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EQUIDISTRIBUTION OF EISENSTEIN SERIES ON CONVEX CO-COMPACT HYPERBOLIC MANIFOLDS

COLIN GUILLARMOU AND FRÉDÉRIC NAUD

ABSTRACT. For convex co-compact hyperbolic manifolds $\Gamma \backslash \mathbb{H}^{n+1}$ for which the dimension of the limit set satisfies $\delta_\Gamma < n/2$, we show that the high-frequency Eisenstein series associated to a point ξ “at infinity” concentrate microlocally on a measure supported by (the closure of) the set of points in the unit cotangent bundle corresponding to geodesics ending at ξ . The average in ξ of these limit measures equidistributes towards the Liouville measure.

1. INTRODUCTION

Since the early work of Schnirelman [25], Colin de Verdière [6] and Zelditch [27], it is a well known fact that on any compact Riemannian manifold X whose geodesic flow is ergodic, one can find a full density sequence $\lambda_j \rightarrow +\infty$ of eigenvalues of the Laplacian Δ_X such that the corresponding normalized eigenfunctions ψ_j are equidistributed i.e. for all $f \in L^2(X)$, we have

$$\lim_{j \rightarrow +\infty} \int_X f(z) |\psi_j(z)|^2 dv(z) = \int_X f(z) dv(z),$$

where dv is the normalized volume measure. Much less is known for non-compact manifolds and the closest analogs of the above celebrated theorem are due to Zelditch [28], Luo-Sarnak [16] and Jakobson [15] in the case of non-compact finite area hyperbolic surfaces. Let us recall their results. By \mathbb{H}^2 we denote the usual hyperbolic plane. Let $X = \Gamma \backslash \mathbb{H}^2$ be a finite area surface where Γ is a non co-compact co-finite Fuchsian group. The non compact ends of X are cusps which can be viewed on the universal cover as fixed points c_j in $\partial \mathbb{H}^2$ of parabolic elements in Γ . The number of non-equivalent cusps is finite. The spectrum of the Laplacian Δ_X consists of two types. The discrete part which corresponds to $L^2(X)$ -eigenfunctions and may be not finite as in the arithmetic case, but is conjectured by Philips-Sarnak [24] to be finite in the generic case. The absolutely continuous part $[1/4, +\infty)$ which is parametrized ($t \in \mathbb{R}$) by the finite set of Eisenstein series $E_X(1/2 + it; z, j)$ related to each cusp c_j . The Eisenstein series $E_X(1/2 + it; z, j)$ are smooth (*non* $L^2(X)$) eigenfunctions

$$\Delta_X E_X(1/2 + it; z, j) = (1/4 + t^2) E_X(1/2 + it; z, j).$$

In the Poincaré upper half-plane model, Eisenstein series are usually defined through the series (absolutely convergent for $\text{Re}(s) > 1$)

$$E_X(s; z, j) = \sum_{\gamma \in \Gamma_j \backslash \Gamma} \left(\text{Im}(\sigma_j^{-1} \gamma z) \right)^s,$$

which enjoys a meromorphic continuation to the whole complex plane and is analytic on the critical line $\{\text{Re}(s) = 1/2\}$. In the above formula, Γ_j is the stabilizer of the cusp c_j and

$\sigma_j \in \mathrm{PSL}_2(\mathbb{R})$ is such that $\sigma_j(\infty) = c_j$. In the case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, there is only one Eisenstein series given by the formula (valid for $\mathrm{Re}(s) > 1$)

$$E_X(s; x + iy) = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{y^s}{|cz + d|^{2s}}.$$

For all $t \in \mathbb{R}$, define the density μ_t by

$$\int_X a(z) d\mu_t(z) := \sum_j \int_X a(z) |E_X(1/2 + it; z, j)|^2 dv(z),$$

where $a \in C_0^\infty(X)$. If we assume that we have only finitely many eigenvalues (believed to be the generic case) then Zelditch's equidistribution result¹ is as follows: for $a \in C_0^\infty(X)$,

$$\frac{1}{s(T)} \int_{-T}^T \left| \int_X a d\mu_t - \partial_t s(t) \int_X a dv \right| dt \rightarrow 0 \quad \text{as } T \rightarrow \infty$$

where $s(t)$ is the scattering phase, this coefficient appearing as a sort of regularization of Eisenstein series due to the fact that the Weyl law involves the continuous spectrum.

On the other hand, for the modular surface $X = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$, Luo and Sarnak [16] showed that as $t \rightarrow +\infty$,

$$\int_X a d\mu_t = \frac{48}{\pi} \log(t) \int_X a dv + o(\log(t)),$$

which is a much stronger statement obtained via sharp estimates on certain L -functions. A microlocal version was later proved by Jakobson [15].

In the present paper, we focus on the case of *infinite volume hyperbolic manifolds* without cusps, more precisely convex co-compact quotients $X = \Gamma \backslash \mathbb{H}^{n+1}$ of the hyperbolic space. Let us recall some basic notations. A discrete group of orientation preserving isometries of \mathbb{H}^{n+1} is said to be convex co-compact if it admits a polygonal, finitely sided fundamental domain whose closure does not intersect the limit set of Γ . The limit set Λ_Γ and the set of discontinuity Ω_Γ are usually defined by

$$\Lambda_\Gamma := \overline{\Gamma \cdot o} \cap S^n, \quad \Omega_\Gamma := S^n \setminus \Lambda_\Gamma,$$

where $o \in \mathbb{H}^{n+1}$ is a point in \mathbb{H}^{n+1} . By a result of Patterson and Sullivan [20, 26], the Hausdorff dimension of Λ_Γ

$$\delta_\Gamma := \dim_{\mathrm{Haus}}(\Lambda_\Gamma)$$

is also the exponent of convergence of the Poincaré series, i.e. for all $m, m' \in \mathbb{H}^{n+1}$ and $s > 0$,

$$(1) \quad \sum_{\gamma \in \Gamma} e^{-sd(\gamma m, m')} < \infty \iff s > \delta_\Gamma,$$

where $d(m, m')$ denotes the hyperbolic distance.

In that case the spectrum of Δ_X has been completely described by Lax-Philips and consists of the absolutely continuous spectrum $[n^2/4, +\infty)$ and a (possibly empty) finite set of eigenvalues in $(0, n^2/4)$. Just like in the finite volume case, there is a way (see next § for full details) to define Eisenstein functions which parametrize the continuous spectrum. These functions replace in this setting the plane wave solutions $e^{\pm isz \cdot \omega}$ of $(\Delta - s^2)u = 0$ in \mathbb{R}_z^n (here $\omega \in S^{n-1}$). Let g denote the usual hyperbolic metric. Convex co-compact quotients $X = \Gamma \backslash \mathbb{H}^{n+1}$ fall in the class of conformally compact manifolds, which means that X compactifies smoothly to a

¹More details about the appropriate normalization can be found in his paper [28] where a microlocal statement is actually proved.

manifold \overline{X} with boundary $\partial\overline{X}$ such that there exists a smooth boundary defining function x with the property that x^2g extends to a smooth metric on \overline{X} and $|dx|_{x^2g} = 1$ on $\partial\overline{X}$. The boundary ∂X is called the conformal boundary and can be identified with $\Gamma\backslash\Omega_\Gamma$. The Eisenstein series² are smooth functions $E_X(s; m, \xi)$ where $s \in \mathbb{C}$ is the spectral parameter as above and $(m, \xi) \in X \times \partial\overline{X}$. They are generalized non- $L^2(X)$ eigenfunctions of the Laplacian. Using the ball model and an appropriate boundary defining function, Eisenstein series lift to \mathbb{H}^{n+1} to

$$E_X(s; m, \xi) = \sum_{\gamma \in \Gamma} \left(\frac{1 - |\gamma m|^2}{4|\gamma m - \xi|^2} \right)^s,$$

which is absolutely convergent for $\operatorname{Re}(s) > \delta_\Gamma$. Our first result is the following.

Theorem 1. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a convex co-compact quotient with $\delta_\Gamma < n/2$. Let $a \in C_0^\infty(X)$ and let $E_X(s; \cdot, \xi)$ be an Eisenstein series as above with a given point $\xi \in \partial\overline{X}$ at infinity. Then we have as $t \rightarrow +\infty$,*

$$\int_X a(m) \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m) = \int_X a(m) E_X(n; m, \xi) dv(m) + \mathcal{O}(t^{2\delta_\Gamma - n}).$$

This result shows individual equidistribution of high energy Eisenstein series. The limit measure on X is given by the harmonic density $E_X(n; m, \xi)$ whose boundary limit is the Dirac mass at $\xi \in \partial X$.

A microlocal extension of this theorem is actually proved in §3. We first need to introduce some adequate notations. Fix any $\xi \in \partial X$. Let \mathcal{L}_ξ^Γ defined by

$$\mathcal{L}_\xi^\Gamma := \overline{\cup_{\gamma \in \Gamma} \mathcal{L}_{\gamma\xi}} \subset S^*X,$$

where $\mathcal{L}_{\gamma\xi}$ are stable Lagrangian submanifolds of the unit cotangent bundle S^*X : the Lagrangian manifold $\mathcal{L}_{\gamma\xi}$ is defined to be the projection on $\Gamma\backslash S^*\mathbb{H}^{n+1}$ of

$$\{(m, \nu_{\gamma\xi}(m)) \in S^*\mathbb{H}^{n+1}; m \in \mathbb{H}^{n+1}\},$$

where $\nu_{\gamma\xi}(m)$ is the unit (co)vector tangent to the geodesic starting at m and pointing toward $\gamma\xi \in S^n$. The set \mathcal{L}_ξ^Γ “fibers” over X , and the fiber over a point $m \in X$ corresponds to the closure of the set of directions $v \in S^*X$ such that the geodesic starting at m with directions v converges to $\xi \in \partial\overline{X}$ as $t \rightarrow +\infty$. Since the closure of the orbit $\Gamma.\xi$ satisfies $\overline{\Gamma.\xi} \supset \Lambda_\Gamma$, we actually have

$$\mathcal{L}_\xi^\Gamma \supset \mathcal{T}_+ := \{(m, \nu) \in S^*X : g_t(m, \nu) \text{ remains bounded as } t \rightarrow +\infty\},$$

where $g_t : S^*X \rightarrow S^*X$ is the geodesic flow. The set \mathcal{T}_+ is often referred as the *forward trapped set*. The Hausdorff dimension of \mathcal{L}_ξ^Γ is $n + \delta_\Gamma + 1$ and satisfies $n + 1 < \delta_\Gamma + n + 1 < 2n + 1$ if Γ is non elementary.

Our phase-space statement is the following, we refer the reader to §3.2 for the required background on pseudo-differential operators.

²They depend on the boundary defining function x ; in the boundary variable ξ they should be thought of as section of a complex power of the conormal bundle to the boundary

Theorem 2. *Let A be a compactly supported 0-th order pseudodifferential operator with principal symbol ³ $a \in C_0^\infty(X, T^*X)$, then as $t \rightarrow +\infty$*

$$\left\langle AE_X\left(\frac{n}{2} + it; \cdot, \xi\right), E_X\left(\frac{n}{2} + it; \cdot, \xi\right) \right\rangle_{L^2(X)} = \int_{S^*X} a \, d\mu_\xi + \mathcal{O}(t^{-\min(1, n-2\delta_\Gamma)})$$

where μ_ξ is a g_t -invariant measure supported on the fractal subset $\mathcal{L}_\xi^\Gamma \subset S^*X$.

Notice that the fractal behaviour of the semi-classical limit μ_ξ can only be observed at the microlocal level. This is to our knowledge the first example in the mathematical literature of a high energy limit of eigenfunctions which is concentrated microlocally on a genuine fractal trapped set.

In the theoretical physics literature, we mention some work of Ishio-Keating [14] on somehow related questions for billiards.

By averaging over the boundary with respect to the volume measure on $\partial\bar{X}$ induced by the defining function x , we obtain as $t \rightarrow +\infty$

$$(2) \quad \int_{\partial\bar{X}} \int_X a(m) \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m) dv_{\partial\bar{X}}(\xi) = \text{vol}(S^n) \int_X a(m) dv(m) + \mathcal{O}(t^{2\delta_\Gamma - n})$$

and

$$\int_{\partial\bar{X}} \left\langle AE_X\left(\frac{n}{2} + it; \cdot, \xi\right), E_X\left(\frac{n}{2} + it; \cdot, \xi\right) \right\rangle_{L^2(X)} dv_{\partial\bar{X}}(\xi) = \int_{S^*X} a \, d\mu + \mathcal{O}(t^{-\min(n-2\delta_\Gamma, 1)})$$

where μ denotes the Liouville measure. This is the perfect analog of the previously known results for the modular surface (actually with a remainder in our case).

It is actually possible to prove a much stronger asymptotic expansion for the averaged case. We show the following.

