Global in time solution to the incompressible Euler Equations on R^n. An elementary approach.

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GLOBAL IN TIME SOLUTION TO THE INCOMPRESSIBLE EULER EQUATIONS ON $\mathbb{R}^n$. AN ELEMENTARY APPROACH.

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ABSTRACT. In this paper we prove a theorem of global time-extension for the local classical solution of Euler's evolution problem in $\mathbb{R}^n$ with $n \geqslant 2$ for incompressible fluids subjected to external forces and regular initial conditions. This will be achieved by expressing the boundedness of the time derivative of the L^∞ solution norm.

1. INTRODUCTION

Let's consider the problem of evolution for the Euler equations in $\mathbb{R}^n$

$$\begin{cases} \partial_t u = -u \cdot \nabla u - \nabla p + f & (x \in \mathbb{R}^n, t \geqslant 0) \\ \nabla \cdot u = 0 & (x \in \mathbb{R}^n, t \geqslant 0) \\ u(x,0) = u_0(x) & (x \in \mathbb{R}^n) \end{cases} \tag{1.1}$$

where:

$$u(x,t), f(x,t) \in \mathbb{R}^n, \quad p(x,t) \in \mathbb{R} \quad u_0 \in C^\infty(\mathbb{R}^n), \quad \nabla \cdot u_0 = 0$$

$$|\partial_x^\alpha u_0(x)| \leqslant C_{\alpha K} (1+|x|)^{-K}, \qquad x \in \mathbb{R}^n, \quad \forall\, \alpha, K \tag{1.2}$$

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leqslant C_{\alpha m K} (1+|x|+t)^{-K}, \qquad (x,t) \in \mathbb{R}^n \times [0,\infty), \forall\, \alpha, m, K \tag{1.3}$$

The Euler equations describe the time evolution of the speed vector field $u(x,t)$ and the pressure $p(x,t)$ of an incompressible inviscid fluid, depending on the initial velocity $u_0(x)$ and the external force $f(x,t)$.

Equation 1.1 represents the Eulerian description of the flow. The Lagrangian formulation of Euler equation 1.1 describes the flow in term of a volume preserving diffeomorphism, the time dependent map $X : \mathbb{R}^n \to \mathbb{R}^n$:

$$\alpha \mapsto X(\alpha, t), \qquad X(\alpha, 0) = \alpha$$

These maps represent marked fluid particle trajectories, α in label of the particle,

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which can be seen as the location of the particle at time $t = 0$. The fact that the particle travels with velocity u is expressed by the system of ordinary differential equations,

$$\frac{\partial}{\partial t}X(\alpha,t) = u(X(\alpha,t),t), \qquad X(\alpha,0) = \alpha \tag{1.4}$$

The Lagrangian formulation of Euler equations and incomprensibility condition given by 1.1 are respectively the Newton's second law and $\det(\nabla_\alpha X) = 1$, where $\det(\nabla_\alpha X)$ denotes the determinant of $\nabla_\alpha X$, the jacobian matrix of X. [AJM02] The map X defined by Equation 1.4 is a volume preserving C^1-diffeomorphism from $\mathbb{R}^n$ on itself, indeed, since $u \in C^1(\mathbb{R}^n \times [0,T])$, then thanks to Cauchy-Lipschitz Theorem, we deduce that the differential equation 1.4 admits a unique solution $X \in C^1(\mathbb{R}^n \times [0,T])$.

We introduce P the Leray's projection operator with components,

$$P = I - \nabla\Delta^{-1}\nabla\cdot$$

The operator P is a projection: $P^2 = P$, annihilates gradients and map into divergence-free vectors; it is a bounded operator on L^p, $1 < p < \infty$ and commutes with translation. P yields to Helmholtz decomposition, indeed for al $v \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ has a unique orthogonal decomposition:

$$v = Pv + \nabla q$$
$$\nabla \cdot (Pv) = 0, \qquad q = \Delta^{-1}\nabla \cdot v$$
$$Pv, \nabla q \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n).$$

2. Local in time Existence Theorems

In this section we report without proof some results on classical mild solutions, obtained for the first time by Kato and Fujita in the Sobolev spaces $H^s(\mathbb{R}^n)$ for $s \geqslant n/2 - 1$ [HF64], Theorems based on Picard's contraction principle.

For a systematic treatment of the theorems of existence with demonstrations, for example, see [LR02] [AJM02] (proofs are given only for the homogeneous case but they are easily extendable in case of external force $f(x,t) \in \mathcal{S}(\mathbb{R}^{n+1})$).

