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Emmanuel Lépinette

# Essential Supremum in a $d$ -dimensional Real Space with Respect to a Random Cone and Applications

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**Abstract** The goal of this paper is to introduce the notion of essential supremum of a family of multi-dimensional random variables with respect to a random convex cone. We give two applications in mathematical finance; We determine the “minimal” portfolio process super-hedging an American claim in the Kabanov discrete-time model with transaction costs. For the same model, we construct a dynamic risk measure in a continuous-time setting. At last, we solve a Skorokhod problem with oblique reflection.

**Mathematics Subject Classification (2000)** 60G44

**JEL Classification** G11 · G13

## 1 Introduction

The primary goal of this paper was the following. Assume that a financial market with proportional transaction costs is modeled by the Kabanov or Campi–Schachermayer framework, [8]. A vector-valued portfolio process, expressed in physical units, represents the number of different risky assets an agent holds. Given a vector-valued process  $h := (h_t)_{t \leq T}$ , we interpret as an American option, the problem is to define the possible endowments (super-hedging prices) we need to start a vector-valued portfolio process which can be liquidated at any instant  $t \in [0, T]$ , paying transaction costs, in such a way that all its components are greater or equal than those of  $h_t$ . In a discrete-time setting, we may reformulate the problem as follows: given a set-valued

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random process  $(K_t(\omega))_{t=0,\dots,T}$ ,  $\omega \in \Omega$ , whose values are closed convex cones in  $\mathbf{R}^d$ ,  $d \geq 1$ , find  $V_0 \in \mathbf{R}^d$  allowing to start a vector-valued portfolio process  $(V_t)_{t=0,\dots,T}$ , i.e.  $\Delta V_t := V_t - V_{t-1} \in -K_t$  a.s., and super-hedging  $h$  in the following sense:  $V_t \geq_{K_t} h_t$ , i.e.  $V_t - h_t \in K_t$  a.s. for all  $t \leq T$ . Papers [2] and [4] provide dual characterizations of all super-hedging prices of a given pay-off referred as the “hedging theorems”. The natural question which arises is to determine the “minimal” (in a sense we shall precise) endowments  $V_0$  we need to super-hedge the payoff  $h$ . Of course the same problem can be posed for a European option, i.e. when we only require to super-hedge the pay-off at the terminal date  $T$ . Without transaction costs, the problem is simpler. A portfolio process is scalar-valued and under the absence of arbitrage opportunities, the minimal prices of a (bounded from below) European claim  $h_T \in \mathbf{R}$  (depending on the price process  $(S_t)_{t \leq T} \in \mathbf{R}^d$ ) is given by the interval  $[\sup_{Q \in \mathcal{M}(S)} e^{-rT} E_Q h_T, \infty)$  where  $T$  is the maturity date of the option,  $r$  is the return of the non-risky asset and  $\mathcal{M}(S)$  is the set of all so-called risk-neutral probability measures under which the discounted prices  $\tilde{S}_t := e^{-rt} S_t$  are martingales, Th. 2.1.11 [8]. It is then possible to define a minimal super-hedging price as the unique element  $\sup_{Q \in \mathcal{M}(S)} e^{-rT} E_Q h_T$  (since  $\mathbf{R}$  is endowed with a total ordering). With transaction costs, the portfolio processes are multi-dimensional and the ordering, defined by random cones  $(K_t)_{t=0,\dots,T}$  in  $\mathbf{R}^d$ ,  $d \geq 1$ , is only partial. The natural problem is to determine the set of all endowments  $V_0 \in \mathbf{R}^d$ , allowing to start a portfolio process super-hedging a given pay-off  $(h_t)_{t \leq T}$ , which are minimal in the following sense: if  $V_0$  is minimal and  $\tilde{V}_0$  is another super-hedging price with  $\tilde{V}_0 \leq_{K_0} V_0$ , then  $\tilde{V}_0 = V_0$ . More generally, among the portfolio processes super-hedging the claim  $(h_t)_{t \leq T}$  at any date  $t$  (with respect to the random cone  $K_t$ ), may we find minimal portfolios? In this paper, we prove the existence of such set of portfolios we call *Essential Supremum*. This is a generalization of the notion of essential supremum of a family of real-valued random variables but it is not a singleton in general. Although, in the two-dimensional space  $\mathbf{R}^2$ , we prove and provide a construction of a unique minimal portfolio. This is of interest in finance. Indeed, recall that  $V_t \geq_{K_t} \tilde{V}_t$  or  $V_t - \tilde{V}_t \in K_t$  a.s. means that we can change  $V_t$  into  $\tilde{V}_t$  paying some transaction costs, i.e.  $\tilde{V}_t$  is “cheaper” than  $V_t$ . One of our goal is then to determine the cheapest portfolios among a set of specific ones, e.g. the portfolios super-hedging an American claim.

We propose to extend the concept of essential supremum of a family of real-valued random variables to vector-valued random variables in  $\mathbf{R}^d$  when the partial ordering is defined by a random cone. The paper is organized as follows; we introduce the definition and the immediate properties. We then give conditions under which the essential supremum can be uniquely defined as a singleton (this is the case in  $\mathbf{R}^2$ ). We end the paper with applications in finance and with a Skorohod problem with oblique relection.

## 2 Essential supremum with respect to a random cone

### 2.1 The model

Let  $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \leq T}, P)$  be a continuous-time stochastic basis verifying the usual conditions. In particular, the filtration  $(\mathcal{F}_t)$  is right continuous. Assume we are given  $t \in [0, T]$  and a  $\mathcal{F}_t$ -adapted closed and convex cone-valued mapping  $G_t$  in  $\mathbf{R}^d$  with

$$G_t := \text{cone}\{\xi_t^1, \dots, \xi_t^N\},$$

where  $\xi_t^1, \dots, \xi_t^N$  are  $\mathcal{F}_t$ -adapted processes we call the generators of  $G_t$ . We suppose that the cone  $G_t$  is proper, i.e.  $G_t \cap (-G_t) = \{0\}$  a.s., and  $\mathbf{R}_+^d \subseteq G_t$ . The positive dual of  $G_t$  is defined as

$$G_t^* := \{x \in \mathbf{R}^d : xy \geq 0, \forall y \in G_t\}$$

where  $xy$  is the Euclidean dot product. This is also a  $\mathcal{F}_t$ -adapted closed convex cone-valued mapping and **we assume** throughout the paper that

$$G_t^* = \text{cone}\{\xi_t^{1*}, \dots, \xi_t^{M*}\}$$

where  $\xi_t^{1*}, \dots, \xi_t^{M*}$  are  $\mathcal{F}_t$ -adapted processes we call the generators of  $G_t^*$ . We recall that the assumptions above hold in the financial models with transaction costs, [8]. In this case,  $G_t$  is the so-called solvency cone which is polyhedral.

We denote by  $\geq_{G_t}$  the partial ordering generated by  $G_t$ . If  $X, Y$  are two random variables in  $\mathbf{R}^d$ , we note  $X \geq_{G_t} Y$  if  $X - Y \in G_t$  a.s. The set of all  $\mathcal{F}_t$ -measurable random variables with values in a random set  $A \subseteq \mathbf{R}^d$  is denoted by  $L^0(A, \mathcal{F}_t)$ . We denote by  $\mathbf{1}_d$  the vector in  $\mathbf{R}^d$  whose each component equals 1 and  $e_i$  the vector in  $\mathbf{R}^d$  whose each component equals 0 except the  $i$ -th one equal to 1.

### 2.2 Definitions

Our goal is to define “the essential supremum” of a collection of  $\mathcal{F}_T$ -measurable random variables  $(\gamma_i)_{i \in I}$  with respect to  $G_t$  at any date  $u \in [t, T]$ . When existence and uniqueness hold,  $\gamma^* = \text{essup}_i(\gamma_i)_u$  is the  $\mathcal{F}_u$ -adapted random variable satisfying the following statements:

$$\gamma^* \geq_{G_t} \gamma_i \text{ a.s., } \forall i \in I, \quad (2.1)$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u), \gamma \geq_{G_t} \gamma_i \text{ a.s., } \forall i \in I, \Rightarrow \gamma \geq_{G_t} \gamma^* \text{ a.s.} \quad (2.2)$$

Recall that the ordering induced by the cone  $G_t$  is only partial. Hence, the existence of such “essential supremum” is not trivial since uniqueness does not necessarily hold. We then define more generally the *Essential supremum* at date  $u \in [t, T]$  as the class (eventually empty)  $\Gamma^* := \text{Essup}_i(\gamma_i)_u$  of all  $\mathcal{F}_u$ -adapted random variables  $\gamma^*$  satisfying the following statements:

$$\gamma^* \geq_{G_t} \gamma_i, \quad \forall i \in I, \quad (2.3)$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) \text{ and } \gamma \geq_{G_t} \gamma_i, \quad \forall i \in I, \Rightarrow \exists \gamma^* \in \Gamma^* \text{ s.t. } \gamma \geq_{G_t} \gamma^* \quad (2.4)$$

$$\gamma_1^*, \gamma_2^* \in \Gamma^* \text{ and } \gamma_1^* \neq \gamma_2^* \Rightarrow P(\gamma_1^* - \gamma_2^* \in G_t \setminus \{0\}) = 0. \quad (2.5)$$

When (2.3) holds, we say that  $\gamma^*$  dominates the family  $(\gamma_i)_{i \in I}$  at date  $t$ .

**Lemma 2.1** *The set  $\Gamma^*$  is unique.*

*Proof.* Let us consider two classes  $C_1$  and  $C_2$  satisfying the axioms (2.3), (2.4) and (2.5). Consider  $\gamma_1^* \in C_1$ . Since  $\gamma_1^*$  dominates the collection  $(\gamma_i)_{i \in I}$ , there exists  $\gamma_2^* \in C_2$  with  $\gamma_1^* \geq_{G_t} \gamma_2^*$ . Similarly, there exists  $\tilde{\gamma}_1^* \in C_1$  satisfying  $\gamma_2^* \geq_{G_t} \tilde{\gamma}_1^*$ . Then,  $\gamma_1^* - \tilde{\gamma}_1^* \in G_t$  a.s. But (2.5) implies that  $\gamma_1^* = \tilde{\gamma}_1^* = \gamma_2^*$ . We deduce that  $C_1 \subset C_2$  and finally  $C_1 = C_2$ .  $\square$

**Lemma 2.2** *The following equivalence holds:*

$\text{essup}_i(\gamma_i)_u$  exists if and only if  $\text{Essup}_i(\gamma_i)_u$  is a singleton.

**Lemma 2.3** *If  $\text{Essup}_i(\gamma_i)_u$  is not a singleton, then  $\text{Essup}_i(\gamma_i)_u$  is empty or infinite.*

*Proof.* Suppose that  $\text{Essup}_i(\gamma_i)_u = \{\gamma_1^*, \gamma_2^*, \dots, \gamma_m^*\}$  where  $\gamma_i^* \neq \gamma_j^*$  if  $i \neq j$ . For each  $k \in \mathbf{N}$ , the random variable  $(1/k)\gamma_1^* + (1 - 1/k)\gamma_2^*$  dominates  $(\gamma_i)_{i \in I}$ . We deduce the existence of an integer  $N_k \in \{3, \dots, m\}$  such that  $(1/k)\gamma_1^* + (1 - 1/k)\gamma_2^* \geq_{G_t} \gamma_{N_k}^*$ . But there exists at least an integer  $j$  in  $\{3, \dots, m\}$  and an infinite subsequence  $(N_{k_n})$  such that  $N_{k_n} = j, \forall n$ . By letting  $n$  tend to  $\infty$ , we deduce that  $\gamma_2^* \geq_{G_t} \gamma_j^*$  so that  $\gamma_2^* = \gamma_j^*$  hence a contradiction.  $\square$

**Lemma 2.4** *The class  $\text{Essup}_i(\gamma_i)_u$  is decomposable, i.e. whatever  $\gamma_1^*, \gamma_2^*$  in  $\text{Essup}_i(\gamma_i)_u$  and  $\Omega_u \in \mathcal{F}_u$ ,  $\gamma_1^* I_{\Omega_u} + \gamma_2^* I_{\Omega_u^c} \in \text{Essup}_i(\gamma_i)_u$ .*

*Proof.* It suffices to consider  $\Gamma = \text{Essup}_i(\gamma_i)_u \cup \{\tilde{\gamma}\}$  where  $\tilde{\gamma}$  is defined as  $\tilde{\gamma} := \gamma_1^* I_{\Omega_u} + \gamma_2^* I_{\Omega_u^c}$ . This set satisfies Axioms (2.3), (2.4) and (2.5) so that  $\Gamma = \text{Essup}_i(\gamma_i)_u$ .  $\square$

**Lemma 2.5** *If  $A \subset B$  then  $\text{Essup} B \subseteq \text{Essup} A$ .*

Similarly, we define  $\text{essinf}_i(\gamma_i)_u := -\text{essup}_i(-\gamma_i)_u$  and

$$\text{Essinf}_i(\gamma_i)_u := -\text{Essup}_i(-\gamma_i)_u.$$

### 3 Existence

In the following section, we show that the Essential supremum of a family of  $\mathcal{F}_T$ -adapted random variables at some date  $u \in [t, T]$  is not empty under mild assumptions. To do so, we prove several lemmas. The conditional expectation we use in the formulation of the lemmas must be understood in the generalized sense. Precisely, denoting a random variable  $\xi$  by  $\xi = \xi^+ - \xi^-$  where  $\xi^+, \xi^- \in \mathbf{R}_+^d$ , we write for a given  $\sigma$ -algebra  $\mathcal{G}$ ,  $E(\xi|\mathcal{G}) = 0$  if  $E(\xi^+|\mathcal{G}) = E(\xi^-|\mathcal{G})$ .