Theorem 3. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold and assume that the limit set of Γ has Hausdorff dimension $\delta_\Gamma < n/2$. Let $a \in C_0^\infty(X)$ and \mathcal{C} be the set of all closed geodesics. Then for all $N \in \mathbb{N}$, one has the expansion as $t \rightarrow +\infty$*

$$(3) \quad \begin{aligned} & \int_{\partial X} \int_X a(m) \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m) dv_{\partial\bar{X}}(\xi) = \text{vol}(S^n) \int_X a(m) dv(m) \\ & + L(t) \sum_{\gamma \in \mathcal{C}} e^{-(\frac{n}{2} + it)\ell(\gamma)} \sum_{k=0}^N \frac{H_k(\ell(\gamma))}{|\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|} t^{-k} \int_\gamma P_k a \, d\mu + \\ & + L(-t) \sum_{\gamma \in \mathcal{C}} e^{-(\frac{n}{2} - it)\ell(\gamma)} \sum_{k=0}^N \frac{H_k(\ell(\gamma))}{|\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|} (-t)^{-k} \int_\gamma P_k a \, d\mu + \mathcal{O}(t^{-N-n-1}). \end{aligned}$$

where $H_k(\ell(\gamma))$ is an explicit bounded function of $\ell(\gamma)$, P_k are differential operators of order $2k$ with coefficient uniformly bounded in terms of $\gamma \in \mathcal{C}$, $d\mu$ is the Riemannian measure induced on γ , $R_\gamma \in \text{SO}(n)$ is the holonomy along γ and

$$L(t) = t^{-(n-1)/2} \frac{2^{\frac{n}{2}+3-2it} |\Gamma(it)|^2}{\Gamma(it + \frac{1}{2}) \Gamma(\frac{n}{2} - it)} = \mathcal{O}(t^{-n})$$

$$P_0 = 1, \quad H_0(\ell(\gamma)) = 2^{\frac{n}{2}} \pi^{\frac{n-1}{2}} e^{i\frac{n-1}{4}\pi}.$$

³ $a \in C_0^\infty(X, T^*X)$ means that $a \in C^\infty(T^*X)$ is compactly supported on the base but not necessarily in the fibers.

Notice that the asymptotic expansion is valid at all orders and, except for the first term, involves oscillating sums over closed geodesics. This formula is similar to what physicists call “the Gutzwiller formula” in the literature related to Quantum Chaos. In the mathematics literature it is of course quite similar to the Selberg trace formula (proved in [22, 3, 12] in this setting, see also [9]) and to trace formulae proved in [5, 4, 7, 2, 18].

Let us describe the organization of the paper. The section §2 is devoted to the necessary analytic and geometric background. The proofs start in §3 and take advantage of the convergence of Poincaré series guaranteed by the hypothesis $\delta_\Gamma < n/2$. We first prove the “configuration space” equidistribution result using a direct approach which is based on an “off-diagonal” cancelation phenomena due to some non-stationnary phase effect. To get a sharp remainder estimates in terms of δ , we need to analyze differently the elements in the group with large and small translation lengths. This technique is robust enough to extend to the pseudo-differential (phase space) equidistribution case. We treat the problem using semi-classical analysis, namely some standard results on the action of h -Pseudos on h -Lagrangian distributions. Theorem 2 is based on a completely different technique and make use of Stone’s formula and the high frequency asymptotic for the resolvent on the universal covering \mathbb{H}^{n+1} . The result follows from a repeated application of stationnary phase expansions (with parameter) and requires some precise estimate in terms of the group elements.

We believe that extensions of these results to non-constant negative curvatures setting can be obtained for manifolds with asymptotically hyperbolic (or asymptotically Euclidean) ends under a condition on the topological pressure of the flow on the trapped set like in [19]. This will be pursued elsewhere.

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2. GENERALITIES

2.1. Definition of Eisenstein series. Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold. The group Γ is a discrete group of orientation preserving isometries of \mathbb{H}^{n+1} with only hyperbolic transformations (i.e. fixing 2 points on the sphere $S^n = \partial\mathbb{H}^{n+1}$). The limit set is $\Lambda_\Gamma := \overline{\Gamma \cdot o} \cap S^n$ where $o \in \mathbb{H}^{n+1}$ is the center in the unit ball model $B_o(1) \subset \mathbb{R}^{n+1}$ of \mathbb{H}^{n+1} . The discontinuity set is the complement $\Omega_\Gamma := S^n \setminus \Lambda_\Gamma$ and Γ acts properly discontinuously on $\mathbb{H}^{n+1} \cup \Omega_\Gamma \subset \overline{B_o(1)}$. The manifold X compactifies smoothly to a manifold \overline{X} with boundary $\partial\overline{X}$ by setting $\overline{X} = \Gamma \backslash (\mathbb{H}^{n+1} \cup \Omega_\Gamma)$. If g denotes the hyperbolic metric on X , there exists a smooth boundary defining function x such that $x^2 g$ extends to a smooth metric on \overline{X} and that $|dx|_{x^2 g} = 1$ on $\partial\overline{X}$. The Laplacian Δ on (X, g) has absolutely continuous spectrum on $[n^2/4, \infty)$ and a possibly non-empty finite set of eigenvalues in $(0, n^2/4)$.

By [17, 11], the resolvent of the Laplacian

$$R_X(s) := (\Delta - s(n-s))^{-1} \quad \text{defined in the half plane } \operatorname{Re}(s) > n/2$$

admits a meromorphic continuation to the whole complex plane \mathbb{C} , with poles of finite rank (i.e. the polar part at a pole is a finite rank operator), as a family of bounded operators

$$R_X(s) : C_0^\infty(X) \rightarrow C^\infty(X).$$

Using [17, 11], the resolvent integral kernel $R_X(s; m, m')$ near the boundary $\partial\overline{X}$ has an asymptotic expansion given as follows: for any $m \in X$ fixed

$$m' \rightarrow R_X(s; m, m')x(m')^{-s} \in C^\infty(\overline{X})$$

and similarly for $m' \in X$ fixed and $m' \rightarrow \partial\overline{X}$. The Eisenstein series are functions in $C^\infty(X \times \partial\overline{X})$ defined by⁴

$$(4) \quad E_X(s; m, \xi) = \frac{2\pi^{\frac{n}{2}}\Gamma(s - \frac{n}{2} + 1)}{2^{-s}\Gamma(s)} [x(m')^{-s} R_X(s; m, m')] |_{m'=\xi}, \quad \xi \in \partial\overline{X}.$$

The terminology ‘series’ will become clear later since they can be obtained by series over the group. Notice that $E_X(s; \cdot, \xi)$ depends a priori on the choice of the boundary defining function x however any other choice $x' = \psi(\xi)x + O(x)$ with $\psi \in C^\infty(\partial\overline{X})$ would simply amount to multiply $E_X(s; m, \xi)$ by $\psi(\xi)^{-s}$. Moreover, they are generalized eigenfunctions of Δ_X

$$(\Delta_X - s(n-s))E_X(s; \cdot, \xi) = 0, \quad \forall \xi \in \partial\overline{X}.$$

The functions $E_X(s, \cdot, \xi)$ replace the plane wave functions $z \rightarrow e^{isz \cdot \xi}$ in \mathbb{R}^n (with $\xi \in S^{n-1}$) which satisfy $(\Delta_{\mathbb{R}^n} - s^2)e^{isz \cdot \xi} = 0$, they provide a spectral representation of the continuous spectrum for Δ_X .

Let us define the constant

$$(5) \quad C(s) = \pi^{-\frac{n}{2}} 2^{-s} \frac{\Gamma(s)}{\Gamma(s - \frac{n}{2})}.$$

The spectral measure $2td\Pi_X(t)$ of the Laplacian Δ_X is related to Eisenstein series by

$$(6) \quad \begin{aligned} 2td\Pi_X(t; m, m') &= \frac{is}{\pi} (R_X(\frac{n}{2} + it; m, m') - R_X(\frac{n}{2} - it; m, m')) \\ &= \frac{|C(\frac{n}{2} + it)|^2}{2\pi} \int_{\partial\overline{X}} E_X(\frac{n}{2} + it; m, \xi) E_X(\frac{n}{2} - it; m', \xi) dv_{\partial\overline{X}}(\xi) \\ 2td\Pi_X(t; m, m) &= \frac{|C(\frac{n}{2} + it)|^2}{2\pi} \int_{\partial\overline{X}} |E_X(\frac{n}{2} + it; m, \xi)|^2 dv_{\partial\overline{X}}(\xi) \end{aligned}$$

with $dv_{\partial\overline{X}}$ the Riemannian measure on $\partial\overline{X}$ induced by the metric $h_0 = (x^2g)|_{T\partial\overline{X}}$. The first identity is Stone’s formula, the second identity is an application of Green’s formula proved in [10, Prop .1] or [23]. We notice that the density $|E_X(\frac{n}{2} + it; m, \xi)|^2 dv_{\partial\overline{X}}(\xi)$ is independent of the choice of boundary defining function x used for the definition (4), as long as h_0 is defined by $x^2g|_{T\partial\overline{X}}$.

2.2. Eisenstein functions on \mathbb{H}^{n+1} . To fix the ideas, and since it will be used later, let us recall the case of \mathbb{H}^{n+1} . The resolvent kernel is given (away from the diagonal $m = m'$) by [11, Sec. 2]:

$$(7) \quad \begin{aligned} R_{\mathbb{H}^{n+1}}(s; m, m') &= \frac{\pi^{-\frac{n}{2}} 2^{-2s-1} \Gamma(s)}{\Gamma(s - \frac{n}{2} + 1)} \cosh^{-2s} \left(\frac{d(m, m')}{2} \right) \\ &\quad \times F \left(s, s - \frac{n-1}{2}, 2s - n + 1; \cosh^{-2} \left(\frac{d(m, m')}{2} \right) \right) \end{aligned}$$

⁴The normalization factor is not the usual one, but it is the natural one for our equidistribution purpose; this corresponds somehow to the normalization given by the Weyl law in the co-compact setting.

where $d(m, m')$ is the hyperbolic distance between m and m' , and for $z \in \mathbb{C} \setminus [1, +\infty)$,

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,$$

is the usual hypergeometric function. In the ball model $B_o(1) := \{m \in \mathbb{R}^{n+1}; |m| < 1\}$ (with center denoted by o), the function

$$(8) \quad x(m) := 2e^{-d(m,o)} = 2\frac{1-|m|}{1+|m|}$$

is a boundary defining function for the closed ball $\overline{B_o(1)}$. Moreover, using

$$(9) \quad \sinh^2\left(\frac{d(m, m')}{2}\right) = \frac{|m - m'|^2}{(1 - |m|^2)(1 - |m'|^2)},$$

we obtain for $E_0(s; m, \xi) := E_{\mathbb{H}^{n+1}}(s; m, \xi)$ defined with the boundary defining function x

$$(10) \quad E_0(s; m, \xi) = \left(\frac{1 - |m|^2}{4|m - \xi|^2}\right)^s = e^{s\phi_\xi(m)} \quad \text{with} \quad \phi_\xi(m) := \log\left(\frac{1 - |m|^2}{4|m - \xi|^2}\right).$$

Let $s_h := n/2 + i/h$ with $h > 0$ small, then as a function of m , $E_0(s_h; m, \xi)$ is a semi-classical Lagrangian distribution associated to the Lagrangian submanifold

$$\mathcal{L}_\xi := \{(m, d\phi_\xi(m)) \in T_m^*\mathbb{H}^{n+1}; m \in \mathbb{H}^{n+1}\} \subset T^*\mathbb{H}^{n+1}.$$

If $H_\xi(m)$ is the horosphere tangent to ξ and containing m , then $\phi_\xi(m) = \pm d(H_\xi(m), H_\xi(o))$ is the hyperbolic distance between $H_\xi(m)$ and $H_\xi(o)$. Moreover, since $\phi_{R\xi}(m) = \phi_\xi(R^{-1}m)$ for any $R \in \text{SO}(n+1)$, the norm $|d\phi_\xi(m)|_g$ with respect to the hyperbolic metric g is independent of ξ . By going through the half-space model $\mathbb{R}_x^+ \times \mathbb{R}_y^n$ of \mathbb{H}^{n+1} , and choosing $\xi = \infty$, we have $\phi_\infty(x, y) = \log(y)$ and thus $|d\phi_\infty(m)|_g = 1$ for all $m \in \mathbb{H}^{n+1}$, which implies that

$$(11) \quad |d\phi_\xi(m)|_g = 1 \text{ for all } \xi \in S^n, m \in \mathbb{H}^{n+1}.$$

The level sets of ϕ_ξ are the horospheres tangent to ξ , and the covector $d\phi_\xi(m)$ is dual via the metric to the gradient $\nabla\phi_\xi(m)$ which is the unit vector tangent to the geodesic going from m to ξ . The manifold \mathcal{L}_ξ is the *stable manifold* associated to ξ .

Let us describe the action of isometries on Eisenstein functions.

Lemma 4. *Let $\gamma \in \text{Isom}_+(\mathbb{H}^{n+1})$ be an isometry of \mathbb{H}^{n+1} , then for all $m \in \mathbb{H}^{n+1}$ and $\xi \in S^n$,*

$$E_0(s; \gamma m, \gamma \xi) = E_0(s; m, \xi) |D\gamma(\xi)|^{-s}$$

where $|D\gamma(\xi)|$ is the Euclidean norm of $D\gamma$ at the point $\xi \in S^n$.

