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# Essential Supremum in a $d$ -dimensional Real Space with Respect to a Random Cone and Applications

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**Abstract** The goal of this paper is to introduce the notion of essential supremum of a family of multi-dimensional random variables with respect to a random convex cone. We give two applications in mathematical finance; We determine the “minimal” portfolio process super-hedging an American claim in the Kabanov discrete-time model with transaction costs. For the same model, we construct a dynamic risk measure in a continuous-time setting. At last, we solve a Skorokhod problem with oblique reflection.

**Mathematics Subject Classification (2000)** 60G44

**JEL Classification** G11 · G13

## 1 Introduction

The primary goal of this paper was the following. Assume that a financial market with proportional transaction costs is modeled by the Kabanov or Campi–Schachermayer framework, [3]. A vector-valued portfolio process, expressed in physical units, represents the number of different risky assets an agent holds. Given a vector-valued process  $h := (h_t)_{t \leq T}$ , we interpret as an American option, the problem is to define the possible endowments (super-hedging prices) we need to start a vector-valued portfolio process which can be liquidated at any instant  $t \in [0, T]$ , paying transaction costs, in such a way that all its components are greater or equal than those of  $h_t$ . In a discrete-time setting, we may reformulate the problem as follows: given a set-valued

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random process  $(K_t(\omega))_{t=0,\dots,T}$  whose values are closed convex cones in  $\mathbf{R}^d$ ,  $d \geq 1$ , find  $V_0 \in \mathbf{R}^d$  allowing to start a vector-valued portfolio process  $(V_t)_{t=0,\dots,T}$ , i.e.  $\Delta V_t := V_t - V_{t-1} \in -K_t$  a.s., and super-hedging  $h$  in the following sense:  $V_t \geq_{K_t} h_t$ , i.e.  $V_t - h_t \in K_t$  a.s. for all  $t \leq T$ . Papers [1] and [2] provide dual characterizations of all super-hedging prices of a given pay-off referred as the “hedging theorems”. The natural question which arises is to determine the “minimal” endowments  $V_0$  we need to super-hedge the payoff  $h$ . Of course the same problem can be posed for a European option, i.e. when we only require to super-hedge the pay-off at the terminal date  $T$ . Without transaction costs, the problem is simpler. A portfolio process is scalar-valued and under the absence of arbitrage opportunities, the minimal prices of a bounded from below European claim  $h_T \in \mathbf{R}$  depending on the price process  $(S_t)_{t \leq T} \in \mathbf{R}^d$  is given by the interval  $[\sup_{Q \in \mathcal{M}(S)} e^{-rT} E_Q h_T, \infty)$  where  $T$  is the maturity date of the option,  $r$  is the return of the non-risky asset and  $\mathcal{M}(S)$  is the set of all so-called risk-neutral probability measures under which the discounted prices  $\tilde{S}_t := e^{-rt} S_t$  are martingales, Th. 2.1.11 [3]. It is then possible to define a minimal super-hedging price as the unique element  $\sup_{Q \in \mathcal{M}(S)} e^{-rT} E_Q h_T$  (since  $\mathbf{R}$  is endowed with a total ordering). With transaction costs, the portfolio processes are multi-dimensional and the ordering, defined by random cones in  $\mathbf{R}^d$ , is only partial. The natural question is to determine the set of all endowments  $V_0 \in \mathbf{R}$  allowing to start a portfolio process super-hedging a given pay-off  $(h_t)_{t \leq T}$  which are minimal in the following sense: if  $V_0$  is minimal and  $\tilde{V}_0$  is another super-hedging price with  $\tilde{V}_0 \leq_{K_0} V_0$ , then  $\tilde{V}_0 = V_0$ . More generally, among the portfolio processes super-hedging the claim  $(h_t)_{t \leq T}$  at any date  $t$  (with respect to the random cone  $K_t$ ), may we find minimal portfolios? In this paper, we prove the existence of such set of portfolios we call *Essential Supremum*. This is a generalization of the notion of essential supremum of a family of real-valued random variables but it is not a singleton in general. Although, in the two-dimensional space  $\mathbf{R}^2$ , we prove and provide a construction of a unique minimal portfolio. This is of great interest in finance. Indeed, recall that  $V_t \geq_{K_t} \tilde{V}_t$  or  $V_t - \tilde{V}_t \in K_t$  a.s. means that we can change  $V_t$  into  $\tilde{V}_t$  paying some transaction costs, i.e.  $\tilde{V}_t$  is “cheaper” than  $V_t$ .

We propose to extend the concept of essential supremum of a family of real-valued random variables to vector-valued random variables when the partial ordering is defined by a random cone. The paper is organized as follows; we introduce the definition and the immediate properties. We then give conditions under which the essential supremum can be uniquely defined as a singleton (this is the case in  $\mathbf{R}^2$ ). We end the paper with applications in finance and with a Skorohod problem with oblique relection.

## 2 Essential supremum with respect to a random cone

### 2.1 The model

Let  $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \leq T}, P)$  be a continuous-time stochastic basis verifying the usual conditions. In particular, the filtration  $(\mathcal{F}_t)$  is right continuous.

Assume we are given  $t \in [0, T]$  and a  $\mathcal{F}_t$ -adapted closed convex cone-valued mapping  $G_t$  in  $\mathbf{R}^d$  with

$$G_t := \text{cone}\{\xi_t^1, \dots, \xi_t^N\}$$

where  $\xi_t^1, \dots, \xi_t^N$  are  $\mathcal{F}_t$ -adapted processes we call the generators of  $G_t$ . We suppose that the cone  $G_t$  is proper, i.e.  $G_t \cap (-G_t) = \{0\}$  a.s., and  $\mathbf{R}_+^d \subseteq G_t$ . The positive dual of  $G_t$  is defined as

$$G_t^* := \{x \in \mathbf{R}^d : xy \geq 0, \forall y \in G_t\}.$$

This is also a  $\mathcal{F}_t$ -adapted closed convex cone-valued mapping and **we assume** throughout the paper that

$$G_t^* = \text{cone}\{\xi_t^{1*}, \dots, \xi_t^{M*}\}$$

where  $\xi_t^{1*}, \dots, \xi_t^{M*}$  are  $\mathcal{F}_t$ -adapted processes we call the generators of  $G_t^*$ . We recall that the assumptions above hold in the financial models with transaction costs [3]. In this case,  $G_t$  is the so-called solvency cone which is polyhedral.

We denote by  $\geq_{G_t}$  the partial ordering generated by  $G_t$ . If  $X, Y$  are two random variables in  $\mathbf{R}^d$ , we note  $X \geq_{G_t} Y$  if  $X - Y \in G_t$  a.s. The set of all  $\mathcal{F}_t$ -measurable random variables with values in a random set  $A \subseteq \mathbf{R}^d$  is denoted by  $L^0(A, \mathcal{F}_t)$ . We denote by  $\mathbf{1}_d$  the vector in  $\mathbf{R}^d$  whose each component equals 1 and  $e_i$  the vector in  $\mathbf{R}^d$  whose each component equals 0 except the  $i$ th one equal to 1.

## 2.2 Definitions

Our goal is to define “the essential supremum” of a collection of  $\mathcal{F}_T$ -measurable random variables  $(\gamma_i)_{i \in I}$  with respect to  $G_t$  at any date  $u \in [t, T]$ . When existence and uniqueness hold,  $\gamma^* = \text{esssup}_i(\gamma_i)_u$  is the  $\mathcal{F}_u$ -adapted random variable satisfying the following statements:

$$\gamma^* \geq_{G_t} \gamma_i \text{ a.s. } \forall i \in I, \quad (2.1)$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u), \gamma \geq_{G_t} \gamma_i \text{ a.s. } \forall i \in I \Rightarrow \gamma \geq_{G_t} \gamma^* \text{ a.s.} \quad (2.2)$$

Recall that the ordering induced by the cone  $G_t$  is only partial. Hence, the existence of such “essential supremum” is not trivial since uniqueness does not necessarily hold. We then define more generally the *Essential supremum* at date  $u \in [t, T]$  as the class (eventually empty)  $\Gamma^* := \text{Essup}_i(\gamma_i)_u$  of all  $\mathcal{F}_u$ -adapted random variables  $\gamma^*$  satisfying the following statements:

$$\gamma^* \geq_{G_t} \gamma_i, \quad \forall i \in I, \quad (2.3)$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) \text{ and } \gamma \geq_{G_t} \gamma_i \forall i \in I \Rightarrow \exists \gamma^* \in \Gamma^* \text{ s.t. } \gamma \geq_{G_t} \gamma^* \quad (2.4)$$

$$\gamma_1^*, \gamma_2^* \in \Gamma^* \text{ and } \gamma_1^* \neq \gamma_2^* \Rightarrow P(\gamma_1^* - \gamma_2^* \in G_t \setminus \{0\}) = 0. \quad (2.5)$$

When (2.3) holds, we say that  $\gamma^*$  dominates the family  $(\gamma_i)_i$  at date  $t$ .

**Lemma 2.1** *The set  $\Gamma^*$  is unique.*

*Proof.* Let us consider two classes  $C_1$  and  $C_2$  satisfying the axioms (2.3), (2.4) and (2.5). Consider  $\gamma_1^* \in C_1$ . Since  $\gamma_1^*$  dominates the collection  $(\gamma_i)_i$ , there exists  $\gamma_2^* \in C_2$  with  $\gamma_1^* \geq_{G_t} \gamma_2^*$ . Similarly, there exists  $\tilde{\gamma}_1^* \in C_1$  satisfying  $\gamma_2^* \geq_{G_t} \tilde{\gamma}_1^*$ . Then,  $\gamma_1^* - \tilde{\gamma}_1^* \in G_t$  a.s. But (2.5) implies that  $\gamma_1^* = \tilde{\gamma}_1^* = \gamma_2^*$ . We deduce that  $C_1 \subset C_2$  and finally  $C_1 = C_2$ .  $\square$

**Lemma 2.2** *The following equivalence holds:*

$\text{essup}_i(\gamma_i)_u$  exists if and only if  $\text{Essup}_i(\gamma_i)_u$  is a singleton.

**Lemma 2.3** *If  $\text{Essup}_i(\gamma_i)_u$  is not a singleton, then  $\text{Essup}_i(\gamma_i)_u$  is empty or infinite.*

*Proof.* Suppose that  $\text{Essup}_i(\gamma_i)_u = \{\gamma_1^*, \gamma_2^*, \dots, \gamma_m^*\}$  where  $\gamma_i^* \neq \gamma_j^*$  if  $i \neq j$ . For each  $k \in \mathbf{N}$ , the random variable  $(1/k)\gamma_1^* + (1-1/k)\gamma_2^*$  dominates  $(\gamma_i)_i$ . We deduce  $N_k \in \{3, \dots, m\}$  such that  $(1/k)\gamma_1^* + (1-1/k)\gamma_2^* \geq_{G_t} \gamma_{N_k}^*$ . But there exists at least an integer  $j \in \{3, \dots, m\}$  and an infinite subsequence  $N_{k_n}$  such that  $N_{k_n} = j, \forall n$ . By letting  $n$  tend to  $\infty$ , we deduce that  $\gamma_2^* \geq_{G_t} \gamma_j^*$  so that  $\gamma_2^* = \gamma_j^*$  hence a contradiction.  $\square$

**Lemma 2.4** *The class  $\text{Essup}_i(\gamma_i)_u$  is decomposable, i.e. whatever  $\gamma_1^*, \gamma_2^*$  in  $\text{Essup}_i(\gamma_i)_u$  and  $\Omega_u \in \mathcal{F}_u$ ,  $\gamma_1^* I_{\Omega_u} + \gamma_2^* I_{\Omega_u^c} \in \text{Essup}_i(\gamma_i)_u$ .*

*Proof.* It suffices to consider  $\Gamma = \text{Essup}_i(\gamma_i)_u \cup \{\tilde{\gamma}\}$  where  $\tilde{\gamma}$  is defined as  $\tilde{\gamma} := \gamma_1^* I_{\Omega_u} + I_{\Omega_u^c} \gamma_2^*$ . This set satisfies Axioms (2.3), (2.4) and (2.5) so that  $\Gamma = \text{Essup}_i(\gamma_i)_u$ .  $\square$

**Lemma 2.5** *If  $A \subset B$  then  $\text{Essup } B \subseteq \text{Essup } A$ .*

Similarly, we define  $\text{essinf}_i(\gamma_i)_u := -\text{essup}_i(-\gamma_i)_u$  and

$$\text{Essinf}_i(\gamma_i)_u := -\text{Essup}_i(-\gamma_i)_u.$$

### 3 Existence

In the following section, we show that the Essential supremum of a family of  $\mathcal{F}_T$ -adapted random variables at some date  $u \in [t, T]$  is not empty under mild assumptions. To do so, we prove several lemmas.