**Lemma 3.1** *Let us consider a collection  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Let us define for any  $\hat{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \hat{\gamma}$ ,  $\forall i \in I$ , the real number

$$a_1(\hat{\gamma}) := \inf_{\gamma \in C_1(\hat{\gamma})} E\xi_t^{1*} \gamma \geq 0,$$

where  $C_1(\hat{\gamma}) := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I \text{ and } \gamma \leq_{G_t} \hat{\gamma}\}$ . Consider a sequence  $(\hat{\gamma}_n) \in C_1(\hat{\gamma})$  such that  $a_1(\hat{\gamma}) = \lim_n E\xi_t^{1*} \hat{\gamma}_n$ . Let us define  $\chi^n := -\hat{\gamma}_n / (1 + |\hat{\gamma}|)$ . Since  $\mathbf{R}_+^d \subseteq G_t$ , it follows that  $\chi^n \geq_{G_t} -\mathbf{1}_d$  and  $\chi^n \in L^0(-G_t, \mathcal{F}_u)$ . Indeed, by assumption  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$  so that  $\gamma \in G_t$  whenever  $\gamma \in C_1(\hat{\gamma})$ . To see it, observe that  $\xi_t^{i*} \gamma \geq \xi_t^{i*} \gamma_{j_0}$  whatever the selector  $\xi_t^{i*} \in L^0(G_t^*, \mathcal{F}_t)$ . It follows that  $\xi_t^{i*} \gamma \geq \xi_t^{i*} E(\gamma_{j_0} | \mathcal{F}_u)$  and  $\xi_t^{i*} \gamma \geq 0$ . From there, we deduce that  $\gamma \in G_t$  a.s.

We split  $\Omega$  into  $\Omega = \Omega_1 \cup \Omega_1^c$  where  $\Omega_1 = \{\liminf_n |\chi^n| = \infty\}$ . On  $\Omega_1$ , there exists a.s.  $(\omega)$  a subsequence  $(n_k(\omega))$  such that  $|\chi^{n_k}| \rightarrow \infty$ . Let us define  $\tilde{\chi}^n = (\chi^n / |\chi^n|) I_{|\chi^n| \neq 0}$ . We may assume by compactity that  $\tilde{\chi}^{n_k(\omega)}(\omega) \rightarrow \tilde{\chi}^\infty(\omega)$  on  $\Omega_1$  where  $|\tilde{\chi}^\infty(\omega)| = 1$ . Moreover,  $\tilde{\chi}^{n_k(\omega)}(\omega) \geq_{G_t} -\mathbf{1}_d / |\chi^{n_k}|$ . By getting  $(n_k)$  converged to  $\infty$ , we deduce that  $\tilde{\chi}^\infty(\omega) \geq_{G_t} 0$  in contradiction with the assumption  $\tilde{\chi}^\infty(\omega) \leq_{G_t} 0$ . Indeed, since  $G_t$  is closed and proper, we get that  $\tilde{\chi}^\infty(\omega) = 0$  in contradiction with  $|\tilde{\chi}^\infty| = 1$ .

We then deduce that  $\Omega = \Omega_1^c$ . According to the lemma on subsequences (Lemma 2.1.2, [8]), there exists a strictly increasing sequence of integer-valued  $\mathcal{F}_u$ -adapted random variables  $(n_k)$  such that  $\chi^{n_k} \rightarrow \chi^\infty$  on  $\Omega$ . It follows that  $\hat{\gamma}_{n_k}$  is also convergent to some random variable  $\bar{\gamma}^1$ . Since  $G_t$  is convex and closed,  $\bar{\gamma}^1 \in C_1(\hat{\gamma})$ . In particular,  $\bar{\gamma}^1 \in L^0(G_t, \mathcal{F}_u)$ . We then may apply the Fatou lemma:

$$E\xi_t^{1*} \bar{\gamma}^1 \leq \liminf_k E\xi_t^{1*} \hat{\gamma}_{n_k} \leq a_1(\hat{\gamma}) \leq E\xi_t^{1*} \bar{\gamma}^1.$$

We then deduce that  $a_1(\hat{\gamma}) = E\xi_t^{1*} \bar{\gamma}^1$ . We repeat the procedure in the following way. Consider

$$a_2(\hat{\gamma}) := \inf_{\gamma \in C_2(\hat{\gamma})} E\xi_t^{2*} \gamma \geq 0,$$

where

$$C_2(\hat{\gamma}) := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I \text{ and } \gamma \leq_{G_t} \bar{\gamma}^1\} \subset C_1(\hat{\gamma}).$$

We get  $\bar{\gamma}^2 \in C_2(\hat{\gamma})$  such that  $a_1(\hat{\gamma}) = E\xi_t^{1*} \bar{\gamma}^2$  and  $a_2(\hat{\gamma}) = E\xi_t^{2*} \bar{\gamma}^2$ . Reiterating the procedure, we finally get a  $\mathcal{F}_u$ -adapted random variable  $\bar{\gamma}^M \in C_i(\hat{\gamma})$ ,  $1 \leq i \leq M$ , satisfying

$$a_i(\hat{\gamma}) = E\xi_t^{i*} \bar{\gamma}^M, \quad \forall i = 1, \dots, M. \quad (3.1)$$

Observe that  $\bar{\gamma}^M \leq_{G_t} \hat{\gamma}$  by definition. We denote by  $\Lambda(\hat{\gamma})$  the set of all random variables  $\bar{\gamma}^M \in C_M(\hat{\gamma})$  verifying (3.1) and we define

$$\text{Essup}_i(\gamma_i)_u := \bigcup_{\hat{\gamma} \in \Theta} \Lambda(\hat{\gamma}) \quad (3.2)$$

where  $\Theta$  is the set of all  $\mathcal{F}_u$ -adapted  $\widehat{\gamma}$  satisfying  $\gamma_i \leq_{G_t} \widehat{\gamma}$ ,  $\forall i \in I$ . Of course, for a given  $\widehat{\gamma}$ ,  $\Lambda(\widehat{\gamma})$  is not necessarily unique so that the union should be understood over all possible sets  $C_M(\widehat{\gamma})$ . It remains to show that the class defined by (3.2) satisfies the statements (2.3), (2.4) and (2.5).

If  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$ , it is obvious that  $\gamma^*$  verifies Assertion (2.3). We claim that for any  $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u)$ , dominating every  $\gamma_i$ ,  $i \in I$ , Assertion (2.4) holds. It suffices to consider an arbitrary  $\gamma^* \in \Lambda(\gamma)$ . At last, consider  $\gamma_1^*, \gamma_2^* \in \text{Essup}_i(\gamma_i)_u$  with  $\gamma_1^* \in \Lambda(\widehat{\gamma}_1)$  and  $\gamma_2^* \in \Lambda(\widehat{\gamma}_2)$ . Assume that  $\gamma_1^* - \gamma_2^* \in G_t \setminus \{0\}$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . Then, there exists  $i \in \{1, \dots, m\}$  and a non null set  $\Theta_u \subset \Omega_u$  such that  $\xi_t^{i*}(\gamma_1^* - \gamma_2^*)I_{\Theta_u} \geq 0$  and  $E\xi_t^{i*}(\gamma_1^* - \gamma_2^*)I_{\Theta_u} > 0$ . It follows that

$$a_i(\widehat{\gamma}_1) = E\xi_t^{i*}\gamma_1^* > E\xi_t^{i*}(\gamma_2^*I_{\Theta_u} + \gamma_1^*I_{\Theta_u^c}). \quad (3.3)$$

But  $\gamma = \gamma_2^*I_{\Theta_u} + \gamma_1^*I_{\Theta_u^c} \leq_{G_t} \gamma_1^* \in C_i(\widehat{\gamma}_1)$  and  $\gamma$  dominates the family  $(\gamma_i)_{i \in I}$ . It follows that  $\gamma \in C_i(\widehat{\gamma}_1)$  which implies that  $E\xi_t^{i*}\gamma \geq a_i(\widehat{\gamma}_1)$  in contradiction with (3.3). Hence, Assertion (2.5) holds.  $\square$

**Corollary 3.2** *Let us consider a family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u)$  exists. Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Let us define, for all  $i \in I$ ,  $\widetilde{\gamma}_i := \gamma_i - E(\gamma_{j_0} | \mathcal{F}_u)$  where  $j_0 \in I$  is fixed. This collection obviously satisfies the conditions of Lemma 3.1. We then conclude with

$$\text{Essup}_i(\gamma_i)_u = \text{Essup}_i(\widetilde{\gamma}_i)_u + E(\gamma_{j_0} | \mathcal{F}_u).$$

$\square$

In the one dimensional case, with  $G_0 = \mathbf{R}_+$ , we may define the essential supremum  $\text{essup}_i(\gamma_i)_u$  as soon as the family  $(\gamma_i)_{i \in I}$  is bounded from above by a real-valued and  $\mathcal{F}_u$ -adapted random variable as shown above. In particular  $\text{essup}_i(\gamma_i)_T$  coincides with the usual essential supremum  $\text{essup}_i(\gamma_i)$  as defined in the literature. Actually, we may extend the definition for random variables taking values in  $(-\infty, \infty]$  as follows:

**Proposition 3.3** *For any family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}, \mathcal{F}_T)$  of scalar random variables (which may take infinite values), there exists a unique  $\mathcal{F}_u$ -adapted random variable  $\text{essup}_i(\gamma_i)_u := \gamma$  verifying:*

- (1)  $\gamma \geq \gamma_i$ ,  $\forall i \in I$ ,
- (2) If  $\gamma' \in L^0(\mathbf{R}, \mathcal{F}_u)$  verifies  $\gamma' \geq \gamma_i$ ,  $\forall i \in I$ , then  $\gamma' \geq \gamma$ .