Proof. We use the fact that the function (9) is invariant with respect to γ , i.e. for all m, m'

$$\frac{|\gamma m - \gamma m'|}{(1 - |\gamma m|^2)(1 - |\gamma m'|^2)} = \frac{|m - m'|}{(1 - |m|^2)(1 - |m'|^2)}$$

and thus multiplying this inequality by $(1 - |m'|^2)$ and letting $m' \rightarrow \xi \in S^n$, we obtain $E_0(s; \gamma m, \gamma \xi)/E_0(s; m, \xi) = \lim_{m' \rightarrow \xi} \left(\frac{1 - |\gamma m'|^2}{1 - |m'|^2}\right)^{-s}$. This limit is given by $|D\gamma(\xi)|^{-s}$. \square

Notice that for any Möbius transformation γ , $|D\gamma(\xi)v| = |D\gamma(\xi)|$ for any v of Euclidean norm 1. Let us finally give the formula for the integral of the hyperbolic Poisson kernel: for all $m \in \mathbb{H}^{n+1}$,

$$(12) \quad \int_{S^n} E_0(n; m, \xi) d\xi = 1/C(n) = \text{vol}(S^n).$$

where $C(n)$ is defined in (5) and $d\xi$ is the canonical measure on S^n .

2.3. Eisenstein series on $\Gamma \backslash \mathbb{H}^{n+1}$.

Lemma 5. *Let Γ be a convex co-compact group of isometries of \mathbb{H}^{n+1} with $\delta_\Gamma < n/2$ and $\pi_\Gamma := \mathbb{H}^{n+1} \rightarrow \Gamma \backslash \mathbb{H}^{n+1}$ be the quotient map. Let $\xi \in \Omega_\Gamma$ and $m \in \mathbb{H}^{n+1}$, then for $\text{Re}(s) > \delta_\Gamma$ the series*

$$\tilde{E}(s; m, \xi) := \sum_{\gamma \in \Gamma} E_{\mathbb{H}^{n+1}}(s; \gamma m, \xi)$$

converges absolutely and is a Γ -automorphic function on \mathbb{H}^{n+1} satisfying

$$E_X(s; \pi_\Gamma(m), \pi_\Gamma(\xi)) = \tilde{E}(s; m, \xi)$$

if $E_X(s; \cdot, \cdot)$ is defined by (4) with a boundary defining function x so that, in a neighbourhood of ξ , $\pi_\Gamma^*(x) = (1 - |m|)/(1 + |m|)$.

Proof. By combining (8) and (10), together with the estimate $|\gamma m - \xi| > \epsilon > 0$ for some ϵ uniform in $\gamma \in \Gamma$, we deduce the existence of a constant C_s uniform in γ so that

$$|E_0(s; \gamma, \xi)| \leq C_s e^{-sd(\gamma m, o)}.$$

This shows that the series converges in $\text{Re}(s) > \delta_\Gamma$ by (1).

The resolvent kernel for $\text{Re}(s) > n/2$ on the quotient is given, for $m, m' \in \mathbb{H}^{n+1}$ so that $\pi_\Gamma(m) \neq \pi_\Gamma(m')$, by the sum

$$R_X(s; \pi_\Gamma(m), \pi_\Gamma(m')) = \sum_{\gamma \in \Gamma} R_{\mathbb{H}^{n+1}}(s; \gamma m, m'),$$

and this series converges absolutely in $\text{Re}(s) > \delta_\Gamma$ by (7) and (1), moreover the convergence is uniform on compact sets in the variable m, m' (in any C^k norms). The proof is standard and is left to the reader. By using (8) and (9), it is easily seen that after we multiply this last equation by $x(\pi_\gamma(m'))^{-s}$, we take the limit $m' \rightarrow \xi$ and it is given by

$$\frac{2s - n}{C(s)} \sum_{\gamma \in \Gamma} \lim_{m' \rightarrow \xi} (R_{\mathbb{H}^{n+1}}(s; \gamma m, m') (2 \frac{1 - |m'|}{1 + |m'|})^{-s}) = \tilde{E}(s; m, \xi).$$

This concludes the proof. \square

2.4. The Liouville measure. If (X, g) is an oriented $n + 1$ dimensional Riemannian manifold, recall that the Liouville form (or measure) on the unit sphere bundle SX is given by the $2n + 1$ form $\mu := \theta \wedge d\theta^n$ where θ is the contact form on SX

$$\theta_u(v) := g(u, \pi_* v)$$

if $\pi : SX \rightarrow X$ is the projection. Isometries γ of X act on SX by $\gamma.(m, v) := (\gamma m, d\gamma(m)v)$ and clearly $\gamma^* \theta = \theta$. In particular, this implies that $\gamma^* \mu = \mu$. Since the group $\text{Isom}_+(\mathbb{H}^{n+1})$ acts transitively on $S\mathbb{H}^{n+1}$, the Liouville form is determined by its value at one point, for instance $(m, v) = (o, e_1)$ where e_1 is a fixed unit vector of $S_o\mathbb{H}^{n+1}$.

Let us define the map $\psi : \mathbb{H}^{n+1} \times S^n \rightarrow S\mathbb{H}^{n+1}$ by

$$(13) \quad \psi : (m, \xi) \mapsto (m, \nabla\phi_\xi(m))$$

where ∇ denotes the gradient with respect to the hyperbolic metric g . This map is a diffeomorphism and it identifies the point $(m, v) \in S\mathbb{H}^{n+1}$ to the point $(m, \xi) \in \mathbb{H}^{n+1} \times S^n$ so that ξ is the end point of the geodesic starting at m with tangent vector v . The group of isometries $\text{Isom}_+(\mathbb{H}^{n+1})$ acts transitively on $\mathbb{H}^{n+1} \times S^n$ by $\gamma \cdot (m, \xi) := (\gamma m, \gamma \xi)$ and using Lemma 4, we obtain for all $\gamma \in \text{Isom}_+(\mathbb{H}^{n+1})$

$$(14) \quad \gamma^* \nu = \nu, \quad \text{where } \nu := E(n; m, \xi) dv(m) \wedge d\xi$$

where $dv(m)$ and $d\xi$ are the canonical volume forms of \mathbb{H}^{n+1} and S^n . But we also have $\psi(\gamma m, \gamma \xi) = \gamma \cdot \psi(m, \xi)$ and therefore we deduce that for all $\gamma \in \text{Isom}_+(\mathbb{H}^{n+1})$

$$\gamma^* \psi^* \mu = \psi^* \gamma^* \mu = \psi^* \mu.$$

By (14), this implies that both ν and $\psi^* \mu$ are preserved by the transitive action of $\text{Isom}_+(\mathbb{H}^{n+1})$, and thus there exists a constant C_n such that $\psi^* \mu$, and an easy computation at $m = 0, \xi = e_1$ gives

$$(15) \quad \psi^* \mu = \nu.$$

The same is obviously true with the Liouville measure on $S^*\mathbb{H}^{n+1}$ by duality, provided ψ is defined with $d\phi_\xi(m)$ instead of $\nabla\phi_\xi(m)$. On a quotient $X = \Gamma \backslash \mathbb{H}^{n+1}$, we also have $S^*X = \Gamma \backslash S^*\mathbb{H}^{n+1} \simeq \Gamma \backslash (\mathbb{H}^{n+1} \times S^n)$ and the Liouville measure on the quotient lifts to the Liouville measure μ on \mathbb{H}^{n+1} .

3. PROOF OF EQUIDISTRIBUTION OF EISENSTEIN SERIES

3.1. Equidistribution on X . In this section, we show equidistribution in space of the Eisenstein series.

Theorem 6. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact quotient with $\delta_\Gamma < n/2$, and let x be a boundary defining function of \overline{X} . Let $a \in C_0^\infty(X)$ and let $E_X(s; \cdot, \xi)$ be the Eisenstein series defined by (4) using the defining function x and a point $\xi \in \partial\overline{X}$ at infinity. Then we have as $t \rightarrow \infty$,*

$$\int_X a(m) \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m) = \int_X a(m) E_X(n; m, \xi) dv(m) + \mathcal{O}(t^{2\delta-n}).$$

Proof. Let us consider the ball model $B_o(1) = \{m \in \mathbb{R}^{n+1}; |m| < 1\}$ of \mathbb{H}^{n+1} , with boundary the sphere S^n . Consider a fundamental domain \mathcal{F} of Γ and let $\xi \in \Omega_\Gamma \cap \overline{\mathcal{F}}$ be a point in the domain of discontinuity. The boundary defining function x of X lifts to an automorphic function \tilde{x} on \mathbb{H}^{n+1} and we define $\eta(\xi) := 2 \lim_{m \rightarrow \xi} (1 - |m|) / \tilde{x}(m)$. By Lemma 5, the Eisenstein series on $X = \Gamma \backslash \mathbb{H}^{n+1}$ at parameter s such that $\text{Re}(s) > \delta_\Gamma$ is given by

$$(16) \quad E_X(s; \pi_\Gamma(m), \pi_\Gamma(\xi)) = \sum_{\gamma \in \Gamma} E_0(s; \gamma m, \xi) \eta(\xi)^s = \sum_{\gamma \in \Gamma} \left(E_0(1; \gamma m, \xi) \eta(\xi) \right)^s$$

$$E_0(1; m, \xi) := \frac{1 - |m|^2}{4|m - \xi|^2},$$

for $m \in \mathbb{H}^{n+1}$. For simplicity, we shall identify $\pi_\Gamma(m)$ with m in the fundamental domain. We are interested in the pointwise norm (for $t \in \mathbb{R}$)

$$\begin{aligned} \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 &= \sum_{\gamma \in \Gamma} \left(E_0(1; \gamma m, \xi) \eta(\xi) \right)^n \\ &\quad + \eta(\xi)^n \sum_{\gamma \neq \gamma'} \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \left(\frac{E_0(1; \gamma m, \xi)}{E_0(1; \gamma' m, \xi)} \right)^{it} \\ \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 &= E_X(n; m, \xi) + \eta(\xi)^n \sum_{\gamma \neq \gamma'} \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \left(\frac{E_0(1; \gamma m, \xi)}{E_0(1; \gamma' m, \xi)} \right)^{it} \end{aligned}$$

Let $a \in C_0^\infty(X)$ and consider the off-diagonal sum $\sum_{\gamma \neq \gamma'} I_{\gamma, \gamma'}(t)$ where

$$(17) \quad I_{\gamma, \gamma'}(t) := \int_X a(m) \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \left(\frac{E_0(1; \gamma m, \xi)}{E_0(1; \gamma' m, \xi)} \right)^{it} dv(m)$$

To end the proof, we need to show the

Lemma 7. *As $t \rightarrow \infty$, the off-diagonal sum satisfies $\sum_{\gamma \neq \gamma'} I_{\gamma, \gamma'}(t) = \mathcal{O}(t^{2\delta_\Gamma - n})$*

Proof. The proof reduces to some non-stationary phase in t on a finite number of terms in each factor of $\Gamma \times \Gamma$, essentially those γ which satisfy $d(o, \gamma o) \leq \log t$. Let us first define the Buseman function

$$(18) \quad B_\xi(m, m') := \log(1 - |m|^2) - \log(|m - \xi|^2) - \log(1 - |m'|^2) + \log(|m' - \xi|^2)$$

which satisfies the cocycle condition $B_\xi(m, m'') = B_\xi(m, m') + B_\xi(m', m'')$ and for any $\gamma \in \text{Isom}(\mathbb{H}^{n+1})$

$$B_{\gamma\xi}(\gamma m, \gamma m') = B_\xi(m, m').$$

One can rewrite (17) as

$$\int_X a(m) \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} e^{itB_\xi(\gamma m, \gamma' m)} dv(m)$$

The phase $B_\xi(\gamma m, \gamma' m) = B_{\gamma^{-1}\xi}(m, \gamma^{-1}o) + B_{\gamma'^{-1}\xi}(\gamma'^{-1}o, m)$ has for gradient (with respect to Euclidean metric) in m

$$\nabla_m B_\xi(\gamma m, \gamma' m) = 2 \left(\frac{m - \gamma^{-1}\xi}{|m - \gamma^{-1}\xi|^2} - \frac{m - \gamma'^{-1}\xi}{|m - \gamma'^{-1}\xi|^2} \right)$$

with Euclidean norm given by

$$(19) \quad \left| \nabla_m B_\xi(\gamma m, \gamma' m) \right| = \frac{2|\gamma^{-1}\xi - \gamma'^{-1}\xi|}{|m - \gamma'^{-1}\xi| \cdot |m - \gamma^{-1}\xi|}.$$

