2-dipath and oriented L(2,1)-labelings of some families of oriented planar graphs

Sagnik Sen

2-dipath and oriented $L(2,1)$-labelings of some families of oriented planar graphs

Sagnik Sen [1,2]

Bordeaux University, LaBRI
351, cours de la Libération
F-33405 Talence Cedex

Abstract

In this paper, we improve the existing bounds [2] for 2-dipath and oriented $L(2,1)$ span for the class of planar graphs with girth $5, 11, 16$, cactus, wheels and leaf independent Halin graphs. Some of these bounds are tight.

Keywords: 2-dipath $L(2,1)$-labeling, oriented $L(2,1)$-labeling, girth, planar graph, cactus, Halin graph.

1 Introduction

In this paper, p and q are non-negative integers. A *k-$L(p,q)$-labeling* of a graph G is a function ℓ from the vertex set $V(G)$ to the set $\{0, 1, ..., k\}$ such that
- $| l(u) - l(v) | \leq p$ if u and v are at distance 1 in G,
- $| l(u) - l(v) | \leq q$ if u and v are at distance 2 in G.

[1] This work is supported by ANR GRATEL project ANR-09-blan-0373-01
[2] Email: `sagnik.sen@labri.fr`

To distinguish close and very close transmitters in a wireless communication system, Griggs and Yeh [6] proposed a variation of the Frequency Assignment Problem (or simply FAP) by introducing the above notion for the special case $(p, q) = (2, 1)$ which was generalized by Georges and Mauro [4] afterwards.

A common feature of graph theoretic models for FAP is that communication is assumed to be possible in both directions (duplex) between two radio transmitters and, therefore, these models are based on undirected graphs. But in reality, to model FAP on directed or oriented graphs could be interesting as pointed by Aardal *et al* [1] in their survey.

An *oriented graph* is a directed graph with no cycle of size 1 or 2.

There are two different oriented versions of $L(p, q)$-labeling, namely, 2-dipath $L(p, q)$-labeling, introduced by Chang *et al* [3], and oriented $L(p, q)$-labeling, introduced by Gonçalves, Raspaud and Shalu [5].

A *2-dipath k-$L(p, q)$-labeling* of an oriented graph $\vec{G}$ is a function l from the vertex set $V(\vec{G})$ to the set $\{0, 1, ..., k\}$ such that

- $| l(u) - l(v) | \leq p$ if u and v are adjacent in $\vec{G}$,

- $| l(u) - l(v) | \leq q$ if u and v are connected by a directed 2-path in $\vec{G}$.

The *2-dipath span* $\vec{\lambda}_{p,q}(\vec{G})$ of an oriented graph $\vec{G}$ is defined as $max\{k \mid \vec{G}$ has a 2-dipath k-$L(p, q)$-labeling $\}$. The *2-dipath span* $\vec{\lambda}_{p,q}(G)$ of an undirected graph G is defined as $max\{\vec{\lambda}_{p,q}(\vec{G}) \mid \vec{G}$ is an orientation of $G\}$. The *2-dipath span* $\vec{\lambda}_{p,q}(\mathcal{F})$ of a family of (oriented or undirected) graphs is defined as $max\{\vec{\lambda}_{p,q}(H) \mid H \in \mathcal{F}\}$.

An *oriented k-$L(p, q)$-labeling* of an oriented graph $\vec{G}$ is a function l from the vertex set $V(\vec{G})$ to the set $\{0, 1, ..., k\}$ such that

- l is a 2-dipath k-$L(p, q)$-labeling of G,

- If $\vec{xy}$ and $\vec{uv}$ are two arcs in $\vec{G}$ then, $l(x) = l(v)$ implies $l(y) \neq l(u)$.

The *oriented spans* $\lambda^o_{p,q}(\vec{G})$, $\lambda^o_{p,q}(G)$ and $\lambda^o_{p,q}(\mathcal{F})$ are defined similarly as above.

The *girth* of a graph is the length of its shortest cycle. We denote by $\mathcal{P}_g$ the family of planar graphs with girth at least g. A *cactus* is a connected graph in which any two cycles can have at most one vertex in common. We denote by $\mathcal{C}$ the family of all cacti.