*Proof.* It suffices to take  $\gamma := \lim_{m \rightarrow \infty} \text{essup}_i(\gamma_i \wedge m)_u$ .  $\square$

**Remark 3.4** *We have  $\text{essup}_i(\gamma_i)_u \geq \text{essup}_i(\gamma_i)_T$ ,  $\forall u \in [0, T]$ .*

**Corollary 3.5** *Assume that the family  $(\gamma_i)_{i \in I}$  satisfies  $\text{essup}_i|\gamma_i|_u < \infty$  a.s. Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Consider the family

$$\tilde{\gamma}_i := \frac{\gamma_i}{1 + \text{essup}_j |\gamma_j|_u}, \quad i \in I.$$

We have  $-\mathbf{1}_d \leq_{G_t} \tilde{\gamma}_i \leq_{G_t} \mathbf{1}_d$ . Applying Corollary 3.2, we deduce that  $\text{Essup}_i(\tilde{\gamma}_i)_u$  is not empty. It follows that  $\text{Essup}_i(\gamma_i)_u$  is not empty and is given by

$$\text{Essup}_i(\gamma_i)_u = (1 + \text{essup}_j |\gamma_j|_u) \text{Essup}_i(\tilde{\gamma}_i)_u. \quad (3.4)$$

Indeed, consider  $\gamma^* \in (1 + \text{essup}_j |\gamma_j|_u) \text{Essup}_i(\tilde{\gamma}_i)_u$ , i.e.  $\gamma^* = (1 + \text{essup}_j |\gamma_j|_u) \tilde{\gamma}^*$  where

$$\tilde{\gamma}^* \geq_{G_t} \frac{\gamma_i}{1 + \text{essup}_j |\gamma_j|_u}, \quad \forall i \in I.$$

We deduce that  $\gamma^* \geq_{G_t} \gamma_i, \forall i \in I$ , i.e. Assertion (2.3) holds. Let us consider a random variable  $\gamma$  verifying  $\gamma \geq_{G_t} \gamma_i, \forall i \in I$ . Then, we also have the inequalities  $\gamma / (1 + \text{essup}_j |\gamma_j|_u) \geq_{G_t} \tilde{\gamma}_i, \forall i \in I$ . We deduce  $\tilde{\gamma}^* \in \text{Essup}_i(\tilde{\gamma}_i)_u$  such that  $\gamma / (1 + \text{essup}_j |\gamma_j|_u) \geq_{G_t} \tilde{\gamma}^*$  and  $\gamma \geq_{G_t} (1 + \text{essup}_j |\gamma_j|_u) \tilde{\gamma}^*$  i.e. Assertion (2.4) holds. Similarly, Assertion (2.5) holds.  $\square$

#### 4 Immediate properties

**Corollary 4.1** *Let us consider a family  $(\gamma_i)_{i \in I}$ . Then,  $\text{essup}_i |\gamma_i|_T < \infty$  (a.s.) if and only if there exists  $\gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that  $\gamma_1 \leq_{G_t} \gamma_i \leq_{G_t} \gamma_2, \forall i$ .*

*Proof.* Assume that  $\text{essup}_i |\gamma_i|_T < \infty$  a.s. Applying Corollary 3.5, we deduce that  $\text{Essinf}_i(\gamma_i)_T$  and  $\text{Essup}_i(\gamma_i)_T$  are not empty so that we can conclude. Reciprocally, suppose that  $\text{essup}_i |\gamma_i|_T = \infty$  on a non null set  $\Lambda$ . We may extract a subsequence such that  $|\gamma_i| \rightarrow \infty$ . We introduce the random variables  $\tilde{\gamma}_i = \gamma_i / |\gamma_i|$ ,  $\tilde{\gamma}_{1i} = \gamma_1 / |\gamma_i|$  and  $\tilde{\gamma}_{2i} = \gamma_2 / |\gamma_i|$ . By assumption, we obtain  $\tilde{\gamma}_{1i} \leq_{G_t} \tilde{\gamma}_i \leq_{G_t} \tilde{\gamma}_{2i}, \forall i \in I$ . By compactity, we may assume that  $\tilde{\gamma}_i \rightarrow \tilde{\gamma}$  with  $|\tilde{\gamma}| = 1$ . On the other hand, the last inequality yields  $\tilde{\gamma} = 0$  on  $\Lambda$  since  $G_t$  is closed and proper. Hence a contradiction.  $\square$

**Proposition 4.2** *Let us consider a family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that  $E(\gamma_{i_0} | \mathcal{F}_u)$  exists for some  $j_0 \in I$  and  $\text{Essup}_i(\gamma_i)_u \neq \emptyset$  where  $u \in [t, T]$ . If  $\lambda_u \in L^0(\mathbf{R}_+, \mathcal{F}_u)$ , then*

$$\text{Essup}_i(\lambda_u \gamma_i)_u = \lambda_u \text{Essup}_i(\gamma_i)_u.$$

*Proof.* We first suppose that there exists  $i_0 \in I$  such that  $E(\gamma_{i_0} | \mathcal{F}_u) = 0$ . It follows that

$$\text{Essup}_i(\lambda_u \gamma_i)_u \cup \text{Essup}_i(\gamma_i)_u \subset L^0(G_t, \mathcal{F}_u). \quad (4.5)$$

Let us consider  $\gamma \in \text{Essup}_i(\lambda_u \gamma_i)_u$ . Since  $\gamma \geq_{G_t} \lambda_u \gamma_i, \forall i \in I$ , we then deduce that for a given  $\gamma^{1*} \in \text{Essup}_i(\gamma_i)_u$ ,

$$\frac{\gamma}{\lambda_u} I_{\{\lambda_u \neq 0\}} + \gamma^{1*} I_{\{\lambda_u = 0\}} \geq_{G_t} \gamma_i, \quad \forall i \in I.$$

We deduce  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$  such that

$$\frac{\gamma}{\lambda_u} I_{\{\lambda_u \neq 0\}} + \gamma^{1*} I_{\{\lambda_u = 0\}} \geq_{G_t} \gamma^*.$$

By virtue of (4.5),  $\gamma \geq_{G_t} \lambda_u \gamma^*$ . Similarly, there exists  $\gamma^1 \in \text{Essup}_i(\lambda_t \gamma_i)_u$  such that  $\lambda_u \gamma^* \geq_{G_t} \gamma^1$ . We then deduce that  $\gamma = \gamma^1 = \lambda_u \gamma^*$ . We have shown

$$\text{Essup}_i(\lambda_u \gamma_i)_u \subseteq \lambda_u \text{Essup}_i(\gamma_i)_u.$$

Reciprocally, if  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$ , then  $\lambda_u \gamma^* \geq_{G_t} \lambda_u \gamma_i, \forall i \in I$ . So, there is  $\gamma \in \text{Essup}_i(\lambda_u \gamma_i)_u$  such that  $\lambda_u \gamma^* \geq_{G_t} \gamma$ . We deduce that

$$\gamma^* \geq_{G_t} \frac{\gamma}{\lambda_u} I_{\lambda_u \neq 0} + \gamma^* I_{\lambda_u = 0} \geq_{G_t} \gamma^{1*} \quad (4.6)$$

where  $\gamma^{1*} \in \text{Essup}_i(\gamma_i)_u$ . From there,  $\gamma^* = \gamma^{1*}$  and  $\lambda_u \gamma^* = \gamma$  if  $\lambda_u \neq 0$ . Otherwise,  $\lambda_u \gamma^* \geq_{G_t} \gamma$  implies that  $\gamma = 0$  since  $\gamma \geq_{G_t} 0$  by virtue of (4.5). We then conclude that  $\lambda_u \gamma^* = \gamma$  and the reverse inclusion is proven.

In the general case, we fix  $i_0 \in I$  and we deduce from above that

$$\begin{aligned} \text{Essup}_i(\lambda_u \gamma_i)_u &= \text{Essup}_i(\lambda_u(\gamma_i - E(\gamma_{i_0} | \mathcal{F}_u)))_u + \lambda_u E(\gamma_{i_0} | \mathcal{F}_u), \\ &= \lambda_u \text{Essup}_i(\gamma_i - E(\gamma_{i_0} | \mathcal{F}_u))_u + \lambda_u E(\gamma_{i_0} | \mathcal{F}_u), \\ &= \lambda_u (\text{Essup}_i(\gamma_i - E(\gamma_{i_0} | \mathcal{F}_u))_u + E(\gamma_{i_0} | \mathcal{F}_u)), \\ &= \lambda_u \text{Essup}_i(\gamma_i)_u. \end{aligned}$$

□

**Proposition 4.3** *Let us consider  $X \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  where  $u \in [t, T]$ . Assume that  $\text{Essup}(X, 0)_u$  is a singleton. Then,*

$$\text{Essup}(X, -X, 0)_u \subset \text{Essup}(X, 0)_u + \text{Essup}(-X, 0)_u.$$

*Proof.* Let us consider  $\gamma^* \in \text{Essup}(X, -X, 0)_u$ . Then,  $\gamma^* + X \geq_{G_t} X$  and  $\gamma^* + X \geq_{G_t} 0$ . We deduce that  $\gamma^* + X \geq_{G_t} \gamma_1$  where  $\gamma_1 := \text{essup}(X, 0)_u$ . We also have  $\gamma^* \geq_{G_t} \gamma_1$ . Then  $\gamma^* - \gamma_1 \geq_{G_t} -X, 0$ . We deduce that  $\gamma^* - \gamma_1 \geq_{G_t} \gamma_2$  where  $\gamma_2 \in \text{Essup}(-X, 0)_u$ . On the other hand,  $\gamma_1 + \gamma_2$  dominates  $X, -X$  and  $0$ . Hence, there is  $\gamma^{1*} \in \text{Essup}(-X, X, 0)_u$  such that  $\gamma^* \geq_{G_t} \gamma_1 + \gamma_2 \geq_{G_t} \gamma^{1*}$ . It follows that  $\gamma^* = \gamma_1 + \gamma_2 = \gamma^{1*}$ . □

**Remark 4.4** *In the case where the three essential supremum sets above are singletons, the inclusion is an equality.*

**Proposition 4.5** *Assume that the cone  $G_t$  is such that the essential supremum of two random variables is necessary a singleton. Let us consider a family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_T)$  and  $u \in [t, T]$ . Then, the set  $\text{Essup}_i(\gamma_i)_u$  is closed in probability.*

*Proof.* Consider  $\Gamma$  the closure in probability of  $\text{Essup}_i(\gamma_i)_u$ . It suffices to check that  $\Gamma$  satisfies the same axioms than  $\text{Essup}_i(\gamma_i)_u$ . First, if  $\gamma^n$  converges in probability to  $\gamma^* \in \Gamma$ , there exists a subsequence  $n_k$  such that  $\gamma^{n_k} \rightarrow \gamma^*$  a.s. From  $\gamma^{n_k} \geq_{G_t} \gamma_i, \forall i \in I$ , we deduce that  $\gamma^* \geq_{G_t} \gamma_i, \forall i$ . Secondly, if  $\delta \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfies  $\delta \geq_{G_t} \gamma_i, \forall i \in I$ , then there exists  $\gamma \in \text{Essup}_i(\gamma_i)_u \subseteq \Gamma$  such that  $\delta \geq_{G_t} \gamma$ . At last, consider  $\gamma_1, \gamma_2 \in \Gamma$  and suppose that  $\gamma_1 - \gamma_2 \in G_t \setminus \{0\}$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . We may assume there exists  $i \in \{1, \dots, M\}$  such that  $\xi_t^{i*} \gamma_1 > \xi_t^{i*} \gamma_2$  on  $\Omega_u$  where, by density,  $\gamma_1, \gamma_2 \in \text{Essup}_i(\gamma_i)_u$ . We assume without loss of generality that  $\text{Essup}_i(\gamma_i)_u \neq \emptyset$ . It follows that the essential supremum is given by Lemma 3.1. In particular, considering the notations of the proof,  $\gamma_1 \in C_M(\widehat{\gamma})$  and, for all  $i$ , the inequality  $a_i(\widehat{\gamma}) = E\xi_t^{i*} \gamma_1 > E\xi_t^{i*} \gamma \geq a_i(\widehat{\gamma})$  holds where  $\gamma := \gamma_1 I_{\Omega \setminus \Omega_u} + \text{essinf}(\gamma_2, \gamma_1) I_{\Omega_u}$ , hence a contradiction. Indeed,  $\text{essinf}(\gamma_2, \gamma_1) \leq_{G_t} \gamma_2$  yields the last strict inequality. Since  $\gamma_2, \gamma_1$  dominate the family  $(\gamma_i)_{i \in I}$ , we deduce that  $\text{essinf}(\gamma_2, \gamma_1) \geq_{G_t} (\gamma_i)_{i \in I}$  with  $\text{essinf}(\gamma_2, \gamma_1) \leq_{G_t} \widehat{\gamma}$ .  $\square$

**Definition 4.6** A subset  $\Lambda_u \subseteq L^0(\mathbf{R}^d, \mathcal{F}_u)$  is said decomposable if  $\Omega_u \in \mathcal{F}_u$  and  $\gamma, \widetilde{\gamma} \in \Lambda_u$  implies  $\gamma I_{\Omega_u} + \widetilde{\gamma} I_{\Omega_u^c} \in \Lambda_u$ .

**Corollary 4.7** Suppose that the conditions of Proposition 4.5 hold. Let us consider a family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_T)$ . Then, there exists a closed set-valued and  $\mathcal{F}_u$ -measurable mapping  $\Gamma_u \subseteq \mathbf{R}^d$  such that

$$\text{Essup}_i(\gamma_i)_u = L^0(\Gamma_u, \mathcal{F}_u).$$

In particular, the graphs of  $\Gamma_u$  are  $\mathcal{F}_u \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable.