**Lemma 3.1** *Let us consider a collection  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \bar{\gamma}, \forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Let us define for any  $\hat{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \hat{\gamma} \forall i \in I$ , the real number

$$a_1(\hat{\gamma}) := \inf_{\gamma \in C_1(\hat{\gamma})} E \xi_t^{1*} \gamma \geq 0$$

where

$$C_1(\hat{\gamma}) := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I \text{ and } \gamma \leq_{G_t} \hat{\gamma}\}.$$

Consider a sequence  $(\hat{\gamma}_n) \in C_1(\hat{\gamma})$  such that  $a_1(\hat{\gamma}) = \lim_n E\xi_t^{1*}\hat{\gamma}_n$ . Let us define  $\chi^n := -\hat{\gamma}_n/(1 + |\hat{\gamma}|)$ . Since  $\mathbf{R}_+^d \subseteq G_t$ , it follows that  $\chi^n \geq_{G_t} -\mathbf{1}_d$  and  $\chi^n \in L^0(-G_t, \mathcal{F}_u)$ . Indeed, by assumption  $E(\gamma_{j_0}|\mathcal{F}_u) = 0$  so that  $\gamma \in G_t$  whenever  $\gamma \in C_1(\hat{\gamma})$ . To see it, observe that  $\xi_t^{i*}\gamma \geq \xi_t^{i*}\gamma_{j_0}$  whatever the selector  $\xi_t^{i*} \in L^0(G_t^*, \mathcal{F}_t)$ . It follows that  $\xi_t^{i*}\gamma \geq \xi_t^{i*}E(\gamma_{j_0}|\mathcal{F}_u)$  and  $\xi_t^{i*}\gamma \geq 0$ . From there, we deduce that  $\gamma \in G_t$  a.s.

We split  $\Omega$  in  $\Omega = \Omega_1 \cup \Omega_1^c$  where  $\Omega_1 = \{\liminf_n |\chi^n| = \infty\}$ . On  $\Omega_1$ , there exists a.s.  $(\omega)$  a subsequence  $n_k(\omega)$  such that  $|\chi^{n_k}| \rightarrow \infty$ . Let us define  $\tilde{\chi}^n = (\chi^n/|\chi^n|)I_{|\chi^n| \neq 0}$ . We may assume by compacity that  $\tilde{\chi}^{n_k(\omega)}(\omega) \rightarrow \tilde{\chi}^\infty(\omega)$  on  $\Omega_1$  where  $|\tilde{\chi}^\infty(\omega)| = 1$ . Moreover,  $\tilde{\chi}^{n_k(\omega)}(\omega) \geq_{G_t} -\mathbf{1}_d/|\chi^{n_k}|$ . By getting  $n_k$  converged to  $\infty$ , we deduce that  $\tilde{\chi}^\infty(\omega) \geq_{G_t} 0$  in contradiction with the assumption  $\tilde{\chi}^\infty(\omega) \leq_{G_t} 0$ . Indeed, since  $G_t$  is closed and proper, we deduce that  $\tilde{\chi}^\infty(\omega) = 0$  in contradiction with  $|\tilde{\chi}^\infty| = 1$ .

We then deduce that  $\Omega = \Omega_1^c$ . According to the lemma on subsequences (Lemma 2.1.2 [3]), there exists a strictly increasing sequence of integer-valued  $\mathcal{F}_u$ -adapted random variables  $n_k$  such that  $\chi^{n_k} \rightarrow \chi^\infty$  on  $\Omega$ . It follows that  $\hat{\gamma}_{n_k}$  is also convergent to some random variable  $\bar{\gamma}^1$ . Since  $G_t$  is convex and closed,  $\bar{\gamma}^1 \in C_1(\hat{\gamma})$ . In particular,  $\bar{\gamma}^1 \in L^0(G_t, \mathcal{F}_u)$ . We then may apply the Fatou lemma:

$$E\xi_t^{1*}\bar{\gamma}^1 \leq \liminf_k E\xi_t^{1*}\hat{\gamma}_{n_k} \leq a_1(\hat{\gamma}) \leq E\xi_t^{1*}\bar{\gamma}^1.$$

We then deduce that  $a_1(\hat{\gamma}) = E\xi_t^{1*}\bar{\gamma}^1$ . We repeat the procedure in the following way. Consider

$$a_2(\hat{\gamma}) := \inf_{\gamma \in C_2(\hat{\gamma})} E\xi_t^{2*}\gamma \geq 0$$

where

$$C_2(\hat{\gamma}) := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I \text{ and } \gamma \leq_{G_t} \bar{\gamma}^1\} \subset C_1(\hat{\gamma}).$$

We get  $\bar{\gamma}^2 \in C_2(\hat{\gamma})$  such that  $a_1(\hat{\gamma}) = E\xi_t^{1*}\bar{\gamma}^2$  and  $a_2(\hat{\gamma}) = E\xi_t^{2*}\bar{\gamma}^2$ . Reiterating the procedure, we finally get a  $\mathcal{F}_u$ -adapted random variable  $\bar{\gamma}^M \in C_i(\hat{\gamma})$ ,  $1 \leq i \leq M$  satisfying

$$a_i(\hat{\gamma}) = E\xi_t^{i*}\bar{\gamma}^M, \quad \forall i = 1, \dots, M. \quad (3.1)$$

Observe that  $\bar{\gamma}^M \leq_{G_t} \hat{\gamma}$  by definition. We denote by  $\Lambda(\hat{\gamma})$  the set of all random variables  $\bar{\gamma}^M \in C_M(\hat{\gamma})$  verifying (3.1) and we define

$$\text{Essup}_i(\gamma_i)_u := \bigcup_{\hat{\gamma} \in \Theta} \Lambda(\hat{\gamma}) \quad (3.2)$$

where  $\Theta$  is the set of all  $\mathcal{F}_u$ -adapted  $\hat{\gamma}$  satisfying  $\gamma_i \leq_{G_t} \hat{\gamma}$ ,  $\forall i \in I$ . Of course, for a given  $\hat{\gamma}$ ,  $\Lambda(\hat{\gamma})$  is not necessarily unique so the union should be understood over all possible sets  $C_M(\hat{\gamma})$ . It remains to show that the class defined by (3.2) satisfies the statements (2.3), (2.4) and (2.5).

If  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$ , it is obvious that  $\gamma^*$  verifies Assertion (2.3). We claim that for any  $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u)$ , dominating every  $\gamma_i$ ,  $i \in I$ , Assertion (2.4) holds. It suffices to consider an arbitrary  $\gamma^* \in \Lambda(\gamma)$ . At last, consider

$\gamma_1^*, \gamma_2^* \in \text{Essup}_i(\gamma_i)_u$  with  $\gamma_1^* \in A(\widehat{\gamma}_1)$  and  $\gamma_2^* \in A(\widehat{\gamma}_2)$ . Assume that  $\gamma_1^* - \gamma_2^* \in G_t \setminus \{0\}$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . Then, there exists  $i \in \{1, \dots, m\}$  and a non null set  $\Theta_u \subset \Omega_u$  such that  $\xi_t^{i*}(\gamma_1^* - \gamma_2^*)I_{\Theta_u} \geq 0$  and  $E\xi_t^{i*}(\gamma_1^* - \gamma_2^*)I_{\Theta_u} > 0$ . It follows that

$$a_i(\widehat{\gamma}_1) = E\xi_t^{i*}\gamma_1^* > E\xi_t^{i*}(\gamma_2^*I_{\Theta_u} + \gamma_1^*I_{\Theta_u^c}). \quad (3.3)$$

But  $\gamma = \gamma_2^*I_{\Theta_u} + \gamma_1^*I_{\Theta_u^c} \leq_{G_t} \gamma_1^* \in C_i(\widehat{\gamma}_1)$  and  $\gamma$  dominates the family  $(\gamma_i)_{i \in I}$ . It follows that  $\gamma \in C_i(\widehat{\gamma}_1)$  which implies that  $E\xi_t^{i*}\gamma \geq a_i(\widehat{\gamma}_1)$  in contradiction with (3.3). Hence, Assertion (2.5) holds.  $\square$

**Corollary 3.2** *Let us consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_t} \bar{\gamma} \forall i \in I$ . Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Let us define, for all  $i \in I$ ,  $\tilde{\gamma}_i := \gamma_i - E(\gamma_{j_0} | \mathcal{F}_u)$  where  $j_0 \in I$  is fixed. This collection obviously satisfies the conditions of Lemma 3.1. We then conclude with

$$\text{Essup}_i(\gamma_i)_u = \text{Essup}_i(\tilde{\gamma}_i)_u + E(\gamma_{j_0} | \mathcal{F}_u).$$

$\square$

In the one dimensional case, with  $G_0 = \mathbf{R}_+$ , we may define the essential supremum  $\text{essup}_i(\gamma_i)_u$  as soon as the family  $(\gamma_i)_{i \in I}$  is bounded from above by a real-valued and  $\mathcal{F}_u$ -adapted random variable as shown above. In particular  $\text{essup}_i(\gamma_i)_T$  coincides with the usual essential supremum  $\text{essup}_i(\gamma_i)$  as defined in the literature. Actually, we may extend the definition for random variables taking values in  $(-\infty, \infty]$  as follows:

**Proposition 3.3** *For any family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}, \mathcal{F}_T)$  of scalar random variables (which may take infinite values), there exists a unique  $\mathcal{F}_u$ -adapted random variable  $\text{essup}_i(\gamma_i)_u := \gamma$  verifying:*

- (1)  $\gamma \geq \gamma_i, \quad \forall i \in I,$
- (2) *If  $\gamma' \in L^0(\mathbf{R}, \mathcal{F}_u)$  verifies  $\gamma' \geq \gamma_i, \forall i \in I$ , then  $\gamma' \geq \gamma$ .*

*Proof.* It suffices to take  $\gamma := \lim_{m \rightarrow \infty} \text{essup}_i(\gamma_i \wedge m)_u$ .  $\square$

**Remark 3.4** *We have  $\text{essup}_i(\gamma_i)_u \geq \text{essup}_i(\gamma_i)_T, \quad \forall u \in [0, T]$ .*

**Corollary 3.5** *Assume that the family  $(\gamma_i)_{i \in I}$  satisfies  $\text{essup}_i|\gamma_i|_u < \infty$  a.s. Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty.*

*Proof.* Consider the family

$$\tilde{\gamma}_i := \frac{\gamma_i}{1 + \text{essup}_j|\gamma_j|_u}, \quad i \in I.$$

We have  $-\mathbf{1}_d \leq_{G_t} \tilde{\gamma}_i \leq \mathbf{1}_d$ . Applying Corollary 3.2, we deduce that  $\text{Essup}_i(\tilde{\gamma}_i)_u$  is not empty. It follows that  $\text{Essup}_i(\gamma_i)_u$  is not empty and is given by

$$\text{Essup}_i(\gamma_i)_u = (1 + \text{essup}_j|\gamma_j|_u)\text{Essup}_i(\tilde{\gamma}_i)_u. \quad (3.4)$$

Indeed, consider  $\gamma^* \in (1 + \text{esssup}_j |\gamma_j|_u) \text{Essup}_i(\tilde{\gamma}_i)_u$ , i.e.  $\gamma^* = (1 + \text{esssup}_j |\gamma_j|_u) \tilde{\gamma}^*$  where

$$\tilde{\gamma}^* \geq_{G_t} \frac{\gamma_i}{1 + \text{esssup}_j |\gamma_j|_u}, \quad \forall i \in I.$$

