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► **To cite this version:**

Ulisse Iotti. Global in time solution to the incompressible Euler Equations on  $R^n$ . An elementary approach.. 2011. hal-00608010v4

**HAL Id: hal-00608010**

**<https://hal.science/hal-00608010v4>**

Preprint submitted on 16 Jul 2011 (v4), last revised 21 Jul 2011 (v5)

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**GLOBAL IN TIME SOLUTION TO THE INCOMPRESSIBLE  
EULER EQUATIONS ON  $\mathbb{R}^n$ . AN ELEMENTARY APPROACH.**

ULISSE IOTTI

ABSTRACT. In this paper we prove a theorem of global time-extension for the local classical solution of Euler's evolution problem in  $\mathbb{R}^n$  with  $n \geq 2$  for incompressible fluids subjected to external forces and regular initial conditions. This will be achieved by expressing the boundedness of the time derivative of the  $L^\infty$  solution norm.

1. INTRODUCTION

Let's consider the problem of evolution for the Euler equations in  $\mathbb{R}^n$

$$(1.1) \quad \begin{cases} \partial_t u = -u \cdot \nabla u - \nabla p + f & (x \in \mathbb{R}^n, t \geq 0) \\ \nabla \cdot u = 0 & (x \in \mathbb{R}^n, t \geq 0) \\ u(x, 0) = u_0(x) & (x \in \mathbb{R}^n) \end{cases}$$

where:

$$u(x, t), f(x, t) \in \mathbb{R}^n, \quad p(x, t) \in \mathbb{R} \quad u_0 \in C^\infty(\mathbb{R}^n), \quad \nabla \cdot u_0 = 0$$

$$(1.2) \quad |\partial_x^\alpha u_0(x)| \leq C_{\alpha K} (1 + |x|)^{-K}, \quad x \in \mathbb{R}^n, \quad \forall \alpha, K$$

$$(1.3) \quad |\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha m K} (1 + |x| + t)^{-K}, \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \forall \alpha, m, K$$

The Euler equations describe the time evolution of the speed vector field  $u(x, t)$  and the pressure  $p(x, t)$  of an incompressible inviscid fluid, depending on the initial velocity  $u_0(x)$  and the external force  $f(x, t)$ .

Equation 1.1 represent the Eulerian description of the flow. The Lagrangian formulation of Euler equation 1.1 describes the flow in term of a volume preserving diffeomorphism, the time dependent map  $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ :

$$\alpha \mapsto X(\alpha, t), \quad X(\alpha, 0) = \alpha$$

These maps represent marked fluid particle trajectories,  $\alpha$  in label of the particle,

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*Date:* 6/07/2011.

*2000 Mathematics Subject Classification.* Primary: 35Q31; Secondary 76D03.

*Key words and phrases.* Euler equations, Navier-Stokes equations, global existence, regularity, nonlinear PDE, flow, fluid dynamics.

which can be seen as the location of the particle at time  $t = 0$ . The fact that the particle travel with velocity  $u$  is expressed in the system of ordinary differential equations,

$$(1.4) \quad \frac{\partial}{\partial t} X(\alpha, t) = u(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha$$

The Lagrangian formulation of Euler equations and incompressibility condition given by 1.1 are respectively the Newton's second law and  $\det(\nabla_\alpha X) = 1$ , where  $\det(\nabla_\alpha X)$  denotes the determinant of  $\nabla_\alpha X$  the jacobian matrix of  $X$ . [AJM02]

The map  $X$  defined by Equation 1.4 is well a volume preserving  $C^1$ -diffeomorphism from  $\mathbb{R}^n$  on itself, indeed, since  $u \in C^1(\mathbb{R}^n \times [0, T])$ , then thanks to Cauchy-Lipschitz Theorem, we deduce that the differential equation 1.4 admits a unique solution  $X \in C^1(\mathbb{R}^n \times [0, T])$ .