*Proof.* We know that  $\text{Essup}_i(\gamma_i)_u$  is closed in probability. Moreover, observe that  $\text{Essup}_i(\gamma_i)_u$  is decomposable. We then conclude applying Proposition 5.4.3 [8].  $\square$

**Proposition 4.8** Let us consider a totally ordered family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that there exists  $\overline{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  with  $\gamma_i \leq_{G_t} \overline{\gamma}, \forall i \in I$ . Then  $\text{Essup}_i(\gamma_i)_t$  is a singleton  $\{\gamma^*\}$ . Moreover, there exists a random subsequence  $(n_k) \in \mathbf{N}$  such that  $(\gamma_{n_k})$  is an increasing sequence converging a.s. to  $\gamma^*$ .

*Proof.* Consider

$$b^j := \sup_{i \in I} E\xi_t^{j*} \gamma_i, \quad j = 1, \dots, M.$$

There exists some subsequences  $(i_k^{(j)})$  such that  $b^j = \lim_k \nearrow E\xi_t^{j*} \gamma_{i_k^{(j)}}$ . Since the family we consider is totally ordered, we deduce that the sequences  $(\gamma_{i_k^{(j)}})_k$  are increasing with respect to the partial ordering induced by  $G_t$ . We then set

$$\gamma_{i_k} := \text{Essup}_{1 \leq j \leq M} (\gamma_{i_k^{(j)}})_t \subseteq \{\gamma_{i_k^{(j)}} : 1 \leq j \leq M\}.$$

It is straightforward that  $(\gamma_{i_k})_k$  is still an increasing sequence and moreover

$$b^j = \lim_k \nearrow E\xi_t^{j*} \gamma_{i_k}, \quad \forall 1 \leq j \leq M.$$

Notice that  $\gamma_{i_1} \leq_{G_t} \gamma_{i_k} \leq_{G_t} \bar{\gamma}$ ,  $k \geq 1$ . We deduce that  $\liminf_k |\gamma_{i_k}|_t < \infty$ . We then deduce a random sequence  $(n_k) \in \mathbf{N}$  such that  $\gamma_{n_k}$  converges a.s. to some  $\gamma^* \leq_{G_t} \bar{\gamma}$ . Recall that  $\gamma_{n_k}$  reads as  $\gamma_{n_k} = \sum_{p=1}^{\infty} \gamma_p I_{\{n_k=p\}}$ . Since the sequences  $(\gamma_{i_k})_k$  and  $(n_k)$  are increasing (a.s.), we then deduce that the sequence  $(\gamma_{n_k})$  is also increasing with respect to the partial ordering induced by  $G_t$ . From  $n_k(\omega) \geq k$  a.s.  $(\omega)$ , we deduce that  $\gamma_{n_m(\omega)} \geq_{G_t} \gamma_k(\omega)$  a.s. if  $m \geq k$ . It follows that  $\gamma^* \geq_{G_t} \gamma_k$  for any  $k$ . From there, we deduce that  $\text{Essup}_i(\gamma_i)_t = \{\gamma^*\}$ . Indeed, if  $\gamma \geq_{G_t} \gamma_k$  for any  $k$ , then  $\gamma \geq_{G_t} \gamma_{n_k}$  and  $\gamma \geq_{G_t} \gamma^*$ , i.e. the singleton  $\{\gamma^*\}$  satisfies the required axioms to be the essential supremum.  $\square$

We immediately deduce the following:

**Corollary 4.9** *Let us consider a family  $(\gamma_i)_{i \in I} \subseteq L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $\text{Essup}_i(\gamma_i)_t \neq \emptyset$ . Assume that  $(\gamma_i)_{i \in I}$  is decomposable and closed in probability. Then  $(\gamma_i)_{i \in I}$  is an inductive set and so contains at least one maximal element (with respect to the partial ordering of the cone  $G_t$ ).*

## 5 When the generators are linearly independent

Recall that  $x \in G_t$  if and only if there are some positive coefficients  $\alpha_i(x) \geq 0$  such that

$$x = \sum_{i=1}^N \alpha_i(x) \xi_t^i.$$

We have the following lemma:

**Lemma 5.1** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Let us consider  $x, y \in G_t$ . Then,  $x \geq_{G_t} y$  if and only if  $\alpha_i(x) \geq \alpha_i(y), \forall i = 1, \dots, N$ .*

*Proof.* If  $x - y \in G_t$  and  $x, y \in G_t$ , then

$$x - y = \sum_{i=1}^N \alpha_i(x - y) \xi_t^i = \sum_{i=1}^N (\alpha_i(x) - \alpha_i(y)) \xi_t^i.$$

We deduce that  $0 \leq \alpha_i(x - y) = \alpha_i(x) - \alpha_i(y), \forall i$ , and we conclude.  $\square$

**Lemma 5.2** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent and we are given a finite family  $\eta_1, \dots, \eta_m \in L^0(G_t, \mathcal{F}_T)$ . Consider  $u \in [t, T]$ . If there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  such that  $(\eta_i)_i \leq_{G_t} \bar{\gamma}$  a.s., then  $\Gamma_u^* := \text{Essup}_i(\eta_i)_u$  is not empty and reduced to the singleton  $\text{essup}_i(\eta_i)_u$ .*

*Proof.* By a measurable selection argument, let us write

$$\eta_i = \sum_{j=1}^N \alpha_j^i \xi_t^j$$

where  $\alpha_j^i$  are positive and  $\mathcal{F}_T$ -adapted random variables. We then define  $\hat{\alpha}_j := \text{essup}_i(\alpha_j^i)_u$  which is a.s. finite. Indeed, recall that  $(\eta_i)_i \leq_{G_t} \bar{\gamma}$  so that we may apply Lemma 5.1. Let us set

$$\text{essup}_i(\eta_i)_u := \gamma^* := \sum_{j=1}^N \hat{\alpha}_j \xi_t^j.$$

By virtue of Lemma 5.1,  $\gamma^*$  dominates  $\eta_i$  for all  $i$ . Moreover, if another  $\mathcal{F}_u$ -adapted random variable  $\eta$  dominates  $\eta_i$  for all  $i$ , then  $\eta \in L^0(G_t, \mathcal{F}_u)$ . Its decomposition

$$\eta = \sum_{j=1}^N \alpha_j \xi_t^j$$

verifies  $\alpha_j \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  and  $\alpha_j \geq \alpha_j^i, \forall i$ , by virtue of Lemma 5.1, i.e.  $\alpha_j \geq \hat{\alpha}_j$  or  $\eta \geq_{G_t} \gamma^*$ .  $\square$

In the same way, we show the following:

**Lemma 5.3** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent and we are given a finite family  $\eta_1, \dots, \eta_m \in L^0(G_t, \mathcal{F}_T)$ . If there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$ ,  $u \geq t$ , such that  $(\eta_i)_i \geq_{G_t} \bar{\gamma}$  a.s., then  $\text{Essinf}_i(\gamma_i)_u$  is not empty and is reduced to the singleton  $\text{essinf}_i(\gamma_i)_u$ .*

**Lemma 5.4** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Suppose we are given a family  $(\gamma_i)_{i \in I}$  and  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$ ,  $u \geq t$ , such that  $\gamma_i \leq_{G_t} \bar{\gamma}, \forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u) = \exists$ . Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty and is reduced to the singleton  $\text{essup}_i(\gamma_i)_u$ .*

*Proof.* We may assume without loss of generality that there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Let us define

$$a_1 := \inf_{\gamma \in C} E \xi_t^{1*} \gamma \geq 0$$

where  $C := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I\}$ . Let us consider a sequence  $(\hat{\gamma}_n)_n \subseteq C$  such that  $a_1 = \lim_n E \xi_t^{1*} \hat{\gamma}_n$ . Let us define

$$\chi^n := -\frac{\hat{\gamma}_n}{1 + |\hat{\gamma}|}.$$

By assumption, we have  $\chi^n \geq_{G_t} -\mathbf{1}_d$ , and  $\chi^n \in L^0(-G_t, \mathcal{F}_u)$ .

As in the proof of Lemma 3.1, there exists a  $\mathcal{F}_u$ -adapted subsequence  $(n_k)$  such that  $(\chi^{n_k})$  is a.s. convergent. Hence  $(\hat{\gamma}_{n_k})$  is also convergent to  $\bar{\gamma}^1$ . Since  $G_t$  is convex and closed,  $\bar{\gamma}^1 \in C$  and belongs to  $L^0(G_t, \mathcal{F}_u)$ . Applying the Fatou lemma:

$$E \xi_t^{1*} \bar{\gamma}^1 \leq \liminf_k E \xi_t^{1*} \hat{\gamma}_{n_k} \leq a_1.$$

We deduce that  $a_1(\hat{\gamma}) = E \xi_t^{1*} \bar{\gamma}^1$ . In the same way, we define  $a_2, \dots, a_M$  and we find  $\bar{\gamma}^2, \dots, \bar{\gamma}^M \in C$  satisfying

$$a_i := \inf_{\gamma \in C} E \xi_t^{i*} \gamma = E \xi_t^{i*} \bar{\gamma}^i, \quad i \leq M.$$

By virtue of Lemma 5.3, let us define  $\gamma^* := \operatorname{ess\,inf}_i(\bar{\gamma}^i)_u$ . By definition,  $\gamma^* \in C$  and

$$a_i := \inf_{\gamma \in C} E\xi_t^{i*} \gamma = E\xi_t^{i*} \gamma^*, \quad i \leq M.$$

Assume that  $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  verifies  $\gamma \geq_{G_t} \gamma_i, \forall i \in I$ , and  $\gamma - \gamma^* \notin G_t$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . Then, there exists  $i \in \{1, \dots, M\}$  and a non null set  $\Theta_u \subset \Omega_u$  such that  $\xi_t^{i*}(\gamma - \gamma^*)I_{\Theta_u} \leq 0$  and  $E\xi_t^{i*}(\gamma - \gamma^*)I_{\Theta_u} < 0$ . It follows that

$$a_i = E\xi_t^{i*} \gamma^* > E\xi_t^{i*}(\gamma I_{\Theta_u} + \gamma^* I_{\Theta_u^c}). \quad (5.7)$$

But  $\tilde{\gamma} = \gamma I_{\Theta_u} + \gamma^* I_{\Theta_u^c} \in C$ . Hence,  $E\xi_t^{i*} \tilde{\gamma} \geq a_i$  and we get a contradiction.  $\square$

**Corollary 5.5** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Let us consider a collection  $(\gamma_i)_{i \in I}$  with  $\operatorname{ess\,sup}_i |\gamma_i|_u < \infty$  a.s. Then,  $\operatorname{Ess\,sup}_i (\gamma_i)_u$  is not empty and equals the singleton  $\operatorname{ess\,sup}_i (\gamma_i)_u$ .*

*Proof.* The proof is similar to that of Corollary 3.5.  $\square$

**Lemma 5.6** *Assume that  $G$  is generated by some linearly independent generators  $\xi_t^1, \dots, \xi_t^N$ . We are given a family of  $\mathcal{F}_T$ -adapted random variables  $(\gamma_i)_i$  such that  $\operatorname{ess\,sup}_i (\gamma_i)_u$  exists, where  $u \in [t, T]$ . Consider an arbitrary selector  $\eta_t$  of  $G^*$ , i.e.  $\eta_t \in L^0(G_t^*, \mathcal{F}_t)$ . Then,*

$$(\operatorname{ess\,sup}_i (\gamma_i)_u) \cdot \eta_t \geq \operatorname{ess\,sup}_i (\gamma_i \cdot \eta_t)_u.$$

*Proof.* Since  $\operatorname{ess\,sup}_i (\gamma_i)_u \geq_{G_t} \gamma_i, \forall i \in I$ , we deduce the inequality  $(\operatorname{ess\,sup}_i (\gamma_i)_u) \cdot \eta_t \geq \gamma_i \cdot \eta_t, \forall i \in I$ . It follows that

$$(\operatorname{ess\,sup}_i (\gamma_i)_u) \cdot \eta_t \geq \operatorname{ess\,sup}_i (\gamma_i \cdot \eta_t)_u.$$