A *Halin graph* is a planar graph constructed from a plane embedding of a tree with at least four vertices and with no vertices of degree 2, by connecting all the leaves with a cycle (called *outercycle*) that passes around the tree in the natural cyclic order defined by the embedding of the tree. An orientation

$\vec{H}$ of a Halin graph H is said to be *leaf independent* if no two vertices on the outercycle are joined by a directed 2-path in the tree. Let $\mathcal{H}_{li}$ denote the family of all oriented leaf independent Halin graphs. A *wheel* is a Halin graph whose tree is a star. Let $\mathcal{W}$ denote the family of all wheels.

Calamoneri and Sinaimeri [2] studied 2-dipath $L(2,1)$-labeling of some families of oriented planar graphs. In particular, they proved that $\vec{\lambda}_{2,1}(\mathcal{P}_{11}) \leq 12$, $\vec{\lambda}_{2,1}(\mathcal{P}_{16}) \leq 8$, $6 \leq \vec{\lambda}_{2,1}(\mathcal{C}) \leq 8$, and $8 \leq \vec{\lambda}_{2,1}(\mathcal{W}) \leq 9$. They also conjectured that $\vec{\lambda}_{2,1}(\mathcal{P}_5) \leq 5$.

In this paper, we disprove this conjecture, improve the above results and give some new results on 2-dipath and oriented $L(2,1)$-labelings.

From the definitions, we have the following:

Lemma 1.1 *For every (undirected or oriented graph) G and every $p, q \geq 0$, $\vec{\lambda}_{p,q}(G) \leq \lambda^o_{p,q}(G)$.*

A *homomorphism* f of an oriented graph $\vec{G}$ to an oriented graph $\vec{H}$ is a mapping $f : V(\vec{G}) \longrightarrow V(\vec{H})$ such that $\vec{xy} \in A(\vec{G})$ implies $\overrightarrow{f(x)f(y)} \in V(\vec{H})$.

Lemma 1.2 *If there is a homomorphism $f : \vec{G} \longrightarrow \vec{H}$ then $\vec{\lambda}_{p,q}(\vec{G}) \leq \vec{\lambda}_{p,q}(\vec{H})$, for every $p, q \geq 0$.*

In this paper, we shall use the following from [7].

A *pattern* Q of length k is a word $Q = q_0 q_1 ... q_{k-1}$ with $q_i \in \{+, -\}$ for every i, $0 \leq i \leq k-1$. A *Q-walk* in a digraph $\vec{G}$ is a walk $P = x_0 x_1 ..... x_k$ such that for every i, $0 \leq i \leq k-1$, $x_i x_{i+1} \in E(\vec{G})$ if $q_i = +$ and $x_{i+1} x_i \in E(G)$ otherwise. For $X \subseteq V(\vec{G})$ we denote by $N_Q(X)$ the set of all vertices y such that there exists a Q-walk going from some vertex $x \in X$ to y. We then say that a digraph $\vec{G}$ is *k-nice* if for every pattern Q of length k and for every vertex $x \in V(\vec{G})$ we have $N_Q(\{x\}) = V(\vec{G})$. In other words, a digraph is k-nice if for all pairs of vertices x, y (allowing $x = y$) there is a k-walk from x to y for each of the 2^k possible oriented patterns. Observe that if a digraph G is k-nice for some k, then it is k'-nice for every $k' \geq k$.

Now we state a theorem from [7].

Theorem 1.3 *Let H_k be a k-nice oriented graph, $k \geq 3$. Every oriented graph whose underlying graph is in $\mathcal{P}_{5k-4}$ admits a homomorphism to H_k.*

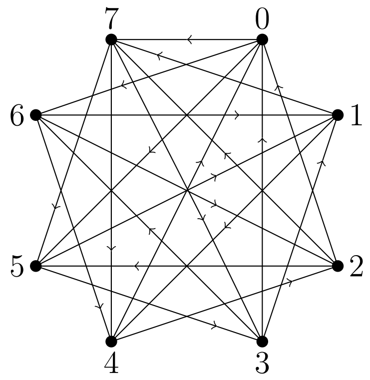

Fig. 1. $\vec{B}$ is a 4-nice graph.

