathcal{P}_{\mathcal{G}_1}(E)$ is defined in Definitions 2.2-2.3, and is associated to a weight function $m_{\mathcal{G}_1}$ and endowed with the metric induced from \mathcal{G}_1 .

(A2') Assumptions for the existence of the statistical and pullback semigroups.

For some $\delta \in (0, 1]$ and some $\bar{a} \in (0, \infty)$ we assume that for any $a \in (\bar{a}, \infty)$:

(i) The equation (2.1) generates a semigroup

$$S_t^{NL} : \mathcal{BP}_{\mathcal{G}_1, a} \rightarrow \mathcal{BP}_{\mathcal{G}_1, a}$$

which is δ -Hölder continuous locally uniformly in time, in the sense that for any $\tau \in (0, \infty)$ there exists $C_\tau \in (0, \infty)$ such that

$$\forall f, g \in \mathcal{BP}_{\mathcal{G}_1, a}, \quad \sup_{t \in [0, \tau]} \|S_t^{NL} f - S_t^{NL} g\|_{\mathcal{G}_1} \leq C_\tau \|f - g\|_{\mathcal{G}_1}^\delta.$$

- (ii) The application Q is bounded and δ -Hölder continuous from $\mathcal{BP}_{\mathcal{G}_1, a}$ into \mathcal{G}_1 .
- (iii) E is a locally compact Polish space and there is \mathcal{F}_1 in duality with \mathcal{G}_1 such that \mathcal{F}_1 is dense in $C_b(E)$ in the sense of uniform convergence on any compact set.

(A3') Convergence of the generators.

For any fixed $\ell \in \mathbb{N}^*$ and any $\varphi \in C_b(E^\ell)$, the sequence

$$G_{\ell+1}^N(\varphi) \in C_b(E^{\ell+1}) \quad \text{satisfies} \quad G_{\ell+1}^N \varphi \xrightarrow{N \rightarrow \infty} G_{\ell+1}^\infty \varphi$$

uniformly on compact sets, where $G_{\ell+1}^\infty \varphi$ satisfies the following *compatibility binary derivation structure*: for any $\varphi = \varphi_1 \otimes \dots \otimes \varphi_\ell \in C_b(E)^{\otimes \ell}$ and any $V = (v_1, \dots, v_{\ell+1}) \in E^{\ell+1}$

$$(8.5) \quad G_{\ell+1}^\infty(\varphi)(V) = \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \varphi_j(v_j) \right) Q^*(\varphi_i)(v_i, v_{\ell+1})$$

where Q^* is related to Q through the duality relation

$$\forall f \in \mathcal{P}_{\mathcal{G}_1}, \forall \psi \in C_b(E), \quad \langle Q(f, f), \psi \rangle = \langle f \otimes f, Q^*(\psi) \rangle.$$

Remark 8.1. The identity (8.5) is called *compatibility binary derivation structure* for the following reasons: *compatibility* since it is a natural condition in order that any solution f_t to the nonlinear Boltzmann provides a tensorized solution to the BBGKY hierarchy (8.3). Indeed, considering such a solution

$$f_t \in C(\mathbb{R}_+; \mathcal{P}_{\mathcal{G}_1}(E)) \quad \text{and} \quad \varphi = \varphi_1 \otimes \dots \otimes \varphi_\ell \in C_b(E)^{\otimes \ell}$$

we compute

$$\begin{aligned} \frac{d}{dt} \langle f_t^{\otimes \ell}, \varphi \rangle &= \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \langle f_t, \varphi_j \rangle \right) \frac{d}{dt} \langle f_t, \varphi_i \rangle = \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \langle f_t, \varphi_j \rangle \right) \langle Q(f_t, f_t), \varphi_i \rangle \\ &= \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \langle f_t, \varphi_j \rangle \right) \langle f_t \otimes f_t, Q^*(\varphi_i) \rangle = \langle f_t^{\otimes \ell+1}, G_{\ell+1}^{\infty} \varphi \rangle. \end{aligned}$$

The word *binary* refers to the fact that $G_{\ell+1}^{\infty}$ decomposes in function acting on *one* variable and adding *one* variable, which corresponds to the binary nature of the collisions. Finally the word *derivation* refers to the fact that the following distributivity property holds

$$G_{\ell+1}^{\infty}(\varphi \otimes \psi) = G_{\ell+1}^{\infty}(\varphi) \otimes \psi + \varphi \otimes G_{\ell+1}^{\infty}(\psi).$$

Let us mention that this distributivity property is at the basis of the original combinatorial proof of Kac [42] of propagation of chaos for the simplified Boltzmann-Kac equation. This structure assumption is also partly inspired from [53].