$\square$

**Lemma 5.7** *Assume that  $G_t$  and  $G_t^*$  are generated by a finite number of linearly independent generators. We are given a family of  $\mathcal{F}_T$ -adapted random variables  $(\gamma_i)_i$  such that  $\operatorname{ess\,sup}_i (\gamma_i)_u$  exists, where  $u \in [t, T]$ . Consider an arbitrary generator  $\eta_t \in L^0(G_t^*, \mathcal{F}_t)$ . Then,*

$$(\operatorname{ess\,sup}_i \gamma_i)_u \cdot \eta_t = \operatorname{ess\,sup}_i (\gamma_i \cdot \eta_t)_u.$$

*Proof.* In sight of the previous lemma, assume that

$$(\operatorname{ess\,sup}_i \gamma_i)_u \cdot \eta_t \not\geq \operatorname{ess\,sup}_i (\gamma_i \cdot \eta_t)_u. \quad (5.8)$$

We denote by  $\eta^{(1)} := \eta_t, \dots, \eta^{(d)}$  the generators of  $G_t^*$  and we recall that they are linearly independent by assumption. By a measurable selection argument, we consider  $\xi_t \in \{\eta^{(2)}, \dots, \eta^{(d)}\}^\perp$  with  $\xi_t \eta^{(1)} \geq 0$  and  $|\xi_t| = 1$ . It is possible; in the contrary case, we have  $\{\eta^{(2)}, \dots, \eta^{(d)}\}^\perp = \{0\}$  and  $\operatorname{lin}(\eta^{(2)}, \dots, \eta^{(d)}) = \mathbf{R}^d$ . We deduce that  $\xi_t \in L^0(G_t, \mathcal{F}_t)$ . Again, by a measurable selection argument, we then deduce from (5.8) the existence of  $\alpha_u \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  small enough such that

$$(\operatorname{ess\,sup}_i (\gamma_i)_u - \alpha_u \xi_t) \cdot \eta_t \geq \operatorname{ess\,sup}_i (\gamma_i \cdot \eta_t)_u.$$

From there, it follows that  $\operatorname{ess\,sup}_i (\gamma_i)_u - \alpha_u \xi_t \geq_{G_t} \gamma_i, \forall i \in I$ , and we get that  $\operatorname{ess\,sup}_i (\gamma_i)_u - \alpha_u \xi_t \geq_{G_t} \operatorname{ess\,sup}_i (\gamma_i)_u$  hence a contradiction.  $\square$

**Lemma 5.8** *Let us consider a collection of  $\mathcal{F}_T$ -measurable random variables  $\gamma = (\gamma_i)_{i \in I}$  such that  $\text{essup}_{i \in I}(\gamma_i)_u$  exists where  $u \in [t, T]$ . Consider the “upward” completion defined as*

$$\bar{\gamma}^{up} := \left\{ \text{essup}_{i_1, \dots, i_n}(\gamma_{i_1}, \dots, \gamma_{i_n}) : n \in \mathbf{N}, i_j \in I \right\}. \quad (5.9)$$

Then

$$\text{essup}_{i \in I}(\gamma_i)_u = \text{essup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

*Proof.* By definition,  $\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \gamma_{i_j}$ , whatever  $i_j \in I$ . We deduce that  $\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \text{essup}_{j \leq n}(\gamma_{i_j})_u$  and then

$$\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \text{essup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

The reverse inequality is obvious so that we can conclude.  $\square$

**Corollary 5.9** *Assume that  $G_t$  is generated by a finite number of linearly independent generators and  $G_t^*$  is generated by a basis of  $\mathbf{R}^d$ . We are given a directed upwards family of  $\mathcal{F}_t$ -adapted random variables  $(\gamma_i)_{i \in I}$  such that  $\text{essup}_i(\gamma_i)_t$  exists. Consider  $\eta_s \in L^0(\mathbf{R}^d, \mathcal{F}_s)$  where  $s \leq t$ . Then, there exists a sequence  $(i_n) \subseteq I$  such that*

$$\left( \text{essup}_i(\gamma_i)_t \right) \cdot \eta_s = \lim_n \gamma_{i_n} \cdot \eta_s.$$

In particular,

$$\left( \text{essup}_i(\gamma_i)_t \right) \cdot \eta_s \leq \text{essup}_i(\gamma_i \cdot \eta_s).$$

*Proof.* Since the generators  $\xi_t^{1*}, \dots, \xi_t^{d*}$  of  $G_t^*$  generate  $\mathbf{R}^d$ , we deduce by a measurable selection argument that  $\eta_s = \sum_{j=1}^d a_j \xi_t^{j*}$  where  $a_j \in L^0(\mathbf{R}, \mathcal{F}_t)$  for all  $j \leq d$ . By virtue of Lemma 5.7,

$$\left( \text{essup}_i(\gamma_i)_t \right) \cdot \eta_s = \sum_{j=1}^d a_j \text{essup}_i \left( \gamma_i \cdot \xi_t^{j*} \right)_t$$

where here  $\text{essup}_i \left( \gamma_i \cdot \xi_t^{j*} \right)_t = \text{essup}_i \left( \gamma_i \cdot \xi_t^{j*} \right)$  is the  $\mathbf{R}$ -valued essential supremum of the litterature. For each  $j \leq d$ , the family  $(\gamma_i \cdot \xi_t^{j*})_i$  is directed upwards. We then deduce the existence of a sequence  $(i_n^j) \in I$  such that  $\text{essup}_i \left( \gamma_i \cdot \xi_t^{j*} \right) = \lim_n \uparrow \gamma_{i_n^j} \cdot \xi_t^{j*}$ , see A.3.3 [8]. But,  $(\gamma_i)_{i \in I}$  being directed upwards, we deduce  $(i_n) \subseteq I$  such that  $\text{essup}_i \left( \gamma_i \cdot \xi_t^{j*} \right) = \lim_n \uparrow \gamma_{i_n} \cdot \xi_t^{j*}$  for all  $j \leq d$  and we conclude.  $\square$

**Lemma 5.10** *Assume that  $G_t$  and  $G_t^*$  are generated by linearly independent generators. We are given a family of  $\mathcal{F}_T$ -adapted random variables  $\gamma = (\gamma_i)_{i \in I}$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\bar{\gamma} \geq_{G_t} \gamma_i, \forall i$ , where  $u \in [t, T]$ . If  $E(\text{essup}_i(\gamma_i)_u | \mathcal{F}_t)$  exists, then*

$$\text{essup}_i E(\gamma_i | \mathcal{F}_t)_t \leq_{G_t} E(\text{essup}_i(\gamma_i)_u | \mathcal{F}_t).$$

Moreover, if  $\gamma$  is directed upward and  $u = T$  then

$$\text{essup}_i E(\gamma_i | \mathcal{F}_t)_t = E(\text{essup}_i(\gamma_i)_T | \mathcal{F}_t).$$

*Proof.* Let us consider an arbitrary selector  $\xi_t^* \in L^\infty(G_t^*, \mathcal{F}_t)$ . Then, for all  $i$ ,  
 $E(\text{esssup}_i(\gamma_i)_u | \mathcal{F}_t) \xi_t^* = E((\text{esssup}_i(\gamma_i)_u) \xi_t^* | \mathcal{F}_t) \geq E(\gamma_i \xi_t^* | \mathcal{F}_t) = E(\gamma_i | \mathcal{F}_t) \xi_t^*$ .

From there, we get that

$$E(\text{esssup}_i(\gamma_i)_u | \mathcal{F}_t) \geq_{G_t} E(\gamma_i | \mathcal{F}_t), \quad \forall i \in I,$$

and we conclude that

$$E(\text{esssup}_i(\gamma_i)_u | \mathcal{F}_t) \geq_{G_t} \text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t.$$

By virtue of Lemma 5.8, recall that

$$\text{esssup}_i(\gamma_i)_u = \text{esssup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

Assume that the inequality

$$E(\text{esssup}_i(\gamma_i)_T | \mathcal{F}_t) \leq_{G_t} \text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t$$

does not hold on a non null set  $A_t \in \mathcal{F}_t$ . Then, there exists a generator  $\xi_t^* \in L^0(G_t^*, \mathcal{F}_t)$  such that the following inequality holds on a subset of  $A_t$ .

$$\begin{aligned} E(\text{esssup}_i(\gamma_i)_T | \mathcal{F}_t) \xi_t^* &> (\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t) \xi_t^* \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t), \\ E((\text{esssup}_i(\gamma_i)_T) \xi_t^* | \mathcal{F}_t) &> (\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t) \xi_t^* \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t). \end{aligned}$$

By virtue of Lemma 5.7, we have on  $A_t$ ,

$$E(\text{esssup}_i(\gamma_i \xi_t^*)_T | \mathcal{F}_t) > \text{esssup}_i E(\gamma_i \xi_t^* | \mathcal{F}_t)_t \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t).$$

But the family  $(\gamma_i \xi_t^*)_i$  is directed upwards by assumption and takes real positive values. By virtue of Proposition A.3.1 [8], we then deduce that

$$E(\text{esssup}_i(\gamma_i \xi_t^*)_T | \mathcal{F}_t) = \text{esssup}_i E(\gamma_i \xi_t^* | \mathcal{F}_t)_t$$

which yields a contradiction.  $\square$

## 6 Generalization and applications

We assume given  $t \in [0, T]$  and a family of  $\mathcal{F}_{t_i}$ -adapted closed convex cones  $\tilde{\mathcal{G}} := (\tilde{G}_{t_i})_{i \in I}$ ,  $t_i \in [t, T]$ . We consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}, \mathcal{F}_T)$ . We define the generalized Essential supremum at date  $u \in [t, T]$  as the class  $\Gamma^* := \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  (eventually empty) of  $\mathcal{F}_u$ -adapted random selectors  $\gamma^*$  satisfying

$$\gamma^* \geq_{\tilde{G}_{t_i}} \gamma_i, \quad \forall i \in I, \tag{6.10}$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u), \gamma \geq_{\tilde{G}_{t_i}} \gamma_i, \forall i \in I, \Rightarrow \exists \gamma^* \in \Gamma^* \text{ s.t. } \gamma \geq_{G_t} \gamma^* \tag{6.11}$$

$$\gamma_1^*, \gamma_2^* \in \Gamma^*, \gamma_1^* \neq \gamma_2^* \Rightarrow \gamma_1^* - \gamma_2^* \notin G_t \setminus \{0\} \text{ a.s.} \tag{6.12}$$

**Remark 6.1** Similarly, we may define  $\text{Essinf}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  replacing the inequality  $\gamma^* \geq_{\tilde{G}_{t_i}} \gamma_i$  by  $\gamma^* \leq_{\tilde{G}_{t_i}} \gamma_i$  and in (6.11) changing  $\gamma \geq_{G_t} \gamma^*$  into  $\gamma \leq_{G_t} \gamma^*$ .

Following the proofs of Section 3, we get similar results:

**Lemma 6.2** Let us consider a collection  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{\tilde{G}_{t_i}} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $\tilde{G}_{t_{j_0}} \subseteq G_t$  and  $E(\gamma_{j_0} | \mathcal{F}_u)$  exists. Then,  $\text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  is not empty.

**Theorem 6.3** Assume that  $G_t$  is generated by a finite number of linearly independent generators. Let us consider a family  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_{t_i}} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $\tilde{G}_{t_{j_0}} \subseteq G_t$  and  $E(\gamma_{j_0} | \mathcal{F}_u)$  exists. Then,  $\text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  is not empty and equals the singleton  $\text{essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$ .