Remark that $\nabla_m B_\xi(\gamma m, \gamma' m) = Q_m(\gamma^{-1}\xi, \gamma'^{-1}\xi) / (|m - \gamma'^{-1}\xi|^2 \cdot |m - \gamma^{-1}\xi|^2)$ for some $Q_m(u, v)$ polynomial in (u, v) and with $Q_m(u, u) = 0$, moreover it is smooth in m in a compact set $K \subset \mathbb{H}^{n+1}$, we then easily deduce that for each $\alpha \in \mathbb{N}^{n+1}$ there is $C_\alpha > 0$ such that for all $m \in K$ and $\alpha \in \mathbb{N}^{n+1}$

$$(20) \quad \left| \partial_m^\alpha \nabla_m B_\xi(\gamma m, \gamma' m) \right| \leq C_\alpha |\gamma^{-1}\xi - \gamma'^{-1}\xi|.$$

Let o be the center of the unit ball in \mathbb{R}^{n+1} . For γ a Möbius transformation preserving the unit ball, we define $a_\gamma = \gamma^{-1}(\infty) \in \mathbb{R}^{n+1}$ the center of the isometric sphere $S(a_\gamma, r_\gamma)$ of γ ,

where $r_\gamma = 1/\sinh(\frac{1}{2}d(o, \gamma o))$ is the radius of $S(a_\gamma, r_\gamma)$ (here $d(\cdot, \cdot)$ is the hyperbolic distance). By §3.5 in Beardon [1], we have for any $\xi, \xi' \in \overline{B_o(1)}$

$$(21) \quad |\gamma\xi - \gamma\xi'| = \frac{1}{\sinh^2(\frac{1}{2}d(o, \gamma o))} \frac{|\xi - \xi'|}{|\xi - a_\gamma||\xi' - a_\gamma|}.$$

Since $|a_\gamma|$ is bounded uniformly, we can set $\xi' = \gamma^{-1} \circ \gamma'\xi$ and we deduce directly that $|\gamma\xi - \gamma'\xi|^2 \geq Ce^{-d(o, \gamma o)}$ and by symmetry in γ, γ' this gives

$$(22) \quad |\gamma\xi - \gamma'\xi| \geq C \max(e^{-d(o, \gamma o)}, e^{-d(o, \gamma' o)})$$

uniformly in $\gamma \neq \gamma'$ and $\xi \in \Omega_\Gamma$. By combining (19), (20) and (22), we deduce that for m in a compact set K of \mathbb{H}^{n+1} one has

$$(23) \quad \left| \partial_m^\alpha \frac{\nabla_m B_\xi(\gamma m, \gamma' m)}{|\nabla_m B_\xi(\gamma m, \gamma' m)|^2} \right| \leq C_\alpha \min(e^{d(o, \gamma o)}, e^{d(o, \gamma' o)}).$$

For each $\gamma, \gamma' \in \Gamma$, we can therefore integrate by part to get

$$\begin{aligned} I_{\gamma, \gamma'}(t) &= \frac{1}{it} \int_X a(m) \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \nabla_m (e^{itB_\xi(\gamma m, \gamma' m)}) \cdot \frac{\nabla_m B_\xi(\gamma m, \gamma' m)}{|\nabla_m B_\xi(\gamma m, \gamma' m)|^2} dv(m) \\ &= \frac{1}{it} \int_X e^{itB_\xi(\gamma m, \gamma' m)} \nabla^* \left(a(m) \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \frac{\nabla_m B_\xi(\gamma m, \gamma' m)}{|\nabla_m B_\xi(\gamma m, \gamma' m)|^2} \right) dv(m) \end{aligned}$$

where ∇^* is the adjoint of ∇ with respect to the Riemannian measure $dv(m)$. For $\xi \in \Omega_\Gamma$, there exists $C > 0$ depending only on K such that for all $m \in K$ and $\gamma \in \Gamma$, one has

$$|\gamma m - \xi| > C, \quad (1 - |\gamma m|^2) \leq Ce^{-d(o, \gamma o)}, \quad |D\gamma(m)| \leq Ce^{-d(o, \gamma o)}.$$

The last identity comes for instance from (21). Combining these estimates with (23), we obtain that there is $C > 0$ such that for all $\gamma \neq \gamma'$ and $m \in K$

$$\begin{aligned} \left| \nabla^* \left(a(m) \left(E_0(1; \gamma m, \xi) E_0(1; \gamma' m, \xi) \right)^{\frac{n}{2}} \frac{\nabla_m B_\xi(\gamma m, \gamma' m)}{|\nabla_m B_\xi(\gamma m, \gamma' m)|^2} \right) \right| &\leq \\ &C e^{-\frac{n}{2}(d(o, \gamma o) + d(o, \gamma' o))} \min(e^{d(o, \gamma o)}, e^{d(o, \gamma' o)}). \end{aligned}$$

We thus have the estimate, which is uniform in $\gamma \neq \gamma'$

$$|I_{\gamma, \gamma'}(t)| \leq Ct^{-1} e^{-\frac{n}{2}(d(o, \gamma o) + d(o, \gamma' o))} \min(e^{d(o, \gamma o)}, e^{d(o, \gamma' o)}).$$

We can iterate this argument and get for $j \in \mathbb{N}$

$$(24) \quad |I_{\gamma, \gamma'}(t)| \leq Ct^{-j} e^{-\frac{n}{2}(d(o, \gamma o) + d(o, \gamma' o))} \min(e^{jd(o, \gamma o)}, e^{jd(o, \gamma' o)}).$$

Let us decompose Γ into

$$\Gamma = \Gamma_{\leq} \sqcup \Gamma_{>} : \{\gamma \in \Gamma; d(o, \gamma) \leq \log(t)\} \sqcup \{\gamma \in \Gamma; d(o, \gamma) > \log(t)\}$$

and then the double sum $\sum_{(\gamma, \gamma') \in \Gamma \times \Gamma \setminus \text{diag}} I_{\gamma, \gamma'}(t)$ into 4 parts involving $\Gamma_\alpha \times \Gamma_\beta \setminus \text{diag}$ for $\alpha, \beta \in \{\leq, >\}$. For the elements in Γ_{\leq} , we can now take advantage of the bound (24) with j taken larger than $n/2 - \delta_\Gamma$. Using the counting estimate (proved by Patterson [21])

$$(25) \quad \#\{\gamma \in \Gamma; d(o, \gamma o) \leq T\} = \mathcal{O}(e^{\delta_\Gamma T})$$

it is easily seen that for each $\alpha, \beta \in \{\leq, >\}$, we have

$$\sum_{(\gamma, \gamma') \in \Gamma_\alpha \times \Gamma_\beta \setminus \text{diag}} I_{\gamma, \gamma'}(t) = \mathcal{O}(t^{2\delta_\Gamma - n}).$$

Let us for instance write the case $\alpha = \beta = \leq$, then by (24) and (25) we have for any $j > n/2 - \delta_\Gamma$ integer

$$\begin{aligned} \sum_{\substack{d(o,\gamma o) \leq \log t \\ d(o,\gamma' o) \leq \log t, \gamma' \neq \gamma}} I_{\gamma,\gamma'}(t) &= \mathcal{O}(t^{2j}) \sum_{d(o,\gamma o) \leq \log t} e^{(j-\frac{n}{2})d(o,\gamma o)} \sum_{d(o,\gamma' o) \leq \log t} e^{(j-\frac{n}{2})d(o,\gamma' o)} \\ &= \mathcal{O}(t^{n-2\delta_\Gamma}) \end{aligned}$$

and the other cases are similar. \square

This achieves the proof of the Theorem. \square

Notice that $E_X(n; m, \xi)$ is a harmonic function, more precisely this is the harmonic function on X with distributional boundary value the Dirac mass δ_ξ at ξ .

As a corollary, we obtain

Corollary 8. *Let $(X, g) = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact quotient with $\delta_\Gamma < n/2$, let x be a boundary defining function of \overline{X} and let $dv_{\partial\overline{X}}$ be the Riemannian measure on the conformal boundary induced by the smooth metric $x^2 g|_{\partial\overline{X}}$. Let $a \in C_0^\infty(X)$ and let $E_X(s; \cdot, \xi)$ be the Eisenstein series defined by (4) using the defining function x and a point $\xi \in \partial\overline{X}$ at infinity. Then, we have as $t \rightarrow \infty$*

$$\int_{\partial\overline{X}} \int_X a(m) \left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m) dv_{\partial\overline{X}}(\xi) = \text{vol}(S^n) \int_X a(m) dv(m) + \mathcal{O}(t^{2\delta_\Gamma - n}).$$

Proof. The statement in Theorem (6) is uniform in $\xi \in \partial\overline{X}$, this is clear from the proof of that Theorem. If \mathcal{F} is a fundamental domain of X and η is defined as in the proof of Theorem (6) in $\overline{\mathcal{F}} \cap S^n$, then the measure $dv_{\partial\overline{X}}$ pulls back to $\overline{\mathcal{F}} \cap S^n$ by π_Γ to the measure $\eta(\xi)^{-n} d\xi$ where $d\xi$ is the canonical measure on S^n . Since $E_X(n; \pi_\gamma(m), \pi_\gamma(\xi)) = \eta(\xi)^n \sum_\Gamma E_0(n; \gamma m, \xi)$, we have

$$\begin{aligned} \int_{\overline{\mathcal{F}} \cap S^n} E_X(n; \pi_\gamma(m), \pi_\gamma(\xi)) \eta(\xi)^{-n} d\xi &= \sum_{\gamma \in \Gamma} \int_{\overline{\mathcal{F}} \cap S^n} E_0(n; m, \gamma^{-1}\xi) |D\gamma(\gamma^{-1}\xi)|^{-n} d\xi \\ &= \sum_{\gamma \in \Gamma} \int_{\gamma(\overline{\mathcal{F}} \cap S^n)} E_0(n; m, \xi) d\xi \\ \int_{\partial\overline{X}} E_X(n; m, \xi) dv_{\partial\overline{X}}(\xi) &= \int_{\Omega_\Gamma} E_0(n; m, \xi) d\xi = \int_{S^n} E_0(n; m, \xi) d\xi = \text{vol}(S^n) \end{aligned}$$

where in the second line we used Lemma 4, in the third we used that the Jacobian of γ at $\xi \in S^n$ is $|D\gamma(\xi)|^n$, the fourth line we used that the set of discontinuity $\Omega_\Gamma = \cup_{\gamma \in \Gamma} \gamma(\overline{\mathcal{F}} \cap S^n)$ has full measure in S^n (for the standard measure on S^n). This ends the proof. \square

3.2. Equidistribution in phase space. In this section, we extend the previous equidistribution result to “microlocal lifts” of the measures

$$\left| E_X\left(\frac{n}{2} + it; m, \xi\right) \right|^2 dv(m).$$

To this end, we recall a few basic facts on pseudo-differential operators, including semi-classical ones. The proof of the micro-local extension will be based on semi-classical quantization and action on Lagrangian states.

3.3. Pseudo-differential and semi-classical pseudo-differential operators. Let X be a complete manifold of dimension d . For $s \in \mathbb{R}$, we define the space $\Psi_c^s(X)$ of compactly supported pseudo-differential operators of order s on X to be the set of bounded operators $A : C^\infty(X) \rightarrow C_0^\infty(X)$, with Schwartz kernel compactly supported in $X \times X$, and which decomposes as a sum $A = A_{\text{sm}} + A_{\text{sg}}$ where A_{sm} has a smooth Schwartz kernel and A_{sg} can be written in any local chart U with coordinate z under the following form: for all $f \in C_0^\infty(X)$ supported in U ,

$$A_{\text{sg}}f(z) = \frac{1}{(2\pi)^d} \int e^{i(z-z')\xi} a(z, \xi) f(z') dz' d\xi$$

with $a(z, \xi)$ is a *classical symbol* of order s in U . A classical symbol a is an element in $C^\infty(U \times \mathbb{R}^d)$ such that for all α, γ , there exists $C > 0$ so that

$$|\partial_z^\gamma \partial_\xi^\alpha a(z, \xi)| \leq C(1 + |\xi|)^{s-|\alpha|}, \quad a(z, \xi) \sim_{|\xi| \rightarrow \infty} \sum_{j=0}^{\infty} a_j(z, \xi)$$

with a_j some homogeneous functions of degree $s - j$ in ξ . The *principal symbol* of A is defined to be a_0 , and it is invariantly defined as a homogeneous section of degree s on T^*X .