2 Subfamilies of Planar graphs

In this section, we state our results. Due to space limititions, almost all the proofs are omitted.

Theorem 2.1 $\vec{\lambda}_{2,1}(\mathcal{P}_5) \geq 6$.

To prove this theorem, we construct a planar graph of order 241 and girth 5 that does not admit any oriented $5 - L(2,1)$-labeling. This results disproves Calamoneri and Sinaimeri's conjecture.

Theorem 2.2 $\vec{\lambda}_{2,1}(\mathcal{P}_{11}) \leq \lambda^o_{2,1}(\mathcal{P}_{11}) \leq 10$.

To prove this theorem we construct a 3-nice graph that admits an oriented 10-$L(2,1)$-labeling. The result then follows from Theoren 1.3, Lemma 1.2 and Lemma 1.1.

Theorem 2.3 $\vec{\lambda}_{2,1}(\mathcal{P}_{16}) \leq \lambda^o_{2,1}(\mathcal{P}_{16}) \leq 7$.

To prove this theorem we show that the graph $\vec{B}$ from Figure 1 is 4-nice. Clearly, $\lambda^o_{2,1}(B) = 7$ and the result follows from Theorem 1.3, Lemma 1.2 and Lemma 1.1.

Theorem 2.4 $\vec{\lambda}_{2,1}(\mathcal{C}) = \lambda^o_{p,q}(\mathcal{C}) = 7$.

Proof (Sketch) To prove this theorem, we first construct an example of a cactus $\vec{C}$ of order 600 with $\vec{\lambda}_{2,1}(\vec{C}) \geq 7$.

We then prove that every oriented cactus admits a homomorphism to the oriented graph $\vec{B}$ from Figure 1. If not, let $\vec{G}$ be a minimal counterexample.

If there is a degree one vertex v in $\vec{G}$ such that $v \in N^+(u)$ (or $v \in N^-(u)$) for some $u \in V(\vec{G})$, then $\vec{G}[V(\vec{G}) \setminus \{v\}]$ is also a cactus. As $\vec{G}$ is a mini-

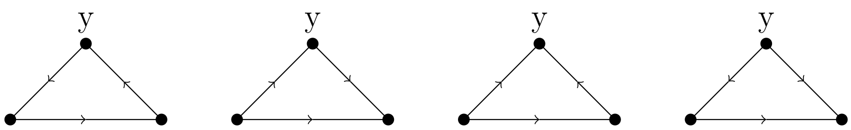

Fig. 2. Four different 3-cycles with respect to the vertex y.

mal counterexample, there is a homomorphism f from $\vec{G}[V(\vec{G} \setminus \{v\}]$ to $\vec{B}$. Now, since all the vertices of $\vec{B}$ have at least one in-neighbour and one out-neighbour, we can map v to any vertex $x \in N^+(f(u))$ (or $x \in N^-(f(u))$).

If there is no degree one vertex in $\vec{G}$ then there is at least one cycle $\vec{C} \subseteq \vec{G}$ such that exactly one vertex z of the cycle $\vec{C}$ has degree greater than 2 (since $\vec{G}$ cannot be a cycle). Now, $\vec{G}[V(\vec{G}) \setminus \{V(\vec{C}) \setminus \{z\}\}]$ is a cactus and, since $\vec{G}$ is a minimal counterexample, there is a homomorphism f from $\vec{G}[V(\vec{G}) \setminus \{V(\vec{C}) \setminus \{z\}\}]$ to $\vec{B}$.

Claim: Let $\vec{O}$ be an oriented cycle. Given any $x \in V(\vec{O})$ and $y \in V(\vec{B})$, there exists a homomorphism $h : \vec{O} \longrightarrow \vec{B}$ such that $h(x) = y$.

If the claim is true then we can extend f to a homomorphism of $\vec{G}$ to $\vec{B}$, which will end the proof.