*Proof.* By virtue of Lemma 6.2, consider  $\gamma_1, \gamma_2 \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u \neq \emptyset$ . Assume that  $\gamma_1 \neq \gamma_2$  so that  $\gamma_2 - \gamma_1, \gamma_1 - \gamma_2 \notin G_t \setminus \{0\}$  a.s. Without loss of generality, we assume that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Then, there exists coefficients  $\alpha_j, \beta_j \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  such that

$$\gamma_1 = \sum_{j=1}^N \alpha_j \xi_t^j, \quad \gamma_2 = \sum_{j=1}^N \beta_j \xi_t^j$$

where  $(\xi_t^j) \subseteq L^0(\mathbf{R}^d, \mathcal{F}_t)$  are the linear independent generators of  $G_t$ . Suppose that  $\alpha_1 > \beta_1$  on a non null set. Then, let us define  $\gamma_1^{(1)} := \gamma_1$  and

$$\gamma_2^{(1)} := \gamma_2 I_{\{\alpha_1 > \beta_1\}} + \gamma_1 I_{\{\alpha_1 \leq \beta_1\}} \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u.$$

Otherwise, we set  $\gamma_1^{(1)} := \gamma_2$  and  $\gamma_2^{(1)} := \gamma_1$ . We then deduce that

$$\gamma_1^{(1)} = \sum_{j=1}^N \alpha_j^{(1)} \xi_t^j, \quad \gamma_2^{(1)} = \sum_{j=1}^N \beta_j^{(1)} \xi_t^j$$

with  $\alpha_1^{(1)} \geq \beta_1^{(1)}$ . Assume we have already defined  $(\gamma_1^{(j)}, \gamma_2^{(j)}) \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$ , for  $j \leq N-1$ , verifying  $\alpha_k^{(j)} \geq \beta_k^{(j)}$  for all  $k \leq j$ . Suppose that  $\beta_{j+1}^{(j)} > \alpha_{j+1}^{(j)}$  on a non null set. We then define  $\gamma_1^{(j+1)} := \gamma_1^{(j)}$  and

$$\gamma_2^{(j+1)} := \gamma_1^{(j)} I_{\{\beta_{j+1}^{(j)} > \alpha_{j+1}^{(j)}\}} + \gamma_2^{(j)} I_{\{\beta_{j+1}^{(j)} \leq \alpha_{j+1}^{(j)}\}} \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u.$$

Otherwise, we set  $(\gamma_1^{(j+1)}, \gamma_2^{(j+1)}) := (\gamma_1^{(j)}, \gamma_2^{(j)})$ . We get that  $\alpha_k^{(j+1)} \geq \beta_k^{(j+1)}$  for all  $k \leq j+1$ . In particular, with  $j = N-1$ , we deduce that  $\gamma_1^{(N)} \geq_{G_t} \gamma_2^{(N)}$ . But  $\gamma_1^{(N)}, \gamma_2^{(N)} \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  implies that  $\gamma_1^{(N)} = \gamma_2^{(N)}$  hence we have the equality chain  $\gamma_1^{(N-1)} = \gamma_2^{(N-1)}, \dots, \gamma_1^{(1)} = \gamma_2^{(1)}$  and finally  $\gamma_1 = \gamma_2$ .  $\square$

## 6.1 Superhedging of American options

Let us consider the discrete-time financial model of Kabanov with proportional transaction costs (see [8] for more details). We are given random cones  $(G_t)_{0 \leq t \leq T}$  satisfying the hypothesis given in introduction. We interpret them as solvency cones, i.e.  $G_t$  is the set of all portfolio positions we can change into positive ones (i.e. in  $\mathbf{R}_+^d$ ) paying transaction costs. A portfolio process  $(V_t)_{0 \leq t \leq T}$  expressed in physical units is a process with the dynamics  $\Delta V_t := V_t - V_{t-1} \in -G_t$  a.s. The goal is the following: given an American claim  $(h_t)_{0 \leq t \leq T}$  (a stochastic process), find a “minimal” portfolio  $(V_t)_{0 \leq t \leq T}$  such that  $V_t \geq_{G_t} h_t$ , for all  $t$ , and  $V_T = h_T$ . Such a portfolio  $V$  is said “minimal” in the sense that if  $W$  is another one satisfying the same property, then  $W_t \geq_{G_t} V_t$  for all  $t$ . Throughout this section, we suppose that the random cones  $G$  and  $G^*$  are generated by linear independent generators. This is the case when  $d = 2$  and the transaction costs coefficients are strictly positive, [8]. We define a “Snell envelop” as follows.

**Proposition 6.4** *Assume there exists  $k \geq 0$  such that  $h_t \leq_{G_t} k \mathbf{1}_d$  for all  $t$ . Then, there exists a unique minimal portfolio process  $(V_t)_{0 \leq t \leq T}$  such that  $V_t \geq_{G_t} h_t$ , for all  $t$ , and  $V_T = h_T$ . Moreover,*

$$V_t = \text{essup}^{G_t} ((h_t, G_t), (V_{t+1}, G_{t+1}))_t, \quad t \leq T - 1.$$

*Proof.* We set  $V_T = h_T$  and inductively we define  $V_t := \text{essup} (h_t, V_{t+1})_t$  where the essential supremum is understood in the generalized sense of this section;  $h_t$  is associated to the cone  $G_t$  and  $V_{t+1}$  is associated to  $G_{t+1}$ . It follows that  $V_t$  is  $\mathcal{F}_t$ -adapted and satisfies  $V_t \geq_{G_t} h_t$ . Since, we also have  $V_t \geq_{G_{t+1}} V_{t+1}$ , it follows that  $\Delta V_{t+1} \in -G_{t+1}$ , i.e.  $(V_t)_{0 \leq t \leq T}$  is a portfolio process. Moreover, consider any portfolio process  $W$  super-replicating  $(h_t)_{0 \leq t \leq T}$ . We necessarily have  $W_t \geq_{G_t} h_t$  and  $W_T = h_T$ . From the dynamics  $\Delta W_{t+1} := W_{t+1} - W_t \in -G_{t+1}$ , we also deduce that  $W_t \geq_{G_{t+1}} W_{t+1}$ . By backward induction and definition, it follows that  $W_t \geq_{G_t} V_t$  for all  $t$ .  $\square$

**Remark 6.5** *The link between this minimal portfolio process and the dual characterization of the super hedging prices of an American claim as formulated in the papers [2] and [4] is left for future research.*

## 6.2 Dynamic risk measure

Assume we are given two constant closed and proper cones  $K$  and  $G$  generated by linear independent generators of  $\mathbf{R}^d$ . We suppose that  $K \subset G$  and  $\text{int } K \neq \emptyset$ . We also suppose that  $\mathbf{R}_+^d \subseteq G$ . In this section,  $G$  is interpreted as the solvency cone in the continuous-time model of Campi–Schachermayer as described in Section 3.6.6, [8]. We are given discounted prices  $(S^i)_{1 \leq i \leq d}$  verifying  $S^i = 1$  we interpret as currencies like Dollars, Euros, Yen, etc. The constant proportional transaction costs coefficients  $(\lambda^{i,j})_{i,j}$  of the model are interpreted as the exchange rate between the currency numbers  $i$  and numbers  $j$ . Therefore,  $G$  is defined as, [8],

$$G := \text{cone}\{(1 + \lambda^{i,j})e_i - e_j : 1 \leq i, j \leq d\}.$$

The problem is the following: we are given a vector-valued process  $X$  with  $X_t \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  which models the value at date  $t$  of some multi-assets financial portfolio, the investment of a company. A dynamic measure of risk  $(\rho_t)$  associates to  $X$  the set of  $\mathcal{F}_t$ -adapted portfolios of reference instruments  $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  which, when added to  $X_t$ , make the total position  $X_t + x_t$  acceptable by the regulator/supervisor. We send the readers to the papers [13] and [6] for static risk measures in models with transaction costs.

In this example, we suppose that the acceptable positions are given by the cones  $K$ , we call the security cone. The process  $(\rho_t(X))$  is defined as a set-valued mapping but, as in the previous example, there exists a “minimal” selector of the risk measure which turns out to be also a risk-measure according to the definition we shall recall below. Let us before show the following.

**Proposition 6.6** *Suppose that  $h$  is an adapted process and there is  $k \geq 0$  such that  $h_t \leq_K k\mathbf{1}_d$  for all  $t \in [0, T]$ . Then, there exists a unique minimal process  $(V_t)_{t \in [0, T]}$  such that  $V_t \geq_K h_t$ , for all  $t$ ,  $V_T = h_T$  and  $V_{t_2} - V_{t_1} \in -G$  for all  $t_1 \leq t_2$ . Moreover,  $V$  is left-continuous and of finite variations. If  $W$  is another such process with  $W_T \geq_K h_T$ , then necessarily we have  $W \geq_G V$ .*

*Proof.* We consider the refining sequence

$$II^n := \{t_i^n := Ti/2^n : i = 0, \dots, 2^n\}$$

of partitions of  $[0, T]$  and set  $II = \cup_n II^n$ . Like in Proposition 6.4, we deduce, for each  $n \in \mathbf{N}$ , the minimal portfolio process  $(V_{t_i^n}^{(n)})$  super-replicating  $(h_{t_i^n})$  at dates  $t \in II^n$  in the following sense:

$$\begin{aligned} V_{t_i^n}^{(n)} &:= \operatorname{essup}^G \left( (h_{t_i^n}, K), (V_{t_{i+1}^n}^{(n)}, G) \right)_{t_i^n}, \quad i \leq 2^n - 1, \\ V_T^{(n)} &:= h_T. \end{aligned}$$

Observe that by definition  $V_{t_i^n}^{(n+1)} \geq_G V_{t_{2i+1}^{n+1}}^{(n+1)}$  and  $V_{t_{2i+1}^{n+1}}^{(n+1)} \geq_G V_{t_{i+1}^n}^{(n)}$ . It follows that  $V_{t_i^n}^{(n+1)} \geq_G V_{t_{i+1}^n}^{(n)}$ . By virtue of the definition of  $V^{(n)}$  as a minimal process, we then deduce that  $k\mathbf{1}_d \geq_G V_t^{(n+1)} \geq_G V_t^{(n)}$  for any  $t \in II^n$ . We then define  $V_t^{(\infty)}$ , for each  $t \in II$ , as the increasing limit of the sequence  $(V_t^{(n)})$  with respect to  $G$  or equivalently as the  $\mathcal{F}_t$ -adapted essential supremum of the family  $(V_t^{(n)})$  with respect to the cone  $G$ . Then, we put  $V_t := \lim_{r \uparrow t, r \in II} V_r^{(\infty)}$  which is left-continuous. Observe that  $V_t$  is a decreasing limit with respect to  $G$ . Indeed,  $V_{t_1}^{(\infty)} - V_{t_2}^{(\infty)} \in G$  if  $t_1 \geq t_2$  and  $t_1, t_2 \in II$ . We then deduce that  $V$  is an adapted process. By virtue of Lemma 7.1, we also deduce that  $V$  is of finite variations, i.e.  $V$  is a portfolio process according to Lemma 3.6.12, [8], with respect to the constant cone  $G$ . Finally, consider any portfolio process  $W$   $K$ -superhedging the payoff  $h$ . Then, the discrete-time process  $(W_{t_i^n})$  verifies  $W_{t_i^n} \geq_K h_{t_i^n}$  and  $W_{t_i^n} \geq_G W_{t_{i+1}^n}$  by assumption. It follows that  $W_t \geq_K V_t^{(n)}$  for all  $t \in II$  and  $n$  such that

$t \in \mathbb{I}^n$ . As  $n \rightarrow \infty$ , we obtain  $W_t \geq_K V_t^{(\infty)}$  for all  $t \in \mathbb{I}$ . Moreover, if  $W$  is left-continuous we get that  $W_t \geq_{G_t} V_t$  for all  $t$ .  $\square$

Throughout the sequence, we suppose that  $X \in D_\rho$ , i.e.  $X_t \geq -k\mathbf{1}_d$  for all  $t \in [0, T]$  where  $k > 0$ . Applying the last lemma to  $h = -X$ , we then deduce a unique process  $\rho_t(X)$  satisfying  $\rho_t(X) + X_t \in K$  which is minimal according to the ordering generated by the cone  $G$ . The dynamic  $\rho_{t_2}(X) - \rho_{t_1}(X) \in -G$  it satisfies means that  $\rho(X)$  is a self-financed portfolio subjected to the transaction costs given by  $(\lambda^{i,j})_{i,j}$ , see [8]. We then easily show the following:

**Proposition 6.7** *The mapping  $\rho$  is a coherent risk measure, i.e. satisfies the following statements:*

- R0.)  $\rho_t(0) = 0$ .
- R1.) If  $X \geq_K 0$ , then  $\rho_t(X) \leq_{G_t} 0$  a.s.,  $\forall t \in [0, T]$ .
- R2.) If  $X, Y \in D_\rho$  then,  $\rho_t(X + Y) \leq_{G_t} \rho_t(X) + \rho_t(Y)$  a.s.,  $\forall t \in [0, T]$ .
- R3.) If  $\lambda \in \mathbf{R}_+$  and  $X \in D_\rho$  then  $\rho_t(\lambda X) = \lambda \rho_t(X)$  a.s.,  $\forall t \in [0, T]$ .
- R4.) If  $X \in D_\rho$  and  $a \in \mathbf{R}$ , then  $\rho_t(X + \bar{a}) = \rho_t(X) + \{-a\}$ ,  $\forall t \in [0, T]$ .

