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► To cite this version:

Youri Davydov, Clément Dombry. On the convergence of Le Page series in Skohorod space. 2011. hal-00607964

HAL Id: hal-00607964

<https://hal.science/hal-00607964>

Preprint submitted on 11 Jul 2011

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On the convergence of Le Page series in Skohorod space

Youri Davydov^{*} and Clément Dombry[†]

Abstract

We consider the problem of the convergence of the so-called Le Page series in the Skohorod space $\mathbb{D}^d = \mathbb{D}([0, 1], \mathbb{R}^d)$ and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an α -stable distribution on \mathbb{D}^d with given spectral measure.

Key words: stable distribution, Le Page series, Skohorod space.

AMS Subject classification. Primary: 60E07 **Secondary:** 60G52.

1 Introduction

We are interested in the convergence in the Skohorod space $\mathbb{D}^d = \mathbb{D}([0, 1], \mathbb{R}^d)$ endowed with the J_1 -topology of random series of the form

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1], \quad (1)$$

where $\alpha \in (0, 2)$ and

- $(\Gamma_i)_{i \geq 1}$ is the increasing enumeration of the points of a Poisson point process on $[0, +\infty)$ with Lebesgue intensity;
- $(\varepsilon_i)_{i \geq 1}$ is an i.i.d. sequence of real random variables;

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- $(Y_i)_{i \geq 1}$ is an i.i.d. sequence of \mathbb{D}^d -valued random variables;
- the sequences (Γ_i) , (ε_i) and (Y_i) are independent.

Note that a more constructive definition for the sequence $(\Gamma_i)_{i \geq 1}$ is given by

$$\Gamma_i = \sum_{j=1}^i \gamma_j, \quad i \geq 1,$$

where $(\gamma_i)_{i \geq 1}$ is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of (ε_i) and (Y_i) .

Series of the form (1) are known as Le Page series. For fixed $t \in [0, 1]$, the convergence in \mathbb{R}^d of the series (1) is ensured as soon as one of the following conditions is satisfied:

- $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$,
- $1 \leq \alpha < 2$, $\mathbb{E}\varepsilon_1 = 0$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$.

Here $|\cdot|$ denotes the usual Euclidean norm on \mathbb{R} or on \mathbb{R}^d . The random variable $X(t)$ has then an α -stable distribution on \mathbb{R}^d . Conversely, it is well known that any α -stable distributions on \mathbb{R}^d admits a representation in terms of Le Page series (see for example Samorodnitsky and Taqqu [9] section 3.9).

There is a vast literature on symmetric α -stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand [7] or Araujo and Giné [1]). In particular, any symmetric α -stable distribution on a separable Banach space can be represented as an almost surely convergent Le Page series (see Corollary 5.5 in [7]). The existence of a symmetric α -stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in [7] for a precise statement. In [3], Davydov, Molchanov and Zuyev consider α -stable distributions in the more general framework of abstract convex cones.

The space \mathbb{D}^d equipped with the norm

$$\|x\| = \sup\{|x_i(t)|, \quad t \in [0, 1], \quad i = 1, \dots, d\}, \quad x = (x_1, \dots, x_d) \in \mathbb{D}^d,$$

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the J_1 -topology. On the other hand, the space \mathbb{D}^d with the J_1 -topology is Polish, i.e. there exists a metric on \mathbb{D}^d compatible with the J_1 -topology that makes \mathbb{D}^d a complete and separable metric space. However, such a metric can not be compatible with the vector space structure since the addition is not continuous in the J_1 -topology. These properties explains why the general theory of stable distributions on separable Banach space can not be applied to the space \mathbb{D}^d .

Nevertheless, in the case when the series (1) converges, the distribution of the sum X defines an α -stable distribution on \mathbb{D}^d . We can determine the associated spectral measure σ on the unit sphere $\mathbb{S}^d = \{x \in \mathbb{D}^d; \|x\| = 1\}$. It is given by

$$\sigma(A) = \frac{\mathbb{E}\left(|\varepsilon_1|^\alpha \|Y_1\|^\alpha \mathbf{1}_{\{\text{sign}(\varepsilon_1)Y_1/\|Y_1\| \in A\}}\right)}{\mathbb{E}(|\varepsilon_1|^\alpha \|Y_1\|^\alpha)}, \quad A \in \mathcal{B}(\mathbb{S}^d).$$

This is closely related to regular variations theory (see Hult and Lindskog [5] or Davis and Mikosch [2]): for all $r > 0$ and $A \in \mathcal{B}(\mathbb{S}^d)$ such that $\sigma(\partial A) = 0$, it holds that

$$\lim_{n \rightarrow \infty} n \mathbb{P}\left(\frac{X}{\|X\|} \in A \mid \|X\| > r b_n\right) = r^{-\alpha} \sigma(A),$$

with

$$b_n = \inf\{r > 0; \mathbb{P}(\|X\| < r) \leq n^{-1}\}, \quad n \geq 1.$$

The random variable X is said to be regularly varying in \mathbb{D}^d with index α and spectral measure σ .