We introduce  $P$  the Leray's projection operator with components,

$$P = I - \nabla \Delta^{-1} \nabla.$$

The operator  $P$  is a projection:  $P^2 = P$ , annihilates gradients and map into divergence-free vectors; it is a bounded operator on  $L^p$ ,  $1 < p < \infty$  and commute with translation.  $P$  yields to Helmholtz decomposition, indeed for al  $v \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$  has a unique orthogonal decomposition:

$$\begin{aligned} v &= Pv + \nabla q \\ \nabla \cdot (Pv) &= 0, \quad q = \Delta^{-1} \nabla \cdot v \\ Pv, \nabla q &\in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n). \end{aligned}$$

## 2. LOCAL IN TIME EXISTENCE THEOREMS

In this section we report without proof some results on classical mild solutions, obtained for the first time by Kato and Fujita in the Sobolev spaces  $H^s(\mathbb{R}^n)$  for  $s \geq n/2 - 1$  [HF64], Theorems based on Picard's contraction principle.

For a systematic treatment of the theorems of existence with demonstrations, for example, see [LR02] [AJM02] (proofs are given only for the homogeneous case but they are easily extendable in case of external force  $f(x, t) \in \mathcal{S}(\mathbb{R}^{n+1})$ ).

**Definition 2.1.** (Divergence-free Sobolev spaces)

Let's define

$$V^s = \{v \in H^s(\mathbb{R}^n) : \nabla \cdot v = 0\} = PH^s(\mathbb{R}^n)$$

**Theorem 2.2** (Existence of local solutions to Euler's problem). *Let  $s > n/2 + 2$ . For all  $u_0 \in V^s(\mathbb{R}^n)$ , there exist  $T^* > 0$  (possibly infinite) and a unique solution*

$$u \in C([0, T^*), V^s(\mathbb{R}^n)) \cap C^1([0, T^*), V^{s-2}(\mathbb{R}^n))$$

and

$$u \in C([0, T^*), C^2(\mathbb{R}^n)) \cap C^1([0, T^*), C(\mathbb{R}^n))$$

for Euler's equation on  $\mathbb{R}^n \times (0, T^*)$  so that  $u(\cdot, 0) = u_0$ .

**Theorem 2.3.** *Solution  $u$  described in Theorem 2.2 is  $u(\cdot, t) \in C(\mathbb{R}^n)$  for  $t \in (0, T^*)$ , furthermore, for the maximal time of existence is  $T^* < \infty$  if and only if*

$$\lim_{t \rightarrow T^*} \|u(\cdot, t)\|_\infty = \infty.$$

**Proposition 2.4.** *From local solution  $u(x, t)$  we gain the pressure term resolving Poisson's problem, which is in turn obtained by taking the divergence from the first Euler equation 1.1.*

$$\begin{cases} \Delta p = -\nabla \cdot [(u \cdot \nabla)u] + \nabla \cdot f = -\sum_{j,k} \partial_j u_k \partial_k u_j + \nabla \cdot f & (x \in \mathbb{R}^n, t \geq 0) \\ \lim_{|x| \rightarrow 0} p \end{cases}$$

### 3. CONTROL OF THE $L^\infty$ NORM

**Theorem 3.1** (Derivative of the  $L^\infty$  norm). *Let  $u$  be a solution in  $[0, T^*)$  of the problem 1.1 described in the theorem 2.2, be  $t \in (0, T^*)$ , then*

$$(3.1) \quad \frac{d \|u(t)\|_\infty}{dt} \leq \|Pf(t)\|_\infty$$

*Proof.* In the problem 1.1 the term  $-\nabla p$  has the task to cancel all non-zero divergence components which come from the nonlinear term and the one due to external force. By applying Helmholtz's decomposition to  $u \cdot \nabla u$  and to  $f$  we'll get