Definition 2.1. (Divergence-free Sobolev spaces)
Let's define

$$V^s = \{v \in H^s(\mathbb{R}^n) : \nabla \cdot v = 0\} = PH^s(\mathbb{R}^n)$$

Theorem 2.2 (Existence of local solutions to Euler's problem)**.** *Let $s > n/2 + 2$. For all $u_0 \in V^s(\mathbb{R}^n)$, there exists $T^* > 0$ (possibly infinite) and a unique solution*

$$u \in C([0,T^*), V^s(\mathbb{R}^n)) \cap C^1([0,T^*), V^{s-2}(\mathbb{R}^n))$$

and

$$u \in C([0,T^*), C^2(\mathbb{R}^n)) \cap C^1([0,T^*), C(\mathbb{R}^n))$$

for Euler's equation on $\mathbb{R}^n \times (0,T^*)$ *so that* $u(\cdot,0) = u_0$.

Theorem 2.3. *Solution* u *described in Theorem 2.2 is* $u(\cdot,t) \in C(\mathbb{R}^n)$ *for* $t \in (0,T^*)$*, furthermore, for the maximal time of existence, it is* $T^* < \infty$ *if and only if*

$$\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = \infty.$$

Proposition 2.4. *From local solution* $u(x,t)$ *we gain the pressure term resolving Poisson's problem, which is in turn obtained by taking the divergence from the first Euler equation 1.1.*

$$\begin{cases} \Delta p = -\nabla \cdot [(u \cdot \nabla)u] + \nabla \cdot f = -\sum\limits_{j,k} \partial_j u_k \partial_k u_j + \nabla \cdot f & (x \in \mathbb{R}^n, t \geqslant 0) \\ \lim\limits_{|x| \to 0} p \end{cases}$$

3. Control of the L^∞ norm

Theorem 3.1 (Derivative of the L^∞ norm)**.** *Let* u *be a solution in* $[0,T^*)$ *of the problem 1.1 described in the theorem 2.2, be* $t \in (0,T^*)$*, then*

$$\frac{d\,\|u(t)\|_\infty}{dt} \leqslant \|Pf(t)\|_\infty \tag{3.1}$$

Proof. In the problem 1.1 the term $-\nabla p$ has the task to cancel all non-zero divergence components which come from the nonlinear term and the ones due to external force. By applying Helmholtz's decomposition to $u \cdot \nabla u$ and to f we'll get

$$u \cdot \nabla u = P(u \cdot \nabla u) + \nabla q_1$$

$$f = P(f) + \nabla q_2$$

and it is

$$-\nabla q_1 + \nabla q_2 = \nabla p$$

We note that where

$$\nabla \cdot (u \cdot \nabla u) = 0 \quad \Rightarrow \quad u \cdot \nabla u = P(u \cdot \nabla u), \quad \nabla q_1 = 0, \quad \nabla p = \nabla q_2$$

particularly, this will be true where $\nabla u = 0$, infact

$$\nabla u = 0 \quad \Rightarrow \quad 0 = \sum_{j,k} \partial_j u_k \partial_k u_j = \nabla \cdot (u \cdot \nabla u)$$

Let now be $t \in (0, T^*)$, u is a function differentiable in all variables, and be $\tilde{x}$ an absolute maximum point for $|u(\cdot, t)|$, then we'll have

$$\nabla u(\tilde{x}, t) = 0$$

To estimate the derivative of u in $(\tilde{x}, t)$ with respect to time, we are going to substitute in the equation 1.1

$$\frac{\partial u(\tilde{x}, t)}{\partial t} = -u(\tilde{x}, t) \cdot \nabla u(\tilde{x}, t) - \nabla p(\tilde{x}, t) + Pf(\tilde{x}, t) + \nabla q_2(\tilde{x}, t) = Pf(\tilde{x}, t)$$

$$\frac{\partial u(\tilde{x}, t)}{\partial t} = Pf(\tilde{x}, t)$$

from which follows

$$\frac{\partial |u(\tilde{x}, t)|}{\partial t} \leqslant |Pf(\tilde{x}, t)|$$

By Theorem 2.2, $u \in C^1([0, T^*), C(\mathbb{R}^n))$; then the time derivative of $\|u(t)\|_\infty$ exists, thus :

$$\frac{d \|u(t)\|_\infty}{dt} = \lim_{h \to 0} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} = \lim_{h \to 0^+} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h}$$

where $|u(\tilde{x}_h, t+h)|$ is the absolute maximum at time $(t+h)$ reached at $\tilde{x}_h \in \mathbb{R}^n$. Evaluating the difference quotient we can write

$$\frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} = \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)| + |u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} =$$

$$\frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)|}{h} + \frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} =$$

$$= L_1 + L_2$$

where $\tilde{x}_h^*$ is such that

$$\frac{\partial}{\partial s}X(\tilde{x}_h^*, s) = u(X(\tilde{x}_h^*, s), s+t), \quad X(\tilde{x}_h^*, h) = \tilde{x}_h, \quad X(\tilde{x}_h^*, 0) = \tilde{x}_h^*$$