### 6.3 Skorohod problem with oblique reflection

This last example is motivated by the consumption-investment problem with friction of Chapter 4, [8], but for Lévy-driven processes as in [5]. Contrarily to the classical Merton problem, [10], constructing the optimal wealth process in presence of friction is far from being obvious. We infer that the optimal pair (i.e. the portfolio process and the control) is the solution of a stochastic differential equation with reflection, usually called stochastic Skorokhod problem, but with jumps. In the case where the price process is continuous, the Skorokhod problem is given by the equations (4.8.24)-(4.8.25), [8]. Due to the continuity of the prices and the oblique reflection on the boundary, the portfolio process  $(V_t)$  remains in the so-called no-transaction cone  $K_0$ . With unpredictable price jumps, the portfolio may jump outside  $K_0$  but we conjecture that an oblique reflection of  $V_t$  should constraint  $V_{t+}$  to stay in  $K_0$  in such a way that the controlled difference  $\Delta V_{t+} := V_{t+} - V_t$  corresponds to transaction costs, i.e.  $\Delta V_{t+} \in -G$  (where  $G$  is the solvency cone as already defined). Of course, that means that we allow transactions “from the right”, i.e. the controls are not assumed to be càdlàg as in [5]. We currently work on an extension of [5] to the larger class of right and left limited controls we believe to be the right one. In the paper [7], the authors propose such an optimal control but, unfortunately, this one does not belong to the class of right-continuous processes they consider.

In the following, we solve the corresponding deterministic Skorokhod problem with oblique reflection using the notion of essential supremum we introduced above and we leave the stochastic version for further research. We send the readers to the paper [1] by Anderson and Orey where they use the method of local coordinates and the papers [11], [12] and [3] for further readings about the classical approaches of the litterature.

Assume we are given two constant closed and proper cones  $K$  and  $G$  generated by linear independent generators of  $\mathbf{R}^2$  with  $K \subset G$ ,  $\partial K \subseteq \text{int } G$  and  $\text{int } K \neq \emptyset$ . We suppose that  $\mathbf{R}_+^2 \subseteq G$ . Let us consider a càdlàg (right-continuous and left-limited) and deterministic function  $(X_t)_{t \geq 0}$  with  $X_0 \in \text{int } K$ .

The goal is to find a left-continuous and right-limited function  $(Y_t)_{t \geq 0}$  of finite variations such that  $Z_t = X_t + Y_t$  verifies:

$$\begin{aligned} Z_t &\in G, Z_{t+} \in K \text{ a.s.}, \\ \dot{Y}_t, \Delta^+ Z_t &\in -\partial G, \\ \Delta^+ Z_t \neq 0 &\Leftrightarrow Z_t \notin K \end{aligned}$$

where  $\dot{Y}_t$  is the optional version of the Radon–Nikodym derivative  $dY^c/d\text{Var}(Y^c)$ ,  $Y^c$  is the continuous part of  $Y$  and  $\text{Var}(Y^c)$  its total variation.

**Definition 6.8** *A process  $A$  is said  $G$ -increasing if  $A_t \geq_G A_s$  for  $t \geq s$ .*

We consider the refining sequence  $\Pi^n := \{t_i^n := Ti/2^n : i = 0, \dots, 2^n\}$  of partitions of  $[0, T]$  and denote  $\Pi = \cup_{n \geq 1} \Pi^n$ . We say that  $h$  is a càdlàg (respectively làdcàg)  $\Pi^n$ -process if  $h$  is constant on each interval  $[t_i^n, t_{i+1}^n[$  (respectively on  $]t_i^n, t_{i+1}^n]$ ). Moreover, a left-continuous process  $(V_t)_{t \leq T}$  is assumed to have a right limit at date  $T$  we denote by  $V_{T+}$  and verifies  $V_0 = V_{0+}$ . We recall in Appendix the projection of a point  $x \in G$  onto  $K$  parallel to  $-G$  we denote by  $\mathbb{P}_{K \rightarrow}^{-G}(x)$ . Given  $\vec{k} \in \text{int } K$ , we fix  $r_k > 0$  such that  $B(\vec{k}, r_k) \subseteq K$ . We denote  $\vec{k}^* := \vec{k}/r_k$ .

**Lemma 6.9**

*Let us consider a function  $(X_t)_{t \leq T}$  and  $\|X\|_u := \sup_{t \leq T} \|X_t\|$ . Then,*

$$-\vec{k}^* \|X\|_u \leq_K X \leq_K \vec{k}^* \|X\|_u.$$

*Proof.* Since  $B(\vec{k}, r_k) \subseteq K$ , we get that  $B(\vec{k}^*, 1) \subseteq K$ . It follows that  $B(0, \|X\|_u) \subseteq K - \vec{k}^* \|X\|_u$ . Since,  $X$  and  $-X$  evolves in  $B(0, \|X\|_u)$  we easily conclude.  $\square$

Observe that there exists a constant  $\beta_K$  only depending on the cone  $K$  and  $\vec{k}^*$  such that  $-\vec{k}^* \leq_K x \leq_K \vec{k}^*$  implies the inequality  $\|x\| \leq \beta_K$ .

Our main result is:

**Theorem 6.10** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg function. Then, there exists a làdcàg function  $\Sigma_1(X) := (\bar{V}_t)_{t \leq T}$  of finite variations which is  $G$ -decreasing such that  $\Sigma_2(X) := X + V \in G$ ,  $\Sigma_2(X)_+ \in K$ ,  $\|V\|_u \leq \beta_K \|X\|_u$  and  $\Delta^+ V \in -\partial G$ . Moreover,  $\dot{V} = dV^c/d\text{Var}(V^c) \in -\partial G$  and  $\dot{V} = \dot{V} \mathbf{1}_{\Sigma_2(X) \notin \text{int } K}$ .*

The proof follows from the following result.

**Theorem 6.11** *Suppose that  $h$  is a deterministic càdlàg function. Suppose that  $h_t \leq_K \overrightarrow{k}$  for all  $t \in [0, T]$  where  $\overrightarrow{k} \in K$ . Then, there exists a làdcàg function  $(V_t)_{t \leq T}$  such that  $V_t \geq_K h_t$ , for all  $t$ . Moreover,  $V$  is of finite variations with  $\Delta^+ V \in -\partial G$  and  $\dot{V}^c \in -\partial G$ . If  $W$  is another such process, then necessarily we have  $W \geq_G V$ .*

To give the proof, we need the following lemmas.

**Proposition 6.12** *Suppose that  $(h_t)_{t \in \Pi^n}$  is a discrete-time process and there is  $\overrightarrow{k} \in K$  such that  $h_t \leq_K \overrightarrow{k}$  for all  $t \in \Pi^n$ . Then, there exists a process  $Z(h) := (Z_t)_{t \in \Pi^n}$  such that  $Z \geq_K h$  and  $Z_{t_{i+1}^n} - Z_{t_i^n} \in -G$  for all  $i \leq 2^n - 1$ . If  $W$  is another such process, then necessarily we have  $W \geq_G V$ .*

*Proof.* Like in Proposition 6.4, it suffices to set  $Z_T = h_T$  and

$$Z_{t_i^n} := \text{essup}^G((h_{t_i^n}, K), (Z_{t_{i+1}^n}, G))_{t_i^n}, \quad i \leq 2^n - 1.$$

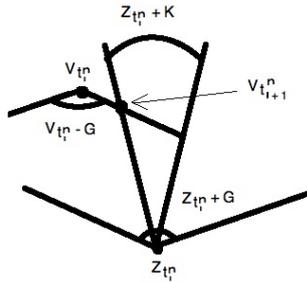
□

**Proposition 6.13** *Suppose that  $(h_t)_{t \leq T}$  is a càdlàg  $\Pi^n$ -process and there is  $\overrightarrow{k} \in K$  such that  $h_t \leq_K \overrightarrow{k}$  for all  $t \leq T$ . Then, there is a unique minimal làdcàg  $\Pi^n$ -process  $(V_t)_{t \leq T}$  such that  $V \geq_G h$ ,  $V_0 = V_{0+}$ ,  $\overrightarrow{k} \geq_G V_+ \geq_K h$  and  $\Delta^+ V \in -\partial G$ ,  $V_{t_{i+1}^n} - h_{t_i^n} = \mathbb{P}_K^{-G}(V_{t_i^n} - h_{t_i^n})$ . If  $W$  is another such process, we have  $W \geq V$ .*

*Proof.* Consider the minimal  $G$ -decreasing process  $(Z_t)_{t \in \Pi^n}$  we get using Proposition 6.12. Let us set  $V_0 = V_{0+} := Z_0 := V_{t_1^n}$  and let us define recursively  $V_{t_{i+1}^n} = V_{t_i^n+}$  as follows. Suppose we have already defined  $V_{t_i^n} = V_{t_{i-1}^n+}$  such that  $V_{t_i^n} \geq_K h_{t_{i-1}^n}$  and  $V_{t_i^n} \geq_G Z_{t_{i-1}^n}$ . Then,  $V_{t_i^n} \geq_G Z_{t_i^n} \geq_G h_{t_i^n}$  so that we may define

$$V_{t_{i+1}^n} = \text{essup}^{-G}((h_{t_i^n}, K); (V_{t_i^n}, -G))_{t_i^n} := \text{essinf}^G((h_{t_i^n}, -K); (V_{t_i^n}, G))_{t_i^n}.$$

Observe that we have  $V_{t_{i+1}^n} - V_{t_i^n} \in -\partial G$ , i.e.  $V_{t_{i+1}^n} = \mathbb{P}_{K+h_{t_i^n}}^{-G}(V_{t_i^n})$ :



To see the uniqueness, let us set  $U_t := V_t - h_t$ ,  $t \in \Pi^n$ . Observe that it is equivalent to define  $U$  instead of  $V$  as follows: the discrete-time process  $U$  is defined as  $U_0 = U_{0+} = Z_0 - h_0$  and

$$U_{t_i-} := U_{t_{i-1}+}, \quad U_{t_i} := U_{t_i-} - \Delta h_{t_i} \in G, \quad U_{t_i+} := \mathbb{P}_K^{-G}(U_{t_i}).$$

Consider another process  $W$  satisfying the same properties than  $V$ . We first get that  $W_+ \geq_G Z$  so that  $W_{t_1} \geq_G V_{t_1}$ . We then consider  $\tilde{U}$  the process associated to  $W$  which is defined as  $U$  and satisfies the same properties. By induction, we get that  $\tilde{U} \geq_G U$  so that  $W \geq_G V$ .  $\square$

**Corollary 6.14** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg  $\Pi^n$ -process. Then, there is a  $G$ -minimal làdcàg  $\Pi^n$ -process  $\Sigma_1(X) := (V_t)_{t \leq T}$  with  $\Sigma_1(X)_0 = Z(-X)_0$ ,  $\Sigma_2(X) := X + \Sigma_1(X) \in G$ ,  $\Sigma_2(X)_+ = \mathbb{P}_K^{-G}(\Sigma_2(X)) \in K$  and  $\Delta^+V \in -\partial G$ . Moreover, we have  $\|\Sigma_1(X)\|_u \leq \beta_K \|X\|_u$  and  $\Delta^+V_t \neq 0$  if and only if  $\Sigma_2(X)_t \notin \text{int } K$ .*

*Proof.* It suffices to apply Proposition 6.13 and Lemma 6.9.  $\square$

**Corollary 6.15** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg function. Then, there exists a làdcàg function  $\Sigma_1(X) := (\bar{V}_t)_{t \leq T}$  of finite variations which is  $G$ -decreasing such that  $\Sigma_2(X) := X + \bar{V} \in G$ ,  $\Sigma_2(X)_+ \in K$ ,  $\|V\|_u \leq C \|X\|_u$  and  $\Delta^+V \in -\partial G$ . Moreover,*

$$\dot{V} = dV^c/d\text{Var}(V^c) \in -\partial G, \quad \dot{V} = \dot{V} \mathbf{1}_{\Sigma_2(X) \notin \text{int } K}.$$