(A4') Differential stability of the limit semigroup.

We consider some Banach space $\mathcal{G}_2 \supset \mathcal{G}_1$ (where \mathcal{G}_1 was defined in (A2)) and the corresponding space of probability measures $\mathcal{P}_{\mathcal{G}_2}(E)$ (see Definitions 2.2-2.3) with the weight function $m_{\mathcal{G}_2}$ and endowed with the metric induced from \mathcal{G}_2 .

We assume that the flow S_t^{NL} is $UC^1(\mathcal{P}_{\mathcal{G}_1}, \mathcal{P}_{\mathcal{G}_2}) \cap UC^{0,1/2}(\mathcal{P}_{\mathcal{G}_1}, \mathcal{P}_{\mathcal{G}_2})$ for any $t \geq 0$, where $UC^{0,1/2}$ is the space of functions \mathcal{S} satisfying (2.7) with Ω_c such that $\Omega_c(s)/s^{1/2} \rightarrow 0$ as $s \rightarrow 0$.

Thanks to (A2'), we know from Lemma 2.11 that for any $\Phi \in UC_b(\mathcal{P}_{\mathcal{G}_1}, \mathbb{R})$ we may define the C_0 -semigroup $T_t^{\infty}[\Phi] \in UC_b(\mathcal{P}_{\mathcal{G}_1})$ by

$$T_t^{\infty}[\Phi](f) = \Phi(S_t^{NL} f),$$

and so that $\Phi_t = T_t^{\infty}[\Phi]$ satisfies the equation

$$\partial_t \Phi = G^{\infty}[\Phi]$$

with a generator G^{∞} which is a closed operator on $UC_b(\mathcal{P}_{\mathcal{G}_1})$ and has domain $\text{Dom}(G^{\infty})$ which contains $UC^1(\mathcal{P}_{\mathcal{G}_1})$, and is defined by

$$G^{\infty}[\Phi](f) = \langle Q(f, f), D\Phi(f) \rangle_{\mathcal{P}_{\mathcal{G}_1}, C_b(\mathcal{P}_{\mathcal{G}_1})}.$$

The evolution corresponds to the following dual evolution equation

$$(8.6) \quad \frac{d}{dt} \langle \pi_t, \Phi \rangle = \langle \pi_t, G^{\infty}[\Phi] \rangle.$$

Our goal is to prove first that the evolution equations (8.3) and (8.6) are identical (or in other words that the generator \bar{G}^{∞} introduced for the hierarchy is well-defined and equal to G^{∞}), and second and most importantly that the solution to these equations is given by the characteristics method

$$(8.7) \quad \forall \Phi \in UC_b(\mathcal{P}_{\mathcal{G}_1}; \mathbb{R}), \quad \langle \pi_t, \Phi \rangle = \langle \pi_0, T_t^{\infty} \Phi \rangle.$$

Let us explain why the relation (8.7) indeed defines uniquely a probability evolution $\bar{\pi}_t \in P(\mathcal{P}_{\mathcal{G}_1})$. For any $\ell \in \mathbb{N}^*$ we define

$$\varphi \in \mathcal{F}^{\otimes \ell} \mapsto \langle \bar{\pi}_t^{\ell}, \varphi \rangle := \langle \pi_0, T_t^{\infty} R_{\varphi}^{\ell} \rangle.$$

That is a positive linear form on $\mathcal{F}^{\otimes \ell}$. Thanks to **(A2')-(iii)**, the Stone-Weierstrass density theorem and the Markov-Riesz representation theorem, we conclude that π_t^ℓ is well defined as a element of $\mathcal{P}_{\mathcal{G}_1}(E)^{\otimes \ell}$. Since now the sequence (π_t^ℓ) is symmetric and compatible, the Hewitt-Savage representation theorem implies that there exists a unique probability measure $\bar{\pi}_t \in P(\mathcal{P}_{\mathcal{G}_1})$ such that for any $\varphi \in \mathcal{F}_1^{\otimes \ell}$

$$(8.8) \quad \langle \bar{\pi}_t, R_\varphi^\ell \rangle := \langle \pi_0, T_t^\infty R_\varphi^\ell \rangle.$$

Theorem 8.2. *Under the assumptions **(A1')-(A2')-(A3')-(A4')**, for any initial datum $\pi_0 \in P(\mathcal{P}_{\mathcal{G}_1})$, the flow $\bar{\pi}_t$ defined from (8.8) is the unique solution in $C([0, \infty); P(\mathcal{P}_{\mathcal{G}_1}))$ to the infinite hierarchy evolution (8.3) starting from π_0 .*

Moreover, if $\pi_0 = \delta_{f_0}$ with $f_0 \in P(E)$ then $\pi_t = \delta_{f_t}$ for any $t \geq 0$, with $f_t := S_t^{NL} f_0$. As a consequence we deduce that if f_0^N is f_0 -chaotic, then f_t^N is $S_t^{NL} f_0$ -chaotic. More generally, if f_0^N converges to π_0 then f_t^N converge to $\bar{\pi}_t$, which is the associated statistical solution.