$$u \cdot \nabla u = P(u \cdot \nabla u) + \nabla q_1$$

$$f = P(f) + \nabla q_2$$

and it is

$$-\nabla q_1 + \nabla q_2 = \nabla p$$

We note that where

$$\nabla \cdot (u \cdot \nabla u) = 0 \quad \Rightarrow \quad u \cdot \nabla u = P(u \cdot \nabla u), \quad \nabla q_1 = 0, \quad \nabla p = \nabla q_2$$

particularly, this will be true where  $\nabla u = 0$ , infact

$$\nabla u = 0 \quad \Rightarrow \quad 0 = \sum_{j,k} \partial_j u_k \partial_k u_j = \nabla \cdot (u \cdot \nabla u)$$

Let now be  $t \in (0, T^*)$ ,  $u$  is a function differentiable in all variables, be  $\tilde{x}$  an absolute maximum point for  $|u(\cdot, t)|$ , then we'll have

$$\nabla u(\tilde{x}, t) = 0$$

To estimate the derivative of  $u$  in  $(\tilde{x}, t)$  with respect to time, we are going to substitute in the equation 1.1

$$\frac{\partial u(\tilde{x}, t)}{\partial t} = -u(\tilde{x}, t) \cdot \nabla u(\tilde{x}, t) - \nabla p(\tilde{x}, t) + Pf(\tilde{x}, t) + \nabla q_2(\tilde{x}, t) = Pf(\tilde{x}, t)$$

$$\frac{\partial u(\tilde{x}, t)}{\partial t} = Pf(\tilde{x}, t)$$

from which follows

$$\frac{\partial |u(\tilde{x}, t)|}{\partial t} \leq |Pf(\tilde{x}, t)|$$

By Theorem 2.2,  $u \in C^1([0, T^*), C(\mathbb{R}^n))$ , then the time derivative of  $\|u(t)\|_\infty$  exist, thus :

$$\frac{d\|u(t)\|_\infty}{dt} = \lim_{h \rightarrow 0} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} = \lim_{h \rightarrow 0^+} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h}$$

where  $|u(\tilde{x}_h, t+h)|$  is the absolute maximum at time  $(t+h)$  reached at  $\tilde{x}_h \in \mathbb{R}^n$ . Evaluating the difference quotient we can write

$$\begin{aligned} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} &= \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)| + |u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} = \\ &= \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}_h^*, t)|}{h} + \frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} = \\ &= L_1 + L_2 \end{aligned}$$

where  $\tilde{x}_h^*$  is such that

$$\frac{\partial}{\partial s} X(\tilde{x}_h^*, s) = u(X(\tilde{x}_h^*, s), s + t), \quad X(\tilde{x}_h^*, h) = \tilde{x}_h, \quad X(\tilde{x}_h^*, 0) = \tilde{x}_h^*$$

Considering  $L_1$ : we can write

$$\begin{aligned} & \left| |u(\tilde{x}_h, t + h)| - |u(\tilde{x}_h^*, t)| \right| \leq \\ & \leq \left| \nabla |u(\tilde{x}_h, t + h)| \frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t + h) \right| |h| + \left| \frac{\partial |u(\tilde{x}_h, t + h)|}{\partial t} \right| |h| + o(|h|) \end{aligned}$$

Being  $\tilde{x}_h$  point of maximum at time  $(t + h)$ , for what we have seen so far, it is

$$\nabla |u(\tilde{x}_h, t + h)| = 0, \quad \frac{\partial |u(\tilde{x}_h, t + h)|}{\partial t} \leq |Pf(\tilde{x}_h, t + h)|$$

furthermore, we have that

$$\frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t + h) \equiv \frac{\partial}{\partial s} X(\tilde{x}_h^*, h) = u(X(\tilde{x}_h^*, h), t + h) = u(\tilde{x}_h, t + h)$$

being the derivative along the flow line passing by  $\tilde{x}_h$  at time  $(t + h)$ .  
 $u \in C(\mathbb{R}^n \times [0, T^*))$  and for  $(t + h) < T^*$  it is

$$|u(\tilde{x}_h, t + h)| < \infty$$

from which

$$\nabla |u(\tilde{x}_h, t + h)| \frac{d\tilde{x}_h}{dt}(\tilde{x}_h, t + h) = 0$$

thus

$$\left| |u(\tilde{x}_h, t + h)| - |u(\tilde{x}_h^*, t)| \right| \leq |Pf(\tilde{x}_h, t + h)| |h| + o(|h|)$$