Let $h \in (0, 1)$ be a small parameter. For $s, k \in \mathbb{R}$, we define the space $\Psi_c^{s,k}(X)$ of compactly supported semi-classical pseudo-differential operators of order s to be the set of h -dependent bounded operators $A_h : C^\infty(X) \rightarrow C_0^\infty(X)$, with Schwartz kernel compactly supported in $X \times X$, and which decomposes as a sum $A_h = A_{h,\text{sm}} + A_{h,\text{sg}}$ where $A_{h,\text{sm}}$ has a smooth Schwartz kernel satisfying $\|A_{h,\text{sm}}\|_{L^2 \rightarrow L^2} = O(h^\infty)$ and $A_{h,\text{sg}}$ can be written in any local chart U with coordinate z under the following form: for all $f \in C_0^\infty(X)$ supported in U ,

$$A_{h,\text{sg}}f(z) = \frac{1}{(2\pi h)^d} \int e^{i\frac{(z-z')\xi}{h}} a(h; z, \xi) f(z') dz' d\xi$$

with $a(h; z, \xi)$ a *semi-classical symbol* of order s in U . A semi-classical symbol $a(h; \cdot, \cdot)$ is an h -dependent element in $C^\infty(U \times \mathbb{R}^d)$ such that for all α, γ and all N , there exists $C > 0$ so that

$$|\partial_z^\gamma \partial_\xi^\alpha a(h; z, \xi)| \leq Ch^{-k}(1 + |\xi|)^{s-|\alpha|}, \quad a(h; z, \xi) = \sum_{j=0}^{N-1} h^{-k+j} a_j(z, \xi) + h^N r_N(h; z, \xi)$$

where a_j, r_N are smooth functions satisfying

$$|\partial_z^\gamma \partial_\xi^\alpha a_j(z, \xi)| \leq C(1 + |\xi|)^{s-|\alpha|}, \quad |\partial_z^\gamma \partial_\xi^\alpha r_N(h; z, \xi)| \leq Ch^{-k}(1 + |\xi|)^{s-N-|\alpha|}.$$

The space $\Psi_c^{0,0}(X)$ is an algebra under composition of operators, and the principal symbol of a composition is the product of their principal symbol. The *semi-classical principal symbol* of A_h is defined to be $h^{-k}a_0$, and it is invariantly defined as a section of T^*X . The *microsupport* $\text{WF}_h(A_h)$ of A_h is the complement of the set of points $(z, \xi) \in T^*X$ such that $|\partial^\alpha a_h| = \mathcal{O}(h^\infty)$ near (z, ξ) , for all α .

Lemma 9. *Let $B \in \Psi_c^0(X)$ and $A_h \in \Psi_c^{0,0}(X)$ with microsupport satisfying*

$$\text{dist}(\text{WF}_h(A_h), T_0^*X) > \epsilon,$$

*for some $\epsilon > 0$ uniform in h , and where T_0^*X is the zero section in T^*X . Let a_0, b_0 be the semi-classical principal symbol and classical principal symbol of A_h and B . Then $BA_h \in \Psi_c^{0,0}(X)$ and the principal symbol of BA_h is b_0a_0 .*

Proof. We decompose $B = B_{\text{sm}} + B_{\text{sg}}$, and using an adequate change of variables B_{sg} is given locally by

$$Bf(z) = \frac{1}{(2\pi h)^d} \int e^{i\frac{(z-z')\cdot\xi}{h}} b(z, \xi/h) f(z') d\xi dz'.$$

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ so that its support does not intersect $\text{WF}_h(A_h)$. Then

$$B_{\text{sg}}^1 : f \mapsto B_{\text{sg}}^1 f(z) = \frac{1}{(2\pi h)^d} \int e^{i\frac{(z-z')\cdot\xi}{h}} (1 - \chi(\xi)) b(z, \xi/h) f(z') d\xi dz'$$

is an operator in $\Psi_c^{0,0}(X)$ with semi-classical symbol $(1 - \chi(\xi))b_0(z, \xi)$. On the other hand, by using non-stationary phase and the fact that $\text{supp}(\chi) \cap \text{WF}_h(A_h) = \emptyset$, we deduce that $(B_{\text{sg}} - B_{\text{sg}}^1)A_h$ has a smooth kernel and is an $\mathcal{O}(h^\infty)$ as an operator on $L^2(X)$. The composition $B_{\text{sm}}A_h$ has the same properties by applying again non-stationary phase (using that B_{sm} has kernel supported away from the diagonal and $a(h; z, \xi) = 0$ near $\xi = 0$). We conclude that $BA_h \in \Psi_c^{0,0}(X)$ and its principal symbol is $(1 - \chi)b_0a_0 = b_0a_0$. \square

3.4. Action of semi-classical operators on semi-classical Lagrangian distributions.

Let us consider a Lagrangian submanifold $\mathcal{L} \subset T^*X$ which is the graph of $d\varphi$ for some smooth function φ on X

$$\mathcal{L} = \{(m, d\varphi(m)) \in T^*X; m \in X\}.$$

A (compactly supported) *semi-classical Lagrangian distribution* with Lagrangian \mathcal{L} is a function on X which can be written, for all $N \in \mathbb{N}$, under the form

$$u(m) = e^{i\frac{\varphi(m)}{h}} \left(\sum_{j=1}^{N-1} a_j(m) h^j + h^N r_{N,h}(m) \right)$$

for some $a_j, r_N \in C_0^\infty(X)$ and $|\partial_m^\alpha r_{N,h}| = \mathcal{O}(1)$ for all α . We call a_0 the principal symbol of u . Semi-classical pseudo-differential operators preserve the space of lagrangian distributions with Lagrangian \mathcal{L} , as is proved for example in [19, Lemma 4.1], which we recall here

Lemma 10. *Let $T \in \Psi_c^{0,0}(X)$ be a compactly supported semi-classical operator with principal symbol α_0 and u be a compactly supported semi-classical Lagrangian distribution with Lagrangian \mathcal{L} and principal symbol a_0 . Then Tu is a compactly supported Lagrangian distribution with Lagrangian \mathcal{L} and with principal symbol $b_0(m) := \alpha_0(m, d\varphi(m))a_0(m)$. More precisely*

$$Tu(m) = e^{i\frac{\varphi(m)}{h}} (b_0(m) + hr_{1,h}(m))$$

with $r_{1,h} \in C_0^\infty(X)$ satisfying $\|r_{1,h}\|_{L^2} \leq C(\|u\|_{L^2} + \|a_0\|_{C^{n+3}})$ for some $C > 0$ depending only on semi norms of the local symbol of T and on semi-norms of φ .

We will use the semi-classical notations in this section: let $s_h = \frac{n}{2} + \frac{i}{h}$, then for $\xi \in S^n$ fixed

$$(26) \quad E_0(s_h; m, \xi) = E_0\left(\frac{n}{2}; m, \xi\right) e^{i\frac{\phi_\xi(m)}{h}} \quad \text{with } \phi_\xi(m) := \log\left(\frac{1 - |m|^2}{4|m - \xi|^2}\right).$$

As a function of m , $E_0(s_h; m, \xi)$ is a semi-classical Lagrangian distribution associated to the Lagrangian submanifold

$$\mathcal{L}_\xi := \{(m, d\phi_\xi(m)) \in T_m^*\mathbb{H}^{n+1}; m \in \mathbb{H}^{n+1}\} \subset T^*\mathbb{H}^{n+1}.$$

Let Γ be a convex co-compact group of isometries of \mathbb{H}^{n+1} . For a given $\xi \in \Omega_\Gamma$, let us define the (closure of) discrete union of Lagrangians

$$(27) \quad \mathcal{L}_\xi^\Gamma := \overline{\bigcup_{\gamma \in \Gamma} \mathcal{L}_{\gamma\xi}}.$$

If $\pi_\Gamma : \mathbb{H}^{n+1} \rightarrow \Gamma \backslash \mathbb{H}^{n+1} = X$ is the projection to the quotient, then

$$\pi_\Gamma(\mathcal{L}_\xi^\Gamma) = \overline{\bigcup_{\gamma \in \Gamma} \pi_\Gamma(\mathcal{L}_{\gamma\xi})}$$

is the closure of a countable superposition of Lagrangian submanifolds of T^*X . Over a point $m \in X$, $\pi_\Gamma(\mathcal{L}_\xi^\Gamma) \cap S_m^*X$ corresponds to the (closure of) set of directions $v(m) \in S_m^*X$ such that the geodesic starting at $(m, v(m))$ tends to $\pi_\Gamma(\xi) \in \Gamma \backslash \Omega_\Gamma = \partial\overline{X}$. We point out that since Γ is discrete, for all $\xi \in \Omega_\Gamma$, we have

$$\overline{\Gamma \cdot \xi} = \left(\bigcup_{\gamma \in \Gamma} \gamma\xi \right) \sqcup \Lambda_\Gamma,$$

see Beardon [1], Theorem 5.3.9. In other words, the set of accumulation points of the orbit $\Gamma \cdot \xi$ is the full limit set Λ_Γ . Since \mathcal{L}_ξ^Γ is locally diffeomorphic to $\mathbb{H}^{n+1} \times \overline{\Gamma \cdot \xi}$, some basic facts of dimension theory show that the Hausdorff dimension of \mathcal{L}_ξ^Γ is $n + 1 + \delta_\Gamma$. We show the following

Theorem 11. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold, and assume $\delta_\Gamma < \frac{n}{2}$. Let $A \in \Psi_c^0(X)$ be a compactly supported classical pseudo-differential operator of order 0 with principal symbol $a \in C_0^\infty(X, T^*X)$. Then for $\xi \in \partial\overline{X} = \Gamma \backslash \Omega_\Gamma$, we have as $h \rightarrow 0$ and for if $s_h := n/2 + i/h$*

$$\langle AE(s_h; \cdot, \xi), E(s_h; \cdot, \xi) \rangle_{L^2(X)} = \int_{S^*X} a d\mu_\xi + \mathcal{O}(h^{\min(1, n-2\delta)})$$

where $d\mu_\xi$ is a measure supported on $\pi_\Gamma(\mathcal{L}_\xi^\Gamma)$ as defined in (27). More precisely, denoting also a and ξ for their lifts to respectively $S^*\mathbb{H}^{n+1}$ and Ω_Γ through π_Γ , and if \mathcal{F} is a fundamental domain of Γ , $\int_{S^*X} a d\mu_\xi$ can be written as a converging series

$$(28) \quad \int_{S^*X} a(m, v) d\mu_\xi(m, v) := \int_{\mathcal{F}} \sum_{\gamma \in \Gamma} a(m, d\phi_{\gamma\xi}(m)) E_0(n; m, \gamma\xi) |D\gamma(\xi)|^n dv(m).$$

Proof. First we study $AE_X(s_h; \cdot, \xi)$. Let $\chi, \chi' \in C_0^\infty(X)$ be functions such that $\chi A \chi = A$ and $\chi' \chi = \chi$, and let $\psi_h \in \Psi_c^{0,0}(X)$ be a semi-classical pseudo-differential operator with $\text{WF}_h(\psi_h) \subset \{(m, v) \in T^*M; |v| - 1 \leq 1/2\}$ and with semi-classical principal symbol ψ equal to 1 on $\{(m, v) \in T^*M; |v| - 1 \leq 1/4\}$. We first claim that $\chi'(1 - \psi_h)\chi E_X(s_h; \cdot, \xi) = \mathcal{O}(h)$ in L^2 . Indeed, let $P_h := h^2(\Delta_X - n^2/4) - 1$, then using that $P_h E_X(s_h; \cdot, \xi) = 0$, we deduce that

$$P_h \chi'(1 - \psi_h)\chi E_X(s_h; \cdot, \xi) = [P_h, \chi'(1 - \psi_h)\chi] E_X(s_h; \cdot, \xi)$$

which has L^2 norm $\mathcal{O}(h)$ since the commutator is in $\Psi_c^{0,-1}(X)$ (the semi-classical symbol of ψ_h being compactly supported) and $\|\chi E_X(s_h; \cdot, \xi)\|_{L^2} = \mathcal{O}(1)$. The claim follows by the invertibility of the semi-classical principal symbol of $P_h(1 - \psi_h)$ on the microsupport of $\chi E_X(s_h; \cdot, \xi)$.

Therefore

$$(29) \quad AE_X(s_h; \cdot, \xi) = A\psi_h \chi E_X(s_h; \cdot, \xi) + \mathcal{O}_{L^2}(h).$$

By Lemma 9, $A_h := A\psi_h \chi' \in \Psi_c^{0,0}(X)$ is a semi-classical operator with semi-classical principal symbol $a(m, v)\psi(m, v)$.

Now we lift all objects to the covering \mathbb{H}^{n+1} and we can assume without loss of generality that A has kernel compactly supported in $\mathcal{F} \times \mathcal{F}$ where \mathcal{F} is a fundamental domain of Γ . Recall that $E_X(s_h; m, \xi)$ can be expressed as a converging series

$$\begin{aligned} E_X(s_h; m, \xi) &= \sum_{\gamma \in \Gamma} E_0\left(\frac{n}{2}; m, \gamma\xi\right) |D\gamma^{-1}(\gamma\xi)|^{-s_h} e^{i\phi_{\gamma\xi}(m)/h} \\ &= \sum_{\gamma \in \Gamma} E_0\left(\frac{n}{2}; m, \gamma\xi\right) |D\gamma(\xi)|^{s_h} e^{i\phi_{\gamma\xi}(m)/h} \end{aligned}$$

For the second line, we have used that γ is a Möbius transform, so that

$$|D\gamma^{-1}(\gamma\xi)|^{-1} = |D\gamma(\xi)|,$$

series being convergent when $\delta_\Gamma < n/2$ since $|D\gamma(\xi)| \leq Ce^{-d(o, \gamma o)}$ uniformly for $\xi \in \overline{F} \cap S^n$, by (21). In particular, one has $\|\chi E_X(s_h; \cdot, \xi)\|_{L^2} = \mathcal{O}(1)$, uniformly in $\xi \in \overline{\partial X}$.