*Proof.* Consider the càdlàg function  $X$  as an element of the Skorokhod space  $\mathbb{D}[0, T]$  endowed with the usual Skorokhod topology. In the following, we use the notations of Chapter VI, [9]. Consider a sequence of càdlàg  $\Pi^n$ -functions  $X^n$  such that  $X^n$  converges uniformly to  $X$  on  $[0, T]$  and  $(\Pi^n)$  is a refining sequence of partitions. For each  $n$ , Corollary 6.14 ensures the existence of  $(\Sigma_1(X^n), \Sigma_2(X^n))$  solving the Skorokhod problem. Observe the inequalities  $\sup_n \|\Sigma_2(X^n)\|_u < \infty$  and  $\sup_n \|\Sigma_1(X^n)\|_u < \infty$  since  $\sup_n \|X^n\|_u < \infty$ . Recall that for each  $n$ ,  $\Sigma_1(X^n)$  is a  $G$ -decreasing process. Using Lemma 3.6.13, [8], (or Lemma 7.1 of this paper) and Corollary 6.14, the Helly theorem asserts that for any sequence of  $\mathcal{A} := \{\text{Var}(\Sigma_1(X^n)_+)\}$ , there exists a subsequence  $\alpha_n \in \mathcal{A}$  such that  $\alpha_n$  converges pointwise to  $\alpha \in \mathbb{D}[0, T]$  at each point  $t \in D$  of continuity of  $\alpha$ . It is straightforward that for any  $t \in D$ , there exists  $t_n \leq t$ , with  $t_n \rightarrow t$ , such that  $\Delta \alpha_n(t_n) \rightarrow \Delta \alpha(t) = 0$  since  $\alpha_n$  is a stepwise function. Moreover  $D$  is dense in  $[0, T]$  as a complement of a countable subset of  $[0, T]$ . By virtue of Lemma VI-2.25, [9], we obtain that  $\alpha_n$  converges to  $\alpha \in \mathbb{D}[0, T]$  for the Skorokhod topology. We then deduce that  $\mathcal{A}$  is relatively compact, i.e.  $\limsup_{\theta \downarrow 0} \sup_{\alpha \in \mathcal{A}} w'_T(\alpha, \theta) = 0$ . We also have  $\limsup_{\theta \downarrow 0} \sup_n w'_T(X^n, \theta) = 0$ . Since  $\Delta \Sigma_2(X^n)_+ = \Delta X^n + \Delta \Sigma_1(X^n)_+$ , we then get that

$$\begin{aligned} w'_T(\Sigma_2(X^n)_+, \theta) &\leq w'_T(X^n, \theta) + w'_T(\Sigma_1(X^n)_+, \theta) \\ &\leq \sup_n w'_T(X^n, \theta) + w'_T(\text{Var}(\Sigma_1(X^n)_+), \theta) \\ &\leq \sup_n w'_T(X^n, \theta) + \sup_{\alpha \in \mathcal{A}} w'_T(\alpha, \theta). \end{aligned}$$

From there, it follows that  $\limsup_{\theta \downarrow 0} \sup_n w'_T(\Sigma_2(X^n)_+, \theta) = 0$  and finally the family  $(\Sigma_2(X^n)_+)$  is relatively compact for the Skorokhod topology. We deduce that  $(\Sigma_2(X^n)_+)$  converges to  $\Sigma_2(X)_+ \in \mathbb{D}[0, T]$  for a subsequence. We then set  $\Sigma_2(X) := \Sigma_2(X)_- + \Delta X$ . By virtue of Proposition VI-2.1, [9], for each  $t \in [0, T]$  there exists  $t_n \rightarrow t$  such that  $\Sigma_2(X^n)_+(t_n) \rightarrow \Sigma_2(X)_+(t)$ ,  $\Sigma_2(X^n)_-(t_n) \rightarrow \Sigma_2(X)_-(t)$  and  $\Delta \Sigma_2(X^n)_+(t_n) \rightarrow \Delta \Sigma_2(X)_+(t)$ . Recall that by assumption,  $\Delta X^n$  converges uniformly to  $\Delta X$ . It follows that

$$\Sigma_2(X^n)(t_n) = \Sigma_2(X^n)_-(t_n) + \Delta X^n(t_n) \rightarrow \Sigma_2(X)(t) \in G.$$

Since  $\Sigma_2(X^n)_+(t_n) = \mathbb{P}_K^{-G}(\Sigma_2(X^n)(t_n))$ , we deduce by continuity that

$$\Sigma_2(X)_+(t) = \mathbb{P}_K^{-G}(\Sigma_2(X)(t)) \in K.$$

We note  $\Sigma_1(X) := \Sigma_2(X) - X$ . From above, we have

$$\Sigma_1(X^n)(t_n) = \Sigma_2(X^n)(t_n) - X^n(t_n) \rightarrow \Sigma_1(X).$$

Since  $\Delta \Sigma_2(X) = \Delta X$ , we deduce that  $\Delta \Sigma_1(X) = 0$  i.e.  $\Sigma_1(X)$  is left-continuous. It is clear that  $\Sigma_1(X)$  is  $G$ -decreasing hence of finite variations:

$$\Sigma_1(X)_t = Z(-X)_0 + \int_0^t \alpha_u d\text{Var} \Sigma_1^c(X)_u + \sum_{s \leq t} \Delta \Sigma_1^+(X)_s$$

where  $\alpha$  is the optional Radon-Nikodym derivative  $d\Sigma_1^c(X)/d\text{Var} \Sigma_1^c(X)$ . From the equalities  $\Delta \Sigma_2(X^n)_+(t_n) = \Delta X^n(t_n) + \Delta^+ \Sigma_1(X^n)(t_n)$  and  $\Delta \Sigma_2(X)_+(t) = \Delta X(t) + \Delta^+ \Sigma_1(X)(t)$ , we deduce that  $\Delta^+ \Sigma_1(X)(t) \in -\partial G$  as  $n \rightarrow \infty$  for all  $t \leq T$ . Assume that  $\Delta^+ \Sigma_1(X)(t) \neq 0$ . Then, we also have  $\Delta^+ \Sigma_1(X^n)(t_n) \neq 0$  if  $n$  is large enough so that  $\Sigma_2(X^n)(t_n) \notin \text{int } K$  hence  $\Sigma_2(X)(t) \notin \text{int } K$ . At last, we may assume, [4], that

$$\alpha = \limsup_j \sum_k \frac{\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j}}{\text{Var}(\Sigma_1(X)_+)_{t_{k+1}^j} - \text{Var}(\Sigma_1(X)_+)_{t_k^j}} \mathbf{1}_{]t_k^j, t_{k+1}^j]}$$

where  $(t_k^j)$  is the same refining sequence of partitions of  $[0, T]$  we use to approximate  $X$ . By construction, we have  $\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j} \in -G$ . Moreover, since  $\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j} \rightarrow \Delta^+ \Sigma_1(X)$ , it follows that

$$\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j} \geq_G \Delta^+ \Sigma_1(X) - \epsilon \mathbf{1}$$

if  $j$  is large enough and  $\epsilon$  is chosen arbitrarily small. Since  $\Delta^+ \Sigma_1(X) \in -\partial G$ , we then deduce as  $\epsilon \rightarrow 0$  that  $\alpha \in (-G) \cap (-\partial G + G) = -\partial G$ . Assume that  $\alpha \neq 0$ , then  $\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j} \neq 0$  if  $j$  is large enough. We deduce  $\tilde{t}_k^j$  such that  $\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{\tilde{t}_k^j} \neq 0$  where  $t_{k+1}^j > \tilde{t}_k^j > t_k^j$ . On the other hand, by virtue of Proposition VI-2.1, we get  $t_n^j$  with  $t_n^j < t_{k+2}^j$  and  $s_n^j$  with  $t_k^j < s_n^j$  converging respectively to  $t_{k+1}^j$  and  $\tilde{t}_k^j$  such that

$$\Sigma_1(X^n)_{t_{k+1}^j} - \Sigma_1(X^n)_{t_k^j} = \Sigma_1(X^n)_{t_n^j} - \Sigma_1(X^n)_{s_n^j} \neq 0.$$

Observe that we necessarily have  $t_n^j > t_{k+1}^j$ . By construction, it follows that  $\Sigma_2(X^n)_{t_n^j} = \Sigma_2(X^n)_{t_{k+1}^j} \notin \text{int } K$ . Taking the limit, we get that

$$\Sigma_2(X)_+(t) \in (\mathbb{R}^2 \setminus \text{int } K) \cap K = \partial K.$$

Since  $\Sigma_2(X)_+(t) = \mathbb{P}_K^{-G}(\Sigma_2(X)(t))$  we then deduce that  $\Sigma_2(X)(t) \notin \text{int } K$ .  $\square$

## 7 Appendix

**Lemma 7.1** *Suppose we are given a constant cone  $G \subseteq \mathbf{R}^d$  generated by a basis of  $\mathbf{R}^d$ . Let us consider a process  $(A_t)_{t \leq T}$  verifying  $A_{t_2} - A_{t_1} \in G$  whatever  $t_2 \geq t_1$ . Then,  $A$  is of finite variations.*

*Proof.* By assumption,  $G = \text{cone}(\xi_i)_{1 \leq i \leq d}$  where  $(\xi_i)$  is a basis of  $\mathbf{R}^d$ . Let us consider the mapping

$$\Phi : \mathbf{R}^d \ni x = \sum_{i=1}^d x_i \xi_i \mapsto \Phi(x) := \sum_{i=1}^d x_i e_i \in \mathbf{R}^d$$

where  $(e_i)$  is the canonical basis of  $\mathbf{R}^d$ . We denote by  $\Phi^{-1}$  the inverse of  $\Phi$ . Recall that  $\Phi$  and  $\Phi^{-1}$  are linear hence continuous. Moreover  $\Phi(G) \subseteq \mathbf{R}_+^d$ . For any subdivision  $(t_i)$  of  $[0, T]$ , we get

$$\begin{aligned} \sum_i |A_{t_{i+1}} - A_{t_i}| &= \sum_i |\Phi^{-1} \circ \Phi(A_{t_{i+1}} - A_{t_i})| \leq \|\Phi^{-1}\| \sum_i |\Phi(A_{t_{i+1}} - A_{t_i})| \\ &\leq C \|\Phi^{-1}\| \sum_i \sum_{j=1}^d |\Phi^j(A_{t_{i+1}}) - \Phi^j(A_{t_i})| \end{aligned}$$

where  $\Phi^j$  is the  $j$ th component of  $\Phi$  and  $C$  is a constant (all the norms being equivalent). By assumption,  $\Phi^j(A_{t_{i+1}}) - \Phi^j(A_{t_i}) \geq 0$  for all  $j$ . It follows that there is another constant  $C$  such that

$$\sum_i |A_{t_{i+1}} - A_{t_i}| \leq C \|\Phi(A_T - A_0)\| \leq C \|\Phi\| |A_T - A_0|.$$

We conclude that  $\text{Var}(A)_T \leq C \|\Phi\| |A_T - A_0|$ .  $\square$

In the following, we assume that  $G \subseteq \mathbf{R}^2$  is a constant cone satisfying the hypothesis of the introduction and  $K \subseteq G$  is a closed cone with  $\partial K \subseteq \text{int } G$ .

**Proposition 7.2 (Projection onto  $K$  parallel to  $-G$ )** *Given  $x \in G$ , there exists a unique  $y := \mathbb{P}_K^{-G}(x) \in K$  such that*

$$\|x - y\| = \min_{k \in K} \{\|x - k\| : x - k \in G\}.$$

We omit the proof which is standard.

**Lemma 7.3** *The mapping  $G \ni x \mapsto \mathbb{P}_K^{-G}(x) \in \mathbf{R}^d$  is continuous.*

*Proof.* We may assume that  $G = \mathbf{R}_+^2$  like in the previous lemma. Then  $\partial K = D_a \cup D_b$  where  $D_a = \{(x_1, y_1) : y_1 = ax_1, x \geq 0\}$  and  $D_b$  is defined similarly with  $b > a > 0$ . It is then easy to show the implication  $x_n \rightarrow x \Rightarrow \mathbb{P}_K^{-G}(x_n) \rightarrow \mathbb{P}_K^{-G}(x)$  since we have an explicit expression of  $\mathbb{P}_K^{-G}(x)$  if  $x \in G$ .  $\square$

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