Shifting to limits

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{\left| |u(\tilde{x}_h, t + h)| - |u(\tilde{x}_h^*, t)| \right|}{|h|} \leq \\ & \leq \lim_{h \rightarrow 0} |Pf(\tilde{x}_h, t + h)| + \lim_{h \rightarrow 0} \frac{o(|h|)}{h} = |Pf(\tilde{x}_0^*, t)| \leq \|Pf(t)\|_\infty \end{aligned}$$

where

$$\tilde{x}_0^* = \lim_{h \rightarrow 0} X(\tilde{x}_h^*, 0)$$

with

$$|u(\tilde{x}_0^*, t)| = |u(\tilde{x}_0, t)|$$

Let's consider now  $L_2$ : being  $|u(\tilde{x}, t)|$  absolute maximum at time  $t$  we'll have

$$\frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} \leq 0 \quad h > 0$$

shifting to limit notation

$$\lim_{h \rightarrow 0^+} \frac{|u(\tilde{x}_h^*, t)| - |u(\tilde{x}, t)|}{h} = \lim_{h \rightarrow 0^+} L_2 \leq 0$$

thus

$$\begin{aligned} \frac{d \|u(t)\|_\infty}{dt} &= \lim_{h \rightarrow 0^+} \frac{|u(\tilde{x}_h, t+h)| - |u(\tilde{x}, t)|}{h} \leq \\ &\leq \lim_{h \rightarrow 0^+} |L_1| + \lim_{h \rightarrow 0^+} L_2 \leq \lim_{h \rightarrow 0^+} |L_1| \leq \|Pf(t)\|_\infty \end{aligned}$$

□

**Theorem 3.2** (Extension theorem). *In the hypotheses of the Theorem 3.1, the solution  $u$  can be extended to the whole real axis of time.*

*Proof.* According to Theorem 2.3, the maximal interval of existence  $T^* < \infty$  if and only if

$$\lim_{t \rightarrow T^*} \|u(\cdot, t)\|_\infty = \infty.$$

Let's suppose  $T^* < \infty$  and  $t \in (0, T^*)$ , integrating the inequality 3.1 we get

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t \|Pf(s)\|_\infty ds$$

$f \in C^\infty(\mathbb{R}^{n+1})$  thus  $Pf \in C^\infty(\mathbb{R}^{n+1})$  so

$$\lim_{t \rightarrow T^*} \int_0^t \|Pf(s)\|_\infty ds \leq T^* \max_{s \in [0, T^*]} \|Pf(s)\|_\infty < \infty$$