Proof of Theorem 8.2. We shall proceed in several steps.

Step 1: Propagation of Dirac mass structure. Let us recall that Hewitt-Savage's theorem [37] implies that for any $\pi \in P(P(E))$ there exists a unique sequence $(\pi^\ell) \in P(E^\ell)$ such that

$$\forall \varphi \in (C_b(E))^{\otimes \ell}, \quad \langle \pi^\ell, \varphi \rangle = \langle \pi, R_\varphi^\ell \rangle.$$

As a consequence, if $\pi_0 = \delta_{f_0}$ and $\bar{\pi}$ satisfies (8.8), then

$$\begin{aligned} \langle \bar{\pi}_{t\ell}, \varphi \rangle &= \langle \bar{\pi}_t, R_\varphi^\ell \rangle = \langle \pi_0, T_t^\infty R_\varphi^\ell \rangle = T_t^\infty [R_\varphi^\ell] (f_0) \\ &= R_\varphi^\ell (S_t^{NL} f_0) = \langle S_t^{NL} f_0, \varphi_1 \rangle \dots \langle S_t^{NL} f_0, \varphi_\ell \rangle, \end{aligned}$$

which means that $\bar{\pi}_{t\ell} = f_t^{\otimes \ell}$, or equivalently $\bar{\pi}_t = \delta_{f_t}$.

Step 2: Equivalence between (8.3) and (8.6).

First let us assume (8.6) and prove (8.3). We start with the following observation. Consider $f \in \mathcal{P}_{\mathcal{G}_1}(E)$ and $\varphi \in \mathcal{F}^{\otimes \ell}$. Then we have $R_\varphi^\ell \in C^{1,1}(\mathcal{P}_{\mathcal{G}_1}(E))$ and we deduce from (8.5) that

$$\langle f^{\otimes \ell+1}, G_{\ell+1}^\infty \varphi \rangle = \langle Q(f, f), DR_\varphi^\ell(f) \rangle = G^\infty [R_\varphi^\ell] (f)$$

which means

$$R_{G_{\ell+1}^\infty \varphi}^{\ell+1} = G^\infty [R_\varphi^\ell].$$

Then, using Hewitt-Savage's Theorem again, (8.6) implies that

$$(8.9) \quad \frac{d}{dt} \langle \pi_{t\ell}, \varphi \rangle = \frac{d}{dt} \langle \pi_t, R_\varphi^\ell \rangle = \langle \pi_t, G^\infty [R_\varphi^\ell] \rangle = \langle \pi_t, R_{G_{\ell+1}^\infty [\varphi]}^{\ell+1} \rangle = \langle \pi_{t(\ell+1)}, G_{\ell+1}^\infty [\varphi] \rangle$$

which means that π_t satisfies (8.3).

Assume conversely that π_t satisfies (8.3) and let us prove (8.6). One needs to prove that one can recover any $\Phi \in UC^1(\mathcal{P}_{\mathcal{G}_1}(E))$ from the previous equation (8.9).

Therefore consider $\Phi \in UC^1(\mathcal{P}_{\mathcal{G}_1}(E))$ and let us define

$$\varphi = \left(\pi_C^\ell \Phi \right) (V) = \Phi \left(\mu_V^\ell \right), \quad V = (v_1, \dots, v_\ell)$$

and let us write (8.9) for this choice of φ :

$$\frac{d}{dt} \langle \pi_t, R_{\pi_C^\ell \Phi}^\ell \rangle = \langle \pi_t, G^\infty [R_{\pi_C^\ell \Phi}^\ell] \rangle = \langle \pi_t, R_{G_{\ell+1}^\infty [\pi_C^\ell \Phi]}^{\ell+1} \rangle.$$

Then, on the one hand, for any $f \in \mathcal{P}_{\mathcal{G}_1}(E)$

$$R_{\pi_C^\ell \Phi}^\ell(f) = \int_{E^\ell} \Phi(\mu_V^\ell) \, df^{\otimes \ell}(V) \xrightarrow{\ell \rightarrow \infty} \Phi(f)$$

by the law of large numbers.