In this framework, convergence of the Le Page series (1) in \mathbb{D}^d is known in some particular cases only:

- When $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}\|Y_1\|^\alpha < \infty$, the series (1) converges almost surely uniformly in $[0, 1]$ (see example 4.2 in Davis and Mikosch [2]);
- When $1 \leq \alpha < 2$, the distribution of the ε_i 's is symmetric, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $Y_i(t) = \mathbf{1}_{[0,t]}(U)$ with $(U_i)_{i \geq 1}$ an i.i.d. sequence of random variables with uniform distribution on $[0, 1]$, the series (1) converges almost surely uniformly on $[0, 1]$ and the limit process X is a symmetric α -stable Lévy process (see Rosinski [8]).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in \mathbb{D}^d of the random series (1). Our main result is the following:

Theorem 1 *Suppose that $1 \leq \alpha < 2$,*

$$\mathbb{E}\varepsilon_1 = 0 \quad , \quad \mathbb{E}|\varepsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}\|Y_1\|^\alpha < \infty.$$

Suppose furthermore that there exist $\beta_1, \beta_2 > \frac{1}{2}$ and F_1, F_2 nondecreasing continuous functions on $[0, 1]$ such that, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}|Y_1(t_2) - Y_1(t_1)|^2 \leq |F_1(t_2) - F_1(t_1)|^{\beta_1}, \quad (2)$$

$$\mathbb{E}|Y_1(t_2) - Y_1(t)|^2 |Y_1(t) - Y_1(t_1)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \quad (3)$$

Then, the Le Page series (1) converges almost surely in \mathbb{D}^d .

The proof of this Theorem is detailed in the next section. We provide hereafter a few cases where Theorem 1 can be applied.

Example 1 The example considered by Davis and Mikosch [2] follows easily from Theorem 1: let U be a random variable with uniform distribution on $[0, 1]$ and consider $Y_1(t) = \mathbf{1}_{[0,t]}(U)$, $t \in [0, 1]$. Then, for $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = t_2 - t_1 \quad \text{and} \quad \mathbb{E}(Y_1(t_2) - Y_1(t))^2 (Y_1(t) - Y_1(t_1))^2 = 0,$$

so that conditions (2) and (3) are satisfied.

Example 2 Example 1 can be generalized in the following way: let $p \geq 1$, $(U_i)_{1 \leq i \leq p}$ independent random variables on $[0, 1]$ and $(R_i)_{1 \leq i \leq p}$ random variables on \mathbb{R}^d . Consider

$$Y_1(t) = \sum_{i=1}^p R_i \mathbf{1}_{[0,t]}(U_i).$$

Assume that for each $i \in \{1, \dots, p\}$, the cumulative distribution function F_i of U_i is continuous on $[0, 1]$. Assume furthermore that there is some $M > 0$ such that for all $i \in \{1, \dots, p\}$

$$\mathbb{E}[R_i^4 \mid \mathcal{F}_U] \leq M \quad \text{almost surely,} \quad (4)$$

where $\mathcal{F}_U = \sigma(U_1, \dots, U_p)$. This is for example the case when the R_i 's are uniformly bounded by $M^{1/4}$ or when the R_i 's have finite fourth moment and are independent of the U_i 's. Simple computations entails that under condition (4), it holds for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 \leq M^{1/2} p^2 |F(t_2) - F(t_1)|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2 (Y_1(t) - Y_1(t_1))^2 \leq M p^4 |F(t_2) - F(t_1)|^4.$$

with $F(t) = \sum_{i=1}^p F_i(t)$. So conditions (2) and (3) are satisfied and Theorem (1) can be applied in this case.

Example 3 A further natural example is the case when $Y_1(t)$ is a Poisson process with intensity $\lambda > 0$ on $[0, 1]$. Then, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = \lambda |t_2 - t_1| + \lambda^2 |t_2 - t_1|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2 (Y_1(t) - Y_1(t_1))^2 = (\lambda |t_2 - t| + \lambda^2 |t_2 - t|^2)(\lambda |t - t_1| + \lambda^2 |t - t_1|^2)$$

and we easily see that conditions (2) and (3) are satisfied.