We deduce that  $\gamma^* \geq_{G_t} \gamma_i, \forall i \in I$ , i.e. Assertion (2.3) holds. Let us consider a random variable  $\gamma$  verifying  $\gamma \geq_{G_t} \gamma_i, \forall i \in I$ . Then, we also have the inequalities  $\gamma / (1 + \text{esssup}_j |\gamma_j|_u) \geq_{G_t} \tilde{\gamma}_i, \forall i \in I$ . We deduce  $\tilde{\gamma}^* \in \text{Essup}_i(\tilde{\gamma}_i)_u$  such that  $\gamma / (1 + \text{esssup}_j |\gamma_j|_u) \geq_{G_t} \tilde{\gamma}^*$  and  $\gamma \geq_{G_t} (1 + \text{esssup}_j |\gamma_j|_u) \tilde{\gamma}^*$  i.e. Assertion (2.4) holds. Similarly, Assertion (2.5) holds.  $\square$

#### 4 Immediate properties

**Corollary 4.1** *Let us consider a family  $(\gamma_i)_{i \in I}$ . Then,  $\text{essup}_i |\gamma_i|_T < \infty$  (a.s.) if and only if there exists  $\gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that  $\gamma_1 \leq_{G_t} \gamma_i \leq_{G_t} \gamma_2, \forall i$ .*

*Proof.* Assume that  $\text{essup}_i |\gamma_i|_T < \infty$  a.s. Applying Corollary 3.5, we deduce that  $\text{Essinf}_i(\gamma_i)_T$  and  $\text{Essup}_i(\gamma_i)_T$  are not empty so that we can conclude. Reciprocally, suppose that  $\text{essup}_i |\gamma_i|_T = \infty$  on a non null set  $\Lambda$ . We may extract a subsequence such that  $|\gamma_i| \rightarrow \infty$ . We introduce the random variables  $\tilde{\gamma}_i = \gamma_i / |\gamma_i|, \tilde{\gamma}_{1i} = \gamma_{1i} / |\gamma_i|$  and  $\tilde{\gamma}_{2i} = \gamma_{2i} / |\gamma_i|$ . By assumption, we obtain  $\tilde{\gamma}_{1i} \leq_{G_t} \tilde{\gamma}_i \leq_{G_t} \tilde{\gamma}_{2i}, \forall i \in I$ . By compactity, we may assume that  $\tilde{\gamma}_i \rightarrow \tilde{\gamma}$  with  $|\tilde{\gamma}| = 1$ . On the other hand, the last inequality yields  $\tilde{\gamma} = 0$  on  $\Lambda$  since  $G_t$  is closed and proper. Hence a contradiction.  $\square$

**Proposition 4.2** *Let us consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that  $\text{Essup}_i(\gamma_i)_u \neq \emptyset$  where  $u \in [t, T]$ . If  $\lambda_u \in L^0(\mathbf{R}_+, \mathcal{F}_u)$ , then*

$$\text{Essup}_i(\lambda_u \gamma_i)_u = \lambda_u \text{Essup}_i(\gamma_i)_u.$$

*Proof.* We first suppose that there exists  $i_0 \in I$  such that  $E(\gamma_{i_0} | \mathcal{F}_u) = 0$ . It follows that

$$\text{Essup}_i(\lambda_u \gamma_i)_u \cup \text{Essup}_i(\gamma_i)_u \subset L^0(G_t, \mathcal{F}_u). \quad (4.5)$$

Let us consider  $\gamma \in \text{Essup}_i(\lambda_u \gamma_i)_u$ . Since  $\gamma \geq_{G_t} \lambda_u \gamma_i, \forall i$ , we then deduce that for a given  $\gamma^{1*} \in \text{Essup}_i(\gamma_i)_u$ ,

$$\frac{\gamma}{\lambda_u} I_{\{\lambda_u \neq 0\}} + \gamma^{1*} I_{\{\lambda_u = 0\}} \geq_{G_t} \gamma_i, \quad \forall i \in I.$$

We deduce  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$  such that

$$\frac{\gamma}{\lambda_u} I_{\{\lambda_u \neq 0\}} + \gamma^{1*} I_{\{\lambda_u = 0\}} \geq_{G_t} \gamma^*.$$

By virtue of (4.5),  $\gamma \geq_{G_t} \lambda_u \gamma^*$ . Similarly, there exists  $\gamma^1 \in \text{Essup}_i(\lambda_t \gamma_i)_u$  such that  $\lambda_u \gamma^* \geq_{G_t} \gamma^1$ . We then deduce that  $\gamma = \gamma^1 = \lambda_u \gamma^*$ . We have shown

$$\text{Essup}_i(\lambda_u \gamma_i)_u \subseteq \lambda_u \text{Essup}_i(\gamma_i)_u.$$



Reciprocally, if  $\gamma^* \in \text{Essup}_i(\gamma_i)_u$ , then  $\lambda_u \gamma^* \geq_{G_t} \lambda_u \gamma_i, \forall i \in I$ . So, there is  $\gamma \in \text{Essup}_i(\lambda_u \gamma_i)_u$  such that  $\lambda_u \gamma^* \geq_{G_t} \gamma$ . We deduce that

$$\gamma^* \geq_{G_t} \frac{\gamma}{\lambda_u} I_{\lambda_u \neq 0} + \gamma^* I_{\lambda_u = 0} \geq_{G_t} \gamma^{1*} \quad (4.6)$$

where  $\gamma^{1*} \in \text{Essup}_i(\gamma_i)_u$ . From there,  $\gamma^* = \gamma^{1*}$  and  $\lambda_u \gamma^* = \gamma$  if  $\lambda_u \neq 0$ . Otherwise,  $\lambda_u \gamma^* \geq_{G_t} \gamma$  implies that  $\gamma = 0$  since  $\gamma \geq_{G_t} 0$  by virtue of (4.5). We then conclude that  $\lambda_u \gamma^* = \gamma$  and the reverse inclusion is proven.

In the general case, we fix  $i_0 \in I$  and we deduce from above that

$$\begin{aligned} \text{Essup}_i(\lambda_u \gamma_i)_u &= \text{Essup}_i(\lambda_u(\gamma_i - E(\gamma_{i_0}|\mathcal{F}_u)))_u + \lambda_u E(\gamma_{i_0}|\mathcal{F}_u), \\ &= \lambda_u \text{Essup}_i(\gamma_i - E(\gamma_{i_0}|\mathcal{F}_u))_u + \lambda_u E(\gamma_{i_0}|\mathcal{F}_u), \\ &= \lambda_u (\text{Essup}_i(\gamma_i - E(\gamma_{i_0}|\mathcal{F}_u))_u + E(\gamma_{i_0}|\mathcal{F}_u)), \\ &= \lambda_u \text{Essup}_i(\gamma_i)_u. \end{aligned}$$

□

**Proposition 4.3** *Let us consider  $X \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  where  $u \in [t, T]$ . Assume that  $\text{Essup}(X, 0)_u$  is a singleton. Then,*

$$\text{Essup}(X, -X, 0)_u \subset \text{Essup}(X, 0)_u + \text{Essup}(-X, 0)_u.$$

*Proof.* Let us consider  $\gamma^* \in \text{Essup}(X, -X, 0)_u$ . Then,  $\gamma^* + X \geq_{G_t} X$  and  $\gamma^* + X \geq_{G_t} 0$ . We deduce that  $\gamma^* + X \geq_{G_t} \gamma_1$  where  $\gamma_1 := \text{essup}^{G_t}(X, 0)_u$ . We also have  $\gamma^* \geq_{G_t} \gamma_1$ . Then  $\gamma^* - \gamma_1 \geq_{G_t} -X, 0$ . We deduce that  $\gamma^* - \gamma_1 \geq_{G_t} \gamma_2$  where  $\gamma_2 \in \text{Essup}^{G_t}(-X, 0)_u$ . On the other hand,  $\gamma_1 + \gamma_2$  dominates  $X, -X$  and  $0$ . Hence there is  $\gamma^{1*} \in \text{Essup}^{G_t}(-X, X, 0)_u$  such that  $\gamma^* \geq_{G_t} \gamma_1 + \gamma_2 \geq_{G_t} \gamma^{1*}$ . It follows that  $\gamma^* = \gamma_1 + \gamma_2 = \gamma^{1*}$ . □

**Remark 4.4** *In the case where the three essential supremum sets above are singletons, the inclusion is an equality.*

**Proposition 4.5** *Assume that the cone  $G_t$  is such that the essential supremum of two random variables is necessary a singleton. Let us consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  and  $u \in [t, T]$ . Then, the set  $\text{Essup}_i(\gamma_i)_u$  is closed in probability.*

*Proof.* Consider  $\Gamma$  the closure in probability of  $\text{Essup}_i(\gamma_i)_u$ . It suffices to check that  $\Gamma$  satisfies the same axioms than  $\text{Essup}_i(\gamma_i)_u$ . First, if  $\gamma^n$  converges in probability to  $\gamma^* \in \Gamma$ , there exists a subsequence  $n_k$  such that  $\gamma^{n_k} \rightarrow \gamma^*$  a.s. From  $\gamma^{n_k} \geq_{G_t} \gamma_i, \forall i \in I$ , we deduce that  $\gamma^* \geq_{G_t} \gamma_i, \forall i$ . Secondly, if  $\delta \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfies  $\delta \geq_{G_t} \gamma_i, \forall i \in I$ , then there exists  $\gamma \in \text{Essup}_i(\gamma_i)_u \subseteq \Gamma$  such that  $\delta \geq_{G_t} \gamma$ . At last, consider  $\gamma_1, \gamma_2 \in \Gamma$  and suppose that  $\gamma_1 - \gamma_2 \in G_t \setminus \{0\}$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . We may assume there exists  $i \in \{1, \dots, M\}$  such that  $\xi_t^{i*} \gamma_1 > \xi_t^{i*} \gamma_2$  on  $\Omega_u$  where, by density,  $\gamma_1, \gamma_2 \in \text{Essup}_i(\gamma_i)_u$ . We assume without loss of generality that  $\text{Essup}_i(\gamma_i)_u \neq \emptyset$ . It follows that the essential supremum is given by Lemma 3.1. In particular, considering the notations of the proof,  $\gamma_1 \in C_M(\hat{\gamma})$  and, for all  $i$ , the inequality  $a_i(\hat{\gamma}) = E\xi_t^{i*} \gamma_1 > E\xi_t^{i*} \gamma \geq a_i(\hat{\gamma})$

holds where  $\gamma := \gamma_1 I_{\Omega \setminus \Omega_u} + \text{essinf}(\gamma_2, \gamma_1) I_{\Omega_u}$  hence a contradiction. Indeed,  $\text{essinf}(\gamma_2, \gamma_1) \leq_{G_t} \gamma_2$  yields the last strict inequality. Since  $\gamma_2, \gamma_1$  dominate the family  $(\gamma_i)_{i \in I}$ , we deduce that  $\text{essinf}(\gamma_2, \gamma_1) \geq_{G_t} (\gamma_i)_{i \in I}$  with  $\text{essinf}(\gamma_2, \gamma_1) \leq_{G_t} \hat{\gamma}$ .  $\square$

**Definition 4.6** A subset  $\Lambda_u \subseteq L^0(\mathbf{R}^d, \mathcal{F}_u)$  is said decomposable if  $\Omega_u \in \mathcal{F}_u$  and  $\gamma, \tilde{\gamma} \in \Lambda_u$  implies  $\gamma I_{\Omega_u} + \tilde{\gamma} I_{\Omega_u^c} \in \Lambda_u$ .

**Corollary 4.7** Suppose that the conditions of Proposition 4.5 hold. Let us consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ . Then, there exists a closed set-valued and  $\mathcal{F}_u$ -measurable mapping  $\Gamma_u \subseteq \mathbf{R}^d$  such that

$$\text{Essup}_i(\gamma_i)_u = L^0(\Gamma_u, \mathcal{F}_u).$$

In particular, the graphs of  $\Gamma_u$  are  $\mathcal{F}_u \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable.