On the other hand, for any $f \in \mathcal{P}_{\mathcal{G}_1}(E)$, we have

$$\begin{aligned} R_{G_{\ell+1}^\infty[\pi_C^\ell \Phi]}^{\ell+1}(f) &= \left\langle f^{\otimes \ell+1}, G_{\ell+1}^\infty(\pi_C^\ell \Phi) \right\rangle \\ &= \left\langle DR_{\pi_C^\ell \Phi}^\ell(f), Q(f, f) \right\rangle \\ &= \sum_{i=1}^{\ell} \int_{E^\ell} \Phi(\mu_V^\ell) \, dQ(f, f)(v_i) \prod_{j \neq i} df(v_j). \end{aligned}$$

For any given $i = 1, \dots, \ell$, we define

$$\phi_{V_i}^{\ell-1} = D\Phi(\mu_{V_i}^{\ell-1}), \quad V_i := (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_\ell)$$

and we write

$$\Phi(\mu_V^\ell) = \Phi(\mu_{V_i}^{\ell-1}) + \left\langle \phi_{V_i}^{\ell-1}, \mu_V^\ell - \mu_{V_i}^{\ell-1} \right\rangle + \mathcal{O}\left(\Omega\left(\left\| \mu_{V_i}^{\ell-1} - \mu_V^\ell \right\|\right)\right),$$

where Ω is the modulus of differentiability of Φ as introduced in Definition 2.7. Observing that

$$\mu_V^\ell - \mu_{V_i}^{\ell-1} = \frac{1}{\ell} \delta_{v_i} - \sum_{j \neq i} \frac{1}{\ell(\ell-1)} \delta_{v_j}$$

and that $\langle Q(f, f), 1 \rangle = 0$ from assumption **(A2')**-(ii), we find

$$\begin{aligned} R_{G_{\ell+1}^\infty(\pi_C^\ell \Phi)}^{\ell+1}(f) &= \sum_{i=1}^{\ell} \int_{E^\ell} \left(\frac{1}{\ell} \phi_{V_i}^{\ell-1}(v_i) + \mathcal{O}(\Omega(\ell^{-1})) \right) dQ(f, f)(v_i) \prod_{j \neq i} df(v_j) \\ &= \sum_{i=1}^{\ell-1} \int_{E^{\ell-1}} \frac{1}{\ell-1} \left\langle Q(f, f), \phi_V^{\ell-1} \right\rangle df^{\otimes(\ell-1)}(V) + \mathcal{O}(\ell \Omega(\ell^{-1})) \\ &= \int_{E^{\ell-1}} \left\langle Q(f, f), D\Phi(\mu_V^{\ell-1}) \right\rangle df^{\otimes(\ell-1)}(V) + \mathcal{O}(\ell \Omega(\ell^{-1})) \\ &\xrightarrow{\ell \rightarrow \infty} \langle D\Phi(f), Q(f, f) \rangle \end{aligned}$$

by the law of large numbers again. This implies (8.6).

Step 3: Uniqueness. Let us prove that any solution of (8.3)-(8.6) satisfies the characteristics equation (8.8), or in other words that $\pi_t = \bar{\pi}_t$. This shall imply uniqueness since we have already seen that the the solution to (8.8) is unique.

The main observation here is that for any $\Phi \in UC^1(\mathcal{P}_{\mathcal{G}_1}(E))$ if we define $\Phi_t := T_t^\infty \Phi$, thanks to assumption **(A4')** and a straightforward extended version of Lemma 2.11, we have

$$\forall t \geq 0, \quad \Phi_t \in UC^1(\mathcal{P}_{\mathcal{G}_1}(E)) \subset \text{Dom}(G^\infty).$$

Then since

$$\tau \in [0, t] \mapsto \langle \pi_\tau, \Phi_{t-\tau} \rangle$$

is C^1 from the fact that $\Phi_{t-\tau} \in UC^1(\mathcal{P}_{\mathcal{G}_1}(E))$ belongs to the domain of G^∞ for any τ , we compute

$$\frac{d}{d\tau} \langle \pi_\tau, \Phi_{t-\tau} \rangle = \langle \pi_\tau, G^\infty[\Phi_{t-\tau}] \rangle - \langle \pi_\tau, G^\infty[\Phi_{t-\tau}] \rangle = 0$$

and we deduce that

$$\langle \pi_t, \Phi_0 \rangle = \langle \pi_0, \Phi_t \rangle$$

which proves that $\pi_t = \bar{\pi}_t$ satisfies (8.8), and concludes the proof. \square