from which

$$\lim_{t \rightarrow T^*} \|u(t)\|_\infty < \infty$$

So, for the theorem 2.3 it must be  $T^* = \infty$

□

#### REFERENCES

- [Ada75] Robert A. Adams. *Sobolev Spaces*. Accademc Press, 1975.
- [AJM02] Andrea L. Bertozzi Andrew J. Majda. *Vorticity and Incompressible Flow*. Cambridge University Press, 2002.
- [Bat00] G.K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge Mathematical Librery, 2000.
- [Can97] M. Cannone. A generalization of a theorem by kato on Navier-Stokes equations. *Rev. Mat. Iberoamericana*, (13):515–541, 1997.
- [CF01] R. Rosa R. Temam C. Foias, O. Manley. *Navier-Stokes Equations and Turbulence*. Cambridge University Press, 2001.
- [Cha05] D. Chae. Remarks on the blow-up criterion of the three-dimensional euler equations. *Nonlinearity*, 18(3), 2005.
- [Con86] P. Constantin. Note on loss of regularity for solutions of the 3d incompressible Euler and related equations. *Commun. Math. Phys.*, 104:311–326, 1986.
- [Con94] P. Constantin. Geometric statistic in turbulence. *SIAM*, 36(73), 1994.
- [Con07] P. Constantin. On the Euler equations of incompressible fluids. *Bulletin of American Mathematical Society*, 44:603–621, 2007.
- [DC99] Hi-Jun Choe Dongho Chae. Regularity of solution to the Navier-Stokes equations. *Electronic Journal of Differential Equations*, (5):1–7, 1999.
- [dV95] H. Beirao da Veiga. A new regularity class for the Navier-Stokes equations. *Chinese Ann. Math. Ser. B*, 16(4):407–412, 1995.
- [dV97] H. Beirao da Veiga. Remarks on the smoothness of the  $L^\infty(0, T; T^3)$ -solution of the 3-d Navier-Stokes equations. *Portugaliae Mathematica*, (54):381–391, 1997.
- [dV04] H. Beirao da Veiga. Direction of vorticity and regularity for solution to the Navier-Stokes under the navier boundary condition. Preprint, 2004.
- [FD00] G. Demengel F. Demengel. *Mesures de Radon et distributions. Théorie et illustrations par les exemples*. Universités Mathématiques. ellipses, 2000.
- [GPG04] Rolf Rannacher Giovanni P. Galdi, Jhon G. Heywood, editor. *Contributions to Current Challenges in Mathematical Fluid Mechanics*. Birkhäuser Verlag, 2004.
- [HBdV87] P. Secchi H. Beirao da Veiga.  $L^p$ -stability for the strong solution of the Navier-Stokes equations in the whole space. *Arch. Rat. Mech. Anal.*, (98):65–70, 1987.
- [HBdV02] Luigi Berselli H. Beirao da Veiga. On regularizing effect of the vorticity direction in incompressible viscous flows. *Differential and Integral Equation*, (3):345–356, 2002.
- [HF62] T. Kato H. Fujita. On the non stationary navier-stokes system. *Rend. Sem. Univ. Padova.*, (32):243–260, 1962.
- [HF64] T. Kato H. Fujita. On the Navier-Stokes initial value problem i. *Arch. Rat. Mech. Anal.*, (16):269–315, 1964.