*Proof.* We know that  $\text{Essup}_i(\gamma_i)_u$  is closed in probability. Moreover, observe that  $\text{Essup}_i(\gamma_i)_u$  is decomposable. We then conclude applying Proposition 5.4.3 [3].  $\square$

**Proposition 4.8** Let us consider a totally ordered family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  with  $\gamma_i \leq_{G_t} \bar{\gamma}, \forall i \in I$ . Then  $\text{Essup}_i(\gamma_i)_t$  is a singleton  $\{\gamma^*\}$ . Moreover, there exists a random subsequence  $n_k \in \mathbf{N}$  such that  $\gamma_{n_k}$  is an increasing sequence converging a.s. to  $\gamma^*$ .

*Proof.* Consider

$$b^j := \sup_{i \in I} E \xi_t^{j*} \gamma_i, \quad j = 1, \dots, M.$$

There exists some subsequences  $(i_k^{(j)})$  such that  $b^j = \lim_k \nearrow E \xi_t^{j*} \gamma_{i_k^{(j)}}$ . Since the family we consider is totally ordered, we deduce that the sequences  $(\gamma_{i_k^{(j)}})_k$  are increasing with respect to the partial ordering induced by  $G_t$ . We then set

$$\gamma_{i_k} := \text{Essup}_{1 \leq j \leq M} (\gamma_{i_k^{(j)}})_t \subseteq \{\gamma_{i_k^{(j)}} : 1 \leq j \leq M\}.$$

It is straightforward that  $(\gamma_{i_k})_k$  is still an increasing sequence and moreover

$$b^j = \lim_k \nearrow E \xi_t^{j*} \gamma_{i_k}, \quad \forall 1 \leq j \leq M.$$

Notice that  $\gamma_{i_1} \leq_{G_t} \gamma_{i_k} \leq_{G_t} \bar{\gamma}, k \geq 1$ . We deduce that  $\liminf_k |\gamma_{i_k}|_t < \infty$ . We then deduce a random sequence  $n_k \in \mathbf{N}$  such that  $\gamma_{n_k}$  converges a.s. to some  $\gamma^* \leq_{G_t} \bar{\gamma}$ . Recall that  $\gamma_{n_k}$  reads as  $\gamma_{n_k} = \sum_{p=1}^{\infty} \gamma_p I_{\{n_k=p\}}$ . Since the sequences  $(\gamma_{i_k})_k$  and  $n_k$  are increasing (a.s.), we then deduce that the sequence  $\gamma_{n_k}$  is also increasing with respect to the partial ordering induced by  $G_t$ . From  $n_k(\omega) \geq k$  a.s.  $(\omega)$ , we deduce that  $\gamma_{n_m(\omega)} \geq_{G_t} \gamma_k(\omega)$  a.s. if  $m \geq k$ . It follows that  $\gamma^* \geq_{G_t} \gamma_k$  for any  $k$ . From there, we deduce that  $\text{Essup}_i(\gamma_i)_t = \{\gamma^*\}$ . Indeed, if  $\gamma \geq_{G_t} \gamma_k$  for any  $k$ , then  $\gamma \geq_{G_t} \gamma_{n_k}$  and  $\gamma \geq_{G_t} \gamma^*$ , i.e. the singleton  $\{\gamma^*\}$  satisfies the required axioms to be the essential supremum.  $\square$

We immediately deduce the following:

**Corollary 4.9** Let us consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $\text{Essup}_i(\gamma_i)_t \neq \emptyset$ . Assume that  $(\gamma_i)_{i \in I}$  is decomposable and closed in probability. Then  $(\gamma_i)_{i \in I}$  is an inductive set and so contains at least one maximal element.

## 5 When the generators are linearly independent

Recall that  $x \in G_t$  if and only if there are some positive coefficients  $\alpha_i(x) \geq 0$  such that

$$x = \sum_{i=1}^N \alpha_i(x) \xi_t^i.$$

We have the following lemma:

**Lemma 5.1** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Let us consider  $x, y \in G_t$ . Then,  $x \geq_{G_t} y$  if and only if  $\alpha_i(x) \geq \alpha_i(y), \forall i = 1, \dots, N$ .*

*Proof.* If  $x - y \in G_t$  and  $x, y \in G_t$ , then

$$x - y = \sum_{i=1}^N \alpha_i(x - y) \xi_t^i = \sum_{i=1}^N (\alpha_i(x) - \alpha_i(y)) \xi_t^i.$$

We deduce that  $0 \leq \alpha_i(x - y) = \alpha_i(x) - \alpha_i(y), \forall i$ , and we conclude.  $\square$

**Lemma 5.2** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent and we are given a finite family  $\eta_1, \dots, \eta_m \in L^0(G_t, \mathcal{F}_T)$ . Consider  $u \in [t, T]$ . If there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  such that  $(\eta_i)_i \leq_{G_t} \bar{\gamma}$  a.s., then  $\Gamma_u^* := \text{Essup}_i(\eta_i)_u$  is not empty and reduced to the singleton  $\text{essup}_i(\eta_i)_u$ .*

*Proof.* By a measurable selection argument, let us write

$$\eta_i = \sum_{j=1}^N \alpha_j^i \xi_t^j$$

where  $\alpha_j^i$  are positive and  $\mathcal{F}_T$ -adapted random variables. We then define  $\hat{\alpha}_j := \text{essup}_i(\alpha_j^i)_u$  which is a.s. finite. Indeed, recall that  $(\eta_i)_i \leq_{G_t} \bar{\gamma}$  so that we may apply Lemma 5.1. Let us set

$$\text{essup}_i(\eta_i)_u := \gamma^* := \sum_{j=1}^N \hat{\alpha}_j \xi_t^j.$$

By virtue of Lemma 5.1,  $\gamma^*$  dominates  $\eta_i$  for all  $i$ . Moreover, if another  $\mathcal{F}_u$ -adapted random variable  $\eta$  dominates  $\eta_i$  for all  $i$ , then  $\eta \in L^0(G_t, \mathcal{F}_u)$ . Its decomposition

$$\eta = \sum_{j=1}^N \alpha_j \xi_t^j$$

verifies  $\alpha_j \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  and  $\alpha_j \geq \alpha_j^i, \forall i$ , by virtue of Lemma 5.1, i.e.  $\alpha_j \geq \hat{\alpha}_j$  or  $\eta \geq_{G_t} \gamma^*$ .  $\square$

In the same way, we show the following:

**Lemma 5.3** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent and we are given a finite family  $\eta_1, \dots, \eta_m \in L^0(G_t, \mathcal{F}_T)$ . If there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  such that  $(\eta_i)_i \geq_{G_t} \bar{\gamma}$  a.s., then  $\text{Essinf}_i(\eta_i)_u$  is not empty and is reduced to the singleton  $\text{essinf}_i(\eta_i)_u$ .*

**Lemma 5.4** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Suppose we are given a family  $(\gamma_i)_{i \in I}$  and  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  such that  $\gamma_i \leq_{G_t} \bar{\gamma}, \forall i \in I$ . Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty and is reduced to the singleton  $\text{essup}_i(\gamma_i)_u$ .*

*Proof.* We may assume without loss of generality that there exists  $j_0 \in I$  such that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Let us define

$$a_1 := \inf_{\gamma \in C} E\xi_t^{1*} \gamma \geq 0$$

where

$$C := \{\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u) : \gamma \geq_{G_t} \gamma_i, \forall i \in I\}.$$

Let us consider a sequence  $(\hat{\gamma}_n)_n \in C$  such that  $a_1 = \lim_n E\xi_t^{1*} \hat{\gamma}_n$ . Let us define

$$\chi^n := -\frac{\hat{\gamma}_n}{1 + |\hat{\gamma}|}.$$

By assumption, we have  $\chi^n \geq_{G_t} -\mathbf{1}_d$ , and  $\chi^n \in L^0(-G_t, \mathcal{F}_u)$ .

As in the proof of Lemma 3.1, there exists a  $\mathcal{F}_u$ -adapted subsequence  $n_k$  such that  $\chi^{n_k}$  is a.s. convergent. Hence  $\hat{\gamma}_{n_k}$  is also convergent to  $\bar{\gamma}^1$ . Since  $G_t$  is convex and closed,  $\bar{\gamma}^1 \in C$  and belongs to  $L^0(G_t, \mathcal{F}_u)$ . Applying the Fatou lemma:

$$E\xi_t^{1*} \bar{\gamma}^1 \leq \liminf_k E\xi_t^{1*} \hat{\gamma}_{n_k} \leq a_1.$$

We deduce that  $a_1(\hat{\gamma}) = E\xi_t^{1*} \bar{\gamma}^1$ . In the same way, we define  $a_2, \dots, a_M$  and we find  $\bar{\gamma}^2, \dots, \bar{\gamma}^M \in C$  satisfying

$$a_i := \inf_{\gamma \in C} E\xi_t^{i*} \gamma = E\xi_t^{i*} \bar{\gamma}^i, \quad i \leq M.$$

By virtue of Lemma 5.3, let us define  $\gamma^* := \text{essinf}_i(\bar{\gamma}^i)_u$ . By definition,  $\gamma^* \in C$  and

$$a_i := \inf_{\gamma \in C} E\xi_t^{i*} \gamma = E\xi_t^{i*} \gamma^*, \quad i \leq M.$$

Assume that  $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  verifies  $\gamma \geq_{G_t} \gamma_i, \forall i \in I$ , and  $\gamma - \gamma^* \notin G_t$  on a non null set  $\Omega_u \in \mathcal{F}_u$ . Then, there exists  $i \in \{1, \dots, M\}$  and a non null set  $\Theta_u \subset \Omega_u$  such that  $\xi_t^{i*}(\gamma - \gamma^*)I_{\Theta_u} \leq 0$  and  $E\xi_t^{i*}(\gamma - \gamma^*)I_{\Theta_u} < 0$ . It follows that

$$a_i = E\xi_t^{i*} \gamma^* > E\xi_t^{i*}(\gamma I_{\Theta_u} + \gamma^* I_{\Theta_u^c}). \quad (5.7)$$

But  $\tilde{\gamma} = \gamma I_{\Theta_u} + \gamma^* I_{\Theta_u^c} \in C$ . Hence,  $E\xi_t^{i*} \tilde{\gamma} \geq a_i$  and we get a contradiction.  $\square$

**Corollary 5.5** *Assume that  $\xi_t^1, \dots, \xi_t^N$  are linearly independent. Let us consider a collection  $(\gamma_i)_{i \in I}$  with  $\text{essup}_i |\gamma_i|_u < \infty$  a.s. Then,  $\text{Essup}_i(\gamma_i)_u$  is not empty and equals the singleton  $\text{essup}_i(\gamma_i)_u$ .*

*Proof.* The proof is similar to that of Corollary 3.5.  $\square$

**Lemma 5.6** *Assume that  $G$  is generated by some linearly independent generators  $\xi_t^1, \dots, \xi_t^N$ . We are given a family of  $\mathcal{F}_T$ -adapted random variables  $(\gamma_i)_i$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\bar{\gamma} \geq_{G_t} \gamma_i, \forall i \in I$ , where  $u \in [t, T]$ . Consider an arbitrary selector  $\eta_t$  of  $G^*$ , i.e.  $\eta_t \in L^0(G_t^*, \mathcal{F}_t)$ . Then,*

$$(\text{essup}_i(\gamma_i)_u) \cdot \eta_t \geq \text{essup}_i(\gamma_i \cdot \eta_t)_u.$$

*Proof.* Since  $\text{essup}_i(\gamma_i)_u \geq_{G_t} \gamma_i, \forall i \in I$ , we deduce the inequality  $(\text{essup}_i(\gamma_i)_u) \cdot \eta_t \geq \gamma_i \cdot \eta_t, \forall i \in I$ . It follows that

$$(\text{essup}_i(\gamma_i)_u) \cdot \eta_t \geq \text{essup}_i(\gamma_i \cdot \eta_t)_u.$$

□

**Lemma 5.7** *Assume that  $G_t$  and  $G_t^*$  are generated by a finite number of linearly independent generators. We are given a family of  $\mathcal{F}_T$ -adapted random variables  $(\gamma_i)_i$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\bar{\gamma} \geq_{G_t} \gamma_i, \forall i$ , where  $u \in [t, T]$ . Consider an arbitrary generator  $\eta_t \in L^0(G_t^*, \mathcal{F}_t)$ . Then,*

$$(\text{essup}_i \gamma_i)_u \cdot \eta_t = \text{essup}_i(\gamma_i \cdot \eta_t)_u.$$