8.4. A remark on stationary statistical solutions. As we have seen:

- The chaoticity of a sequence of symmetric N -particle distributions $f^N \in P(E^N)$, $N \geq 1$ is equivalent to the fact that the associated $\pi \in P(P(E))$ is a Dirac at some $f_0 \in P(E)$: $\pi = \delta_{f_0}$. Hence, in view of Hewitt-Savage's theorem, non-chaoticity can be reframed as saying that π is a superposition of *several*, instead of one, chaotic states.
- We have recalled the result in [42, 10] stating that a chaotic (tensorized) sequence is asymptotically concentrated on the energy sphere, which is an effect of the law of large numbers.
- The N -particle dynamics leaves the energy spheres invariant and relaxes on each energy spheres to the uniform measure. This is a consequence of the energy conservation laws: at the level of the particle system, the dynamics is layered according to the value of this conservation law.

One deduces from these considerations that there is room for *non-chaotic stationary states* of the N -particle system, namely superposition of *several* stationary states on different energy spheres. Let us make this more precise.

Lemma 8.3. *There exists a non-chaotic stationary solutions to the statistical Boltzmann equation. In other words, there exists $\pi \in P(P(\mathbb{R}^d))$ such that $\pi \neq \delta_p$ for some $p \in P(\mathbb{R}^d)$ and $A_{\ell+1}^\infty(\pi_{\ell+1}) = 0$ for any $\ell \in \mathbb{N}$.*

Proof of Lemma 8.3. It is clear that any function on the form

$$V \in \mathbb{R}^{d(\ell+1)} \mapsto \pi_{\ell+1}(V) = \phi(|V|^2)$$

is a stationary solution for the equation (8.3), that is $A_{\ell+1}(\pi_{\ell+1}) = 0$ for any $\ell \geq 1$. Now we define, with $d = 1$ for the sake of simplicity, the sequence

$$\forall \ell \geq 1, \quad V \in \mathbb{R}^\ell \mapsto \pi_\ell(V) = \frac{c_\ell}{(1 + |V|^2)^{m+\ell/2}}$$

where the sequence of positive constants c_ℓ is inductively constructed in the following way.

- First c_1 is chosen in a unique way so that π_1 is a probability measure.
- Then, once c_1, \dots, c_ℓ are constructed, $c_{\ell+1}$ is constructed so that $\Pi_\ell[\pi_{\ell+1}] = \pi_\ell$, which means

$$\forall V \in \mathbb{R}^\ell, \quad \int_{v_* \in \mathbb{R}} \frac{c_{\ell+1}}{(1 + |V|^2 + |v_*|^2)^{m+\ell/2+1/2}} dv_* = \frac{c_\ell}{(1 + |V|^2)^{m+\ell/2}}.$$

This is always possible since

$$\begin{aligned} & \int_{v_* \in \mathbb{R}} \frac{c_{\ell+1}}{(1 + |V|^2 + |v_*|^2)^{m+\ell/2+1/2}} dv_* \\ &= \frac{c_{\ell+1}}{(1 + |V|^2)^{m+\ell/2+1/2}} \int_{v_* \in \mathbb{R}} \frac{1}{\left(1 + \frac{|v_*|^2}{(1+|V|^2)}\right)^{m+\ell/2+1/2}} dv_* \\ &= \frac{c_{\ell+1}}{(1 + |V|^2)^{m+\ell/2}} \int_{v_* \in \mathbb{R}} \frac{1}{(1 + |v_*|^2)^{m+\ell/2+1/2}} dv_* \end{aligned}$$

which concludes the induction.

We then deduce that the sequence π_ℓ , $\ell \geq 1$, satisfies (8.3) since every terms only depends on the energy, and also satisfies the compatibility condition $\Pi_\ell[\pi_{\ell+1}] = \pi_\ell$. This concludes the proof. \square

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STÉPHANE MISCHLER

UNIVERSITÉ PARIS-DAUPHINE
 CEREMADE, UMR CNRS 7534
 PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY 75775 PARIS CEDEX 16
 FRANCE

E-MAIL: mischler@ceremade.dauphine.fr

CLÉMENT MOUHOT

UNIVERSITY OF CAMBRIDGE
 DPMMS, CENTRE FOR MATHEMATICAL SCIENCES
 WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UK

E-MAIL: C.Mouhot@dpmms.cam.ac.uk