The action of A_h on each term of the series, ie. $A_h(\chi e^{i\phi_{\gamma\xi}/h} E_0(\frac{n}{2}; \cdot, \gamma\xi))$, is described by Lemma 10, $A_h(\chi e^{i\phi_{\gamma\xi}/h} E_0(\frac{n}{2}; \cdot, \gamma\xi))$ is a Lagrangian distribution on \mathbb{H}^{n+1} associated to the Lagrangian $\mathcal{L}_{\gamma\xi}$, and compactly supported in \mathcal{F} :

$$(30) \quad \begin{aligned} A_h(\chi e^{i\phi_{\gamma\xi}/h} E_0(\frac{n}{2}; \cdot, \gamma\xi)) &= e^{i\phi_{\gamma\xi}/h} \left(\chi(m) E_0\left(\frac{n}{2}; m, \gamma\xi\right) a(m, d\phi_{\gamma\xi}(m)) \psi(m, d\phi_{\gamma\xi}(m)) \right. \\ &\quad \left. + hr_h(m, \gamma\xi) \right) \end{aligned}$$

with $\|r_h(\cdot, \gamma\xi)\|_{L^2} = \mathcal{O}(1)$ bounded by a constant times finitely many semi-norms of a , $E_0(\frac{n}{2}; \cdot, \xi)$ and of the phase $\phi_{\gamma\xi}$. But it is easily seen that for all $\alpha \in \mathbb{N}^{n+1}$ there exists $C_\alpha > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \overline{F} \cap S^n$,

$$(31) \quad \sup_{m \in \text{supp}(\chi)} |\partial_m^\alpha \phi_{\gamma\xi}(m)| \leq C_\alpha, \quad \sup_{m \in \text{supp}(\chi)} |\partial_m^\alpha E_0\left(\frac{n}{2}; m, \gamma\xi\right)| \leq C_\alpha$$

therefore $\|r_h(\cdot, \gamma\xi)\|_{L^2}$ is uniformly bounded with respect to h, γ and ξ . Notice, by (11) and since $\psi = 1$ on $\{(m, v) \in T^*X; |v| - 1| < 1/4\}$, we have $\psi(m, d\phi_{\gamma\xi}(m)) = 1$. We thus deduce that there exists $C > 0$ uniform in h, γ, ξ such that

$$\left\| e^{i\phi_{\gamma\xi}/h} \left(\chi E_0\left(\frac{n}{2}; \cdot, \gamma\xi\right) a(\cdot, d\phi_{\gamma\xi}(\cdot)) + hr_h(\cdot, \gamma\xi) \right) \right\|_{L^2} \leq C.$$

Consequently, using again $\delta_\Gamma < n/2$ with $|D\gamma(\xi)| \leq Ce^{-d(o, \gamma o)}$,

$$\sum_{\gamma \in \Gamma} \left\| e^{i\phi_{\gamma\xi}/h} |D\gamma(\xi)|^{s_h} \left(\chi E_0\left(\frac{n}{2}; \cdot, \gamma\xi\right) a(\cdot, d\phi_{\gamma\xi}(\cdot)) + hr_h(\cdot, \gamma\xi) \right) \right\|_{L^2} < \infty$$

and we deduce that

$$\begin{aligned} A_h \chi E_X(s_h; m, \xi) &= \sum_{\gamma \in \Gamma} A_h(\chi e^{i\phi_{\gamma\xi}/h} E_0\left(\frac{n}{2}; \cdot, \gamma\xi\right))(m) |D\gamma(\xi)|^{s_h} \\ &= \sum_{\gamma \in \Gamma} e^{i\frac{\phi_{\gamma\xi}(m)}{h}} |D\gamma(\xi)|^{s_h} \left(E_0\left(\frac{n}{2}; m, \gamma\xi\right) a(m, d\phi_{\gamma\xi}(m)) + hr_h(m, \gamma\xi) \right). \end{aligned}$$

Integrating this against $\overline{E_X(s_h, \cdot, \xi)} = E_X(\overline{s_h}, \cdot, \xi)$, we have

$$(32) \quad \begin{aligned} & \langle A_h \chi E_X(s_h; m, \xi), E_X(s_h, \cdot, \xi) \rangle \\ &= \int_{\mathcal{F}} \sum_{\gamma \in \Gamma} |D\gamma(\xi)|^n E_0(n; m, \gamma\xi) a(m, d\phi_{\gamma\xi}) dv(m) + \mathcal{O}(h) \\ &+ \int_{\mathcal{F}} \sum_{\gamma \neq \gamma'} e^{i \frac{B_\xi(\gamma m, \gamma' m)}{h}} |D\gamma(\xi)|^{s_h} |D\gamma'(\xi)|^{\overline{s_h}} \left(E_0(1; m, \gamma\xi) E_0(1; m, \gamma'\xi) \right)^{\frac{n}{2}} a(m, d\phi_{\gamma\xi}(m)) dv(m) \end{aligned}$$

where $B_\xi(m, m')$ is the Buseman function of (18) and the $\mathcal{O}(h)$ term comes from the remainder $h \langle \sum_{\gamma \in \Gamma} e^{i\phi_{\gamma\xi}/h} r_h(\cdot, \gamma\xi) |D\gamma(\xi)|^{s_h}, E_X(s_h, \cdot, \xi) \rangle$ which is uniformly bounded in $\xi \in \partial\overline{X}$ by the discussion above. Now it remains to apply Lemma 7 to deal with the off-diagonal term in the second line of (32), this gives that this term is a $\mathcal{O}(h^{n-2\delta_\Gamma})$. Notice that the amplitude $a(m, d\phi_{\gamma\xi}(m))$ here depends on γ , but this does not affect the argument of Lemma 7, since the derivatives of the amplitude are uniformly bounded in γ, ξ by (31). The proof of the Theorem is then finished by combining this with (29). \square

As a corollary, we also get

Corollary 12. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold with $\delta_\Gamma < \frac{n}{2}$. Let $E_X(s; \cdot, \xi)$ and $dv_{\partial\overline{X}}$ be the Eisenstein series and the Riemannian measure on $\partial\overline{X}$ defined with a boundary defining function x as before. If $A \in \Psi_c^0(X)$ is a compactly supported classical pseudo-differential operator of order 0 with principal symbol $a \in C_0^\infty(X, T^*X)$, then setting $s_h = n/2 + i/h$, we have as $h \rightarrow 0$*

$$\int_{\partial\overline{X}} \langle AE_X(s_h; \cdot, \xi), E_X(s_h; \cdot, \xi) \rangle_{L^2(X)} dv_{\partial\overline{X}}(\xi) = \int_{S^*X} a(m, v) d\mu(m, v) + \mathcal{O}(h^{\min(n-2\delta, 1)})$$

where S^*X is the unit cotangent bundle and $d\mu$ the Liouville measure.

Proof. We integrate (28) in the $\xi \in \partial\overline{X}$ variable with respect to $dv_{\partial\overline{X}}$ like in the proof of Corollary 8 by lifting to the covering: we have, modulo $\mathcal{O}(h^{\min(1, n-2\delta)})$,

$$\begin{aligned} & \int_{\partial\overline{X}} \langle AE_X(s_h; \cdot, \xi), E_X(s_h; \cdot, \xi) \rangle_{L^2(X)} dv_{\partial\overline{X}}(\xi) = \\ & \int_{\mathcal{F}} \sum_{\gamma \in \Gamma} \int_{\xi \in \overline{\mathcal{F}} \cap S^n} a(m, d\phi_{\gamma\xi}(m)) E_0(n; m, \gamma\xi) |D\gamma(\xi)|^n dv(m) d\xi = \\ & \int_{\mathcal{F}} \int_{\xi \in \Omega_\Gamma} a(m, d\phi_\xi(m)) E_0(n; m, \xi) dv(m) d\xi \end{aligned}$$

and the last term is exactly $\int_{S^*X} a(m, v) d\mu(m, v)$ by using (14) and (15). \square

4. GUTZWILLER TYPE FORMULA

In this section, we shall give a more precise description of the equidistribution in space when the Eisenstein series are averaged over the boundary. Through the Stone identity (6), this reduces to a sort of Gutzwiller trace formula. We prove the following

Theorem 13. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold and assume that the limit set of Γ has Hausdorff dimension $\delta_\Gamma < n/2$. Let $a \in C_0^\infty(X)$ and \mathcal{C} be the set of all*

closed geodesics, then for all $N \in \mathbb{N}$, one has the expansion as $t \rightarrow \infty$,

$$(33) \quad \begin{aligned} & \int_{\partial \bar{X}} \int_X a(m) |E(\frac{n}{2} + it; m, \xi)|^2 dm d\xi_{\partial \bar{X}} = \text{vol}(S^n) \int_X a dv \\ & + L(t) \sum_{\gamma \in \mathcal{C}} e^{-(\frac{n}{2} + it)\ell(\gamma)} \sum_{k=0}^N \frac{H_k(\ell(\gamma))}{|\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|} t^{-k} \int_\gamma P_k a d\mu + \\ & + L(-t) \sum_{\gamma \in \mathcal{C}} e^{-(\frac{n}{2} - it)\ell(\gamma)} \sum_{k=0}^N \frac{H_k(\ell(\gamma))}{|\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|} (-t)^{-k} \int_\gamma P_k a d\mu + \mathcal{O}(t^{-N-n-1}). \end{aligned}$$

where $H_k(\ell(\gamma))$ is an explicit bounded function of $\ell(\gamma)$, P_k are differential operators of order $2k$ with coefficients uniformly bounded in terms of $\gamma \in \mathcal{C}$, $d\mu$ is the Riemannian measure induced on γ , $R_\gamma \in \text{SO}(n)$ is the holonomy along γ and

$$\begin{aligned} L(t) &= t^{-(n-1)/2} \frac{2^{\frac{n}{2}+3-2it} |\Gamma(it)|^2}{\Gamma(it + \frac{1}{2}) \Gamma(\frac{n}{2} - it)} = \mathcal{O}(t^{-n}) \\ P_0 &= 1, \quad H_0(\ell(\gamma)) = 2^{\frac{n}{2}} \pi^{\frac{n-1}{2}} e^{i\frac{n-1}{4}\pi} \end{aligned}$$

Proof. We write $\mu_t(a)$ for the left hand side of (3). We will assume, without losing generality, that the support of a is small enough so that there exists a fundamental domain $\mathcal{F} \subset \mathbb{H}^{n+1}$ with $\partial \mathcal{F} \cap \text{supp}(a) = \emptyset$. We have by (6)

$$\begin{aligned} \mu_t(a) &= \int_X \int_{\partial X} a(m) E_X(\frac{n}{2} + it; m, y) E_X(\frac{n}{2} - it; m, y) dv(m) dv_{\partial \bar{X}}(y) \\ &= \frac{4\pi t}{|C(\frac{n}{2} + it)|^2} \int_X a(m) d\Pi_X(t; m, m) dv(m) \\ \mu_t(a) &= \frac{4\pi t}{|C(\frac{n}{2} + it)|^2} \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} a(m) d\Pi_0(t; \gamma m, m) dv(m). \end{aligned}$$

where $d\Pi_X$ is the spectral measure of Δ on X and $d\Pi_0$ the spectral measure of Δ on \mathbb{H}^{n+1} . For the $\gamma = \text{Id}$ term, the spectral measure on \mathbb{H}^{n+1} restricted on the diagonal is a constant function given by

$$\alpha(t) := 4\pi t d\Pi_0(t; m, m) = \pi^{-\frac{n}{2}} \frac{\Gamma(n/2) \Gamma(\frac{n}{2} + it) \Gamma(\frac{n}{2} - it)}{\Gamma(n) \Gamma(it) \Gamma(-it)} = |C(\frac{n}{2} + it)|^2 \text{vol}(S^n).$$

The terms with $\gamma \neq \text{Id}$ are actually lower order as $t \rightarrow \infty$: we claim that

$$(34) \quad \sum_{\gamma \in \Gamma \setminus \text{Id}} \int_{\mathcal{F}} a(m) t d\Pi_0(t; \gamma m, m) dv(m) = \mathcal{O}(1)$$

and we will actually obtain a much better description.