2 Proof

For the sake of clarity, we divide the proof of Theorem 1 into five steps.

Step 1. According to Lemma 1.5.1 in [9], it holds almost surely that for k large enough

$$|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}| \leq 2\alpha^{-1} k^{-1/\alpha} \sqrt{\frac{\ln \ln k}{k}}. \quad (5)$$

This implies the a.s. convergence of the series

$$\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}| |\varepsilon_i| \|Y_i\| < \infty. \quad (6)$$

The series (6) has indeed nonnegative terms, and (5) implies that the following conditionnal expectation is finite,

$$\mathbb{E} \left[\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}| |\varepsilon_i| \|Y_i\| \mid \mathcal{F}_\Gamma \right] = \mathbb{E} |\varepsilon_1| \mathbb{E} \|Y_1\| \sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|$$

where $\mathcal{F}_\Gamma = \sigma(\Gamma_i, i \geq 1)$.

This proves that (6) holds true and it is enough to prove the a.s. convergence in \mathbb{D}^d of the series

$$Z(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0, 1], \quad (7)$$

Step 2. Next, consider

$$\tilde{Z}(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \quad (8)$$

with

$$\tilde{\varepsilon}_i = \varepsilon_i \mathbf{1}_{\{|\varepsilon_i|^\alpha \leq i\}}, \quad i \geq 1.$$

We prove that the series (7) and (8) differ only by a finite number of terms. We have indeed

$$\sum_{i=1}^{\infty} \mathbb{P}(\tilde{\varepsilon}_i \neq \varepsilon_i) = \sum_{i=1}^{\infty} \mathbb{P}(|\varepsilon_i|^\alpha > i) \leq \mathbb{E} |\varepsilon_1|^\alpha < \infty$$

and the Borel-Cantelli Lemma implies that almost surely $\tilde{\varepsilon}_i = \varepsilon_i$ for i large enough. So, both series (7) and (8) have the same nature and it is enough to prove the convergence in \mathbb{D}^d of the series (8).

Step 3. As a preliminary for step 4, we prove several estimates involving the moments of the random variables $(\tilde{\varepsilon}_i)_{i \geq 1}$. First, for all $m > \alpha$,

$$C(\alpha, m) := \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(|\tilde{\varepsilon}_i|^m) < \infty. \quad (9)$$

We have indeed

$$\begin{aligned} C(\alpha, m) &= \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(|\varepsilon_i|^m \mathbf{1}_{\{|\varepsilon_i| \leq i^{1/\alpha}\}}) \\ &= \mathbb{E} \left(|\varepsilon_1|^m \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbf{1}_{\{i \geq |\varepsilon_1|^\alpha\}} \right) \\ &\leq C \mathbb{E}(|\varepsilon_1|^m |\varepsilon_1|^{\alpha-m}) = C \mathbb{E}(|\varepsilon_1|^\alpha) < \infty \end{aligned}$$

where the constant $C = \sup_{x>0} x^{m/\alpha-1} \sum_{i \geq x} i^{-m/\alpha}$ is finite since for $m > \alpha$

$$\lim_{x \rightarrow \infty} x^{m/\alpha-1} \sum_{i \geq x} i^{-m/\alpha} = \frac{\alpha}{m - \alpha}.$$

Similarly, we also have

$$C(\alpha, 1) := \sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| < \infty. \quad (10)$$

Indeed, the assumption $\mathbb{E}\varepsilon_i = 0$ implies $\mathbb{E}(\tilde{\varepsilon}_i) = \mathbb{E}(\varepsilon_i \mathbf{1}_{\{|\varepsilon_i|^\alpha > i\}})$. Hence,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| &\leq \sum_{i=1}^{\infty} i^{-1/\alpha} \mathbb{E}(|\varepsilon_1| \mathbf{1}_{\{|\varepsilon_1|^\alpha > i\}}) \\ &= \mathbb{E}\left(|\varepsilon_1| \sum_{i=1}^{[\lceil |\varepsilon_1|^\alpha \rceil]} i^{-1/\alpha}\right) \\ &\leq \mathbb{E}\left(|\varepsilon_1| C' (|\varepsilon_1|^\alpha)^{1-1/\alpha}\right) = C' \mathbb{E}|\varepsilon_1|^\alpha < \infty \end{aligned}$$

where the constant $C' = \sup_{x>0} x^{1/\alpha-1} \sum_{i=1}^{[x]} i^{-1/\alpha}$ is finite.