*Proof.* In sight of the previous lemma, assume that

$$(\text{essup}_i \gamma_i)_u \cdot \eta_t \not\geq \text{essup}_i(\gamma_i \cdot \eta_t)_u. \quad (5.8)$$

We denote by  $\eta^{(1)} := \eta_t, \dots, \eta^{(d)}$  the generators of  $G_t^*$  and we recall that they are linearly independent by assumption. By a measurable selection argument, we consider  $\xi_t \in \{\eta^{(2)}, \dots, \eta^{(d)}\}^\perp$  with  $\xi_t \eta^{(1)} \geq 0$  and  $|\xi_t| = 1$ . It is possible; in the contrary case, we have  $\{\eta^{(2)}, \dots, \eta^{(d)}\}^\perp = \{0\}$  and  $\text{lin}(\eta^{(2)}, \dots, \eta^{(d)}) = \mathbf{R}^d$ . We deduce that  $\xi_t \in L^0(G_t, \mathcal{F}_t)$ . Again, by a measurable selection argument, we then deduce from (5.8) the existence of a  $\alpha_u \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  small enough such that

$$(\text{essup}_i(\gamma_i)_u - \alpha_u \xi_t) \cdot \eta_t \geq \text{essup}_i(\gamma_i \cdot \eta_t)_u.$$

From there, it follows that  $\text{essup}_i(\gamma_i)_u - \alpha_u \xi_t \geq_{G_t} \gamma_i, \forall i \in I$ , and we get that  $\text{essup}_i(\gamma_i)_u - \alpha_u \xi_t \geq_{G_t} \text{essup}_i(\gamma_i)_u$  hence a contradiction. □

**Lemma 5.8** *Let us consider a collection of  $\mathcal{F}_T$ -measurable random variables  $\gamma = (\gamma_i)_{i \in I}$  such that  $\text{essup}_{i \in I}(\gamma_i)_u$  exists where  $u \in [t, T]$ . Consider the “upward” completion defined as*

$$\bar{\gamma}^{up} := \left\{ \text{essup}_{i_1, \dots, i_n}(\gamma_{i_1}, \dots, \gamma_{i_n}) : n \in \mathbf{N}, i_j \in I \right\}. \quad (5.9)$$

*Then*

$$\text{essup}_{i \in I}(\gamma_i)_u = \text{essup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

*Proof.* By definition,  $\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \gamma_{i_j}$ , whatever  $i_j \in I$ . We deduce that  $\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \text{essup}_{j \leq n}(\gamma_{i_j})_u$  and then

$$\text{essup}_{i \in I}(\gamma_i)_u \geq_{G_t} \text{essup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

The reverse inequality is obvious so that we can conclude. □

**Corollary 5.9** *Assume that  $G_t$  is generated by a finite number of linearly independent generators and  $G_t^*$  is generated by a basis of  $\mathbf{R}^d$ . We are given a directed upwards family of  $\mathcal{F}_t$ -adapted random variables  $(\gamma_i)_{i \in I}$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  satisfying  $\bar{\gamma} \geq_{G_t} \gamma_i, \forall i$ . Consider  $\eta_s \in L^0(\mathbf{R}^d, \mathcal{F}_s)$  where  $s \leq t$ . Then, there exists a sequence  $i_n \in I$  such that*

$$(\text{esssup}_i \gamma_i)_t \cdot \eta_s = \lim_n \gamma_{i_n} \cdot \eta_s.$$

In particular,

$$(\text{esssup}_i \gamma_i)_t \cdot \eta_s \leq \text{esssup}_i (\gamma_i \cdot \eta_s).$$

*Proof.* Since the generators  $\xi_t^{1*}, \dots, \xi_t^{d*}$  of  $G_t^*$  generate  $\mathbf{R}^d$ , we deduce by a measurable selection argument that  $\eta_s = \sum_{j=1}^d a_j \xi_t^{j*}$  where  $a_j \in L^0(\mathbf{R}, \mathcal{F}_t)$  for all  $j \leq d$ . By virtue of Lemma 5.7,

$$(\text{esssup}_i \gamma_i)_t \cdot \eta_s = \sum_{j=1}^d a_j \text{esssup}_i (\gamma_i \cdot \xi_t^{j*})_t$$

where here  $\text{esssup}_i (\gamma_i \cdot \xi_t^{j*})_t = \text{esssup}_i (\gamma_i \cdot \xi_t^{j*})$  is the  $\mathbf{R}$ -valued essential supremum of the litterature. For each  $j \leq d$ , the family  $(\gamma_i \cdot \xi_t^{j*})_i$  is directed upwards. We then deduce the existence of a sequence  $i_n^j \in I$  such that  $\text{esssup}_i (\gamma_i \cdot \xi_t^{j*}) = \lim_n \uparrow \gamma_{i_n^j} \cdot \xi_t^{j*}$ , see A.3.3 [3]. But,  $(\gamma_i)_{i \in I}$  being directed upwards, we deduce  $i_n \in I$  such that  $\text{esssup}_i (\gamma_i \cdot \xi_t^{j*}) = \lim_n \uparrow \gamma_{i_n} \cdot \xi_t^{j*}$  for all  $j \leq d$  and we conclude.  $\square$

**Lemma 5.10** *Assume that  $G_t$  and  $G_t^*$  are generated by linearly independent generators. We are given a family of  $\mathcal{F}_T$ -adapted random variables  $\gamma = (\gamma_i)_{i \in I}$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\bar{\gamma} \geq_{G_t} \gamma_i, \forall i$ , where  $u \in [t, T]$ . If  $E(\text{esssup}_i (\gamma_i)_u | \mathcal{F}_t)$  exists, then*

$$\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t \leq_{G_t} E(\text{esssup}_i (\gamma_i)_u | \mathcal{F}_t).$$

Moreover, if  $\gamma$  is directed upward and  $u = T$  then

$$\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t = E(\text{esssup}_i (\gamma_i)_T | \mathcal{F}_t).$$

*Proof.* Let us consider an arbitrary selector  $\xi_t^* \in L^\infty(G_t^*, \mathcal{F}_t)$ . Then, for all  $i$ ,  $E(\text{esssup}_i (\gamma_i)_u | \mathcal{F}_t) \xi_t^* = E((\text{esssup}_i (\gamma_i)_u) \xi_t^* | \mathcal{F}_t) \geq E(\gamma_i \xi_t^* | \mathcal{F}_t) = E(\gamma_i | \mathcal{F}_t) \xi_t^*$ .

From there, we get that

$$E(\text{esssup}_i (\gamma_i)_u | \mathcal{F}_t) \geq_{G_t} E(\gamma_i | \mathcal{F}_t), \quad \forall i \in I,$$

and we conclude that

$$E(\text{esssup}_i (\gamma_i)_u | \mathcal{F}_t) \geq_{G_t} \text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t.$$

By virtue of Lemma 5.8, recall that

$$\text{esssup}_i(\gamma_i)_u = \text{esssup}_{\bar{\gamma} \in \bar{\gamma}^{up}}(\bar{\gamma})_u.$$

Assume that the inequality

$$E(\text{esssup}_i(\gamma_i)_T | \mathcal{F}_t) \leq_{G_t} \text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t$$

does not hold on a non null set  $\Lambda_t \in \mathcal{F}_t$ .

Then, there exists a generator  $\xi_t^* \in L^0(G_t^*, \mathcal{F}_t)$  such that the following inequality holds on a subset of  $\Lambda_t$ .

$$\begin{aligned} E(\text{esssup}_i(\gamma_i)_T | \mathcal{F}_t) \xi_t^* &> (\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t) \xi_t^* \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t), \\ E((\text{esssup}_i(\gamma_i)_T) \xi_t^* | \mathcal{F}_t) &> (\text{esssup}_i E(\gamma_i | \mathcal{F}_t)_t) \xi_t^* \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t). \end{aligned}$$

By virtue of Lemma 5.7, we have on  $\Lambda_t$ ,

$$E(\text{esssup}_i(\gamma_i \xi_t^*)_T | \mathcal{F}_t) > \text{esssup}_i E(\gamma_i \xi_t^* | \mathcal{F}_t)_t \geq E(\gamma_i \cdot \xi_t^* | \mathcal{F}_t).$$

But the family  $(\gamma_i \xi_t^*)_i$  is directed upwards by assumption and takes real positive values. By virtue of Proposition A.3.1 [3], we then deduce that

$$E(\text{esssup}_i(\gamma_i \xi_t^*)_T | \mathcal{F}_t) = \text{esssup}_i E(\gamma_i \xi_t^* | \mathcal{F}_t)_t$$

which yields a contradiction.  $\square$

## 6 Generalization and applications

We assume given  $t \in [0, T]$  and a family of  $\mathcal{F}_{t_i}$ -adapted closed convex cones  $\tilde{\mathcal{G}} := (\tilde{G}_{t_i})_{i \in I}$ ,  $t_i \in [t, T]$ . We consider a family  $(\gamma_i)_{i \in I} \in L^0(\mathbf{R}, \mathcal{F}_T)$ . We define the generalized Essential supremum at date  $u \in [t, T]$  as the class  $\Gamma^* := \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  (eventually empty) of  $\mathcal{F}_u$ -adapted random selectors  $\gamma^*$  satisfying

$$\gamma^* \geq_{\tilde{G}_{t_i}} \gamma_i, \quad \forall i \in I, \quad (6.10)$$

$$\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_u), \quad \gamma \geq_{\tilde{G}_{t_i}} \gamma_i, \quad \forall i \in I, \Rightarrow \exists \gamma^* \in \Gamma^* \text{ s.t. } \gamma \geq_{G_t} \gamma^* \quad (6.11)$$

$$\gamma_1^*, \gamma_2^* \in \Gamma^*, \quad \gamma_1^* \neq \gamma_2^* \Rightarrow \gamma_1^* - \gamma_2^* \notin G_t \setminus \{0\} \text{ a.s.} \quad (6.12)$$

**Remark 6.1** Similarly, we may define  $\text{Essinf}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  replacing the inequality  $\gamma^* \geq_{\tilde{G}_{t_i}} \gamma_i$  by  $\gamma^* \leq_{\tilde{G}_{t_i}} \gamma_i$  and in (6.11) changing  $\gamma \geq_{G_t} \gamma^*$  into  $\gamma \leq_{G_t} \gamma^*$ .

Following the proofs of Section 3, we get similar results:

**Lemma 6.2** Let us consider a collection  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{\tilde{G}_{t_i}} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $t_{j_0} = t$  and  $\tilde{G}_{t_{j_0}} \subseteq G_t$ . Then,  $\text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$  is not empty.