From the explicit formula of the resolvent for the Laplacian on hyperbolic space in (7), we have that

$$4\pi t d\Pi_0(t; \gamma m, m) = -4t \text{Im}(F_t(\sigma(d(m, \gamma m))))$$

where $\sigma(r) = 1/\cosh(r)$, $d(\cdot, \cdot)$ denotes the hyperbolic distance and

$$F_t(\sigma) := M(t)\sigma^{\frac{n}{2}+it} \int_0^1 (u(1-u))^{it-\frac{1}{2}}(\sigma(1-2u)+1)^{-\frac{n}{2}-it} du$$

$$M(t) := \pi^{-\frac{n+1}{2}} 2^{1-\frac{n}{2}-it} \frac{\Gamma(\frac{n}{2}+it)}{\Gamma(\frac{1}{2}+it)} = \mathcal{O}(t^{\frac{n-1}{2}})$$

Lemma 14. *Let $\epsilon > 0$, and $\sigma(r) = \cosh(r)^{-1}$, then there exist some smooth functions $c_k \in C^\infty([0, 1-\epsilon])$ such that for all N , we have as $t \rightarrow \infty$*

$$F_t(\sigma(r)) = M(t)2^{-it} e^{-itr} \sigma(r)^{\frac{n}{2}} \left(\sum_{k=0}^N t^{-\frac{1}{2}-k} c_k(\sigma(r)) + t^{-\frac{1}{2}-N-1} R_N(\sigma(r), t) \right).$$

with the remainder R_N satisfying $|\partial_\sigma^\ell R_N(\sigma, t)| \leq C_\ell$, for all $\ell \in \mathbb{N}_0$, $\sigma \in [0, 1-\epsilon]$, $t > t_0 > 0$, and $c_0(\sigma) = \pi^{\frac{1}{2}} e^{-i\pi/4} (1-\sigma^2)^{-n/4}$.

Proof. Assuming that $\sigma < 1-\epsilon$, then in the integral defining $F_t(u)$, one can use the stationary phase method (e.g. [8, Th 3.10]) to get an asymptotic expansion in $t \rightarrow \infty$: the phase has a unique nondegenerate critical point in $(0, 1)$ given by $u_\sigma = \frac{1}{2\sigma}(1+\sigma-\sqrt{1-\sigma^2})$, the Hessian is a smooth function of $\sigma \in [0, 1-\epsilon]$ which is bounded from below by a positive constant depending only on $\epsilon > 0$. Using a cutoff function near the point u_σ and using integration by parts for the non-stationary part, one deduces as $t \rightarrow \infty$

$$\sigma^{\frac{n}{2}+it} \int_0^1 (u(1-u))^{it-\frac{1}{2}}(\sigma(1-2u)+1)^{-\frac{n}{2}-it} du \sim \sigma^{\frac{n}{2}} \sum_{k=0}^{\infty} t^{-\frac{1}{2}-k} c_k(\sigma) \left(\frac{1-\sqrt{1-\sigma^2}}{2\sigma} \right)^{it}$$

with $c_k(\sigma)$ some smooth function of $\sigma \in [0, 1-\epsilon]$ and a straightforward computation gives

$$(35) \quad c_0(\sigma) = \pi^{\frac{1}{2}} e^{-i\pi/4} (1-\sigma^2)^{-n/4}.$$

Since $\frac{1-\sqrt{1-\sigma(r)^2}}{2\sigma(r)} = \frac{1}{2}e^{-r}$, we have an expansion, for all $N \in \mathbb{N}$, as $t \rightarrow \infty$

$$(36) \quad F_t(\sigma(r)) = M(t)2^{-it} e^{-itr} \sigma(r)^{\frac{n}{2}} \left(\sum_{k=0}^N t^{-\frac{1}{2}-k} c_k(\sigma(r)) + \mathcal{O}_\sigma(t^{-\frac{1}{2}-N-1}) \right).$$

with a remainder uniform in $\sigma < 1-\epsilon$, as well as its derivatives with respect to σ . \square

Notice that here $d(m, \gamma m) > C_0$ for some $C_0 > 0$ uniform in m (by the convex co-compactness of Γ), then $\sigma(d(m, \gamma m)) < 1-\epsilon$ for some uniform ϵ and thus $F_t(\sigma(m, \gamma m))$ is smooth in $m \in K$ for any fixed compact set $K \subset \mathbb{H}^{n+1}$.

The displacement function $r(m, \gamma m)$, or more precisely $\sinh(\frac{1}{2}r(m, \gamma m))$, is easily computed in dimension $n+1=2$, it is given by $\sinh(\frac{1}{2}r(m, \gamma m)) = \cosh(d(m, \text{axis}(\gamma))) \sinh(\frac{1}{2}\ell(\gamma))$. In higher dimension it is more complicated. Consider the half plane model $\{x > 0, y \in \mathbb{R}^n\}$ of \mathbb{H}^{n+1} . We can assume that ∞ is not in the limit set of Γ , so that a fundamental domain can be taken in a ball $x^2 + |y|^2 \leq R$ for some R . For all $\gamma \in \Gamma$, there is an isometry h_γ such that

$$(37) \quad h_\gamma \circ \gamma \circ h_\gamma^{-1}(x, y) = e^{\ell(\gamma)}(x, R_\gamma y)$$

where $\ell(\gamma) > 0$ is the translation length of γ and $R_\gamma \in \text{SO}(n)$ the holonomy associated to γ . Since $\sigma(d(m, \gamma m)) = \sigma(d(h_\gamma m, h_\gamma \circ \gamma \circ h_\gamma^{-1} h_\gamma m))$, one has

$$(38) \quad \sigma(d(h_\gamma^{-1} m, \gamma \circ h_\gamma^{-1} m)) = e^{-\ell(\gamma)} \frac{2x^2}{x^2(1 + e^{-2\ell(\gamma)}) + |(e^{-\ell(\gamma)} \text{Id} - R_\gamma)y|^2}$$

where $m = (x, y)$. A consequence of this and Lemma 14 is

Corollary 15. *With h_γ defined by (37), we have as $t \rightarrow \infty$*

$$(39) \quad \int_{\mathcal{F}} a(m) F_t(\sigma(d(m, \gamma m))) dv(m) = \frac{M(t)}{2^{it}} \sum_{k=0}^N t^{-k-\frac{1}{2}} \int_{h_\gamma(F)} e^{-itr_\gamma(m)} b_k(\sigma_\gamma(m)) a(h_\gamma^{-1}(m)) \frac{dm}{x^{n+1}} \\ + \mathcal{O}\left(\max_{m \in \text{supp}(a)} e^{-\frac{n}{2}d(m, \alpha m)} t^{-N+n-2}\right)$$

where $r_\gamma(m) := d(h_\gamma^{-1} m, \gamma \circ h_\gamma^{-1} m)$, $\sigma_\gamma(m) := \sigma(d(h_\gamma^{-1} m, \gamma \circ h_\gamma^{-1} m))$ and $b_k(\sigma) := \sigma^{\frac{n}{2}} c_k(\sigma)$.

We now want to use stationary phase in the space variable $m \in \mathcal{F}$ in the integral (39). Let us define the following function $\Phi_\gamma(m) := \log(\sigma(d(h_\gamma^{-1} m, \gamma \circ h_\gamma^{-1} m))$, so that

$$(40) \quad dr_\gamma(m) = \frac{d\Phi_\gamma(m)}{\tanh(r_\gamma(m))}.$$

We first calculate the critical points of the phase $r_\gamma(m)$, which in turn are the critical points of the function $\Phi_\gamma(m)$ in the $m = (x, y)$ coordinates of \mathbb{H}^{n+1}

$$(41) \quad d\Phi_\gamma(x, y) = \left(\frac{2|(e^{-\ell(\gamma)} \text{Id} - R_\gamma)y|^2}{x^2(1 + e^{-2\ell(\gamma)}) + |(e^{-\ell(\gamma)} \text{Id} - R_\gamma)y|^2} \right) \frac{dx}{x} \\ - x \left(\frac{2(e^{-\ell(\gamma)} \text{Id} - R_\gamma)^T (e^{-\ell(\gamma)} \text{Id} - R_\gamma)y}{x^2(1 + e^{-2\ell(\gamma)}) + |(e^{-\ell(\gamma)} \text{Id} - R_\gamma)y|^2} \right) \cdot \frac{dy}{x}.$$

Therefore the critical points are located on the line $L = \{y = 0\}$ which is the image of the axis of γ by h_γ . For each slice $x = \text{cst}$, we will integrate in the y variable in (39) by using the stationary phase, we then first need to check that $\det(\text{Hess}(\Phi_\gamma(x, 0))) \neq 0$ for all x , where the Hessian is with respect to the $y \in \mathbb{R}^n$ variable. The Hessian (in y variable) at $y = 0$ is given by

$$\text{Hess}(\Phi_\gamma(x, 0)) = -2 \frac{(e^{-\ell(\gamma)} \text{Id} - R_\gamma)^T (e^{-\ell(\gamma)} \text{Id} - R_\gamma)}{x^2(1 + e^{-2\ell(\gamma)})}$$

and thus, using that $r_\gamma(x, 0) = \ell(\gamma)$,

$$(42) \quad \text{Hess}(-r_\gamma(x, 0)) = \frac{2}{x^2(1 - e^{-2\ell(\gamma)})} (e^{-\ell(\gamma)} \text{Id} - R_\gamma)^T (e^{-\ell(\gamma)} \text{Id} - R_\gamma).$$

with $|\det(e^{-\ell(\gamma)} \text{Id} - R_\gamma)|^2 = |\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|^2 > 0$.

Therefore the phase is non-stationary if the axis of γ does not intersect the support of $a(m)$, contained in the fundamental domain \mathcal{F} , and the integral is then a $\mathcal{O}(t^{-\infty})$ in this case (constants are depending on γ). We shall split the sum over $\gamma \in \Gamma$ into conjugacy classes $[\gamma]$

in the group, as is usual in Selberg type analysis. For each primitive element γ , the conjugacy class of γ corresponds to a primitive geodesic on X , which is given by

$$\pi_\Gamma \left(\bigcup_{\beta\gamma\beta^{-1} \in [\gamma]} \beta(\text{axis}_\gamma) \cap \mathcal{F} \right)$$

where $\pi_\Gamma : \mathbb{H}^{n+1} \rightarrow \Gamma \backslash \mathbb{H}^{n+1}$ is the natural projection, $[\gamma]$ denotes the conjugacy class of γ and axis_γ is the axis of γ . Moreover, there are only finitely many geodesics $\beta(\text{axis}_\gamma)$ that intersect \mathcal{F} , we then partition $[\gamma]$ into $A_1(\gamma) \cup A_2(\gamma)$ so that $\alpha = \beta\gamma\beta^{-1} \in A_2(\gamma)$ iff $d(\beta(\text{axis}_\gamma), \mathcal{F}) > 1$. If γ is not primitive, one can do the same thing with γ_0^k for some primitive γ_0 .

We first show the

Lemma 16. *Let $\gamma \in \Gamma$ be a primitive element, then that for all $N > 0$ there is C_N depending only on N such that for all $\alpha \in A_2(\gamma)$*

$$(43) \quad \left| \int_{\mathcal{F}} a(m) F_t(\sigma(d(m, \alpha m))) dv(m) \right| \leq C_N t^{-N+(n-1)/2} \max_{m \in \text{supp}(a)} e^{-\frac{n}{2}d(m, \alpha m)}.$$

Proof. For $\alpha \in A_2(\gamma)$, we then use non-stationary phase to evaluate (39), i.e. integration by parts:

$$(44) \quad \begin{aligned} & \int_{h_\alpha(\mathcal{F})} e^{-itr_\alpha(m)} b_k(\sigma_\alpha(m)) a(h_\alpha^{-1}(m)) \frac{dx dy}{x^{n+1}} \\ &= -\frac{1}{it} \int_{h_\alpha(\mathcal{F})} \frac{\partial_x(e^{-itr_\alpha(m)})}{\partial_x \Phi_\alpha(m)} \tanh(r_\alpha(m)) b_k(\sigma_\alpha(m)) a(h_\alpha^{-1}(m)) \frac{dx dy}{x^{n+1}} \\ &= \frac{1}{it} \int_{h_\alpha(\mathcal{F})} e^{-itr_\alpha(m)} (x \partial_x - n) \left(\frac{\tanh(r_\alpha(m)) b_k(\sigma_\alpha(m)) a(h_\alpha^{-1}(m))}{x \partial_x \Phi_\alpha(m)} \right) \frac{dx dy}{x^{n+1}} \\ &= \frac{1}{(it)^N} \int_{h_\alpha(\mathcal{F})} e^{-itr_\alpha(m)} \left((x \partial_x - n) \frac{\tanh(r_\alpha(m))}{x \partial_x \Phi_\alpha} \right)^N \left(b_k(\sigma_\alpha(m)) a(h_\alpha^{-1}(m)) \right) \frac{dx dy}{x^{n+1}} \end{aligned}$$

where $N \in \mathbb{N}$ will be chosen later. To estimate the terms in the last integral, we need an expression for h_α : if $p_\alpha^1, p_\alpha^2 \in \mathbb{R}^n$ are the two fixed points of α , define $p_\alpha := p_\alpha^2 - p_\alpha^1$, then h_α will be the composition of the translation $t_\alpha : m \rightarrow m - p_\alpha^1$ with the reflection in the sphere $\{m; |m - p_\alpha| = |p_\alpha|\}$ followed by the dilation $m \rightarrow |p_\alpha|^{-1}m$

$$(45) \quad g_\alpha : m \mapsto \frac{p_\alpha}{|p_\alpha|} + |p_\alpha| \frac{(m - p_\alpha)}{|m - p_\alpha|^2}, \quad p_\alpha = p_\alpha^2 - p_\alpha^1$$

and this defines $h_\alpha := g_\alpha \circ t_\alpha$. This map is an orientation reversing isometry of \mathbb{H}^{n+1} which maps p_α^1 to 0 and p_α^2 to ∞ , it conjugates α to a model form (37). If $m = (x, y) \in \mathbb{R}^+ \times \mathbb{R}^n$ are coordinates on \mathbb{H}^{n+1} , then

$$(46) \quad x(h_\alpha(m)) = \frac{|p_\alpha|x}{|m - p_\alpha^2|^2}, \quad y(h_\alpha(m)) = \frac{p_\alpha}{|p_\alpha|} + |p_\alpha| \frac{(y - p_\alpha^2)}{|m - p_\alpha^2|^2}.$$