Let's consider L_1. We can write

$$||u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)|| \leqslant$$

$$\leqslant \left| \nabla |u(\tilde{x}_h, t+h)| \frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t+h) \right| |h| + \left| \frac{\partial |u(\tilde{x}_h, t+h)|}{\partial t} \right| |h| + o(|h|)$$

Being $\tilde{x}_h$ a point of maximum at time $(t+h)$, for what we have seen so far, it is

$$\nabla |u(\tilde{x}_h, t+h)| = 0, \qquad \frac{\partial |u(\tilde{x}_h, t+h)|}{\partial t} \leqslant |Pf(\tilde{x}_h, t+h)|$$

furthermore, we have

$$\frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t+h) \equiv \frac{\partial}{\partial s}X(\tilde{x}_h^*, h) = u(X(\tilde{x}_h^*, h), t+h) = u(\tilde{x}_h, t+h)$$

being the derivative along the flow line passing by $\tilde{x}_h$ at time $(t+h)$.
$u \in C(\mathbb{R}^n \times [0, T^*))$ and for $(t+h) < T^*$ it is

$$|u(\tilde{x}_h, t+h)| < \infty$$

from which

$$\nabla |u(\tilde{x}_h, t+h)| \frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t+h) = 0$$

thus

$$||u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)|| \leqslant |Pf(\tilde{x}_h, t+h)| \, |h| + o(|h|)$$

Shifting to limits

$$\lim_{h\to 0} \frac{||u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)||}{|h|} \leqslant$$

$$\leqslant \lim_{h\to 0} |Pf(\tilde{x}_h, t+h)| + \lim_{h\to 0} \frac{o(|h|)}{h} = |Pf(\tilde{x}_0^*, t)| \leqslant \|Pf(t)\|_\infty$$

where

$$\tilde{x}_0^* = \lim_{h\to 0} X(\tilde{x}_h^*, 0)$$

with

$$|u(\tilde{x}_0^*, t)| = |u(\tilde{x}_0, t)|$$

Let's consider now L_2. Being $|u(\tilde{x}, t)|$ absolute maximum at time t we'll have

$$\frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} \leqslant 0 \qquad h > 0$$

shifting to limit notation

$$\lim_{h\to 0^+} \frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} = \lim_{h\to 0+} L_2 \leqslant 0$$

thus

$$\frac{d\left\|u(t)\right\|_\infty}{dt} = \lim_{h\to 0^+} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} \leqslant$$

$$\leqslant \lim_{h\to 0^+} |L_1| + \lim_{h\to 0^+} L_2 \leqslant \lim_{h\to 0^+} |L_1| \leqslant \left\|Pf(t)\right\|_\infty$$

□

Theorem 3.2 (Extension theorem)**.** *In the hypotheses of the Theorem 3.1, the solution u can be extended to the whole real axis of time.*

Proof. According to Theorem 2.3, the maximal interval of existence $T^* < \infty$ if and only if

$$\lim_{t\to T^*} \left\|u(\cdot, t)\right\|_\infty = \infty.$$

Let's suppose $T^* < \infty$ and $t \in (0, T^*)$, integrating the inequality 3.1 we get

$$\left\|u(t)\right\|_\infty \leqslant \left\|u_0\right\|_\infty + \int_0^t \left\|Pf(s)\right\|_\infty ds$$

$f \in C^\infty(\mathbb{R}^{n+1})$ thus $Pf \in C^\infty(\mathbb{R}^{n+1})$ so

$$\lim_{t\to T^*} \int_0^t \left\|Pf(s)\right\|_\infty ds \leqslant T^* \max_{s\in[0,T^*]} \left\|Pf(s)\right\|_\infty < \infty$$

from which

$$\lim_{t \to T^*} \|u(t)\|_\infty < \infty$$

So, for the theorem 2.3 it must be $T^* = \infty$ □

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