Step 4. For $n \geq 1$, consider the partial sum

$$\tilde{Z}_n(t) = \sum_{i=1}^n i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \quad (11)$$

We prove that the sequence of processes $(\tilde{Z}_n)_{n \geq 1}$ is tight in \mathbb{D}^d . Following Theorem 3 in Gikhman and Skohorod [4] chapter 6 section 3, it is enough to show that there exists $\beta > 1/2$ and a non decreasing continuous function F on $[0, 1]$ such that

$$\mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t)|^2 |\tilde{Z}_n(t) - \tilde{Z}_n(t_1)|^2 \leq |F(t_2) - F(t_1)|^{2\beta}, \quad (12)$$

for all $0 \leq t_1 \leq t \leq t_2 \leq 1$. Remark that in Gikhman and Skohorod [4], the result is stated only for $F(t) \equiv t$. However, the case of a general continuous non decreasing function F follows easily from a simple change of variable.

We use the notations $Y(t) = (Y^p(t))_{1 \leq p \leq d}$, $\llbracket 1, n \rrbracket = \{1, \dots, n\}$ and

$\mathbf{i} = (i_1, i_2, i_3, i_4) \in \llbracket 1, n \rrbracket^4$. We have

$$\begin{aligned} & \mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t)|^2 |\tilde{Z}_n(t) - \tilde{Z}_n(t_1)|^2 \\ &= \mathbb{E} \left| \sum_{i=1}^n i^{-1/\alpha} \tilde{\varepsilon}_i(Y_i(t) - Y_i(t_1)) \right|^2 \left| \sum_{j=1}^n j^{-1/\alpha} \tilde{\varepsilon}_j(Y_j(t_2) - Y_j(t)) \right|^2 \\ &= \sum_{1 \leq p, q \leq d} \sum_{\mathbf{i} \in \llbracket 1, n \rrbracket^4} (i_1 i_2 i_3 i_4)^{-1/\alpha} \mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4}) \mathbb{E}[(Y_{i_1}^p(t) - Y_{i_1}^p(t_1)) \end{aligned} \quad (13)$$

$$(Y_{i_2}^p(t) - Y_{i_2}^p(t_1))(Y_{i_3}^q(t_2) - Y_{i_3}^q(t))(Y_{i_4}^q(t_2) - Y_{i_4}^q(t))] \quad (14)$$

$$\leq d^2 \sum_{\mathbf{i} \in \llbracket 1, n \rrbracket^4} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4})| D_{\mathbf{i}}(t, t_1, t_2) \quad (15)$$

where

$$D_{\mathbf{i}}(t, t_1, t_2) = \mathbb{E}|Y_{i_1}(t) - Y_{i_1}(t_1)| |Y_{i_2}(t) - Y_{i_2}(t_1)| |Y_{i_3}(t_2) - Y_{i_3}(t)| |Y_{i_4}(t_2) - Y_{i_4}(t)|.$$

Consider $\sim_{\mathbf{i}}$ the equivalence relation on $\{1, \dots, 4\}$ defined by

$$j \sim_{\mathbf{i}} j' \quad \text{if and only if} \quad i_j = i_{j'}.$$

Let \mathcal{P} be the set of all partitions of $\{1, \dots, 4\}$ and $\tau(\mathbf{i})$ be the partition of $\{1, 2, 3, 4\}$ given by the equivalence classes of $\sim_{\mathbf{i}}$. We introduce these definitions because, since the Y_i 's are i.i.d., the term $D_{\mathbf{i}}(t, t_1, t_2)$ depends on \mathbf{i} only through the associated partition $\tau(\mathbf{i})$. For example, if $\tau(\mathbf{i}) = \{1, 2, 3, 4\}$, i.e. if $i_1 = i_2 = i_3 = i_4$, then

$$D_{\mathbf{i}}(t, t_1, t_2) = \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)|^2.$$

Or if $\tau(\mathbf{i}) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, i.e. if the indices i_1, \dots, i_4 are pairwise distinct, then

$$D_{\mathbf{i}}(t, t_1, t_2) = (\mathbb{E}|Y_1(t) - Y_1(t_1)| \mathbb{E}|Y_1(t_2) - Y_1(t)|)^2.$$

For $\tau \in \mathcal{P}$, we denote by $D_{\tau}(t, t_1, t_2)$ the common value of the terms $D_{\mathbf{i}}(t, t_1, t_2)$ corresponding to indices \mathbf{i} such that $\tau(\mathbf{i}) = \tau$. Define also

$$S_{n, \tau} = \sum_{\mathbf{i} \in \{1, \dots, n\}^4; \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4})|.$$