In particular, one has $\max_{m \in \text{supp}(a)} |h_\alpha(m)| \leq R$ for some R depending only on $\text{supp}(a)$. We notice that there exists $C > 0$ uniform in γ, α such that for all γ, α

$$(47) \quad \min\{|y|; m = (x, y) \in h_\alpha(\text{supp}(a)), \alpha \in A_2(\gamma)\} \geq C.$$

Indeed, by definition of $A_2(\gamma)$ and the fact that h_α is an isometry, the hyperbolic distance between $h_\alpha(\text{supp}(a))$ and the line $y = 0$ is bounded below by 1, but $C_0 > x(h_\alpha(m)) > C_1 |p_\alpha|$ for some $C_0, C_1 > 0$ depending only on the support of $\text{supp}(a)$. Then for $\epsilon > 0$ small and α

such that $|p_\alpha| \geq \epsilon > 0$, there is a constant C depending on ϵ such that (47) holds for those α . For the remaining α for which $|p_\alpha| \leq \epsilon$, it is easily seen from the expression (45) that $|h_\alpha(m)| = 1 + O(\epsilon)$ for $m \in \text{supp}(a)$ and therefore if ϵ is small enough $|y(h_\alpha(m))| > 1/2$.

We now want to bound the terms in the right hand side of (44). Using that $|m|$ is uniformly (with respect to α) bounded on $h_\alpha(\text{supp}(a))$ and using (41), (47), we deduce that there exists $C_N > 0$ depending only on N and $C > 0$ such that for all $[\gamma]$, all $\alpha \in [\gamma]$, and all $j \leq N$

$$\begin{aligned} \min_{m \in h_\alpha(\text{supp}(a))} |(x\partial_x)\Phi_\alpha(m)| &\geq C, & \sup_{m \in h_\alpha(\text{supp}(a))} |Y_1 \dots Y_j \Phi_\alpha(m)| &\leq C_N, \\ \sup_{m \in h_\alpha(\text{supp}(a))} |Y_1 \dots Y_j \sigma_\alpha(m)^{\frac{n}{2}}| &\leq C_N \sup_{\text{supp}(a)} \sigma^{\frac{n}{2}}(d(m, \alpha m)) \\ \sup_{m \in h_\alpha(\text{supp}(a))} |Y_1 \dots Y_j \sigma_\alpha(m)| &\leq C_N \sup_{\text{supp}(a)} \sigma(d(m, \alpha m)) \end{aligned}$$

if $Y_i \in \{x\partial_x, x\partial_y\}$ (recall that $\sigma_\alpha = e^{\Phi_\alpha}$). Since $g_\alpha^{-1}(m) = |p_\alpha|g_\alpha(|p_\alpha|m)$ and

$$dg_\alpha(m) = |p_\alpha| \left(\frac{\text{Id}}{|m - p_\alpha|^2} - 2 \frac{(m - p_\alpha)\langle m - p_\alpha, \cdot \rangle}{|m - p_\alpha|^4} \right)$$

then we have

$$dg_\alpha^{-1}(g_\alpha(m)) = |p_\alpha|^{-1} (|m - p_\alpha|^2 \text{Id} - 2(m - p_\alpha)\langle m - p_\alpha, \cdot \rangle)$$

and combining with (46) and using $dh_\alpha(m) = dg_\alpha(m - p_\alpha^1)$

$$(48) \quad h_\alpha^*(xY)(m) = Y - 2 \frac{(m - p_\alpha^2)}{|m - p_\alpha^2|} \left\langle \frac{m - p_\alpha^2}{|m - p_\alpha^2|^2}, Y \right\rangle, \quad Y \in \{\partial_x, \partial_y\}.$$

Combining with (39), this shows that there is C_N depending only on N such that for all $\alpha \in A_2(\gamma)$

$$\begin{aligned} &\left| \int_{\text{supp}(a)} a(m) F_t(\sigma(d(m, \alpha m))) dv(m) \right| \\ &\leq C_N t^{-N+(n-1)/2} \int_{\text{supp}(a)} \sigma(d(m, \alpha m))^{\frac{n}{2}} \max_{j \leq N} |(h_\alpha^*(x\partial_x))^j a(m)| dv(m). \end{aligned}$$

and by (48) we have $\sup_{m \in \text{supp}(a)} |(h_\alpha^*(x\partial_x))^j a(m)| \leq C_N$ for some $C_N > 0$ independent of α, γ . We thus have proved the Lemma. \square

We need to consider now the finitely many $\alpha \in A_1(\gamma)$ which contributes to the stationary phase (notice that there are roughly $\mathcal{O}(\ell(\gamma))$ elements in $A_1(\gamma)$). Notice that, by the arguments above, the compact set $h_\alpha(\text{supp}(a))$ remains in a bounded region $\{C \geq |y|, C > |x| > 1/C\}$ of \mathbb{H}^{n+1} , uniformly in $\gamma, \alpha \in A_1(\gamma)$. This follows since the distance in the boundary between the fixed points of α is uniformly bounded below if the axis of α intersects $\text{supp}(a)$. For the same reason, we obtain from the expression (45) that for $\alpha \in A_1(\gamma)$

$$(49) \quad \max_{m \in h_\alpha(\text{supp}(a))} |\partial_x^\mu \partial_y^\nu h_\alpha^{-1}(m)| \leq C$$

for some C depending on μ, ν but independent of γ, α . The phase in the integral

$$I_{k,\alpha}(t) := \int_{h_\alpha(\mathcal{F})} e^{-itr_\gamma(x,y)} b_k(\sigma_\alpha(x,y)) a(h_\alpha^{-1}(x,y)) \frac{dx dy}{x^{n+1}},$$

is stationary at $y = 0$, and thus integrating on each slice $x = \text{cst}$, we have by stationary phase ([13, Th. 7.7.5])

$$(50) \quad \left| I_{k,\alpha}(t) - e^{-it\ell(\gamma)} \sum_{j=0}^{N-1} t^{-\frac{n}{2}-j} \int_{\{y=0\} \cap h_\alpha(\mathcal{F})} A_{2j}(x, y, \partial_y) [a(h_\alpha^{-1}(x, y)) b_k(\sigma_\alpha(x, y))] \frac{dx}{x^{n+1}} \right| \\ \leq C t^{-\frac{n}{2}-N} \sum_{|\beta| \leq 2N+2n} \|\partial^\beta (b_k \circ \sigma_\alpha \circ a \circ h_\alpha^{-1})\|_{L^\infty}.$$

for some smooth differential operator $A_{2j}(x, y, \partial_y)$ of order $2j$. By [13, Th. 7.7.5], the constant C is shown to be bounded uniformly with respect to α, γ if

$$\|r_\alpha\|_{C^{3N+3n}(h_\alpha(\text{supp}(a)))} \quad \text{and} \quad \sup_{(x,y) \in h_\alpha(\text{supp}(a))} \frac{|y|}{|dr_\alpha(x, y)|}$$

are bounded uniformly with respect to $\alpha \in A_1(\gamma)$ and with respect to $[\gamma]$. The differential operators A_{2j} have coefficients bounded uniformly in α, γ if

$$|\det(\text{Hess}(r_\alpha))|^{-1} \quad \text{and} \quad \|r_\alpha\|_{C^{3N+3n}(h_\alpha(\text{supp}(a)))}$$

are bounded uniformly. The uniform boundedness of semi-norms of r_α in a fixed compact set of \mathbb{H}^{n+1} (containing $h_\alpha(\text{supp}(a))$) follows directly from the expressions (40) and (41), while determinant of Hessian boundedness is clear from (42) since $\ell(\gamma)$ is bounded below by a positive constant (the radius of injectivity). By the explicit formula (41), and using (40), it is direct that $|dr_\alpha(x, y)| \geq C|y|$ for some C uniform in α and (x, y) in a fixed compact set of \mathbb{H}^{n+1} (using again that $\ell(\gamma)$ is bounded below by a uniform positive constant). The constant C in (50) is therefore uniform in $\alpha \in A_1(\gamma)$ and the conjugacy class $[\gamma]$.

Now we claim that the amplitudes $\|\partial^\beta (b_k \circ \sigma_\alpha \circ a \circ h_\alpha^{-1})\|_{L^\infty} \leq C \max_{m \in \text{supp}(a)} e^{-\frac{n}{2}d(m, \alpha m)}$ with some C uniform in $[\gamma], \alpha \in A_1(\gamma)$. By (49), the terms involving $a \circ h_\alpha$ are clearly bounded uniformly. Moreover, since $b_k(\sigma) = \sigma^{\frac{n}{2}} c_k(\sigma)$, it suffices to bound the derivatives of $\Phi_\alpha = \log(\sigma_\alpha)$. By using that $h_\alpha \text{supp}(a)$ is in a uniform compact set of \mathbb{H}^{n+1} with respect to $\alpha \in A_1(\gamma)$ and $[\gamma]$, we have $|\partial^\beta d\Phi| \leq C$ and thus

$$\sup_{m \in h_\alpha(\text{supp}(a))} |\partial^\beta b_k(\sigma_\alpha(m))| \leq \sup_{m \in \text{supp}(a)} e^{-\frac{n}{2}(d(m, \alpha m))}.$$

Similarly all terms in the left hand side of (50) is bounded by $\sup_{m \in \text{supp}(a)} e^{-\frac{n}{2}(d(m, \alpha m))}$. Notice that $\sigma_\alpha(m) = \cosh^{-1}(\ell(\gamma))$ is constant on $m \in \{y = 0\}$, and thus we have proved

Lemma 17. *With $I_{k,\alpha}$ defined in (50), we have that for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that for all $[\gamma]$, all $\alpha \in A_1(\gamma)$, and all $t > 1$*

$$\left| I_{k,\alpha}(t) - \frac{c_k((\cosh \ell(\gamma))^{-1})}{(\cosh \ell(\gamma))^{\frac{n}{2}}} e^{-it\ell(\gamma)} \sum_{j=0}^{N-1} t^{-\frac{n}{2}-j} \int_{\text{axis}(\gamma) \cap \mathcal{F}} Q_{2j} a(m) d\mu \right| \\ \leq C t^{-\frac{n}{2}-N} \sup_{m \in \text{supp}(a)} e^{-\frac{n}{2}(d(m, \alpha m))}.$$

for some differential operator Q_{2j} with coefficients bounded uniformly in $\alpha, [\gamma]$, and $d\mu$ is the measure induced by the riemannian measure on $\text{axis}(\gamma)$.

In particular from (50) and the usual expression of $A_0(x, y, \partial_y)$ (see Th 3.14 in [8]) and (42), we deduce

$$I_{k,\alpha}(t) = e^{-i(t+\frac{n}{2})\ell(\gamma)} t^{-\frac{n}{2}} \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{n\pi}{4}} (\tanh \ell(\gamma))^{\frac{n}{2}} c_k (\cosh(\ell(\gamma))^{-1})}{|\det(\text{Id} - e^{-\ell(\gamma)} R_\gamma^{-1})|} \int_{\text{axis}(\alpha) \cap \mathcal{F}} a(m) d\mu \\ + \mathcal{O}\left(\sup_{m \in \text{supp}(a)} e^{-\frac{n}{2}d(m,\alpha m)} t^{-n/2-1}\right).$$

where $d\mu$ is the measure induced by the riemannian measure on $\text{axis}(\alpha)$. From (35), one also has

$$c_0(\cosh(\ell(\gamma))^{-1}) = \sqrt{\pi} e^{-i\pi/4} (\tanh \ell(\gamma))^{-n/2}.$$

Summing Lemma 17 and 16 over all $\alpha \in [\gamma]$, we obtain the integrals over the closed geodesic associated to the conjugacy class $[\gamma]$, and then summing over the set of conjugacy classes $[\gamma]$, we obtain the result claimed in the Theorem. The sums over the group converges since each estimate involving $\alpha \in [\gamma]$ has a bound in terms of $\sup_{\text{supp}(a)} e^{-\frac{n}{2}(d(m,\alpha m))}$, and the series

$$\sum_{[\gamma]} \sum_{\alpha \in [\gamma]} e^{-\frac{n}{2}(d(m,\alpha m))}$$

converges uniformly for m in compact subsets of \mathbb{H}^{n+1} . This ends the proof. \square

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