**Theorem 6.3** *Assume that  $G_t$  is generated by a finite number of linearly independent generators. Let us consider a family  $(\gamma_i)_{i \in I} \subset L^0(\mathbf{R}^d, \mathcal{F}_T)$  such that there exists  $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F}_u)$  satisfying  $\gamma_i \leq_{G_{t_i}} \bar{\gamma}$ ,  $\forall i \in I$ . Suppose there exists  $j_0 \in I$  such that  $t_{j_0} = t$  and  $\tilde{G}_{t_{j_0}} \subseteq G_t$ . Then,  $\text{Essup}_i^{G_t}(\gamma_i, G_{t_i})_u$  is not empty and equals the singleton  $\text{essup}_i^{G_t}(\gamma_i, G_{t_i})_u$ .*

*Proof.* By virtue of Lemma 6.2, consider  $\gamma_1, \gamma_2 \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u \neq \emptyset$ . Assume that  $\gamma_1 \neq \gamma_2$  so that  $\gamma_2 - \gamma_1, \gamma_1 - \gamma_2 \notin G_t \setminus \{0\}$  a.s. Without loss of generality, we assume that  $E(\gamma_{j_0} | \mathcal{F}_u) = 0$ . Then, there exists coefficients  $\alpha_j, \beta_j \in L^0(\mathbf{R}_+, \mathcal{F}_u)$  such that

$$\gamma_1 = \sum_{j=1}^N \alpha_j \xi_t^j, \quad \gamma_2 = \sum_{j=1}^N \beta_j \xi_t^j$$

where  $(\xi_t^j) \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  are the linear independent generators of  $G_t$ . Suppose that  $\alpha_1 > \beta_1$  on a non null set. Then, let us define  $\gamma_1^{(1)} := \gamma_1$  and

$$\gamma_2^{(1)} := \gamma_2 I_{\{\alpha_1 > \beta_1\}} + \gamma_1 I_{\{\alpha_1 \leq \beta_1\}} \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u.$$

Otherwise, we set  $\gamma_1^{(1)} := \gamma_2$  and  $\gamma_2^{(1)} := \gamma_1$ . We then deduce that

$$\gamma_1^{(1)} = \sum_{j=1}^N \alpha_j^{(1)} \xi_t^j, \quad \gamma_2^{(1)} = \sum_{j=1}^N \beta_j^{(1)} \xi_t^j$$

with  $\alpha_1^{(1)} \geq \beta_1^{(1)}$ . Assume we have already defined  $(\gamma_1^{(j)}, \gamma_2^{(j)}) \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u$ , for  $j \leq N-1$ , verifying  $\alpha_k^{(j)} \geq \beta_k^{(j)}$  for all  $k \leq j$ . Suppose that  $\beta_{j+1}^{(j)} > \alpha_{j+1}^{(j)}$  on a non null set. We then define  $\gamma_1^{(j+1)} := \gamma_1^{(j)}$  and

$$\gamma_2^{(j+1)} := \gamma_1^{(j)} I_{\{\beta_{j+1}^{(j)} > \alpha_{j+1}^{(j)}\}} + \gamma_2^{(j)} I_{\{\beta_{j+1}^{(j)} \leq \alpha_{j+1}^{(j)}\}} \in \text{Essup}_i^{G_t}(\gamma_i, \tilde{G}_{t_i})_u.$$

Otherwise, we set  $(\gamma_1^{(j+1)}, \gamma_2^{(j+1)}) := (\gamma_1^{(j)}, \gamma_2^{(j)})$ . We get that  $\alpha_k^{(j+1)} \geq \beta_k^{(j+1)}$  for all  $k \leq j+1$ . In particular, with  $j = N-1$ , we deduce that  $\gamma_1^{(N)} \geq_{G_t} \gamma_2^{(N)}$ . But  $\gamma_1^{(N)}, \gamma_2^{(N)} \in \text{Essup}_i^{G_t}(\gamma_i)_u$  implies that  $\gamma_1^{(N)} = \gamma_2^{(N)}$  hence we have the equality chain  $\gamma_1^{(N-1)} = \gamma_2^{(N-1)}, \dots, \gamma_1^{(1)} = \gamma_2^{(1)}$  and finally  $\gamma_1 = \gamma_2$ .  $\square$

## 6.1 Superhedging of American options

Let us consider the discrete-time financial model of Kabanov with proportional transaction costs (see [3] for more details). We are given random cones  $(G_t)_{0 \leq t \leq T}$  satisfying the hypothesis given in introduction. We interpret them as solvency cones, i.e.  $G_t$  is the set of all portfolio positions we can change into positive ones (i.e. in  $\mathbf{R}_+^d$ ) paying transaction costs. A portfolio process  $(V_t)_{0 \leq t \leq T}$  expressed in physical units is a process with the dynamics:



$\Delta V_t := V_t - V_{t-1} \in -G_t$  a.s. The goal is the following: given an American claim  $(h_t)_{0 \leq t \leq T}$ , find a “minimal” portfolio  $(V_t)_{0 \leq t \leq T}$  such that  $V_t \geq_{G_t} h_t$ , for all  $t$ , and  $V_T = h_T$ . Such a portfolio  $V$  is said “minimal” in the sense that if  $W$  is another one satisfying the same property, then  $W_t \geq_{G_t} V_t$  for all  $t$ . Throughout this section, we suppose that the random cones  $G$  and  $G^*$  are generated by linear independent generators. This is the case when  $d = 2$  and the transaction costs coefficients are strictly positive, [3]. We define a “Snell envelop” as follows.

**Proposition 6.4** *Assume there exists  $k \geq 0$  such that  $h_t \leq_{G_t} k \mathbf{1}_d$  for all  $t$ . Then, there exists a unique minimal portfolio process  $(V_t)_{0 \leq t \leq T}$  such that  $V_t \geq_{G_t} h_t$ , for all  $t$ , and  $V_T = h_T$ . Moreover,*

$$V_t = \text{essup}^{G_t} ((h_t, G_t), (V_{t+1}, G_{t+1}))_t, \quad t \leq T - 1.$$

*Proof.* We set  $V_T = h_T$  and inductively we define  $V_t := \text{essup}(h_t, V_{t+1})_t$  where the essential supremum is understood in the generalized sense of this section;  $h_t$  is associated to the cone  $G_t$  and  $V_{t+1}$  is associated to  $G_{t+1}$ . It follows that  $V_t$  is  $\mathcal{F}_t$ -adapted and satisfies  $V_t \geq_{G_t} h_t$ . Since, we also have  $V_t \geq_{G_{t+1}} V_{t+1}$ , it follows that  $\Delta V_{t+1} \in -G_{t+1}$ , i.e.  $(V_t)_{0 \leq t \leq T}$  is a portfolio process. Moreover, consider any portfolio  $W$  process super-replicating  $(h_t)_{0 \leq t \leq T}$ . We necessary have  $W_t \geq_{G_t} h_t$  and  $W_T = h_T$ . From the dynamics  $\Delta W_{t+1} := W_{t+1} - W_t \in -G_{t+1}$ , we also deduce that  $W_t \geq_{G_{t+1}} W_{t+1}$ . By backward induction and definition, it follows that  $W_t \geq_{G_t} V_t$  for all  $t$ .  $\square$

## 6.2 Dynamic risk measure

Assume we are given two constant closed and proper cones  $K$  and  $G$  generated by linear independent generators of  $\mathbf{R}^d$ . We suppose that  $K \subset G$  and  $\text{int } K \neq \emptyset$ . We also suppose that  $\mathbf{R}_+^d \subseteq G$ . In this section,  $G$  is interpreted as the solvency cone in the continuous-time model of Campi–Schachermayer as described in Section 3.6.6 [3]. We are given discounted prices  $(S^i)_{1 \leq i \leq d}$  verifying  $S^i = 1$  we interpret as currencies like Dollars, Euros, Yen, etc. The constant proportional transaction costs coefficients  $(\lambda^{i,j})_{i,j}$  of the model are interpreted as the exchange rate between the currency number  $i$  and number  $j$ . Therefore,  $G$  is defined as, [3],

$$G := \text{cone}\{(1 + \lambda^{i,j})e_i - e_j : 1 \leq i, j \leq d\}.$$

The problem is the following: we are given a vector-valued process  $X$  with  $X_t \in L^0(\mathbf{R}^d, \mathcal{F}_T)$  which models the value at date  $t$  of some multi-assets financial portfolio, the investment of a company. A dynamic measure of risk  $(\rho_t)$  associates to  $X$  the set of  $\mathcal{F}_t$ -adapted portfolios of reference instruments  $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  which, when added to  $X_t$ , make the total position  $X_t + x_t$  acceptable by the regulator/supervisor. In this example, we suppose that the acceptable positions are given by the cones  $K$ , we call the security cone. The process  $(\rho_t(X))$  is defined as a set-valued mapping but, as in the previous example, there exists a “minimal” selector of the risk measure which turns out to be also a risk-measure according to the definition we shall recall below. Let us before show the following.

**Proposition 6.5** *Suppose that  $h$  is an adapted process and there is  $k \geq 0$  such that  $h_t \leq_K k \mathbf{1}_d$  for all  $t \in [0, T]$ . Then, there exists a unique minimal process  $(V_t)_{t \in [0, T]}$  such that  $V_t \geq_K h_t$ , for all  $t$ ,  $V_T = h_T$  and  $V_{t_2} - V_{t_1} \in -G$  for all  $t_1 \leq t_2$ . Moreover,  $V$  is left-continuous and of finite variations. If  $W$  is another such process with only  $W_T \geq_K h_T$ , then necessarily we have  $W \geq_G V$ .*

*Proof.* We consider the refining sequence

$$\Pi^n := \{t_i^n := Ti/2^n : i = 0, \dots, 2^n\}$$

of partitions of  $[0, T]$  and set  $\Pi = \cup_n \Pi^n$ . Like in Proposition 6.4, we deduce, for each  $n \in \mathbf{N}$ , the minimal portfolio process  $(V_{t_i^n}^{(n)})$  super-replicating  $(h_{t_i^n})$  at dates  $t \in \Pi^n$  in the following sense:

$$\begin{aligned} V_{t_i^n}^{(n)} &:= \operatorname{esssup}^G \left( (h_{t_i^n}, K), (V_{t_{i+1}^n}^{(n)}, G) \right)_{t_i^n}, \quad i \leq 2^n - 1, \\ V_T^{(n)} &:= h_T. \end{aligned}$$

Observe that by definition  $V_{t_i^n}^{(n+1)} \geq_G V_{t_{2i+1}^{n+1}}^{(n+1)}$  and  $V_{t_{2i+1}^{n+1}}^{(n+1)} \geq_G V_{t_{i+1}^n}^{(n)}$ . It follows that  $V_{t_i^n}^{(n+1)} \geq_G V_{t_{i+1}^n}^{(n)}$ . By virtue of the definition of  $V^{(n)}$  as a minimal process, we then deduce that  $k \mathbf{1}_d \geq_G V_t^{(n+1)} \geq_G V_t^{(n)}$  for any  $t \in \Pi^n$ . We then define  $V_t^{(\infty)}$ , for each  $t \in \Pi$ , as the increasing limit of the sequence  $(V_t^{(n)})_n$  with respect to  $G$  or equivalently as the  $\mathcal{F}_t$ -adapted essential supremum of the family  $(V_t^{(n)})_n$  with respect to the cone  $G$ . Then, we put  $V_t := \lim_{r \uparrow t, r \in \Pi} V_r^{(\infty)}$  which is left-continuous. Observe that  $V_t$  is a decreasing limit with respect to  $G$ . Indeed,  $V_{t_1}^{(\infty)} - V_{t_2}^{(\infty)} \in G$  if  $t_1 \geq t_2$  with  $t_1, t_2 \in \Pi$ . We then deduce that  $V$  is an adapted process. By virtue of Lemma 7.1, we also deduce that  $V$  is of finite variations, i.e.  $V$  is a portfolio process according to Lemma 3.6.12 [3] with respect to the constant cone  $G$ . Finally, consider any portfolio process  $W$   $K$ -superhedging the payoff  $h$ . Then, the discrete-time process  $(W_{t_i^n})_i$  verifies  $W_{t_i^n} \geq_K h_{t_i^n}$  and  $W_{t_i^n} \geq_G W_{t_{i+1}^n}$  by assumption. It follows that  $W_t \geq_K V_t^{(n)}$  for all  $t \in \Pi$  and  $n$  such that  $t \in \Pi^n$ . As  $n \rightarrow \infty$ , we obtain  $W_t \geq_K V_t^{(\infty)}$  for all  $t \in \Pi$ . Moreover, if  $W$  is left-continuous we get that  $W_t \geq_{G_t} V_t$  for all  $t$ .  $\square$

Throughout the sequence, we suppose that  $X \in D_\rho$ , i.e.  $X_t \geq -k \mathbf{1}_d$  for all  $t \in [0, T]$  where  $k > 0$ . Applying the last lemma to  $h = -X$ , we then deduce a unique process  $\rho_t(X)$  satisfying  $\rho_t(X) + X_t \in K$  which is minimal according to the cone process  $G$ . The dynamic  $\rho_{t_2}(X) - \rho_{t_1}(X) \in -G$  it satisfies means that  $\rho(X)$  is a self-financed portfolio subjected to the transaction costs given by  $(\lambda^{i,j})_{i,j}$ , see [3]. We then easily show the following:

**Proposition 6.6** *The mapping  $\rho$  is a coherent risk measure, i.e. satisfies the following statements:*

$$R0.) \quad \rho_t(0) = 0.$$

- R1.) If  $X \geq_K 0$ , then  $\forall t, \rho_t(X) \leq_{G_t} 0$  a.s.  
R2.) If  $X, Y \in D_\rho$  then  $\forall t, \rho_t(X + Y) \leq_{G_t} \rho_t(X) + \rho_t(Y)$  a.s.  
R3.) If  $\lambda \in \mathbf{R}_+$  and  $X \in D_\rho$  then  $\rho_t(\lambda X) = \lambda \rho_t(X)$  a.s.  
R4.) If  $X \in D_\rho$  and  $a \in \mathbf{R}$ , then  $\rho_t(X + \bar{a}) = \rho_t(X) + \{-a\}$ .