With these notations, equation (15) can be rewritten as

$$\mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t)|^2 |\tilde{Z}_n(t) - \tilde{Z}_n(t_1)|^2 \leq d^2 \sum_{\tau \in \mathcal{P}} S_{n, \tau} D_{\tau}(t, t_1, t_2). \quad (16)$$

Under conditions (2) and (3), we will prove that for each $\tau \in \mathcal{P}$, there exist $\beta_{\tau} > 1/2$, a non decreasing continuous function F_{τ} on $[0, 1]$ and a constant $S_{\tau} > 0$ such that

$$D_{\tau}(t, t_1, t_2) \leq |F_{\tau}(t_1) - F_{\tau}(t_2)|^{2\beta_{\tau}}, \quad 0 \leq t_1 \leq t \leq t_2, \quad (17)$$

and

$$S_{n,\tau} \leq S_\tau, \quad n \geq 1. \quad (18)$$

Equations (16),(17) and (18) together imply inequality (12) for some suitable choices of $\beta > 1/2$ and F .

It remains to prove inequalities (17) and (18). If $\tau = \{1, 2, 3, 4\}$,

$$D_\tau(t, t_1, t_2) \leq \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}$$

and

$$S_n^\tau = \sum_{i=1}^n i^{-4/\alpha} \mathbb{E} \tilde{\varepsilon}_i^4 \leq C(\alpha, 4).$$

If $\tau = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, Cauchy-Schwartz inequality entails

$$D_\tau(t, t_1, t_2) \leq (\mathbb{E}|Y_1(t) - Y_1(t_1)| \mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \leq |F_1(t_2) - F_1(t_1)|^{2\beta_1}$$

and

$$S_n^\tau \leq \sum_{\mathbf{i} \in \{1, \dots, n\}^4; \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E} \tilde{\varepsilon}_{i_1}| |\mathbb{E} \tilde{\varepsilon}_{i_2}| |\mathbb{E} \tilde{\varepsilon}_{i_3}| |\mathbb{E} \tilde{\varepsilon}_{i_4}| \leq C(\alpha, 1)^4.$$

Similarly, for $\tau = \{1, 2, 3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)| \mathbb{E}|Y_1(t_2) - Y_1(t)| \\ &\leq |F_1(t) - F_1(t_1)|^{\beta_1/2} |F_2(t_2) - F_2(t_1)|^{\beta_2} |F_1(t_2) - F_1(t)|^{\beta_1/2} \\ &\leq |(F_1 + F_2)(t_2) - (F_1 + F_2)(t_1)|^{\beta_1 + \beta_2} \end{aligned}$$

and

$$S_n^\tau \leq \sum_{1 \leq i \neq j \leq n} (i^3 j)^{-1/\alpha} \mathbb{E} |\tilde{\varepsilon}_i|^3 |\mathbb{E} \tilde{\varepsilon}_j| \leq C(\alpha, 3) C(\alpha, 3).$$

or for $\tau = \{1, 2\} \cup \{3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 (\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \\ &\leq |F_1(t) - F_1(t_1)|^{\beta_1} |F_1(t_2) - F_1(t)|^{\beta_1} \\ &\leq |F_1(t_2) - F_1(t_1)|^{2\beta_1} \end{aligned}$$

and

$$S_n^\tau \leq \sum_{1 \leq i \neq j \neq k \leq n} (i^2 j k)^{-1/\alpha} \mathbb{E} |\tilde{\varepsilon}_i|^2 |\mathbb{E} \tilde{\varepsilon}_j| |\mathbb{E} \tilde{\varepsilon}_k| \leq C(\alpha, 2) C(\alpha, 1)^2.$$

Similar computations can be checked in all remaining cases. The cardinality of \mathcal{P} is equal to 13.

Step 5. We prove Theorem 1. For each fixed $t \in [0, 1]$, Kolmogorov's three-series Theorem implies that $\tilde{Z}_n(t)$ converge almost surely as $n \rightarrow \infty$. So the finite-dimensional distributions of $(\tilde{Z}_n)_{n \geq 1}$ converge. The tightness in \mathbb{D}^d of the sequence has already been proved in step 4, so $(\tilde{Z}_n)_{n \geq 1}$ weakly convergence in \mathbb{D}^d as $n \rightarrow \infty$. We then apply Theorem 1 in Kallenberg [6] and deduce that \tilde{Z}_n converges almost surely in \mathbb{D}^d . In view of step 1 and step 2, this yields the almost sure convergence of the series (1). \square

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