- [HK94] T. Ogawa H. Kozono. Global strong solutions and its decay properties for the Navier-Stokes equations in three dimensional domains with non compact boundaries. *Math. Z.*, (216):1–30, 1994.
- [HK99] H. Sohr H. Kozono. Global strong solution of the Navier-Stokes equations in 4 and 5 dimensional unbounded domanins. *Annales de l’I. H. P. section C*, tome 16(5):535–561, 1999.
- [HK01] D. Tataru H. Koch. Well-posedness for the Navier-Stokes equations. *Advances in Mathematics*, (157):22–35, 2001.
- [HOK89] Jeans Lorenz Heinz-Otto Kreiss. *Initial-Boundary Value Problems and the Navier-Stokes Equations*. Academic Press, 1989.
- [Hop51] E. Hopf. Über die anfangswertaufgabe für die hydrodynamischen grundgleichungen. *Math. Nachr.*, (4):213–231, 1951.
- [JD05] X. Yu J. Deng, T. Y. Hou. Geometric proprerties and nonblowup of 3d incompressible Euler flow. *Commun. PDEs*, 30(225-243), 2005.
- [JD06] X. Yu J. Deng, T. Y. Hou. Improved geometric conditions for non-blowup of 3d incompressible Euler equation. *Commun. PDEs*, 31:293–306, 2006.
- [Kat84] T. Kato. Strong  $L^p$ -solution of the Navier-Stokes equations in  $\mathbb{R}^m$ , with application to weak solution. *Math. Z.*, (187):471–480, 1984.
- [Ler34] J. Leray. Sur le mouvements d’un liquide visqueux emplissant l’espace. *Acta. Math.*, (63):193–248, 1934.
- [Lie96] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific, 1996.
- [LR02] P G Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. CHAPMAN & ALL/CRC, 2002.
- [MC04a] G. Karch M. Cannone. Smooth or singula solution to the Navier-Stokes equations? Preprint, 2004.
- [MC04b] M. Schonbek M. Canone, F. Planchon. Strong solution to the incompressible Navier-Stokes equations in the half-space. Preprint, 2004.
- [MC05] Grzegorz Karch Marco Cannone. On the validity of Picard algorithm for nonlinear parabolic equations. *Proceedings of the Royal Society of Edinburgh 135A*, 135A, 2005.
- [MS04] Stephen Montgomery-Smith. Condition implying regularity of the three dimensional Navier-Stokes equations. *Applications of Mathematics*, (5):451–464, 2004.
- [PC93a] C. Fefferman P. Constantin. Direction of vorticity and the problem of global regularity for the 3-d Navier-Stokes equations. *Indiana Univ. Math.*, 42:775–789, 1993.
- [PC93b] Charles Fefferman Peter Constantin. Direction of vorticity and the problem of global regularity for the NavierStokes equations. *Indiana Univ. Math. J.*, 43(3):775–789, 1993.
- [PC96] A. J. Majda P. Constantin, C. Fefferman. Geometric constraints on potentially singular solutions for the 3-d Euler equations. *Commun. PDEs*, 21:559–571, 1996.
- [Pro59] G. Prodi. Un teorema di unicITÀ per le equazioni di navier-stokes. *Annali di matematica pura ed applicata. Springer Berlin / Heidelberg*, 48(1):173–182, 1959.
- [RJD87] A. J. Majda R. J. DiPerna. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.*, 108:667–689, 1987.
- [Rud95] Walter Rudin. *Analyse fonctionnelle*. Ediscience International, 1995.
- [SA01] Wade Ramey Sheldon Axler, Paul Bourdon. *Harmonic Function Theory*. Springer, 2001.
- [Ser62] J. Serrin. On the interior regularity of weak solutions of Navier-Stokes equations. *Arch. Rat. Mech. Anal.*, (9):187–196, 1962.

- [Ser63] J. Serrin. The initial value problem for the Navier-Stokes equations. *Univ. Wisconsin Press, Nonlinear problems, Ed. R. Langer*, 1963.
- [Soh01] Hermann Sohr. *Navier-Stokes Equations. An Elementary Functional Analytic Approach*. Birkhäuser Verlag, 2001.
- [Ste70] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1970.
- [Tay97a] Michael E. Taylor. *Partial Differential Equations I. Basic Theory*. Springer, 1997.
- [Tay97b] Michael E. Taylor. *Partial Differential Equations II. Qualitative studies of Linear Equations*. Springer, 1997.
- [Tay97c] Michael E. Taylor. *Partial Differential Equations III. Nonlinear Equations*. Springer, 1997.
- [TK86] G. Ponce T. Kato. Commutator estimates and the Euler and Navier-Stokes equations in the lebesgue spaces  $L^p_s(\mathbb{R}^2)$ . *Revista Math. Iberoamericana*, 2:73–88, 1986.
- [Vla81] Vasilij Sergeevic Vladimirov. *Le distribuzioni nella fisica matematica*. Edizioni Mir, 1981.
- [WB87] H. Sohr W. Borchers. On the semigroup of the stokes operator for exsterior domains in  $L^q$ -spaces. *Math. Z.*, (196):415–425, 1987.
- [Zie89] William P. Ziemer. *Weakly Differentiable Functions*. Springer-Verlag, 1989.

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