### 6.3 Skorohod problem with oblique reflection

Assume we are given two constant closed and proper cones  $K$  and  $G$  generated by linear independent generators of  $\mathbf{R}^2$  with  $K \subset G$ ,  $\partial K \subseteq \text{int } G$  and  $\text{int } K \neq \emptyset$ . We suppose that  $\mathbf{R}_+^2 \subseteq G$ . Let us consider a càdlàg (right-continuous and left-limited) and deterministic function  $(X_t)_{t \geq 0}$  with  $X_0 \in \text{int } K$ .

The goal is to find a left-continuous and right-limited function  $(Y_t)_{t \geq 0}$  of finite variations such that  $Z_t = X_t + Y_t$  verifies:

$$\begin{aligned} Z_t &\in G, Z_{t+} \in K \text{ a.s.}, \\ \dot{Y}_t, \Delta^+ Z_t &\in -\partial G, \\ \Delta^+ Z_t \neq 0 &\Leftrightarrow Z_t \notin K \end{aligned}$$

where  $\dot{Y}_t$  is the option version of the Radon-Nikodym derivative  $dY^c/d\text{Var}(Y^c)$ ,  $Y^c$  is the continuous part of  $Y$  and  $\text{Var}(Y^c)$  its total variation.

**Definition 6.7** A process  $A$  is said  $G$ -increasing if  $A_t \geq_G A_s$  for  $t \geq s$ .

We consider the refining sequence  $\Pi^n := \{t_i^n := Ti/2^n : i = 0, \dots, 2^n\}$  of partitions of  $[0, T]$  and denote  $\Pi = \cup_{n \geq 1} \Pi^n$ . We say that  $h$  is a càdlàg (respectively làdcàg)  $\Pi^n$ -process if  $h$  is constant on each interval  $[t_i^n, t_{i+1}^n[$  (respectively on  $]t_i^n, t_{i+1}^n]$ ). Moreover, a left-continuous process  $(V_t)_{t \leq T}$  is assumed to have a right limit at date  $T$  we denote by  $V_{T+}$  and verifies  $V_0 = V_{0+}$ . We recall in Appendix the projection of a point  $x \in G$  onto  $K$  parallel to  $-G$  we denote by  $\mathbb{P}_K^{-G}(x)$ . Given  $\vec{k} \in \text{int } K$  we fix  $r_k > 0$  such that  $B(\vec{k}, r_k) \subseteq K$ . We denote  $\vec{k}^* := \vec{k}/r_k$ .

#### Lemma 6.8

Let us consider a function  $(X_t)_{t \leq T}$  and  $\|X\|_u := \sup_{t \leq T} \|X_t\|$ . Then,

$$-\vec{k}^* \|X\|_u \leq_K X \leq_K \vec{k}^* \|X\|_u.$$

*Proof.* Since  $B(\vec{k}, r_k) \subseteq K$ , we get that  $B(\vec{k}^*, 1) \subseteq K$ . It follows that  $B(0, \|X\|_u) \subseteq K - \vec{k}^* \|X\|_u$ . Since,  $X$  and  $-X$  evolves in  $B(0, \|X\|_u)$  we easily conclude.  $\square$

Observe that there exists a constant  $\beta_K$  only depending on the cone  $K$  and  $\vec{k}^*$  such that  $-\vec{k}^* \leq_K x \leq_K \vec{k}^*$  implies  $\|x\| \leq \beta_K$ .

Our main result is:

**Theorem 6.9** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg function. Then, there exists a làdcàg function  $\Sigma_1(X) := (V_t)_{t \leq T}$  of finite variations which is  $G$ -decreasing such that  $\Sigma_2(X) := X + V \in G$ ,  $\Sigma_2(X)_+ \in K$ ,  $\|V\|_u \leq \beta_K \|X\|_u$  and  $\Delta^+ V \in -\partial G$ . Moreover,  $\dot{V} = dV^c/d\text{Var}(V^c) \in -\partial G$  and  $\dot{V} = \dot{V} \mathbf{1}_{\Sigma_2(X) \notin \text{int } K}$ .*

The proof follows from the following result.

**Theorem 6.10** *Suppose that  $h$  is a deterministic càdlàg function. Suppose  $h_t \leq_K \vec{k}$  for all  $t \in [0, T]$  where  $\vec{k} \in K$ . Then, there exists a làdcàg function  $(V_t)_{t \leq T}$  such that  $V_t \geq_K h_t$ , for all  $t$ . Moreover,  $V$  is of finite variations with  $\Delta^+ V \in -\partial G$  and  $\dot{V}^c \in -\partial G$ . If  $W$  is another such process, then necessarily we have  $W \geq_G V$ .*

To give the proof, we need the following lemmas.

**Proposition 6.11** *Suppose that  $(h_t)_{t \in \Pi^n}$  is a discrete-time process and there is  $\vec{k} \in K$  such that  $h_t \leq_K \vec{k}$  for all  $t \leq T$ . Then, there exists a process  $Z(h) := (Z_t)_{t \in \Pi^n}$  such that  $Z \geq_K h$  and  $Z_{t_{i+1}^n} - Z_{t_i^n} \in -G$  for all  $i \leq 2^n - 1$ . If  $W$  is another such process, then necessarily we have  $W \geq_G V$ .*

*Proof.* Like in Proposition 6.4, it suffices to set  $Z_T = h_T$  and

$$Z_{t_i^n} := \text{essup}^G((h_{t_i^n}, K), (Z_{t_{i+1}^n}, G))_{t_i^n}, \quad i \leq 2^n - 1.$$

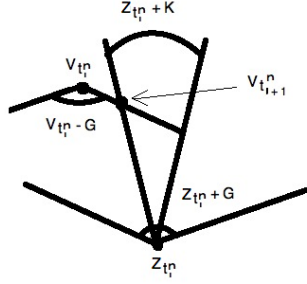
□

**Proposition 6.12** *Suppose that  $(h_t)_{t \leq T}$  is a càdlàg  $\Pi^n$ -process and there is  $\vec{k} \in K$  such that  $h_t \leq_K \vec{k}$  for all  $t \leq T$ . Then, there a unique minimal làdcàg  $\Pi^n$ -process  $(V_t)_{t \leq T}$  such that  $V \geq_G h$ ,  $V_0 = V_{0+}$ ,  $\vec{k} \geq_G V_+ \geq_K h$  and  $\Delta^+ V \in -\partial G$ ,  $V_{t_{i+1}^n} - h_{t_i} = \mathbb{P}_K^{-G}(V_{t_i} - h_{t_i})$ . If  $W$  is another such process, we have  $W \geq V$ .*

*Proof.* Consider the minimal  $G$ -decreasing process  $(Z_t)_{t \in \Pi^n}$  we get using Proposition 6.11. Let us set  $V_0 = V_{0+} := Z_0 := V_{t_1^n}$  and let us define recursively  $V_{t_{i+1}^n} = V_{t_i^n+}$  as follows. Suppose we have already defined  $V_{t_i^n} = V_{t_{i-1}^n+}$  such that  $V_{t_i^n} \geq_K h_{t_{i-1}^n}$  and  $V_{t_i^n} \geq_G Z_{t_{i-1}^n}$ . Then,  $V_{t_i^n} \geq_G Z_{t_i^n} \geq_G h_{t_i^n}$  so that we may define

$$V_{t_{i+1}^n} = \text{essup}^{-G}((h_{t_i^n}, K); (V_{t_i^n}, -G))_{t_i^n} := \text{essinf}^G((h_{t_i^n}, -K); (V_{t_i^n}, G))_{t_i^n}.$$

Observe that we have  $V_{t_{i+1}^n} - V_{t_i^n} \in -\partial G$ , i.e.  $V_{t_{i+1}^n} = \mathbb{P}_{K+h_{t_i}}^{-G}(V_{t_i^n})$ :



To see the uniqueness, let us set  $U_t := V_t - h_t$ ,  $t \in \Pi^n$ . Observe that it is equivalent to define  $U$  instead of  $V$  as follows: the discrete-time process  $U$  is defined as  $U_0 = U_{0+} = Z_0 - h_0$  and

$$U_{t_i-} := U_{t_{i-1}+}, \quad U_{t_i} := U_{t_i-} - \Delta h_{t_i} \in G, \quad U_{t_i+} := \mathbb{P}_K^{-G}(U_{t_i}).$$

Consider another process  $W$  satisfying the same properties than  $V$ . We first get that  $W_+ \geq_G Z$  so that  $W_{t_1} \geq_G V_{t_1}$ . We then consider  $\tilde{U}$  the process associated to  $W$  which is defined as  $U$  and satisfies the same properties. By induction, we get that  $\tilde{U} \geq_G U$  so that  $W \geq_G V$ .  $\square$

**Corollary 6.13** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg  $\Pi^n$ -process. Then, there is a  $G$ -minimal ladcàg  $\Pi^n$ -process  $\Sigma_1(X) := (V_t)_{t \leq T}$  with  $\Sigma_1(X)_0 = Z(-X)_0$ ,  $\Sigma_2(X) := X + \Sigma_1(X) \in G$ ,  $\Sigma_2(X)_+ = \mathbb{P}_K^{-G}(\Sigma_2(X)) \in K$  and  $\Delta^+ V \in -\partial G$ . Moreover, we have  $\|\Sigma_1(X)\|_u \leq \beta_K \|X\|_u$  and  $\Delta^+ V_t \neq 0$  if and only if  $\Sigma_2(X)_t \notin \text{int } K$ .*

*Proof.* It suffices to apply Proposition 6.12 and Lemma 6.8.  $\square$

**Corollary 6.14** *Suppose that  $(X_t)_{t \leq T}$  is a càdlàg function. Then, there exists a ladcàg function  $\Sigma_1(X) := (\bar{V}_t)_{t \leq T}$  of finite variations which is  $G$ -decreasing such that  $\Sigma_2(X) := X + \Sigma_1(X) \in G$ ,  $\Sigma_2(X)_+ \in K$ ,  $\|V\|_u \leq C \|X\|_u$  and  $\Delta^+ V \in -\partial G$ . Moreover,*

$$\dot{V} = dV^c/d\text{Var}(V^c) \in -\partial G, \quad \dot{V} = \dot{V} \mathbf{1}_{\Sigma_2(X) \notin \text{int } K}.$$

*Proof.* Consider the càdlàg function  $X$  as an element of the Skorokhod space  $\mathbb{D}[0, T]$  endowed with the usual Skorokhod topology. In the following, we use the notations of [4] Chapter VI. Consider a sequence of càdlàg  $\Pi^n$ -functions  $X^n$  such that  $X^n$  converges uniformly to  $X$  on  $[0, T]$  and  $(\Pi^n)$  is a refining sequence of partitions. For each  $n$ , Corollary 6.13 ensures the existence of  $(\Sigma_1(X^n), \Sigma_2(X^n))$  solving the Skorokhod problem. Observe the inequalities  $\sup_n \|\Sigma_2(X^n)\|_u < \infty$  and  $\sup_n \|\Sigma_1(X^n)\|_u < \infty$  since  $\sup_n \|X^n\|_u < \infty$ . Recall that for each  $n$ ,  $\Sigma_1(X^n)$  is a  $G$ -decreasing process. Using Lemma 3.6.13