Proof of the claim: We know that $\vec{B}$ is 4-nice. It is then enough to prove that for any oriented 3-cycle $\vec{T}$ and given any $x \in V(\vec{T})$ and $y \in V(\vec{B})$, there exists a homomorphism $h : \vec{O} \longrightarrow \vec{B}$ such that $h(x) = y$. In other words, we need to show that for each $y \in V(\vec{B})$, the 3-cycles in Figure 2 are subgraphs of $\vec{B}$, which can easily be checked.

Clearly, $\lambda^o_{p,q}(B) = 7$. Then, we use Lemma 1.1 and Lemma 1.2 to conclude the proof. □

Theorem 2.5 $\vec{\lambda}_{2,1}(\mathcal{W}) = 8$.

To prove this, we partition $\mathcal{W}$ in three different subfamilies and define for each of them a procedure to construct a 2-dipath 8-$L(2,1)$-labeling of any graph in it. This will prove that $\vec{\lambda}_{2,1}(\mathcal{W}) \leq 8$. The lower bound was proved in [2].

Theorem 2.6 $7 \leq \vec{\lambda}_{2,1}(\mathcal{H}_{li}) \leq 8$.

Proof (Sketch) For any $\vec{G} \in \mathcal{H}_{li}$ with outercycle $\vec{O}$ we give a 4-$\vec{\lambda}_{2,1}$-labeling f of $\vec{O}$ with $\mid \{v \in V(\vec{O}) | f(v) = 3\} \mid \leq 2$. Notice that $\vec{T} = \vec{G}[V(\vec{G}) \setminus \{x \in V(\vec{O}) | f(x) \neq 4\}]$ is a tree. We give a 2-dipath $L(2,1)$-labeling f' of $\vec{T}$ using only labels from $\{4, 6, 8\}$ such that f' and f restricted to $\{x \in V(\vec{O}) | f(x) \neq 4\}$ together gives a 8-$\vec{\lambda}_{2,1}$-labeling of $\vec{G}$. This gives the upper bound. For the lower

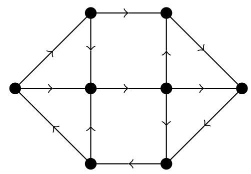

Fig. 3. The leaf independent Halin graph $\vec{H}_{li}$

bound, we show that $\vec{\lambda}_{2,1}(\vec{H}_{li}) = 7$ ($\vec{H}_{li}$ is the graph in Figure 3).

□

3 Conclusion

In this paper we consider the problem of determining $\vec{\lambda}_{2,1}$ and $\lambda^o_{2,1}$ of some families of planar graphs. In some cases, we provide improved bounds, some of them being tight. However, many problems remain open and there is room for improvements and many other interesting questions on this topic.

References

[1] Aardal, K., S. van Hoesel, A. Koster, C. Mannino and A. Sassano, *Models and solution techniques for frequency assignment problems*, Annals of Operations Research **153** (2007), pp. 79–129.

[2] Calamoneri, T. and B. Sinaimeri, *$L(2,1)$-Labeling of Oriented Planar Graphs (Extended Abstract)*, in: *Proc. 12th Italian Conference on Theoretical Computer Science*, (2010).

[3] Chang, G., J. Chen, D. Kuo and S. Liaw, *Distance-two labelings of digraphs*, Discrete Applied Mathematics **155** (2007), pp. 1007–1013.

[4] Georges, J. and D. Mauro, *Generalized vertex labelings with a condition at distance two*, Congressus Numerantium (1995), pp. 141–160.

[5] Gonçalves, D., M. Shalu and A. Raspaud, *On Oriented Labelling Parameters*, Formal Models, Languages and Applications **66**, pp. 34–45.

[6] Griggs, J. and R. Yeh, *Labelling graphs with a condition at distance 2*, SIAM Journal on Discrete Mathematics **5** (1992), pp. 586–595.

[7] Hell, P., A. Kostochka, A. Raspaud and E. Sopena, *On nice graphs*, Discrete Mathematics **234** (2001), pp. 39–51.