[3] (or Lemma 7.1 of this paper) and Corollary 6.13, the Helly theorem asserts that for any sequence of  $\mathcal{A} := \{\text{Var}(\Sigma_1(X^n)_+)\}$  there exists a subsequence  $\alpha_n \in \mathcal{A}$  such that  $\alpha_n$  converges pointwise to  $\alpha \in \mathbb{D}[0, T]$  at each point  $t \in D$  of continuity of  $\alpha$ . It is straightforward that for any  $t \in D$ , there exists  $t_n \leq t$ , with  $t_n \rightarrow t$ , such that  $\Delta\alpha_n(t_n) \rightarrow \Delta\alpha(t) = 0$  since  $\alpha_n$  is a stepwise function. Moreover  $D$  is dense in  $[0, T]$  as a complement of a countable subset of  $[0, T]$ . By virtue of Lemma VI-2.25 [4], we deduce that  $\alpha_n$  converges to  $\alpha \in \mathbb{D}[0, T]$  for the Skorokhod topology. We then deduce that  $\mathcal{A}$  is relatively compact, i.e.  $\limsup_{\theta \downarrow 0} \sup_{\alpha \in \mathcal{A}} w'_T(\alpha, \theta) = 0$ . We also have  $\limsup_{\theta \downarrow 0} \sup_n w'_T(X^n, \theta) = 0$ . Since  $\Delta\Sigma_2(X^n)_+ = \Delta X^n + \Delta\Sigma_1(X^n)_+$ , we then deduce that

$$\begin{aligned} w'_T(\Sigma_2(X^n)_+, \theta) &\leq w'_T(X^n, \theta) + w'_T(\Sigma_1(X^n)_+, \theta) \\ &\leq \sup_n w'_T(X^n, \theta) + w'_T(\text{Var}(\Sigma_1(X^n)_+), \theta) \\ &\leq \sup_n w'_T(X^n, \theta) + \sup_{\alpha \in \mathcal{A}} w'_T(\alpha, \theta). \end{aligned}$$

From there, we get that  $\limsup_{\theta \downarrow 0} \sup_n w'_T(\Sigma_2(X^n)_+, \theta) = 0$  and finally the family  $(\Sigma_2(X^n)_+)$  is relatively compact for the Skorokhod topology. We deduce that  $(\Sigma_2(X^n)_+)$  converges to  $\Sigma_2(X)_+ \in \mathbb{D}[0, T]$  for a subsequence. We then set  $\Sigma_2(X) := \Sigma_2(X)_- + \Delta X$ . By virtue of Proposition VI-2.1 [4], for each  $t \in [0, T]$  there exists  $t_n \rightarrow t$  such that  $\Sigma_2(X^n)_+(t_n) \rightarrow \Sigma_2(X)_+(t)$ ,  $\Sigma_2(X^n)_-(t_n) \rightarrow \Sigma_2(X)_-(t)$  and  $\Delta\Sigma_2(X^n)_+(t_n) \rightarrow \Delta\Sigma_2(X)_+(t)$ . Recall that by assumption,  $\Delta X^n$  converges uniformly to  $\Delta X$ . It follows that

$$\Sigma_2(X^n)(t_n) = \Sigma_2(X^n)_-(t_n) + \Delta X^n(t_n) \rightarrow \Sigma_2(X)(t) \in G.$$

Since  $\Sigma_2(X^n)_+(t_n) = \mathbb{P}_K^{-G}(\Sigma_2(X^n)(t_n))$ , we deduce by continuity that

$$\Sigma_2(X)_+(t) = \mathbb{P}_K^{-G}(\Sigma_2(X)(t)) \in K.$$

We note  $\Sigma_1(X) := \Sigma_2(X) - X$ . From above, we have

$$\Sigma_1(X^n)(t_n) = \Sigma_2(X^n)(t_n) - X^n(t_n) \rightarrow \Sigma_1(X).$$

Since  $\Delta\Sigma_2(X) = \Delta X$ , we deduce that  $\Delta\Sigma_1(X) = 0$  i.e.  $\Sigma_1(X)$  is left-continuous. It is clear that  $\Sigma_1(X)$  is  $G$ -decreasing hence of finite variations:

$$\Sigma_1(X)_t = Z(-X)_0 + \int_0^t \alpha_u d\text{Var} \Sigma_1^c(X)_u + \sum_{s \leq t} \Delta\Sigma_1^+(X)_s$$

where  $\alpha$  is the optional Radon-Nikodym derivative  $d\Sigma_1^c(X)/d\text{Var} \Sigma_1^c(X)$ . From the equalities  $\Delta\Sigma_2(X^n)_+(t_n) = \Delta X^n(t_n) + \Delta^+\Sigma_1(X^n)(t_n)$  and  $\Delta\Sigma_2(X)_+(t) = \Delta X(t) + \Delta^+\Sigma_1(X)(t)$  we deduce that  $\Delta^+\Sigma_1(X)(t) \in -\partial G$  as  $n \rightarrow \infty$  for all  $t \leq T$ . Assume that  $\Delta^+\Sigma_1(X)(t) \neq 0$ . Then, we also have  $\Delta^+\Sigma_1(X^n)(t_n) \neq 0$  if  $n$  is large enough so that  $\Sigma_2(X^n)(t_n) \notin \text{int} K$  hence  $\Sigma_2(X)(t) \notin \text{int} K$ . At last, we may assume, [2], that

$$\alpha = \limsup_j \sum_k \frac{\Sigma_1(X)_{t_{k+1}^j} - \Sigma_1(X)_{t_k^j}}{\text{Var}(\Sigma_1(X)_+)_{t_{k+1}^j} - \text{Var}(\Sigma_1(X)_+)_{t_k^j}} \mathbf{1}_{]t_k^j, t_{k+1}^j]}$$

where  $(t_k^j)$  is the same refining sequence of partitions of  $[0, T]$  we use to approximate  $X$ . By construction, we have  $\Sigma_1(X)_{t_{k+1}^j+} - \Sigma_1(X)_{t_k^j+} \in -G$ . Moreover, since  $\Sigma_1(X)_{t_{k+1}^j+} - \Sigma_1(X)_{t_k^j+} \rightarrow \Delta^+ \Sigma_1(X)$ , it follows that

$$\Sigma_1(X)_{t_{k+1}^j+} - \Sigma_1(X)_{t_k^j+} \geq_G \Delta^+ \Sigma_1(X) - \epsilon \mathbf{1}$$

if  $j$  is large enough and  $\epsilon$  is chosen arbitrarily small. Since  $\Delta^+ \Sigma_1(X) \in -\partial G$ , we then deduce as  $\epsilon \rightarrow 0$  that  $\alpha \in (-G) \cap (-\partial G + G) = -\partial G$ . Assume that  $\alpha \neq 0$ , then  $\Sigma_1(X)_{t_{k+1}^j+} - \Sigma_1(X)_{t_k^j+} \neq 0$  if  $j$  is large enough. We deduce  $\tilde{t}_k^j$  such that  $\Sigma_1(X)_{t_{k+1}^j+} - \Sigma_1(X)_{\tilde{t}_k^j} \neq 0$  where  $t_{k+1}^j > \tilde{t}_k^j > t_k^j$ . On the other hand, by virtue of Proposition V1-2.1, we get  $t_n^j$  with  $t_n^j < t_{k+2}^j$  and  $s_n^j$  with  $t_k^j < s_n^j$  converging respectively to  $t_{k+1}^j$  and  $\tilde{t}_k^j$  such that

$$\Sigma_1(X^n)_{t_{k+1}^j+} - \Sigma_1(X^n)_{t_k^j+} = \Sigma_1(X^n)_{t_n^j+} - \Sigma_1(X^n)_{s_n^j+} \neq 0.$$

Observe we necessarily have  $t_n^j > t_{k+1}^j$ . By construction, it follows that  $\Sigma_2(X^n)_{t_n^j} = \Sigma_2(X^n)_{t_{k+1}^j} \notin \text{int } K$ . Taking the limit, we get that

$$\Sigma_2(X)_+(t) \in (\mathbb{R}^2 \setminus \text{int } K) \cap K = \partial K.$$

Since  $\Sigma_2(X)_+(t) = \mathbb{P}_K^{-G}(\Sigma_2(X)(t))$  we then deduce that  $\Sigma_2(X)(t) \notin \text{int } K$ .  $\square$

## 7 Appendix

**Lemma 7.1** *Suppose we are given a constant cone  $G \subseteq \mathbf{R}^d$  generated by a basis of  $\mathbf{R}^d$ . Let us consider a process  $(A_t)_{t \leq T}$  verifying  $A_{t_2} - A_{t_1} \in G$  whatever  $t_2 \geq t_1$ . Then,  $A$  is of finite variations.*

*Proof.* By assumption,  $G = \text{cone}(\xi_i)_{1 \leq i \leq d}$  where  $(\xi_i)$  is a basis of  $\mathbf{R}^d$ . Let us consider the mapping

$$\Phi : \mathbf{R}^d \ni x = \sum_{i=1}^d x_i \xi_i \mapsto \Phi(x) := \sum_{i=1}^d x_i e_i \in \mathbf{R}^d$$

where  $(e_i)_i$  is the canonical basis of  $\mathbf{R}^d$ . We denote by  $\Phi^{-1}$  the inverse of  $\Phi$ . Recall that  $\Phi$  and  $\Phi^{-1}$  are linear hence continuous. Moreover  $\Phi(G) \subseteq \mathbf{R}_+^d$ . For any subdivision  $(t_i)$  of  $[0, T]$ , we get

$$\begin{aligned} \sum_i |A_{t_{i+1}} - A_{t_i}| &= \sum_i |\Phi^{-1} \circ \Phi(A_{t_{i+1}} - A_{t_i})| \leq \|\Phi^{-1}\| \sum_i |\Phi(A_{t_{i+1}} - A_{t_i})| \\ &\leq C \|\Phi^{-1}\| \sum_i \sum_{j=1}^d |\Phi^j(A_{t_{i+1}}) - \Phi^j(A_{t_i})| \end{aligned}$$

where  $\Phi^j$  is the  $j$ -th component of  $\Phi$  and  $C$  is a constant (all the norms being equivalent). By assumption,  $\Phi^j(A_{t_{i+1}}) - \Phi^j(A_{t_i}) \geq 0$  for all  $j$ . It follows that there is another constant  $C$  such that

$$\sum_i |A_{t_{i+1}} - A_{t_i}| \leq C \|\Phi(A_T - A_0)\| \leq C \|\Phi\| |A_T - A_0|.$$

We conclude that  $\text{Var}(A)_T \leq C \|\Phi\| |A_T - A_0|$ .  $\square$

In the following, we assume that  $G \subseteq \mathbf{R}^2$  is a constant cone satisfying the hypothesis of the introduction and  $K \subseteq G$  is a closed cone with  $\partial K \subseteq \text{int } G$ .

**Proposition 7.2 (Projection onto  $K$  parallel to  $-G$ )** *Given  $x \in G$ , there exists a unique  $y := \mathbb{P}_K^{-G}(x) \in K$  such that*

$$\|x - y\| = \min_{k \in K} \{\|x - k\| : x - k \in G\}.$$

We omit the proof which is standard.

**Lemma 7.3** *The mapping  $G \ni x \mapsto \mathbb{P}_K^{-G}(x) \in \mathbf{R}^d$  is continuous.*

*Proof.* We may assume that  $G = \mathbf{R}_+^2$  like in the previous lemma. Then  $\partial K = D_a \cup D_b$  where  $D_a = \{(x_1, y_1) : y_1 = ax_1, x \geq 0\}$  and  $D_b$  is defined similarly with  $b > a > 0$ . It is then easy to show the implication  $x_n \rightarrow x \Rightarrow \mathbb{P}_K^{-G}(x_n) \rightarrow \mathbb{P}_K^{-G}(x)$  since we have an explicit expression of  $\mathbb{P}_K^{-G}(x)$  if  $x \in G$ .  $\square$

## References

1. Bouchard B., Chassagneux J.F. Representation of continuous linear forms on the set of laddag processes and the pricing of American claims under proportional costs. *Electronic Journal of Probability*, (2009), 14, 612-632.
2. De Vallière D., Denis E., Kabanov Yu. Hedging of American options under transaction costs. *Finance and Stochastics*, 13 (2009), 1, 105-119.
3. Kabanov Y., Safarian M. Markets with Transaction Costs. *Mathematical Theory*. Springer-Verlag, 2009.
4. Jacod J., Shiryaev A.N. *Limit Theorems for Stochastic Processes*. Springer, Berlin-Heidelberg-New York, (1987).